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Spaces becoming noncommutative

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Noncommutative geometry has developed over the last three decades building up a large collection of examples and tools which have deep connections with various areas of mathematics and physics. In many situations geometric information which is difficult to encode with traditional tools might be understood by enlarging the class of spaces in consideration in such a way as to allow the presence of "noncommutative coordinates". The noncommutative spaces thus obtained provide a very rich and fruitful framework which often sheds new light on relevant problems and structures.

Lessons from quantum mechanics

To understand what we mean by a noncommutative space it is good to look back at some of the fundamental ideas underlying the early development of quantum mechanics. In classical mechanics the possible states of a system are described by points in a space which encodes the different possible values for the position and the velocity of its particles. Physical observables are given by functions on this phase space with values in the field of real numbers $\mathbb{R}$. Taking into account the algebraic structure of $\mathbb{R}$ it is possible in particular to add and multiply physical observables. A system consisting of one particle can thus be described by a two-dimensional space whose geometry is determined by the constraints on the motion of this particle. Physical quantities, like the energy of the system, will be given by functions which can be locally expressed in terms of two coordinates q and p representing the position and momentum of the particle. The coordinates q and p of a point in the phase space of a system are itself observables and, multiplication in $\mathbb{R}$ being commutative, we have that

$$pq - qp \quad = \quad 0.$$

In developing the mathematical foundations for quantum mechanics Heisenberg was led to the formulation of a framework in which the position and momentum of a particle where represented not by real valued functions on a classical geometric space but rather by infinite matrices $\hat{q}$ and $\hat{p}$ which did no longer commute but satisfied the relation

$$\hat{p}\hat{q} - \hat{q}\hat{p} \quad = \quad \frac{h}{2\pi}\sqrt{-1}$$

where h is Planck's constant. Heisenberg also showed that this relation implied the uncertainty principle that nowadays bears his name.

The information encoded in the infinite dimensional matrices associated to observables in the quantum mechanical formalism correspond to the quantum modes of these observables which manifest themselves experimentally as spectra (one of the motivations of Heisenberg's work came from the problem of calculating the spectral lines of hydrogen).

Figure 1: Emission spectrum of Hydrogen.

This hints to the fact that a purely spectral framework for geometry, based on ideas analogous to those exploited by Heisenberg, might be possible. As a first remark about the relevance of such a framework we note the fact that wavelengths of helium-neon laser light are used nowadays to obtain the most accurate possible realization of the international unit of length (the meter).

The appearance of the factor $i = \sqrt{-1}$ in Heisenberg's commutation relation makes essential the passage from considering scalars in the field of real numbers $\mathbb{R}$ to scalars in the field of complex numbers $\mathbb{C}$. A deeper level of abstraction is needed to formalize the idea of infinite matrices. Finite dimensional matrices correspond to linear operators between finite dimensional vector spaces. When passing to vector spaces of infinite dimension new tools have to be developed in order to study the properties of the corresponding operators. The language furnished by functional analysis gives us the desired tools.

From spaces to algebras (aka noncommutative spaces)

The notion of a geometric space as we understand it today can be traced back to the works of Carl Friedrich Gauss and Bernhard Riemann. They laid the foundations that, together with contributions of many other great mathematicians, shaped modern geometry.

When we look at our intuitive picture of a geometric space, generally modeled on curves and surfaces on three dimensional euclidean space, it is possible to discern the different levels of abstraction that might be needed in order to understand its properties. It is important in particular to be able to extend notions common from multivariable calculus. Various interrelated areas of mathematics provide the necessary tools; extending concepts like integration, continuity, differentiability and distance to the required generality leads to structures that are the object of study of measure theory, topology, differential geometry and Riemannian geometry respectively.

This hierarchy of structures leads also to the analysis of particular classes of functions on the space being studied. The collection of all functions from a set X to the field of complex numbers $\mathbb{C}$ inherits a nice algebraic structure from that of $\mathbb{C}$. On the one hand we can add functions and multiply them by "scalars", that is elements in $\mathbb{C}$. $Func(X)$ is therefore a vector space over $\mathbb{C}$. Moreover, since the product of two functions is a function and the product operation thus obtained is bilinear with respect to the vector space structure in $Func(X)$ we obtain a structure known as an *algebra*, this is just the mathematical term for such an object. If the set X has some additional structure as those considered in geometry it is then natural to consider the subclass of functions singled out by this structure. It turns out that in some cases the set X together with this extra structure can be recovered form this algebra of functions. For instance, if the set X is a topological space, i.e. if we know which subsets of X are open, we can talk about the set of continuous functions $C(X) \subset Func(X)$. If the topological space X has nice properties[1] then the rich structure of the algebra $C(X)$ determines X together with its topology (this is a consequence of a deep result by the Russian mathematician Israel Gelfand).

Now, there are many geometric constructions that can be reformulated in terms not of points in the space X but in terms of some algebra of functions $A \subset Func(X)$. An advantage of this point of view comes from the fact that many of these constructions make no use of the commutativity of the product in the algebra A and thus still make sense when applied to a noncommutative algebra. That is, given a noncommutative algebra A over $\mathbb{C}$ (i.e. a complex vector space A together with a bilinear product $A \times A \to A$), possibly with some extra structure,

[1] In precise terms, if X is a compact and Hausdorff topological space.

it is possible to develop the tools of geometry with A playing the role of an algebra of functions on some "noncommutative space" X_A for which $A \subset Func(X_A)$. It is important to note that the noncommutative space X_A is defined only as a dual object of the algebra A since for any classical space its algebra of functions would be commutative.
Looking at an example might clarify the situation. Denote the circle by S^1, a two dimensional torus can be described as the cartesian product of two circles $\mathbb{T} = S^1 \times S^1$. If we denote the coordinates of a point $x \in \mathbb{T}$ by the corresponding two angles $x = (\alpha_1, \alpha_2) \in S^1 \times S^1$ then Fourier theory tells us that any continuous function f from $\mathbb{T}$ to $\mathbb{C}$ can be described as series

$$f(x) = \sum_{m,n} a_{m,n}(e^{i\alpha_1})^n(e^{i\alpha_1})^m.$$

If we write u for the function $e^{i\alpha_1}$ and v for the function $e^{i\alpha_2}$ we see that the algebra $A_0 = C(X)$ is generated by u and v. The functions u and v are unitary in the sense that their complex conjugates are their inverses. It can be shown moreover that these conditions, together with commutativity, characterize the algebra of continuous functions on the torus[2]:

$A_0 =$ algebra generated by two unitaries u, v such that $uv = vu$

This, being a purely algebraic definition, can be modified as follows in order to obtain a noncommutative algebra which will model for us a noncommutative torus. Choose a real parameter $0 < \theta < 1$ and define:

$A_\theta =$ algebra generated by two unitaries u, v such that $uv = e^{2\pi i\theta}vu$

By duality we can think of the algebra A_θ as defining for us a "noncommutative space" for which A_θ plays the role of an algebra of functions, this is the so called noncommutative torus $\mathbb{T}_\theta$. This important example arises naturally in different contexts and plays a role in areas that range from the study of the quantum hall effect to conjectural applications in the arithmetic of real quadratic fields.
The noncommutative torus $\mathbb{T}_\theta$ is a prototypical example of a space arising from considering a quotient of a classical space by an equivalence relation. In many situations in geometry we are led to consider certain points in a geometric space as being equivalent. Depending on the nature of this equivalence relation the traditional tools may fail to describe the situation in a satisfactory manner. In these cases noncommutative algebras arise naturally and the tools of noncommutative geometry become available. We can for instance look at the circle and impose the equivalence relation that identifies points which differ from each other by an integer multiple of a fixed angle β. If β is irrational the set of points equivalent to a given one is a dense subset of the circle and the space resulting from identifying equivalent points collapses to a "bad quotient". On the other hand we can encode this equivalence relation in the algebra A_θ for $\theta = \beta$ leading to the rich geometry of the noncommutative space $\mathbb{T}_\theta$.
Another example for which the tools of noncommutative geometry can be applied comes from considering aperiodic tilings of the plane. The problem can be reduced to the study of tilings of the plane by two basic tiles (the so called "kites" and "darts"). Aperiodic tilings of the plane by these two tiles are known as Penrose tilings (see Figure 2). The space of Penrose tillings is pathological when considered

[2] The precise statement specifying also the structure of the algebra thus generated, that of a C^*-algebra.

from a classical point of view since every finite patch of tiles in a tiling does occur, and infinitely many times, in any other tiling.

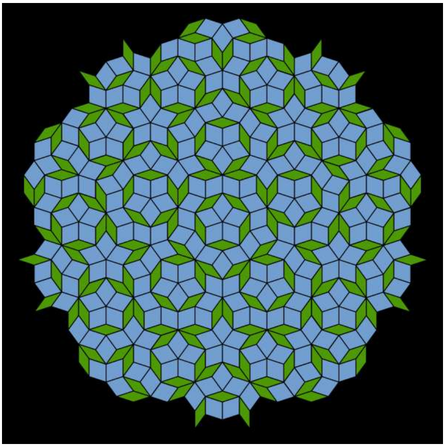

Figure 2: A Penrose tiling

It is important to note that the passage from commutative to noncommutative spaces is far from being a mere translation of relevant notions and constructions into a framework allowing noncommutative algebras of functions. There are many new phenomena which arise naturally. It can be shown for example that noncommutative spaces come equipped with a natural time evolution, that is to say, noncommutative variables are dynamic. Moreover viewing classical spaces as noncommutative spaces sometimes provides new tools and results (the Julia set in Figure 3 provides an example of this situation).

Noncommutative geometry, the beginnings

The prehistory of noncommutative geometry is vast but has certain landmarks which are worth mentioning in order to understand how the theory developed. Functional analysis plays a prominent role and provides the basic language underlying many of the constructions. Noncommutative algebras can in many cases be represented as algebras of operators (think of matrix algebras, possibly infinite dimensional). The corresponding theory of operator algebras owes much of its existence to a series of articles from the late thirties and early forties by Murray and von Neumann as well as the work of Gel'fand and Naimark more or less in the same period (c.f. [1]). The final push to noncommutative geometry from this direction came from the work of Takesaki tree decades later. Simultaneously the celebrated index theorem of Atiyah and Singer provided a fundamental link between topology and the analysis of operators associated to geometric spaces.

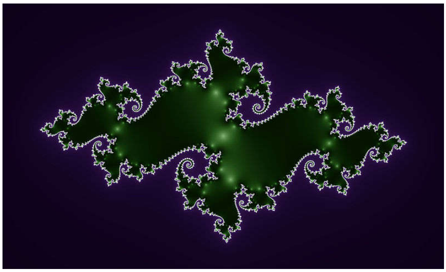

Figure 3: A Julia set.

The birth of noncommutative geometry as such can be traced back to the work of Alain Connes in the late seventies and early eighties. Connes won the fields medal in 1982 for his work culminating in the classification of a class of operator algebras known as factors. Connes' work also provided operator theoretic tools appropriate for the study of foliations (spaces made up of "leaves") which in particular extended to a noncommutative framework tools from differential topology. The development of the theory since then has been linked to a big extent to his work and the work of his collaborators.

It would be unrealistic to try to survey in a couple of paragraphs the almost tree decades of developments that since then have occurred in noncommutative geometry. Giving a short list of relevant names will hardly do any justice to the amount of work done in the subject. The interested reader might refer to Connes' book [1] and his own review on the subject [2] (for a view of the more recent developments see [3] and the introduction [4]). I will just mention by a way of example few of the areas in which research has been and still is more active, needless to say the richness of the theory and the spectrum of applications of its tools make it a very promising area whose full power is still to be seen:

- Physics: renormalization; the standard model in particle physics.
- Number theory: trace formulas and the Riemann hypothesis; class field theory.
- Differential topology: K-theory; conjectures of Novikov type.

I will end up this note quoting Yuri Manin's words on the subject (see his foreword to the book [5] by Matilde Marcolli): *"Noncommutative geometry nowadays looks as a vast building site"... "practitioners of noncommutative geometry (or geometries) already built up a large and swiftly growing body of exciting mathematics, challenging traditional boundaries and subdivisions."*

Referenties

[1] A. Connes, *Noncommutative geometry.* Academic Press (1994).

[2] A. Connes, *A short survey of noncommutative geometry.* J. Math. Phys. **41** (2000) 3832-3866.

[3] A. Connes, M. Marcolli, *Noncommutative geometry, quantum fields and motives.* American Mathematical Society (2008).

[4] M. Khalkhali, M. Marcolli (eds.) *An invitation to noncommutative geometry.* World Scientific Publishing Co (2008).

[5] M. Marcolli, *Arithmetic noncommutative geometry.* American Mathematical Society (2005).