

GKZ Hypergeometric Structures

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Abstract. This text is based on lectures by the author in the Summer School *Algebraic Geometry and Hypergeometric Functions* in Istanbul in June 2005. It gives a review of some of the basic aspects of the theory of hypergeometric structures of Gelfand, Kapranov and Zelevinsky, including Differential Equations, Integrals and Series, with emphasis on the latter. The Secondary Fan is constructed and subsequently used to describe the ‘geography’ of the domains of convergence of the Γ -series. A solution to certain Resonance Problems is presented and applied in the context of Mirror Symmetry. Many examples and some exercises are given throughout the paper.

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1. Introduction

GKZ stands for *Gelfand*, *Kapranov* and *Zelevinsky*, who discovered fascinating generalizations of the classical hypergeometric structures of Euler, Gauss, Appell, Lauricella, Horn [10, 12, 14]. The main ingredient for these new hypergeometric structures is a finite subset $\mathcal{A} \subset \mathbb{Z}^{k+1}$ which generates \mathbb{Z}^{k+1} as an abelian group and for which there exists a group homomorphism $h : \mathbb{Z}^{k+1} \rightarrow \mathbb{Z}$ such that $h(\mathcal{A}) = \{1\}$. The latter condition means that \mathcal{A} lies in a k -dimensional affine hyperplane in \mathbb{Z}^{k+1} . Figure 1 shows \mathcal{A} (the black dots) sitting in this hyperplane for some classical hypergeometric structures. In [12, 14] these new structures were called *\mathcal{A} -hypergeometric systems*. Nowadays many authors call them *GKZ hypergeometric systems*. The original name indeed seems somewhat unfortunate, since *\mathcal{A} -hypergeometric* sounds negative, like ‘αγεομετρικος μη εισιτω’ (a non-geometer should not enter), written over the entrance of Plato’s academy and in the logo of the American Mathematical Society. Besides the set \mathcal{A} the construction of GKZ hypergeometric structures requires a vector $\mathbf{c} \in \mathbb{C}^{k+1}$.

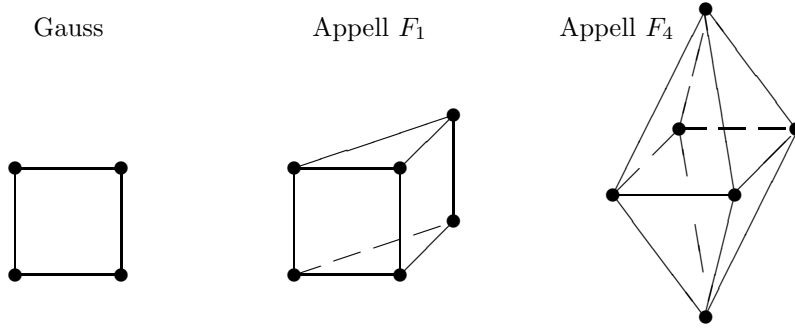


FIGURE 1.

In these notes we report on the basic theory of GKZ hypergeometric structures and show how the traditional aspects *differential equations*, *integrals*, *series* are attached to the data \mathcal{A} , \mathbf{c} . In Section 2 we introduce the GKZ differential equations and give examples of GKZ hypergeometric integrals. In Section 3 we discuss GKZ hypergeometric series (so-called Γ -series). We have put details of the GKZ theory for Lauricella's F_D together in Section 7, so that the reader can compare results and view-points on F_D for various lectures in this School (e.g. [18]).

The beautiful insight of Gelfand, Kapranov and Zelevinsky was that hypergeometric structures greatly simplify if one introduces extra variables and balances this with an appropriate torus action. More precisely the variables in GKZ theory are the natural coordinates on the space $\mathbb{C}^{\mathcal{A}} := \text{Maps}(\mathcal{A}, \mathbb{C})$ of maps from \mathcal{A} to \mathbb{C} . The torus $\mathbb{T}^{k+1} := \text{Hom}(\mathbb{Z}^{k+1}, \mathbb{C}^*)$ of group homomorphisms from \mathbb{Z}^{k+1} to \mathbb{C}^* , acts naturally on $\mathbb{C}^{\mathcal{A}}$ and on functions on $\mathbb{C}^{\mathcal{A}}$: for $\sigma \in \mathbb{T}^{k+1}$ and $\Phi : \mathbb{C}^{\mathcal{A}} \rightarrow \mathbb{C}$

$$(\sigma \cdot \mathbf{u})(\mathbf{a}) = \sigma(\mathbf{a})\mathbf{u}(\mathbf{a}), \quad (\Phi \cdot \sigma)(\mathbf{u}) = \Phi(\sigma \cdot \mathbf{u}), \quad \forall \mathbf{a} \in \mathcal{A}, \forall \mathbf{u} \in \mathbb{C}^{\mathcal{A}}. \quad (1)$$

The GKZ hypergeometric functions associated with \mathcal{A} and \mathbf{c} are defined on open domains in $\mathbb{C}^{\mathcal{A}}$, but they are not invariant under the action of \mathbb{T}^{k+1} , unless $\mathbf{c} = 0$. Rather, for $\mathbf{c} \in \mathbb{Z}^{k+1}$ they transform according to the character of \mathbb{T}^{k+1} given by \mathbf{c} . For $\mathbf{c} \notin \mathbb{Z}^{k+1}$ there is only an infinitesimal analogue of this transformation behavior, encoded in one part of the GKZ system of differential equations (see (20)). On the other hand, the quotient of any two GKZ hypergeometric functions with a common domain of definition associated with \mathcal{A} and \mathbf{c} is always \mathbb{T}^{k+1} -invariant (see (21)).

The important role of the \mathbb{T}^{k+1} -action in GKZ hypergeometric structures motivates a study of the orbit space. Without going into details, this can be described as follows. First take the complement of the coordinate hyperplanes $(\mathbb{C}^*)^{\mathcal{A}} := \text{Maps}(\mathcal{A}, \mathbb{C}^*) = \{\mathbf{u} \in \mathbb{C}^{\mathcal{A}} \mid \mathbf{u}(\mathbf{a}) \neq 0, \forall \mathbf{a} \in \mathcal{A}\}$. The above action of \mathbb{T}^{k+1} preserves this set. In fact $(\mathbb{C}^*)^{\mathcal{A}}$ is a complex torus and \mathbb{T}^{k+1} can be identified

with a subtorus, acting by left multiplication. The quotient is the torus

$$(\mathbb{C}^*)^{\mathcal{A}} / \mathbb{T}^{k+1} = \text{Hom}(\mathbb{L}, \mathbb{C}^*), \quad (2)$$

where \mathbb{L} is the lattice (= free abelian group) of linear relations in \mathcal{A} . It is often convenient to fix an numbering for the elements of \mathcal{A} , i.e. $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_N\}$. Then \mathbb{L} can be described as

$$\mathbb{L} := \{(\ell_1, \dots, \ell_N) \in \mathbb{Z}^N \mid \ell_1 \mathbf{a}_1 + \dots + \ell_N \mathbf{a}_N = 0\}. \quad (3)$$

The rank of \mathbb{L} and the dimension of the torus in (2) are $d := N - k - 1$. In order to obtain the natural space on which the GKZ hypergeometric structures live one must compactify the complex torus in (2). For this purpose Gelfand, Kapranov and Zelevinsky developed the theory of the *Secondary Fan*. This is a complete fan of rational polyhedral cones in the real vector space $\mathbb{L}_{\mathbb{R}}^{\vee} := \text{Hom}(\mathbb{L}, \mathbb{R})$. Sections 4 and 5 give full details about the Secondary Fan and the associated toric variety $\mathcal{V}_{\mathcal{A}}$. Since the Secondary Fan has interesting applications outside the theory of hypergeometric systems Sections 4 and 5 are written so that they can be read independently of other sections. The toric variety $\mathcal{V}_{\mathcal{A}}$ provides a very clear picture of the ‘geography’ for the domains of convergence of the various GKZ hypergeometric series, since these match exactly with discs about the special points of $\mathcal{V}_{\mathcal{A}}$ coming from the maximal cones in the secondary fan (see Proposition 7). For the examples in Figure 1 the toric varieties and special points associated with the maximal cones in the secondary fan are: for Gauss the projective line \mathbb{P}^1 with points $[1, 0]$, $[0, 1]$, for F_4 the projective plane \mathbb{P}^2 with points $[1, 0, 0]$, $[0, 1, 0]$, $[0, 0, 1]$, for F_1 the projective plane blown up in the three points $[1, 0, 0]$, $[0, 1, 0]$, $[0, 0, 1]$ equipped with the six points of intersection of the exceptional divisors and the proper transforms of the coordinate axes in \mathbb{P}^2 .

For most \mathcal{A} the dimension of local solution spaces for the GKZ differential equations equals the volume of the k -dimensional polytope $\Delta_{\mathcal{A}} := \text{convex hull of } \mathcal{A}$ (see Section 2.7); here the volume is normalized as $k! \times$ the Euclidean volume. Thus for the examples in Figure 1 the local solution spaces have dimension 2, 3, 4, respectively. For generic \mathcal{A} and \mathbf{c} the Γ -series provide bases of local solutions for the GKZ differential equations. However, in some exceptional, but very important, cases there are not enough Γ -series, due to a phenomenon called *resonance*. In Section 6 we discuss resonance and demonstrate how one *sometimes* can obtain enough solutions by considering infinitesimal deformations of Γ -series. *Sometimes* here means under the severe restrictions that $\mathbf{c} = 0$ and that one works in the neighborhood of a point on $\mathcal{V}_{\mathcal{A}}$ which corresponds to a unimodular triangulation of the polytope $\Delta_{\mathcal{A}}$. Very recently Borisov and Horja [5] found a way to obtain enough solutions for any $\mathbf{c} \in \mathbb{Z}^{k+1}$ and any triangulation. Their method is close in spirit to the method in Section 6 and [5] gives an up-to-date presentation of this aspect of GKZ hypergeometric structures.

In the 1980’s, while Gelfand, Kapranov and Zelevinsky were working on new hypergeometric structures, physicists discovered fascinating new structures in string theory: the so-called *string dualities*. One of these string dualities, known

as *Mirror Symmetry*, soon attracted the attention of mathematicians, because it claimed very striking consequences for enumerative geometry. Especially the paper [7] of Candelas, de la Ossa, Green and Parkes with a detailed study of the quintic in \mathbb{P}^4 played a pivotal role. Batyrev [1] pointed out that many examples of the Mirror Symmetry phenomenon dealt with pairs of families of Calabi-Yau hypersurfaces in toric varieties coming from dual polytopes. In [3] Batyrev and Borisov extended this kind of Mirror Symmetry to Calabi-Yau complete intersections in toric varieties. Batyrev ([2] thm.14.2) also noticed that the solutions to the differential equations which appeared in Mirror Symmetry, were solutions to GKZ hypergeometric systems constructed from the same data as the toric varieties. The converse is, however, not true: the GKZ system can have solutions which are not solutions to the system of differential equations in Mirror Symmetry. This means that the latter system contains extra differential equations in addition to those of the underlying GKZ system (see [16] §3.3). On the other hand, the solutions to the differential equations which one encounters in Mirror Symmetry, can all be obtained by a few differentiations from solutions to extremely resonant GKZ hypergeometric systems with $\mathbf{c} = 0$. Thus we do not need those extended GKZ systems. In Section 8 we discuss some examples of this intriguing application of GKZ hypergeometric structures to String Theory.

The quotient of two solutions of a GKZ system of differential equations associated with \mathcal{A} and \mathbf{c} is \mathbb{T}^{k+1} -invariant. So one can define (at least locally) a *Schwarz map* from the toric variety $\mathcal{V}_{\mathcal{A}}$ to the projectivization of the vector space of (local) solutions. For Gauss's system, and more generally for Lauricella's F_D , the toric variety $\mathcal{V}_{\mathcal{A}}$ and the projectivized solution space have the same dimension, equal to the rank of \mathbb{L} . The Schwarz map for Gauss's system and Lauricella's F_D is discussed extensively in other lectures in this school, e.g. [18]. Quite in contrast with F_D is the situation for GKZ systems associated with families of Calabi-Yau threefolds. For these the toric variety $\mathcal{V}_{\mathcal{A}}$ has dimension equal to $\text{rank } \mathbb{L}$, but the projectivized local solution space has dimension $1 + 2 \text{rank } \mathbb{L}$. The discussion about the *canonical coordinates* and the *pre-potential* in Section 8.5 can be seen as a description of the image of the (local) Schwarz map. This is closely related to what in the (physics) literature is called *Special Kähler Geometry*.

All this basically concerns only local aspects of GKZ systems of differential equations. About singularities, global solutions or global monodromy of the system not much seems to be known, except for classically studied systems like Gauss's and Lauricella's F_D .

Since these notes are intended as an introduction to GKZ hypergeometric structures, we have included throughout the text many examples and a few exercises. On the other hand we had to omit many topics. One of these omissions concerns *\mathcal{A} -discriminants*. These come up when one identifies $\mathbb{C}^{\mathcal{A}}$ with the space of Laurent polynomials in $k + 1$ variables with exponents in \mathcal{A} ,

$$\mathbf{u} \in \mathbb{C}^{\mathcal{A}} \quad \leftrightarrow \quad \sum_{\mathbf{a} \in \mathcal{A}} \mathbf{u}_{\mathbf{a}} \mathbf{x}^{\mathbf{a}},$$

and then wonders about the Laurent polynomials with singularities, i.e. for which there is a point at which all partial derivatives vanish. For the \mathcal{A} -discriminant and its relation to the secondary fan we refer to [13]. Another omission concerns *symplectic geometry* in connection with the secondary fan. We recommend Guillemin's book [15] for further reading on this topic.

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Contents

1. Introduction	1
2. GKZ systems via examples.	6
2.1. Roots of polynomial equations.	6
2.2. Integral with polynomial integrand	7
2.3. Integral with k -variable Laurent polynomial integrand	8
2.4. Generalized Euler integrals	9
2.5. General GKZ systems of differential equations	9
2.6. Gauss's hypergeometric differential equation as a GKZ system	11
2.7. Dimension of the solution space of a GKZ system	11
3. Γ -series	12
3.1. The Γ -function	12
3.2. Examples of Γ -series	13
3.2.1. Gauss's hypergeometric series	13
3.2.2. The hypergeometric series ${}_pF_{p-1}$	14
3.2.3. The case ${}_1F_0$.	14
3.2.4. The Appell-Lauricella hypergeometric series	14
3.3. Growth of coefficients of Γ -series	15
3.4. Γ -series and power series	16
3.5. Fourier Γ -series	17
3.6. Γ -series and GKZ differential equations.	18
4. The Secondary Fan	18
4.1. Construction of the secondary fan	19
4.2. Alternative descriptions for secondary fan constructions.	22
4.2.1. Piecewise linear functions associated with \mathcal{A}	22
4.2.2. Regular triangulations	24
4.3. The Secondary Polytope	25
5. The toric variety associated with the Secondary Fan	27
5.1. Construction of the toric variety for the secondary fan	27

5.2. Convergence of Fourier Γ -series and the secondary fan	30
5.3. Solutions of GKZ differential equations and the secondary fan	30
6. Extreme resonance in GKZ systems	33
7. GKZ for Lauricella's F_D	39
7.1. Series, \mathbb{L} , \mathcal{A} and the primary polytope $\Delta_{\mathcal{A}}$	39
7.2. Integrals and differential equations for F_D	40
7.3. Triangulations of $\Delta_{\mathcal{A}}$, secondary polytope and fan for F_D	42
8. A glimpse of Mirror Symmetry	44
8.1. GKZ data from Calabi-Yau varieties	44
8.2. The quintic in \mathbb{P}^4	46
8.3. The intersection of two cubics in \mathbb{P}^5	49
8.4. The hypersurface of degree $(3, 3)$ in $\mathbb{P}^2 \times \mathbb{P}^2$	51
8.5. The Schwarz map for some extended GKZ systems	53
References	57

2. GKZ systems via examples.

2.1. Roots of polynomial equations.

It is clear that in general the zeros of a polynomial

$$P_{\mathbf{u}}(x) := u_0 + u_1x + u_2x^2 + \dots + u_nx^n \quad (4)$$

are functions of the coefficients $\mathbf{u} = (u_0, \dots, u_n)$. One wonders: *What kind of functions?* For instance, it has been known since ancient times that the zeros of a quadratic polynomial $ax^2 + bx + c$ are $\frac{1}{2a}(-b \pm \sqrt{b^2 - 4ac})$. Similar formulas exist for polynomials of degrees 3 and 4, but, according to Galois theory, the zeros of a general polynomial of degree ≥ 5 can not be obtained from the polynomial's coefficients by a finite number of algebraic operations. Changing the point of view K. Mayr proved that the roots of polynomials are solutions of certain systems of differential equations:

Theorem 1. (Mayr [19]) *If all roots of the equation $P_{\mathbf{u}}(\xi) = 0$ are simple, then a root ξ satisfies the differential equations: for $i_1 + \dots + i_r = j_1 + \dots + j_r$:*

$$\frac{\partial^r \xi}{\partial u_{i_1} \dots \partial u_{i_r}} = \frac{\partial^r \xi}{\partial u_{j_1} \dots \partial u_{j_r}}.$$

Proof. By differentiating the equation $P_{\mathbf{u}}(\xi) = 0$ with respect to u_i we find $P'_{\mathbf{u}}(\xi) \frac{\partial \xi}{\partial u_i} + \xi^i = 0$. This implies $\frac{\partial \xi}{\partial u_i} = \xi^i \frac{\partial \xi}{\partial u_0} = \frac{1}{1+i} \frac{\partial \xi^{1+i}}{\partial u_0}$. Induction now gives

$$\frac{\partial^r \xi}{\partial u_{i_1} \dots \partial u_{i_r}} = \frac{1}{1 + i_1 + \dots + i_r} \frac{\partial^r \xi^{1+i_1+\dots+i_r}}{\partial u_0^r}.$$

□

It obviously suffices to use only those Mayr's differential equations for which $\{i_1, \dots, i_r\} \cap \{j_1, \dots, j_r\} = \emptyset$. These can also be written as

$$\prod_{\ell_i < 0} \left(\frac{\partial}{\partial u_i} \right)^{-\ell_i} \xi = \prod_{\ell_i > 0} \left(\frac{\partial}{\partial u_i} \right)^{\ell_i} \xi \quad \text{if} \quad \sum_{i=0}^n \ell_i \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (5)$$

A second system of differential equations, satisfied by the roots of polynomials, follows from the easily checked fact that for all $s \in \mathbb{C}^*$:

$$\xi(su_0, su_1, \dots, su_n) = \xi(u_0, \dots, u_n), \quad \xi(u_0, su_1, \dots, s^n u_n) = s^{-1} \xi(u_0, \dots, u_n).$$

When we differentiate this with respect to s and set $s = 1$, we find:

$$\begin{aligned} u_0 \frac{\partial \xi}{\partial u_0} + u_1 \frac{\partial \xi}{\partial u_1} + u_2 \frac{\partial \xi}{\partial u_2} + \dots + u_n \frac{\partial \xi}{\partial u_n} &= 0, \\ 0 u_0 \frac{\partial \xi}{\partial u_0} + 1 u_1 \frac{\partial \xi}{\partial u_1} + 2 u_2 \frac{\partial \xi}{\partial u_2} + \dots + n u_n \frac{\partial \xi}{\partial u_n} &= -\xi, \end{aligned}$$

This can be written more transparently as:

$$\xi(tu_0, tsu_1, ts^2 u_2, \dots, ts^n u_n) = s^{-1} \xi(u_0, \dots, u_n) \quad \text{for} \quad (t, s) \in (\mathbb{C}^*)^2, \quad (6)$$

$$\sum_{i=0}^n \begin{bmatrix} 1 \\ i \end{bmatrix} u_i \frac{\partial \xi}{\partial u_i} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \xi. \quad (7)$$

For more on zeros of 1-variable polynomials and hypergeometric functions see [20].

2.2. Integral with polynomial integrand

Consider the integral

$$I_\sigma^{(m)} = I_\sigma^{(m)}(u_0, \dots, u_n) := \int_\sigma P_{\mathbf{u}}(x)^m \frac{dx}{x}$$

with $m \in \mathbb{Z}$, $P_{\mathbf{u}}(x)$ as in (4) and σ a circle in \mathbb{C} , with radius > 0 , centred at 0, independent of u_0, \dots, u_n , not passing through any zero of $P_{\mathbf{u}}(x)$.

By differentiating under the integral sign we see

$$\frac{\partial I_\sigma^{(m)}}{\partial u_i} = m \int_\sigma x^i P_{\mathbf{u}}(x)^{m-1} \frac{dx}{x}$$

and hence, if $i_1 + \dots + i_r = j_1 + \dots + j_r$, then

$$\frac{\partial^r I_\sigma^{(m)}}{\partial u_{i_1} \dots \partial u_{i_r}} = \frac{\partial^r I_\sigma^{(m)}}{\partial u_{j_1} \dots \partial u_{j_r}}.$$

As before this can also be written as

$$\prod_{\ell_i < 0} \left(\frac{\partial}{\partial u_i} \right)^{-\ell_i} I_\sigma^{(m)} = \prod_{\ell_i > 0} \left(\frac{\partial}{\partial u_i} \right)^{\ell_i} I_\sigma^{(m)} \quad \text{if} \quad \sum_{i=0}^n \ell_i \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (8)$$

For $s \in \mathbb{C}^*$ close to 1 one checks: $I_\sigma^{(m)}(su_0, \dots, su_n) = s^m I_\sigma^{(m)}(u_0, \dots, u_n)$ and

$$I_\sigma^{(m)}(u_0, su_1, \dots, s^n u_n) = \int_\sigma P_{\mathbf{u}}(sx)^m \frac{dx}{x} = \int_{s\sigma} P_{\mathbf{u}}(x)^m \frac{dx}{x} = I_\sigma^{(m)}(u_0, \dots, u_n).$$

More transparently: for $(t, s) \in (\mathbb{C}^*)^2$ sufficiently close to $(1, 1)$

$$I_\sigma^{(m)}(tu_0, tsu_1, ts^2 u_2 \dots, ts^n u_n) = t^m I_\sigma^{(m)}(u_0, \dots, u_n). \quad (9)$$

By differentiating (9) with respect to t and s and setting $t = s = 1$ we find, similar to (7),

$$\sum_{i=0}^n \begin{bmatrix} 1 \\ i \end{bmatrix} u_i \frac{\partial I_\sigma^{(m)}}{\partial u_i} = \begin{bmatrix} m \\ 0 \end{bmatrix} I_\sigma^{(m)}. \quad (10)$$

Note the fundamental role of the set $\mathcal{A} = \left\{ \begin{bmatrix} 1 \\ i \end{bmatrix} \mid i = 0, 1, \dots, n \right\}$ in (5)–(10).

Notice also the torus action (1) on the left hand sides of (6) and (9).

2.3. Integral with k -variable Laurent polynomial integrand

Let us take a Laurent polynomial in k variables

$$P_{\mathbf{u}}(x_1, x_2, \dots, x_k) := \sum_{\mathbf{a} \in \mathbf{A}} u_{\mathbf{a}} x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k} \quad (11)$$

where $\mathbf{a} = (a_1, a_2, \dots, a_k)$ and $\mathbf{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_N\}$ is a finite subset of \mathbb{Z}^k . Consider the integral

$$I_\sigma^{(m)}(\mathbf{u}) := \int_\sigma P_{\mathbf{u}}(x_1, \dots, x_k)^m \frac{dx_1}{x_1} \cdots \frac{dx_k}{x_k} \quad (12)$$

with $\mathbf{u} = (u_{\mathbf{a}})_{\mathbf{a} \in \mathbf{A}}$, $m \in \mathbb{Z}$ and with $\sigma = \sigma_1 \times \dots \times \sigma_k$ a product of k circles $\sigma_1, \dots, \sigma_k$ in \mathbb{C} , centred at 0, independent of \mathbf{u} , so that $P_{\mathbf{u}}(x_1, \dots, x_k) \neq 0$ for all $(x_1, \dots, x_k) \in \sigma_1 \times \dots \times \sigma_k$.

By differentiating under the integral sign we see, for $\mathbf{a} = (a_1, \dots, a_k)$,

$$\frac{\partial I_\sigma^{(m)}(\mathbf{u})}{\partial u_{\mathbf{a}}} = m \int_\sigma x_1^{a_1} \cdots x_k^{a_k} P_{\mathbf{u}}(x_1, \dots, x_k)^{m-1} \frac{dx_1}{x_1} \cdots \frac{dx_k}{x_k}.$$

From this one derives that for every vector $(\ell_1, \dots, \ell_N) \in \mathbb{Z}^N$ which satisfies

$$\ell_1 + \dots + \ell_N = 0, \quad \ell_1 \mathbf{a}_1 + \dots + \ell_N \mathbf{a}_N = 0, \quad (13)$$

the following differential equation holds

$$\prod_{\ell_i < 0} \left(\frac{\partial}{\partial u_i} \right)^{-\ell_i} I_\sigma^{(m)}(\mathbf{u}) = \prod_{\ell_i > 0} \left(\frac{\partial}{\partial u_i} \right)^{\ell_i} I_\sigma^{(m)}(\mathbf{u}); \quad (14)$$

for simplicity of notation we write here and henceforth u_i instead of $u_{\mathbf{a}_i}$.

For $s \in \mathbb{C}^*$ sufficiently close to 1 and for $i = 1, \dots, k$ one calculates:

$$\begin{aligned} I_\sigma^{(m)}(s^{a_{i1}} u_1, s^{a_{i2}} u_2, \dots, s^{a_{iN}} u_N) &= \int_\sigma P_{\mathbf{u}}(x_1, \dots, sx_i, \dots, x_k)^m \omega = \\ &= \int_{\sigma_1 \times \dots \times s\sigma_i \times \dots \times \sigma_k} P_{\mathbf{u}}(x_1, \dots, x_k)^m \omega = \int_{\sigma_1 \times \dots \times \sigma_i \times \dots \times \sigma_k} P_{\mathbf{u}}(x_1, \dots, x_k)^m \omega = I_\sigma^{(m)}(u_1, \dots, u_N); \end{aligned}$$

here a_{ij} denotes the i -th coordinate of the vector \mathbf{a}_j and $\omega = \frac{dx_1}{x_1} \dots \frac{dx_k}{x_k}$. This together with $I_\sigma^{(m)}(su_1, \dots, su_N) = s^m I_\sigma^{(m)}(u_1, \dots, u_N)$ can also be written as:

$$I_\sigma^{(m)}(ts_1^{a_{11}} s_2^{a_{21}} \dots s_k^{a_{k1}} u_1, \dots, ts_1^{a_{1N}} s_2^{a_{2N}} \dots s_k^{a_{kN}} u_N) = t^m I_\sigma^{(m)}(u_0, \dots, u_n) \quad (15)$$

for $(t, s_1, \dots, s_k) \in (\mathbb{C}^*)^{k+1}$ close to $(1, \dots, 1)$. By differentiating with respect to t, s_1, \dots, s_k and setting $t = s_1 = \dots = s_k = 1$ we find

$$\begin{bmatrix} 1 \\ \mathbf{a}_1 \end{bmatrix} u_1 \frac{\partial I_\sigma^{(m)}(\mathbf{u})}{\partial u_1} + \dots + \begin{bmatrix} 1 \\ \mathbf{a}_N \end{bmatrix} u_N \frac{\partial I_\sigma^{(m)}(\mathbf{u})}{\partial u_N} = \begin{bmatrix} m \\ 0 \end{bmatrix} I_\sigma^{(m)}(\mathbf{u}). \quad (16)$$

Note the appearance of the set $\mathcal{A} = \left\{ \begin{bmatrix} 1 \\ \mathbf{a} \end{bmatrix} \in \mathbb{Z}^{k+1} \mid \mathbf{a} \in \mathbf{A} \right\}$ in (13) and (16). Notice also the torus action (1) on the left hand side of (15).

Remark For $m > 0$ one can evaluate $I_\sigma^{(m)}(\mathbf{u})$ using the multinomial and residue theorems. One finds that $I_\sigma^{(m)}(\mathbf{u})$ is actually a polynomial:

$$\frac{1}{(2\pi i)^k} I_\sigma^{(m)}(\mathbf{u}) = \sum_{(m_1, \dots, m_N)} \frac{m!}{(m_1)! \dots (m_N)!} u_1^{m_1} \dots u_N^{m_N} \quad (17)$$

where the sum runs over all N -tuples of non-negative integers (m_1, \dots, m_N) satisfying $m_1 + \dots + m_N = m$ and $m_1 \mathbf{a}_1 + \dots + m_N \mathbf{a}_N = 0$.

In Section 8 one can find explicit examples of these integrals with $m = -1$.

2.4. Generalized Euler integrals

In [12, 14] Gelfand, Kapranov and Zelevinsky investigate integrals of the form

$$\int_\sigma \prod_i P_i(x_1, \dots, x_k)^{\alpha_i} x_1^{\beta_1} \dots x_k^{\beta_k} dx_1 \dots dx_k, \quad (18)$$

which they call *generalized Euler integrals*. Here the P_i are Laurent polynomials, α_i and β_j are complex numbers and σ is a k -cycle. Since the integrand can be multivalued and can have singularities one must carefully give the precise meaning of Formula (18) (see [12] §2.2). Having dealt with the technicalities of the precise definition Gelfand, Kapranov and Zelevinsky view the integrals (18) as functions of the coefficients of the Laurent polynomials P_i . Using the same arguments as we used in Section 2.3 they then verify that these functions satisfy a system of differential equations (19)-(20) for the appropriate data \mathcal{A} and \mathbf{c} . Examples can be found in Sections 7.2 and 8.3.

2.5. General GKZ systems of differential equations

The systems of differential equations (5)-(7), (8)-(10) and (14)-(16) found in the preceding examples are special cases of systems of differential equations discovered by Gelfand, Kapranov and Zelevinsky [10, 12, 14]. The general GKZ system for functions Φ of N variables u_1, \dots, u_N is constructed from a vector $\mathbf{c} \in \mathbb{C}^{k+1}$ and an N -element subset $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_N\} \subset \mathbb{Z}^{k+1}$ which generates \mathbb{Z}^{k+1} as an abelian

group and for which there exists a group homomorphism $h : \mathbb{Z}^{k+1} \rightarrow \mathbb{Z}$ such that $h(\mathbf{a}) = 1$ for all $\mathbf{a} \in \mathcal{A}$. Let $\mathbb{L} \subset \mathbb{Z}^N$ denote the lattice of relations in \mathcal{A} :

$$\mathbb{L} := \{(\ell_1, \dots, \ell_N) \in \mathbb{Z}^N \mid \ell_1 \mathbf{a}_1 + \dots + \ell_N \mathbf{a}_N = \mathbf{0}\}.$$

Note that the condition $h(\mathbf{a}) = 1$ for all $\mathbf{a} \in \mathcal{A}$, implies that $\ell_1 + \dots + \ell_N = 0$ for every $(\ell_1, \dots, \ell_N) \in \mathbb{L}$.

Definition 1. *The GKZ system associated with \mathcal{A} and \mathbf{c} consists of*

- *for every $(\ell_1, \dots, \ell_N) \in \mathbb{L}$ one differential equation*

$$\prod_{\ell_i < 0} \left(\frac{\partial}{\partial u_i} \right)^{-\ell_i} \Phi = \prod_{\ell_i > 0} \left(\frac{\partial}{\partial u_i} \right)^{\ell_i} \Phi, \quad (19)$$

- *the system of $k+1$ differential equations*

$$\mathbf{a}_1 u_1 \frac{\partial \Phi}{\partial u_1} + \dots + \mathbf{a}_N u_N \frac{\partial \Phi}{\partial u_N} = \mathbf{c} \Phi. \quad (20)$$

Remark. It is natural to view u_1, \dots, u_N as coordinates on the space $\mathbb{C}^{\mathcal{A}} := \text{Maps}(\mathcal{A}, \mathbb{C})$. Then the left hand side of the equation (20) is the infinitesimal version of the torus action (1). If Φ_1 and Φ_2 are two solutions of (20) on some open set $U \subset \mathbb{C}^{\mathcal{A}}$, their quotient satisfies

$$\mathbf{a}_1 u_1 \frac{\partial}{\partial u_1} \left(\frac{\Phi_1}{\Phi_2} \right) + \dots + \mathbf{a}_N u_N \frac{\partial}{\partial u_N} \left(\frac{\Phi_1}{\Phi_2} \right) = 0 \quad (21)$$

and is therefore constant on the intersections of U with the \mathbb{T}^{k+1} -orbits.

Thus a basis Φ_1, \dots, Φ_r of the solution space of (19)-(20) induces map from the orbit space $\mathbb{T}^{k+1} \cdot U / \mathbb{T}^{k+1}$ into the projective space \mathbb{P}^{r-1} , like the Schwarz map for Gauss's hypergeometric systems.

Another simple, but nevertheless quite useful, consequence of the GKZ differential equations is:

Proposition 1. *If function Φ satisfies the differential equations (19)-(20) for \mathcal{A} and \mathbf{c} , then $\frac{\partial \Phi}{\partial u_j}$ satisfies the differential equations (19)-(20) for \mathcal{A} and $\mathbf{c} - \mathbf{a}_j$.*

Proof. The derivation $\frac{\partial}{\partial u_j}$ commutes with all derivations involved in (19). On the other hand by applying $\frac{\partial}{\partial u_j}$ to both sides of (20) we get

$$\mathbf{a}_1 u_1 \frac{\partial}{\partial u_1} \left(\frac{\partial \Phi}{\partial u_j} \right) + \dots + \mathbf{a}_N u_N \frac{\partial}{\partial u_N} \left(\frac{\partial \Phi}{\partial u_j} \right) + \mathbf{a}_j \frac{\partial \Phi}{\partial u_j} = \mathbf{c} \frac{\partial \Phi}{\partial u_j}.$$

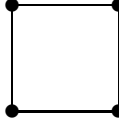
□

2.6. Gauss's hypergeometric differential equation as a GKZ system

The most classical hypergeometric differential equation, due to Euler and Gauss, is:

$$z(z-1)F'' + ((a+b+1)z-c)F' + abF = 0. \quad (22)$$

Here F is a function of one variable z , $' = \frac{d}{dz}$ and a, b, c are additional complex parameters. It is reproduced in the GKZ formalism by $\mathbf{c} = (1-c, -a, -b)$ and

$$\mathcal{A} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \subset \mathbb{Z}^3$$


and, hence, $\mathbb{L} = \mathbb{Z}(1, 1, -1, -1) \subset \mathbb{Z}^4$. Indeed, for these data the GKZ system boils down to the following four differential equations for a function Φ of four variables (u_1, u_2, u_3, u_4) :

$$\begin{aligned} \frac{\partial^2 \Phi}{\partial u_1 \partial u_2} &= \frac{\partial^2 \Phi}{\partial u_3 \partial u_4} \\ u_1 \frac{\partial \Phi}{\partial u_1} - u_2 \frac{\partial \Phi}{\partial u_2} &= (1-c) \Phi \\ u_1 \frac{\partial \Phi}{\partial u_1} + u_3 \frac{\partial \Phi}{\partial u_3} &= -a \Phi \\ u_1 \frac{\partial \Phi}{\partial u_1} + u_4 \frac{\partial \Phi}{\partial u_4} &= -b \Phi \end{aligned}$$

From the second equation we get

$$\frac{\partial^2 \Phi}{\partial u_1 \partial u_2} = u_2^{-1} \left(u_1 \frac{\partial^2 \Phi}{\partial u_1^2} + c \frac{\partial \Phi}{\partial u_1} \right)$$

From the third and fourth equations we get

$$\frac{\partial^2 \Phi}{\partial u_3 \partial u_4} = u_3^{-1} u_4^{-1} \left(-u_1 \frac{\partial}{\partial u_1} - a \right) \left(-u_1 \frac{\partial}{\partial u_1} - b \right) \Phi$$

Together with the first equation this yields

$$u_3^{-1} u_4^{-1} \left(u_1^2 \frac{\partial^2 \Phi}{\partial u_1^2} + (1+a+b)u_1 \frac{\partial \Phi}{\partial u_1} + ab\Phi \right) = u_2^{-1} \left(u_1 \frac{\partial^2 \Phi}{\partial u_1^2} + c \frac{\partial \Phi}{\partial u_1} \right).$$

Setting $u_2 = u_3 = u_4 = 1$, $u_1 = z$ and $F(z) = \Phi(z, 1, 1, 1)$ we find that F satisfies the differential equation (22).

2.7. Dimension of the solution space of a GKZ system

The spaces of (local) solutions of the GKZ differential equations (19)-(20) are complex vector spaces. Theorems 2 and 5 in [10] state that the dimension of the space of (local) solutions of (19)-(20) near a generic point is equal to the normalized volume of the k -dimensional polytope $\Delta_{\mathcal{A}} := \text{convex hull}(\mathcal{A})$; here ‘normalized volume’ means $k!$ times the usual Euclidean volume. In [11] it is pointed out that the proof in [10] requires an additional condition on \mathcal{A} . Corollary 8.9 and

Proposition 13.15 in [23] show that this additional condition is satisfied if the polytope $\Delta_{\mathcal{A}}$ admits a unimodular triangulation. Triangulations of $\Delta_{\mathcal{A}}$ and their importance in GKZ hypergeometric structures are discussed in Section 4.2.2.

3. Γ -series

As before we consider a subset $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_N\} \subset \mathbb{Z}^{k+1}$ which generates \mathbb{Z}^{k+1} as an abelian group and for which there exists a group homomorphism $h : \mathbb{Z}^{k+1} \rightarrow \mathbb{Z}$ such that $h(\mathbf{a}) = 1$ for all $\mathbf{a} \in \mathcal{A}$. And, still as before, we write:

$$\mathbb{L} := \{(\ell_1, \dots, \ell_N) \in \mathbb{Z}^N \mid \ell_1 \mathbf{a}_1 + \dots + \ell_N \mathbf{a}_N = 0\}.$$

The condition $h(\mathbf{a}) = 1$ for all $\mathbf{a} \in \mathcal{A}$, implies that $\ell_1 + \dots + \ell_N = 0$ for every $(\ell_1, \dots, \ell_N) \in \mathbb{L}$. With \mathbb{L} and a vector $\underline{\gamma} = (\gamma_1, \dots, \gamma_N) \in \mathbb{C}^N$ Gelfand, Kapranov and Zelevinsky [10] associate what they call a Γ -series:

Definition 2. The Γ -series associated with \mathbb{L} and $\underline{\gamma} = (\gamma_1, \dots, \gamma_N) \in \mathbb{C}^N$ is

$$\Phi_{\mathbb{L}, \underline{\gamma}}(u_1, \dots, u_N) = \sum_{(\ell_1, \dots, \ell_N) \in \mathbb{L}} \prod_{j=1}^N \frac{u_j^{\gamma_j + \ell_j}}{\Gamma(\gamma_j + \ell_j + 1)}. \quad (23)$$

Here Γ is the Γ -function; its definition and main properties are recalled in Section 3.1. In Section 3.2 we demonstrate how the classical hypergeometric series of Gauss, Appell and Lauricella appear in the Γ -series format. In Section 3.3 we give estimates for the growth of the coefficients in (23). Formula (23) requires for $\underline{\gamma} \notin \mathbb{Z}^{k+1}$ choices of logarithms for u_1, \dots, u_N . By carefully manoeuvring conditions on $\underline{\gamma}$ and substitutions setting some u_j equal to 1, we can avoid problems and show in Section 3.6 how a Γ -series can be viewed as a power series in $d = N - k - 1$ variables with positive radii of convergence. Nevertheless, a formula avoiding choices of logarithms is desirable. For that reason we introduce Fourier Γ -series in Section 3.5. In Section 3.6 we prove that $\Phi_{\mathbb{L}, \underline{\gamma}}(u_1, \dots, u_N)$ can be viewed as a function on some domain in (u_1, \dots, u_N) -space and satisfies the GKZ differential equations.

3.1. The Γ -function

The Γ -function is defined for complex numbers s with $\Re s > 0$ by the integral

$$\Gamma(s) := \int_0^\infty t^{s-1} e^{-t} dt. \quad (24)$$

Using partial integration one immediately checks $\Gamma(s+1) = s\Gamma(s)$ and, hence, for $n \in \mathbb{Z}$, $n > 0$

$$\Gamma(s+n) = s(s+1) \dots (s+n-1) \Gamma(s). \quad (25)$$

Formulas (24) and (25) imply in particular

$$\Gamma(1) = 1, \quad \Gamma(n+1) = n! \quad \text{for } n \in \mathbb{N}. \quad (26)$$

One can extend the Γ -function to a meromorphic function on all of \mathbb{C} by setting

$$\Gamma(s) = \frac{\Gamma(s+n)}{s(s+1)\dots(s+n-1)} \quad \text{with } n \in \mathbb{Z}, n > -\Re s. \quad (27)$$

The functional equation (25) shows that this does not depend on the choice of n . Formula (24) shows $\Gamma(s) \neq 0$ if $\Re s > 1$ and hence Formula (27) shows that the extended Γ -function is holomorphic on $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ and has at $s = -m \in \mathbb{Z}_{\leq 0}$ a first order pole with residue

$$\text{Res}_{s=-m} \Gamma(s) = \frac{(-1)^m}{m!}. \quad (28)$$

The function $\frac{1}{\Gamma(s)}$ is holomorphic on the whole complex plane. Its zero set is $\mathbb{Z}_{\leq 0}$ and its Taylor series at $-m \in \mathbb{Z}_{\leq 0}$ starts like

$$\frac{1}{\Gamma(s-m)} = (-1)^m m! s + \dots \quad (29)$$

The coefficients of (classical) hypergeometric series are usually expressed in terms of *Pochhammer symbols* $(s)_n$. These are defined by $(s)_n = s(s+1)\dots(s+n-1)$ and can be rewritten as quotients of Γ -values:

$$(s)_n = s(s+1)\dots(s+n-1) = \frac{\Gamma(s+n)}{\Gamma(s)} = (-1)^n \frac{\Gamma(1-s)}{\Gamma(1-n-s)}. \quad (30)$$

Note, however, that for integer values of s the Pochhammer symbol $(s)_n$ is perfectly well defined, while some of the individual Γ -values in (30) may become ∞ .

3.2. Examples of Γ -series

3.2.1. Gauss's hypergeometric series. As in the example of Gauss's hypergeometric differential equation (Section 2.6) we take $\mathbb{L} = \mathbb{Z}(1, 1, -1, -1)$ in \mathbb{Z}^4 and $\underline{\gamma} = (0, c-1, -a, -b) \in \mathbb{C}^4$. If c is not an integer ≤ 0 , then, by (23) and (30),

$$\begin{aligned} \Phi_{\mathbb{L}, \underline{\gamma}}(u_1, u_2, u_3, u_4) &= \sum_{n \in \mathbb{Z}} \frac{u_1^n u_2^{c-1+n} u_3^{-a-n} u_4^{-b-n}}{\Gamma(1+n)\Gamma(c+n)\Gamma(1-n-a)\Gamma(1-n-b)} \\ &= \frac{u_2^{c-1} u_3^{-a} u_4^{-b}}{\Gamma(c)\Gamma(1-a)\Gamma(1-b)} \sum_{n \geq 0} \frac{(a)_n (b)_n}{n! (c)_n} (u_1 u_2 u_3^{-1} u_4^{-1})^n \end{aligned}$$

and, hence, $\Phi_{\mathbb{L}, \underline{\gamma}}(z, 1, 1, 1) = \frac{1}{\Gamma(c)\Gamma(1-a)\Gamma(1-b)} F(a, b, c|z)$ with

$$F(a, b, c|z) := \sum_{n \geq 0} \frac{(a)_n (b)_n}{n! (c)_n} z^n.$$

The power series $F(a, b, c|z)$ is Gauss's hypergeometric series. Note that if a or b is a positive integer the Γ -series is 0, but Gauss's hypergeometric series is not 0.

3.2.2. The hypergeometric series ${}_pF_{p-1}$. Quite old generalizations of Gauss's hypergeometric series are the series

$${}_pF_{p-1} \left(\begin{matrix} a_1 & \dots & a_p \\ c_1 & \dots & c_{p-1} \end{matrix} \middle| z \right) := \sum_{n \geq 0} \frac{(a_1)_n \cdots (a_p)_n}{n! (c_1)_n \cdots (c_{p-1})_n} z^n.$$

Like for Gauss's series one easily finds that the series ${}_pF_{p-1}$ match (up to a constant factor) the Γ -series for $\mathbb{L} = \mathbb{Z}(1, \dots, 1, -1, \dots, -1)$ with $p-1$'s and $p-(-1)$'s.

3.2.3. The case ${}_1F_0$. The simplest, yet not totally trivial, case of a Γ -series arises for $\mathbb{L} = \mathbb{Z}(1, -1) \subset \mathbb{Z}^2$. The Γ -series with $\gamma = (0, a)$, $a \in \mathbb{C}$ is

$$\Phi_{\mathbb{Z}(1, -1), (0, a)}(u_1, u_2) = \sum_{n \in \mathbb{Z}} \frac{u_1^n u_2^{a-n}}{\Gamma(1+n)\Gamma(1+a-n)} = \frac{1}{\Gamma(1+a)} (u_1 + u_2)^a;$$

here we use the generalized binomial theorem and (30):

$$\binom{a}{n} = \frac{a(a-1) \cdots (a-n+1)}{n!} = \frac{\Gamma(1+a)}{\Gamma(1+n)\Gamma(1+a-n)}.$$

Remark. Note that $\mathbb{L} = \mathbb{Z}(1, -1)$ implies that the two elements of \mathcal{A} are equal. The GKZ differential equations in this case imply that the hypergeometric functions are in fact just functions of the single variable $u_1 + u_2$. This illustrates a general fact: when setting up the theory of GKZ hypergeometric systems one could take for \mathcal{A} a list of vectors in \mathbb{Z}^{k+1} instead of just a subset (i.e. the elements may occur more than once). But any such apparently more general set up, arises from a case with \mathcal{A} a genuine set by simply replacing a variable by a sum of new variables. So by allowing for \mathcal{A} a list instead of a set one does not get a seriously more general theory. Therefore we ignore this option in these notes.

3.2.4. The Appell-Lauricella hypergeometric series. These are generalizations of Gauss's series to n variables defined by Appell for $n = 2$ and Lauricella for general n . With the notations $\mathbf{z}^{\mathbf{m}} := z_1^{m_1} \cdots z_n^{m_n}$, $(\mathbf{x})_{\mathbf{m}} := (x_1)_{m_1} \cdots (x_n)_{m_n}$, $\mathbf{m}! := m_1! \cdots m_n!$, $|\mathbf{m}| := m_1 + \dots + m_n$ for n -tuples of complex numbers $\mathbf{z} = (z_1, \dots, z_n)$, $\mathbf{x} = (x_1, \dots, x_n)$ and of non-negative integers $\mathbf{m} = (m_1, \dots, m_n)$, the four Lauricella series are

$$\begin{aligned} F_A(a, \mathbf{b}, \mathbf{c}|\mathbf{z}) &:= \sum_{\mathbf{m}} \frac{(a)_{|\mathbf{m}|} (\mathbf{b})_{\mathbf{m}}}{(\mathbf{c})_{\mathbf{m}} \mathbf{m}!} \mathbf{z}^{\mathbf{m}} \\ F_B(\mathbf{a}, \mathbf{b}, c|\mathbf{z}) &:= \sum_{\mathbf{m}} \frac{(\mathbf{a})_{\mathbf{m}} (\mathbf{b})_{\mathbf{m}}}{(c)_{|\mathbf{m}|} \mathbf{m}!} \mathbf{z}^{\mathbf{m}} \\ F_C(a, b, \mathbf{c}|\mathbf{z}) &:= \sum_{\mathbf{m}} \frac{(a)_{|\mathbf{m}|} (b)_{|\mathbf{m}|}}{(\mathbf{c})_{\mathbf{m}} \mathbf{m}!} \mathbf{z}^{\mathbf{m}} \\ F_D(a, \mathbf{b}, c|\mathbf{z}) &:= \sum_{\mathbf{m}} \frac{(a)_{|\mathbf{m}|} (\mathbf{b})_{\mathbf{m}}}{(c)_{|\mathbf{m}|} \mathbf{m}!} \mathbf{z}^{\mathbf{m}} \end{aligned}$$

in the summations \mathbf{m} runs over $\mathbb{Z}_{\geq 0}^n$ and the c -parameters are not integers ≤ 0 . In Appell's notation (for $n = 2$) these series are called F_2, F_3, F_4, F_1 respectively.

One can use (30) to explore the relations between the Lauricella series and Γ -series. For the coefficients in F_D , for instance, we find

$$\frac{(a)_{|\mathbf{m}|}(\mathbf{b})_{\mathbf{m}}}{(c)_{|\mathbf{m}|}\mathbf{m}!} = \Gamma(1-a)\Gamma(c) \prod_{j=1}^n \Gamma(1-b_j) \cdot \prod_{j=1}^N \frac{1}{\Gamma(1+\gamma_j+\ell_j)}$$

with $N = 2n + 2$, $\underline{\gamma} = (\gamma_1, \dots, \gamma_N) = (c-1, -b_1, \dots, -b_n, -a, 0, \dots, 0)$,

$$(\ell_1, \dots, \ell_N) = (m_1, \dots, m_n) \begin{pmatrix} 1 & -1 & 0 & \dots & 0 & -1 & 1 & 0 & \dots & 0 \\ 1 & 0 & -1 & \ddots & \vdots & -1 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots & \vdots & \ddots & \ddots & 0 \\ 1 & 0 & \dots & 0 & -1 & -1 & 0 & \dots & 0 & 1 \end{pmatrix}$$

So for \mathbb{L} we take the lattice which is spanned by the rows of the above $n \times N$ -matrix. Substuting $u_j = 1$ for $1 \leq j \leq n+2$ and $u_j = z_{j-n-2}$ for $n+3 \leq j \leq 2n+2$ turns the Γ -series into a power series:

$$\Phi_{\mathbb{L}, \underline{\gamma}}(1, \dots, 1, z_1, \dots, z_n) = \left(\prod_{j=1}^{n+2} \Gamma(1+\gamma_j)^{-1} \right) F_D(a, \mathbf{b}, c | \mathbf{z}).$$

Exercise Note that the matrix describing \mathbb{L} for Lauricella's F_D is $(\mathbf{1}_n, -\mathbb{I}_n, -\mathbf{1}_n, \mathbb{I}_n)$ where $\mathbf{1}_n$ is the column vector with n components 1 and \mathbb{I}_n is the $n \times n$ -identity matrix. Now find the lattice \mathbb{L} for the Lauricella functions F_A, F_B and F_C .

3.3. Growth of coefficients of Γ -series

Here is first a simple lemma about the growth behavior of the Γ -function.

Lemma 1. *For every $C \in \mathbb{C} \setminus \mathbb{Z}$ there are real constants $P, R, \kappa_1, \kappa_2 > 0$ (depending on C) such that for all $M \in \mathbb{Z}_{\geq 0}$:*

$$|\Gamma(C+M)| \geq \kappa_1 R^M M^M \quad \text{and} \quad |\Gamma(C-M)| \geq \kappa_2 P^{-M} M^{-M}. \quad (31)$$

Proof. From (25) and the triangle inequalities one derives

$$\begin{aligned} |\Gamma(C+M)| &\geq \prod_{j=0}^{M-1} ||C| - j| \cdot |\Gamma(C)| \geq Q^M M! |\Gamma(C)| \geq \kappa R^M M^M |\Gamma(C)|, \\ |\Gamma(C-M)| &\geq \prod_{j=1}^M (|C| + j)^{-1} \cdot |\Gamma(C)| \geq \frac{|\Gamma(C)|}{(|C| + M)^M} \geq P^{-M} M^{-M} |\Gamma(C)| \end{aligned}$$

with $Q := \min_{k \in \mathbb{N}} \frac{|1+|C|-k|}{k}$, $R := Qe^{-1}$, $P := 2 \min(1, |C|)$ and some constant κ (from Stirling's formula). \square

Now consider the coefficient $\prod_{j=1}^N \Gamma(\gamma_j + \ell_j + 1)^{-1}$ in the Γ -series (23). Set $\gamma'_j = \gamma_j$ if $\gamma_j \notin \mathbb{Z}$ and $\gamma'_j = \gamma_j - \frac{1}{2}$ if $\gamma_j \in \mathbb{Z}$. Note that $\Gamma(k - \frac{1}{2}) = (k - \frac{3}{2}) \cdots \frac{1}{2} \Gamma(\frac{1}{2}) \leq (k-1)! \Gamma(\frac{1}{2}) = \Gamma(k) \sqrt{\pi}$ for $k \in \mathbb{Z}_{\geq 1}$. Then, using the above lemma, one sees that there are real constants $K, S > 0$ such that

$$\left| \prod_{j=1}^N \frac{1}{\Gamma(\gamma_j + \ell_j + 1)} \right| \leq \left| \prod_{j=1}^N \frac{\sqrt{\pi}}{\Gamma(\gamma'_j + \ell_j + 1)} \right| \leq K S^D \prod_{j=1}^N |\ell_j|^{-\ell_j}$$

with $D := \frac{1}{2} \sum_{j=1}^N |\ell_j| = \sum_{\ell_j > 0} \ell_j = -\sum_{\ell_j < 0} \ell_j$. Since $\prod_{\ell_j < 0} |\ell_j|^{-\ell_j} \leq D^D$ and $\prod_{\ell_j > 0} |\ell_j|^{-\ell_j} \leq N^D D^{-D}$, our final estimate becomes:

Proposition 2. *There are real numbers $K, T > 0$, depending on $\underline{\gamma} = (\gamma_1, \dots, \gamma_N)$, but independent of $\underline{\ell} = (\ell_1, \dots, \ell_N)$, such that*

$$\left| \prod_{j=1}^N \frac{1}{\Gamma(\gamma_j + \ell_j + 1)} \right| \leq K T^{\sum_{j=1}^N |\ell_j|}. \quad (32)$$

□

3.4. Γ -series and power series

Let $J \subset \{1, \dots, N\}$ be a set with $k+1$ elements, such that the vectors \mathbf{a}_j with $j \in J$ are linearly independent. Write $J' := \{1, \dots, N\} \setminus J$. Let $\underline{\gamma} = (\gamma_1, \dots, \gamma_N) \in \mathbb{C}^N$ be such that $\gamma_j \in \mathbb{Z}$ for $j \in J'$. Since $\frac{1}{\Gamma(s)} = 0$ if $s \in \mathbb{Z}_{\leq 0}$, the Γ -series (23) constructed with such a $\underline{\gamma}$ involves only terms from the set

$$\mathbb{L}_{J, \underline{\gamma}} := \{(\ell_1, \dots, \ell_N) \in \mathbb{L} \mid \gamma_j + \ell_j \geq 0 \text{ if } j \in J'\}. \quad (33)$$

The substitution

$$u_j = z_j \text{ if } j \in J', \quad u_j = 1 \text{ if } j \in J \quad (34)$$

therefore turns the Γ -series into the power series

$$\sum_{(\ell_1, \dots, \ell_N) \in \mathbb{L}_{J, \underline{\gamma}}} \left(\prod_{j=1}^N \frac{1}{\Gamma(\gamma_j + \ell_j + 1)} \right) \prod_{j \in J'} z_j^{\gamma_j + \ell_j}. \quad (35)$$

The following lemma is needed to convert (32) into estimates for the radii of convergence of this power series.

Lemma 2. *Let $J \subset \{1, \dots, N\}$ be a set with $k+1$ elements, such that the vectors \mathbf{a}_j with $j \in J$ are linearly independent. Write $J' := \{1, \dots, N\} \setminus J$. Then there is a positive real constant β such that for every $(\ell_1, \dots, \ell_N) \in \mathbb{L}$*

$$|\ell_1| + \dots + |\ell_N| \leq \beta \sum_{j \in J'} |\ell_j|. \quad (36)$$

Proof. Take any $d \times N$ -matrix \mathbf{B} whose rows form a \mathbb{Z} -basis of \mathbb{L} . Let $\mathbf{b}_1, \dots, \mathbf{b}_N$ be its columns. Let $\mathbf{B}_{J'}$ denote the $d \times d$ -matrix with columns \mathbf{b}_j ($j \in J'$). Then the matrix $\mathbf{B}_{J'}$ is invertible over \mathbb{Q} ; indeed, if it were not, its rows would be linearly dependent and there would be a vector $(\ell_1, \dots, \ell_N) \in \mathbb{L}$ such that $\ell_j = 0$ for $j \in J'$; the relation $\ell_1 \mathbf{a}_1 + \dots + \ell_N \mathbf{a}_N = 0$ would contradict the linear independence of the vectors \mathbf{a}_j with $j \in J$. Now we have the equality of row vectors for every $(\ell_1, \dots, \ell_N) \in \mathbb{L}$

$$(\ell_1, \dots, \ell_N) = (\ell)_{J'} (\mathbf{B}_{J'})^{-1} \mathbf{B}$$

where $(\ell)_{J'}$ is the row vector with components ℓ_j ($j \in J'$). So for β in (36) one can take the maximum of the absolute values of the entries of the matrix $(\mathbf{B}_{J'})^{-1} \mathbf{B}$. \square

Proposition 3. *Let $J \subset \{1, \dots, N\}$ be a set with $k+1$ elements, such that the vectors \mathbf{a}_j with $j \in J$ are linearly independent. Let $\underline{\gamma} = (\gamma_1, \dots, \gamma_N) \in \mathbb{C}^N$ be such that $\gamma_j \in \mathbb{Z}$ for $j \in J' := \{1, \dots, N\} \setminus J$. Then there is an $R \in \mathbb{R}_{>0}$ such that the power series (35) converges on the polydisc given by $|z_j| < R$ for $j = 1, \dots, d$.*

Proof. This follows, with $R = T^{-\beta}$, from Proposition 2 and Lemma 2. \square

3.5. Fourier Γ -series

The substitutions in (34) depend too rigidly on the choice of the set J and make it difficult to combine series constructed with different J 's. In order to get a more flexible framework we make in the Γ -series (23) the substitution of variables $u_j = e^{2\pi i w_j}$ for $j = 1, \dots, N$. We write $\mathbf{w} = (w_1, \dots, w_N)$, $\underline{\gamma} = (\gamma_1, \dots, \gamma_N)$ and $\underline{\ell} = (\ell_1, \dots, \ell_N)$. We also use the dot-product:

$$\mathbf{w} \cdot \underline{\ell} = w_1 \ell_1 + w_2 \ell_2 + \dots + w_N \ell_N.$$

With these new variables and notations the Γ -series (23) becomes

$$\Psi_{\mathbb{L}, \underline{\gamma}}(\mathbf{w}) = \sum_{\underline{\ell} \in \mathbb{L}} \frac{e^{2\pi i \mathbf{w} \cdot (\underline{\gamma} + \underline{\ell})}}{\prod_{j=1}^N \Gamma(\gamma_j + \ell_j + 1)}. \quad (37)$$

As in Section 3.4 we take a set $J \subset \{1, \dots, N\}$ with $k+1$ elements, such that the vectors \mathbf{a}_j with $j \in J$ are linearly independent and let $\underline{\gamma} = (\gamma_1, \dots, \gamma_N) \in \mathbb{C}^N$ be such that $\gamma_j \in \mathbb{Z}$ for $j \in J' := \{1, \dots, N\} \setminus J$. The vector $\sum_{i \in J'} \mathbf{a}_i$ is a \mathbb{Z} -linear combination of the vectors \mathbf{a}_j with $j \in J$. Such a relation is an element of \mathbb{L} . Thus one sees that \mathbb{L} contains an element $\underline{\ell} = (\ell_1, \dots, \ell_N)$ with $\ell_j = 1$ for all $j \in J'$. Since Γ -series do not change if one adds to $\underline{\gamma}$ an element of \mathbb{L} , we can assume without loss of generality that $\underline{\gamma} = (\gamma_1, \dots, \gamma_N) \in \mathbb{C}^N$ is such that $\gamma_j \in \mathbb{Z}_{\leq 0}$ for $j \in J' := \{1, \dots, N\} \setminus J$. Then the series $\Psi_{\mathbb{L}, \underline{\gamma}}(\mathbf{w})$ in (23) involves only terms from the set

$$\mathbb{L}_J := \{(\ell_1, \dots, \ell_N) \in \mathbb{L} \mid \ell_j \geq 0 \text{ if } j \in J'\}. \quad (38)$$

Using the estimates (32) we see that the series $\Psi_{\mathbb{L}, \underline{\gamma}}(\mathbf{w})$ converges if the imaginary part $\Im \mathbf{w}$ of \mathbf{w} satisfies $\Im \mathbf{w} \cdot \underline{\ell} > \frac{\log T}{2\pi}$ for every non-zero $\underline{\ell} \in \mathbb{L}_J$. We return to this issue and put it an appropriate perspective in Section 5.2.

3.6. Γ -series and GKZ differential equations.

As in the previous section we consider a $k+1$ -element subset $J \subset \{1, \dots, N\}$ such that the vectors \mathbf{a}_j with $j \in J$ are linearly independent, and a vector $\underline{\gamma} = (\gamma_1, \dots, \gamma_N) \in \mathbb{C}^N$ be such that $\gamma_j \in \mathbb{Z}$ for $j \in J' := \{1, \dots, N\} \setminus J$. The Γ -series (23) constructed with such a $\underline{\gamma}$ involves only terms from the set $\mathbb{L}_{J, \underline{\gamma}}$ (see (33)). For $M \in \mathbb{N}$ we define the M -th partial Γ -series $\Phi_{\mathbb{L}, \underline{\gamma}, M}(u_1, \dots, u_N)$ to be the subseries of (23) consisting of the terms with $|\ell_1| + \dots + |\ell_N| \leq M$. Then it follows, as in Proposition 3 from Proposition 2 and Lemma 2, that the sequence $\{\Phi_{\mathbb{L}, \underline{\gamma}, M}(u_1, \dots, u_N)\}_{M \in \mathbb{N}}$ converges for $M \rightarrow \infty$ to $\Phi_{\mathbb{L}, \underline{\gamma}}(u_1, \dots, u_N)$ if $|u_j| \leq (2T)^{-\beta}$ for $j \in J'$ and $\frac{1}{2} \leq |u_j| \leq 2$ for $j \in J$. So on this domain the Γ -series $\Phi_{\mathbb{L}, \underline{\gamma}}(u_1, \dots, u_N)$ becomes a function of (u_1, \dots, u_N) that can be differentiated term by term. This shows

- for $(\lambda_1, \dots, \lambda_N) \in \mathbb{L}$

$$\begin{aligned} \prod_{\lambda_i < 0} \left(\frac{\partial}{\partial u_i} \right)^{-\lambda_i} \Phi_{\mathbb{L}, \underline{\gamma}} &= \sum_{(\ell_1, \dots, \ell_N) \in \mathbb{L}} \prod_{j=1}^N \frac{u_j^{\gamma_j + \ell_j + \min(0, \lambda_j)}}{\Gamma(\gamma_j + \ell_j + 1 + \min(0, \lambda_j))} = \\ &= \sum_{(\ell_1, \dots, \ell_N) \in \mathbb{L}} \prod_{j=1}^N \frac{u_j^{\gamma_j + \ell_j + \lambda_j - \max(0, \lambda_j)}}{\Gamma(\gamma_j + \ell_j + \lambda_j + 1 - \max(0, \lambda_j))} = \prod_{\lambda_i > 0} \left(\frac{\partial}{\partial u_i} \right)^{\lambda_i} \Phi_{\mathbb{L}, \underline{\gamma}}. \end{aligned}$$

- for $(a_1, \dots, a_N) \in \mathbb{Z}^N$ such that $\sum_{j=1}^N a_j \ell_j = 0$ for every $(\ell_1, \dots, \ell_N) \in \mathbb{L}$:

$$\begin{aligned} \sum_{j=1}^N a_j u_j \frac{\partial \Phi_{\mathbb{L}, \underline{\gamma}}}{\partial u_j} &= \sum_{(\ell_1, \dots, \ell_N) \in \mathbb{L}} \left(\sum_{j=1}^N a_j (\gamma_j + \ell_j) \right) \prod_{j=1}^N \frac{u_j^{\gamma_j + \ell_j}}{\Gamma(\gamma_j + \ell_j + 1)} \\ &= \left(\sum_{j=1}^N a_j \gamma_j \right) \Phi_{\mathbb{L}, \underline{\gamma}} \end{aligned}$$

The latter system of differential equations is equivalent with the system (20) with $\mathbf{c} = \sum_{j=1}^N \gamma_j \mathbf{a}_j$. This shows:

Proposition 4. *As a function on its domain of convergence $\Phi_{\mathbb{L}, \underline{\gamma}}$ satisfies all differential equations of the GKZ system associated with \mathcal{A} and $\mathbf{c} = \sum_{j=1}^N \gamma_j \mathbf{a}_j$. \square*

Note that the Γ -series $\Phi_{\mathbb{L}, \underline{\gamma}}$ does not change if one adds to $\underline{\gamma}$ an element of \mathbb{L} whereas the differential equations (20) with $\mathbf{c} = \sum_{j=1}^N \gamma_j \mathbf{a}_j$ do not change if one adds to $\underline{\gamma}$ an element of $\mathbb{L} \otimes \mathbb{C}$.

4. The Secondary Fan

As before we consider a subset $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_N\} \subset \mathbb{Z}^{k+1}$ which generates \mathbb{Z}^{k+1} as an abelian group and for which there exists a group homomorphism $h : \mathbb{Z}^{k+1} \rightarrow \mathbb{Z}$

such that $h(\mathbf{a}) = 1$ for all $\mathbf{a} \in \mathcal{A}$. Still as before, we write

$$\mathbb{L} := \{(\ell_1, \dots, \ell_N) \in \mathbb{Z}^N \mid \ell_1 \mathbf{a}_1 + \dots + \ell_N \mathbf{a}_N = \mathbf{0}\},$$

and note that $\ell_1 + \dots + \ell_N = 0$ for every $(\ell_1, \dots, \ell_N) \in \mathbb{L}$. In order to better keep track of the various spaces involved we write \mathbb{M} instead of \mathbb{Z}^{k+1} . Thus the input data is a short exact sequence

$$0 \longrightarrow \mathbb{L} \longrightarrow \mathbb{Z}^N \longrightarrow \mathbb{M} \longrightarrow 0. \quad (39)$$

The vectors $\mathbf{a}_1, \dots, \mathbf{a}_N \in \mathbb{M}$ are the images of the standard basis vectors of \mathbb{Z}^N . We set $d := \text{rank } \mathbb{L}$ and $k+1 := \text{rank } \mathbb{M} = N - d$.

Apart from the common input data this section is independent of the sections on GKZ-systems and Γ -series. It concentrates on geometric and combinatorial structures associated with \mathcal{A} (or equivalently \mathbb{L}).

4.1. Construction of the secondary fan

We write $\mathbb{L}_{\mathbb{R}}^{\vee} := \text{Hom}(\mathbb{L}, \mathbb{R})$, $\mathbb{M}_{\mathbb{R}}^{\vee} := \text{Hom}(\mathbb{M}, \mathbb{R})$ and identify $\text{Hom}(\mathbb{Z}^N, \mathbb{R})$ and \mathbb{R}^N via the standard bases. The \mathbb{R} -dual of the exact sequence (39) is

$$0 \longrightarrow \mathbb{M}_{\mathbb{R}}^{\vee} \longrightarrow \mathbb{R}^N \xrightarrow{\pi} \mathbb{L}_{\mathbb{R}}^{\vee} \longrightarrow 0. \quad (40)$$

Let $\mathcal{P} := \{(x_1, \dots, x_N) \in \mathbb{R}^N \mid x_i \geq 0, \forall i\}$ be the positive orthant in \mathbb{R}^N and let

$$\hat{\pi} : \mathcal{P} \longrightarrow \mathbb{L}_{\mathbb{R}}^{\vee} \quad (41)$$

denote the restriction of π . Since the vector $(1, 1, \dots, 1)$ lies in $\ker \pi$ the map $\hat{\pi}$ is also surjective.

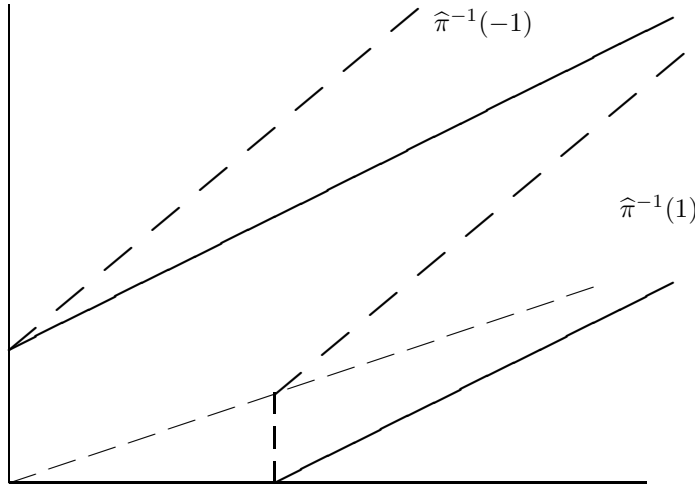


FIGURE 2. Fibres $\hat{\pi}^{-1}(1)$ and $\hat{\pi}^{-1}(-1)$ for $\mathbb{L} = \mathbb{Z}(-2, 1, 1) \subset \mathbb{R}^3$

Example. Take $\mathbb{L} = \mathbb{Z}(-2, 1, 1) \subset \mathbb{R}^3$. Then π can be identified with the map

$$\pi : \mathbb{R}^3 \longrightarrow \mathbb{R}, \quad \pi(x_1, x_2, x_3) = -2x_1 + x_2 + x_3.$$

For $t \in \mathbb{R}$ the polytope $\widehat{\pi}^{-1}(t)$ is the intersection of the positive octant and the plane with equation $-2x_1 + x_2 + x_3 = t$. Figure 2 illustrates this for $t = 1$ and $t = -1$ (with the x_1 -axis drawn vertically).

Let $\mathbf{b}_1, \dots, \mathbf{b}_N \in \mathbb{L}_{\mathbb{R}}^{\vee}$ be the images of the standard basis vectors of \mathbb{R}^N under the map π . Then, for $\mathbf{t} \in \mathbb{L}_{\mathbb{R}}^{\vee}$,

$$(x_1, \dots, x_N) \in \widehat{\pi}^{-1}(\mathbf{t}) \iff \mathbf{t} = x_1 \mathbf{b}_1 + \dots + x_N \mathbf{b}_N \quad \text{and} \quad x_i \geq 0, \forall i.$$

We see that the fiber $\widehat{\pi}^{-1}(\mathbf{t})$ is a convex (unbounded) polyhedron.

Lemma 3. $\mathbf{v} = (v_1, \dots, v_N) \in \mathcal{P}$ is a vertex of $\widehat{\pi}^{-1}(\mathbf{t})$ if and only if $\mathbf{t} = \sum_{j=1}^N v_j \mathbf{b}_j$ and the vectors \mathbf{b}_j with $v_j \neq 0$ are linearly independent over \mathbb{R} .

Proof. Suppose $\mathbf{t} = \sum_{j=1}^N v_j \mathbf{b}_j$, all $v_j \geq 0$ and the vectors \mathbf{b}_j with $v_j \neq 0$ are linearly dependent over \mathbb{R} . Then there is a non-trivial relation $\sum_{j=1}^N x_j \mathbf{b}_j = 0$ with $|x_j| \leq v_j$ for all j and the whole interval $\{\mathbf{v} + s(x_1, \dots, x_N) \mid |s| \leq 1\}$ lies in $\widehat{\pi}^{-1}(\mathbf{t})$. Therefore \mathbf{v} , being the midpoint of this interval, can not be a vertex of $\widehat{\pi}^{-1}(\mathbf{t})$.

Suppose $\mathbf{v} = (v_1, \dots, v_N) \in \widehat{\pi}^{-1}(\mathbf{t})$ is not a vertex of $\widehat{\pi}^{-1}(\mathbf{t})$. Then there is a non-zero vector $\mathbf{x} = (x_1, \dots, x_N) \in \mathbb{R}^N$ such that the interval $\{\mathbf{v} + s\mathbf{x} \mid |s| \leq 1\}$ lies in $\widehat{\pi}^{-1}(\mathbf{t}) = \mathcal{P} \cap \pi^{-1}(\mathbf{t})$. This implies $|x_j| \leq v_j$ for all j and $\sum_{j=1}^N x_j \mathbf{b}_j = 0$. Consequently, the vectors \mathbf{b}_j with $v_j \neq 0$ are linearly dependent over \mathbb{R} . \square

For a vertex $\mathbf{v} = (v_1, \dots, v_N)$ of $\widehat{\pi}^{-1}(\mathbf{t})$ we set

$$I_{\mathbf{v}} := \{i \mid v_i = 0\} \subset \{1, 2, \dots, N\}. \quad (42)$$

In this way every $\mathbf{t} \in \mathbb{L}_{\mathbb{R}}^{\vee}$ yields a list of subsets of $\{1, 2, \dots, N\}$:

$$T_{\mathbf{t}} := \{I_{\mathbf{v}} \mid \mathbf{v} \text{ vertex of } \widehat{\pi}^{-1}(\mathbf{t})\}. \quad (43)$$

Since $\pi^{-1}(\mathbf{t})$ has dimension $N - d$, the cardinality of each $I_{\mathbf{v}}$ must be at least $N - d$.

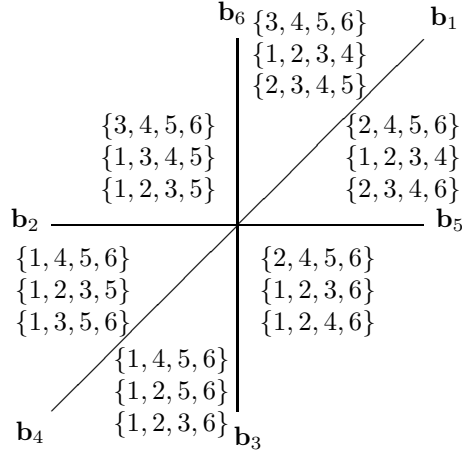
The above lemma provides an alternative description of the list $T_{\mathbf{t}}$:

Corollary 1. A subset $I \subset \{1, \dots, N\}$ is on the list $T_{\mathbf{t}}$ if and only if the vectors \mathbf{b}_j with $j \notin I$ are linearly independent over \mathbb{R} and $\mathbf{t} = \sum_{j \notin I} \tau_j \mathbf{b}_j$ with all $\tau_j \in \mathbb{R}_{>0}$. \square

We now define an equivalence relation on $\mathbb{L}_{\mathbb{R}}^{\vee}$ by: $\mathbf{t} \sim \mathbf{t}' \iff T_{\mathbf{t}} = T_{\mathbf{t}'}$. From Corollary 1 one sees that the equivalence class containing \mathbf{t} is

$$\mathcal{C} = \bigcap_{I \in T_{\mathbf{t}}} (\text{positive span of } \{\mathbf{b}_i\}_{i \notin I}). \quad (44)$$

So the equivalence classes are strongly convex polyhedral cones in $\mathbb{L}_{\mathbb{R}}^{\vee}$.

FIGURE 3. Secondary fan for F_1

Definition 3. *This collection of cones is called the secondary fan of \mathcal{A} (or \mathbb{L}).*

For an equivalence class \mathcal{C} we set $T_{\mathcal{C}} := T_t$ for any $t \in \mathcal{C}$. It follows from (44) that an equivalence class \mathcal{C} is an open cone of dimension d if and only if all sets on the list $T_{\mathcal{C}}$ have exactly $N - d$ elements.

Example. In the example of $\mathbb{L} = \mathbb{Z}(-2, 1, 1) \subset \mathbb{R}^3$ (see Figure 2) the vertices are given by the lists

$$T_t = \begin{cases} \{\{2, 3\}\} & \text{if } t < 0 \\ \{\{1, 2, 3\}\} & \text{if } t = 0 \\ \{\{1, 2\}, \{1, 3\}\} & \text{if } t > 0 \end{cases}$$

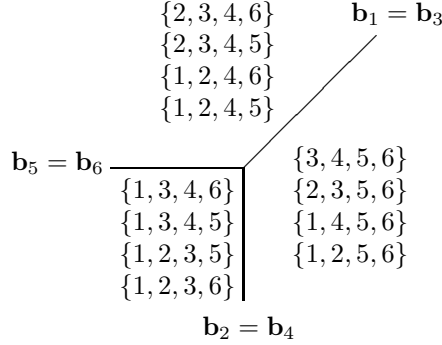
Example. For Gauss's hypergeometric structures $\mathbb{L} = \mathbb{Z}(1, 1, -1, -1) \subset \mathbb{R}^4$ and, hence, $\mathbf{b}_1 = \mathbf{b}_2 = 1$, $\mathbf{b}_3 = \mathbf{b}_4 = -1$ in \mathbb{R} . Corollary 1 now yields the lists

$$T_t = \begin{cases} \{\{1, 2, 3\}, \{1, 2, 4\}\} & \text{if } t < 0 \\ \{\{1, 2, 3, 4\}\} & \text{if } t = 0 \\ \{\{2, 3, 4\}, \{1, 3, 4\}\} & \text{if } t > 0 \end{cases}$$

Example. For Appell's F_1 the lattice $\mathbb{L} \subset \mathbb{Z}^6$ has rank 2 and is generated by the two vectors $(1, 1, 0, -1, -1, 0)$ and $(1, 0, 1, -1, 0, -1)$ which express that the three vertical segments in Figure 1 are parallel. The vectors $\mathbf{b}_1, \dots, \mathbf{b}_6 \in \mathbb{Z}^2$ are therefore

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathbf{b}_4 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \mathbf{b}_5 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{b}_6 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Figure 3 shows the secondary fan for F_1 and gives for each maximal cone \mathcal{C} the corresponding list $T_{\mathcal{C}}$ according to Corollary 1.

FIGURE 4. Secondary fan for F_4

Example. For Appell's F_4 the lattice $\mathbb{L} \subset \mathbb{Z}^6$ has rank 2 and is generated by the two vectors $(1, -1, 1, -1, 0, 0)$ and $(1, 0, 1, 0, -1, -1)$ which express that the three diagonals in Figure 1 intersect at the centre. The vectors $\mathbf{b}_1, \dots, \mathbf{b}_6 \in \mathbb{Z}^2$ are

$$\mathbf{b}_1 = \mathbf{b}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{b}_2 = \mathbf{b}_4 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad \mathbf{b}_5 = \mathbf{b}_6 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

Figure 4 shows the secondary fan for F_4 and gives for each maximal cone \mathcal{C} the corresponding list $T_{\mathcal{C}}$ according to Corollary 1.

Example/exercise. The reader is invited to apply the techniques demonstrated in the previous examples to the examples in Section 8.1 Table 1.

4.2. Alternative descriptions for secondary fan constructions.

We are going to present geometrically appealing alternative descriptions for the polyhedra $\hat{\pi}^{-1}(\mathbf{t})$ and for the lists $T_{\mathcal{C}}$ associated with the maximal cones in the secondary fan. Whereas the constructions in Section 4.1 were completely presented in terms of \mathbb{L} , the alternative descriptions use \mathcal{A} only.

4.2.1. Piecewise linear functions associated with \mathcal{A} . The vectors $\mathbf{a}_1, \dots, \mathbf{a}_N \in \mathbb{M}$ are linear functions on the space $\mathbb{M}_{\mathbb{R}}^{\vee} := \text{Hom}(\mathbb{M}, \mathbb{R})$. Let $\mathbb{M}_{\mathbb{R}} := \mathbb{M} \otimes \mathbb{R}$ and denote the pairing between $\mathbb{M}_{\mathbb{R}}$ and $\mathbb{M}_{\mathbb{R}}^{\vee}$ by $\langle \cdot, \cdot \rangle$. The inclusion $\mathbb{M}_{\mathbb{R}}^{\vee} \hookrightarrow \mathbb{R}^N$ is then given by

$$\mathbb{M}_{\mathbb{R}}^{\vee} \hookrightarrow \mathbb{R}^N, \quad \mathbf{v} \mapsto (\langle \mathbf{a}_1, \mathbf{v} \rangle, \dots, \langle \mathbf{a}_N, \mathbf{v} \rangle). \quad (45)$$

For an N -tuple $\underline{\alpha} = (\alpha_1, \dots, \alpha_N) \in \mathbb{R}^N$ one has the polyhedron

$$K_{\underline{\alpha}} := \{\mathbf{v} \in \mathbb{M}_{\mathbb{R}}^{\vee} \mid \langle \mathbf{a}_j, \mathbf{v} \rangle \geq -\alpha_j, \forall j\}. \quad (46)$$

Recall that $\mathbf{b}_1, \dots, \mathbf{b}_N \in \mathbb{L}_{\mathbb{R}}^{\vee}$ denote the images of the standard basis vectors of \mathbb{R}^N under the map π .

Proposition 5. For $\underline{\alpha} = (\alpha_1, \dots, \alpha_N) \in \mathbb{R}^N$ set $\mathbf{t} = \alpha_1 \mathbf{b}_1 + \dots + \alpha_N \mathbf{b}_N$. Then

$$\hat{\pi}^{-1}(\mathbf{t}) = \underline{\alpha} + K_{\underline{\alpha}}.$$

Proof. Since $\underline{\alpha}$ is in $\pi^{-1}(\mathbf{t})$ a point \mathbf{x} is in $\pi^{-1}(\mathbf{t})$ if and only if $\mathbf{x} - \underline{\alpha}$ is in $\ker \pi = \mathbb{M}_{\mathbb{R}}^{\vee}$. By definition, a point $\mathbf{x} = (x_1, \dots, x_N) \in \pi^{-1}(\mathbf{t})$ lies in $\widehat{\pi}^{-1}(\mathbf{t})$ if and only if $x_j \geq 0$ for all j . Thus, in view of (45),

$$\mathbf{x} \in \widehat{\pi}^{-1}(\mathbf{t}) \iff \mathbf{v} := \mathbf{x} - \underline{\alpha} \text{ satisfies } \langle \mathbf{a}_j, \mathbf{v} \rangle + \alpha_j \geq 0, \forall j.$$

□

Recall that throughout these notes we assume the existence of a group homomorphism $h : \mathbb{Z}^{k+1} \rightarrow \mathbb{Z}$ such that $h(\mathbf{a}) = 1$ for all $\mathbf{a} \in \mathcal{A}$. In the present terminology this amounts to the existence of an element $\mathbf{h} \in \mathbb{M}_{\mathbb{R}}^{\vee}$ such that $\langle \mathbf{a}_j, \mathbf{h} \rangle = 1$ for $j = 1, \dots, N$. Now fix a direct sum decomposition of real vector spaces

$$\mathbb{M}_{\mathbb{R}}^{\vee} = \mathbb{M}_{\mathbb{R}}^{\circ} \oplus \mathbb{R}\mathbf{h} \quad (47)$$

and consider the function

$$\mu_{\underline{\alpha}} : \mathbb{M}_{\mathbb{R}}^{\circ} \longrightarrow \mathbb{R}, \quad \mu_{\underline{\alpha}}(\mathbf{u}) = \min_j (\langle \mathbf{a}_j, \mathbf{u} \rangle + \alpha_j). \quad (48)$$

Proposition 6. *For every $\mathbf{u} \in \mathbb{M}_{\mathbb{R}}^{\circ}$ the vector $\mathbf{u} - \mu_{\underline{\alpha}}(\mathbf{u})\mathbf{h}$ lies in the boundary $\partial K_{\underline{\alpha}}$ of $K_{\underline{\alpha}}$. In other words $\partial K_{\underline{\alpha}}$ is the graph of the function $-\mu_{\underline{\alpha}}$ on $\mathbb{M}_{\mathbb{R}}^{\circ}$.*

Proof. Take $\mathbf{u} \in \mathbb{M}_{\mathbb{R}}^{\circ}$. Then one checks for every j

$$\langle \mathbf{a}_j, \mathbf{u} - \mu_{\underline{\alpha}}(\mathbf{u})\mathbf{h} \rangle = \langle \mathbf{a}_j, \mathbf{u} \rangle - \mu_{\underline{\alpha}}(\mathbf{u}) \geq \langle \mathbf{a}_j, \mathbf{u} \rangle - (\langle \mathbf{a}_j, \mathbf{u} \rangle + \alpha_j) = -\alpha_j.$$

So $\mathbf{u} - \mu_{\underline{\alpha}}(\mathbf{u})\mathbf{h}$ lies in $K_{\underline{\alpha}}$. If j is such that $\mu_{\underline{\alpha}}(\mathbf{u}) = \langle \mathbf{a}_j, \mathbf{u} \rangle + \alpha_j$, then the above computation shows $\langle \mathbf{a}_j, \mathbf{u} - \mu_{\underline{\alpha}}(\mathbf{u})\mathbf{h} \rangle = -\alpha_j$. Therefore $\mathbf{u} - \mu_{\underline{\alpha}}(\mathbf{u})\mathbf{h}$ lies in $\partial K_{\underline{\alpha}}$. □

If $\mu_{\underline{\alpha}}(\mathbf{u}) = \langle \mathbf{a}_j, \mathbf{u} \rangle + \alpha_j$, then the point $\mathbf{u} - \mu_{\underline{\alpha}}(\mathbf{u})\mathbf{h}$ lies in the affine hyperplane

$$\mathcal{H}_j^{\underline{\alpha}} := \{\mathbf{v} \in \mathbb{M}_{\mathbb{R}}^{\vee} \mid \langle \mathbf{a}_j, \mathbf{v} \rangle = -\alpha_j\}, \quad j = 1, \dots, N. \quad (49)$$

For generic $\mathbf{u} \in \mathbb{M}_{\mathbb{R}}^{\circ}$ (i.e. outside some codimension 1 closed subset) the minimum in (48) is attained for exactly one j . Therefore each codimension 1 face of the polyhedron $K_{\underline{\alpha}}$ lies in some unique hyperplane $\mathcal{H}_j^{\underline{\alpha}}$.

Remark. $K_{\underline{\alpha}}$ can also be described as the closure of that connected component of $\mathbb{M}_{\mathbb{R}}^{\vee} \setminus \bigcup_{j=1}^N \mathcal{H}_j^{\underline{\alpha}}$ that contains the points $t\mathbf{h}$ for sufficiently large t .

Example. Figure 5 shows (a piece of) the polyhedron $K_{\underline{\alpha}}$ for

$$\mathcal{A} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\} \quad (50)$$

and $\underline{\alpha} = (21, 35, 35, 14, 21, 28)$. Matching the faces of $K_{\underline{\alpha}}$ with the vectors in \mathcal{A} one checks that the list of vertices is $\{\{1, 2, 5\}, \{1, 4, 5\}, \{1, 3, 4\}, \{4, 5, 6\}, \{3, 4, 6\}\}$.

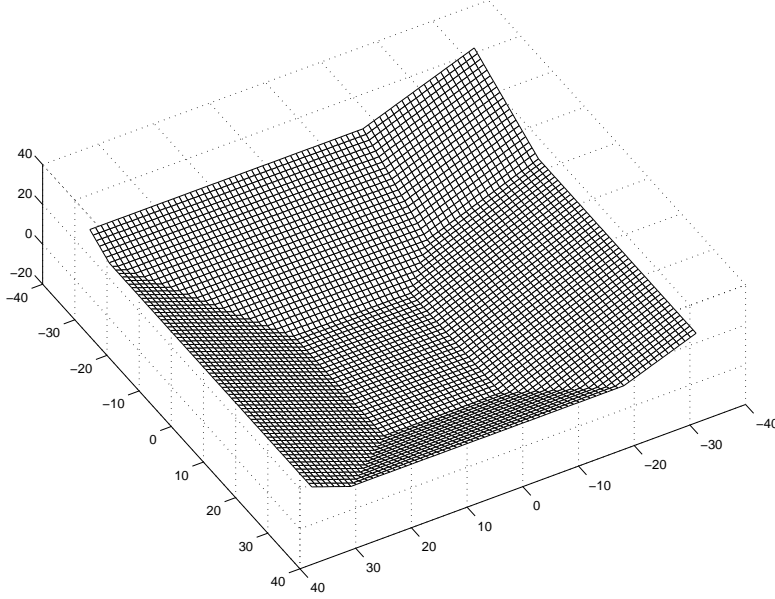


FIGURE 5. Example of a polyhedron $K_{\underline{\alpha}}$ for \mathcal{A} as in (50).

4.2.2. Regular triangulations. Assume $\underline{\alpha} = (\alpha_1, \dots, \alpha_N) \in \mathbb{R}^N$ with all $\alpha_j > 0$. Then the dual of the polyhedron $K_{\underline{\alpha}}$ in (46) is, by definition

$$K_{\underline{\alpha}}^{\vee} := \{\mathbf{w} \in \mathbb{M}_{\mathbb{R}} \mid \langle \mathbf{w}, \mathbf{v} \rangle > -1, \forall \mathbf{v} \in K_{\underline{\alpha}}\}. \quad (51)$$

Lemma 4. $K_{\underline{\alpha}}^{\vee} = \text{convex hull} \{0, \frac{1}{\alpha_1} \mathbf{a}_1, \dots, \frac{1}{\alpha_N} \mathbf{a}_N\}$.

Proof. The inclusion \supset follows directly from the definition of $K_{\underline{\alpha}}$ in (46). Now suppose that the two polyhedra are not equal. Then there is a point \mathbf{p} in $K_{\underline{\alpha}}^{\vee}$ which is separated by an affine hyperplane from $0, \frac{1}{\alpha_1} \mathbf{a}_1, \dots, \frac{1}{\alpha_N} \mathbf{a}_N$. That means that there is a vector $\mathbf{v} \in \mathbb{M}_{\mathbb{R}}^{\vee}$, perpendicular to the hyperplane, such that $\langle \mathbf{p}, \mathbf{v} \rangle < -1$ and $\langle \frac{1}{\alpha_j} \mathbf{a}_j, \mathbf{v} \rangle > -1$ for $j = 1, \dots, N$. The last N inequalities imply according to (46) that \mathbf{v} is in $K_{\underline{\alpha}}$, but then the first inequality contradicts $\mathbf{p} \in K_{\underline{\alpha}}^{\vee}$. So we conclude that the two polyhedra are equal. \square

Next we use the projection from the point 0 to project $K_{\underline{\alpha}}^{\vee}$ into the hyperplane with equation $\langle \mathbf{p}, \mathbf{h} \rangle = 1$. This maps $K_{\underline{\alpha}}^{\vee}$ onto the polytope

$$\Delta_{\mathcal{A}} := \text{convex hull}\{\mathbf{a}_1, \dots, \mathbf{a}_N\}. \quad (52)$$

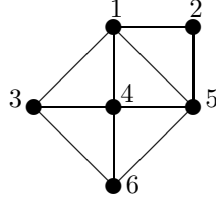


FIGURE 6. Triangulation corresponding with Figure 5

The images of the codimension 1 faces of $K_{\underline{a}}^{\vee}$ which do not contain the vertex 0 induce a subdivision of $\Delta_{\mathcal{A}}$ by the polytopes

$$\text{convex hull}\{\mathbf{a}_i\}_{i \in I} \quad \text{for } I \in T_{\mathbf{t}}, \quad (53)$$

where $\mathbf{t} = \alpha_1 \mathbf{b}_1 + \dots + \alpha_N \mathbf{b}_N$ as in Proposition 5 and $T_{\mathbf{t}}$ is the corresponding list of vertices of $\hat{\pi}^{-1}(\mathbf{t})$ as in (43).

If the point \mathbf{t} lies in some maximal cone \mathcal{C} of the secondary fan, all members of the list $T_{\mathbf{t}} = T_{\mathcal{C}}$ have $N - d = k + 1$ elements. The polytopal subdivision of $\Delta_{\mathcal{A}}$ is then a *triangulation*; i.e. all polytopes in the subdivision (53) are k -dimensional simplices.

Definition 4. *The triangulations of $\Delta_{\mathcal{A}}$ obtained in this way are called regular triangulations.*

Definition 5. *One defines the volume of a k -dimensional simplex with vertex set $\{\mathbf{a}_i\}_{i \in I}$ to be*

$$\text{volume}(\{\mathbf{a}_i\}_{i \in I}) = |\det((\mathbf{a}_i)_{i \in I})|. \quad (54)$$

A regular triangulation of $\Delta_{\mathcal{A}}$ is said to be unimodular if all k -dimensional simplices in the triangulation have volume equal to 1.

By abuse of language we will just say “the triangulation $T_{\mathcal{C}}$ ” instead of “the triangulation corresponding with the maximal cone \mathcal{C} ”. Note that useful information about distances between vertices of $\hat{\pi}^{-1}(\mathbf{t})$ gets lost in the passage to the (purely combinatorial) triangulation $T_{\mathcal{C}}$.

Remark. In general there can be triangulations of $\Delta_{\mathcal{A}}$ with vertices in $\{\mathbf{a}_1, \dots, \mathbf{a}_N\}$, which do not arise from the above construction and are therefore not regular.

4.3. The Secondary Polytope

The k -dimensional polytope $\Delta_{\mathcal{A}}$ defined in (52) is sometimes called the *primary polytope* associated with \mathcal{A} . By definition, the regular triangulations of $\Delta_{\mathcal{A}}$ correspond bijectively with the maximal cones of the secondary fan. To a regular triangulation $T_{\mathcal{C}}$ we assign the point $q_{\mathcal{C}} \in \mathbb{R}^N$ with

$$j^{\text{th}}\text{-coordinate of } q_{\mathcal{C}} = \sum_{I \in T_{\mathcal{C}} \text{ s.t. } j \in I} \text{volume}(\{\mathbf{a}_i\}_{i \in I}),$$

i.e. the sum of the volumes of the simplices in T_C of which \mathbf{a}_j is a vertex.

Definition 6. *The secondary polytope associated with \mathcal{A} is*

$$\text{Sec}(\mathcal{A}) = \text{convex hull} \{q_C \mid T_C \text{ regular triangulation of } \Delta_{\mathcal{A}}\}$$

The map $\mathbb{R}^N \rightarrow \mathbb{M}_{\mathbb{R}}$ maps the j -th standard basis vector of \mathbb{R}^N to \mathbf{a}_j . Thus the point q_C is mapped to

$$\begin{aligned} \sum_{j=1}^N \sum_{I \in T_C \text{ s.t. } j \in I} \text{volume}(\{\mathbf{a}_i\}_{i \in I}) \mathbf{a}_j &= \sum_{I \in T_C} \text{volume}(\{\mathbf{a}_i\}_{i \in I}) \left(\sum_{j \in I} \mathbf{a}_j \right) \\ &= (k+1) \times \text{volume}(\Delta_{\mathcal{A}}) \times \text{barycenter}(\Delta_{\mathcal{A}}). \end{aligned}$$

So the whole secondary polytope is mapped to one point. Therefore, after some translation in \mathbb{R}^N we find the secondary polytope in \mathbb{L} :

$$\text{Sec}(\mathcal{A}) \subset \mathbb{L} \otimes \mathbb{R}.$$

As for the relation between secondary fan and secondary polytope we mention the following theorem, which is in a slightly different formulation proven in [13].

Theorem 2. ([13] p.221 thm.1.7) *The secondary fan, which lies in $\mathbb{L}_{\mathbb{R}}^{\vee}$, is in fact the fan of outward pointing vectors perpendicular to the faces of $\text{Sec}(\mathcal{A})$. \square*

Example. In the example of $\mathbb{L} = \mathbb{Z}(-2, 1, 1) \subset \mathbb{Z}^3$ there are two maximal cones: $\mathbb{R}_{>0}$ and $\mathbb{R}_{<0}$. The corresponding triangulations are:



The secondary polytope is the line segment between the points $(0, 2, 2)$ and $(2, 1, 1)$ in \mathbb{R}^3 .

Example. For Gauss's hypergeometric structures $\mathbb{L} = \mathbb{Z}(1, 1, -1, -1) \subset \mathbb{Z}^4$. From this one sees that $\mathbb{L}_{\mathbb{R}}^{\vee} \simeq \mathbb{R}$ and that there are two maximal cones: $\mathbb{R}_{>0}$ and $\mathbb{R}_{<0}$. The corresponding triangulations are:



The secondary polytope is the line segment between the points $(2, 2, 1, 1)$ and $(1, 1, 2, 2)$ in \mathbb{R}^4 .

Example: For $\mathbb{L} = \mathbb{Z}(-3 \ 1 \ 1 \ 1) \subset \mathbb{Z}^4$ we have $\mathbb{L}_{\mathbb{R}}^{\vee} \simeq \mathbb{R}$ and $\mathbf{b}_1 = -3, \mathbf{b}_2 = \mathbf{b}_3 = \mathbf{b}_4 = 1$. Corollary 1 shows for $t \in \mathbb{L}_{\mathbb{R}}^{\vee} \simeq \mathbb{R}$:

$$T_t = \begin{cases} \{ \{1, 3, 4\}, \{1, 2, 4\}, \{1, 2, 3\} \} & \text{if } t > 0 \\ \{ \{1, 2, 3, 4\} \} & \text{if } t = 0 \\ \{ \{2, 3, 4\} \} & \text{if } t < 0 \end{cases}$$

So there are two maximal cones: $\mathbb{R}_{>0}$ and $\mathbb{R}_{<0}$. In terms of triangulations:

$$\mathcal{A} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$$

The secondary polytope is the line segment between the points $(0, 3, 3, 3)$ and $(3, 2, 2, 2)$ in \mathbb{R}^4 .

Example: Figure 7 shows the secondary polytope with at each vertex the corresponding regular triangulation of $\Delta_{\mathcal{A}}$ for \mathcal{A} as in (50).

5. The toric variety associated with the Secondary Fan

5.1. Construction of the toric variety for the secondary fan

The secondary fan is a complete fan of strongly convex polyhedral cones in $\mathbb{L}_{\mathbb{R}}^{\vee} := \text{Hom}(\mathbb{L}, \mathbb{R})$ which are generated by vectors from the lattice $\mathbb{L}_{\mathbb{Z}}^{\vee} := \text{Hom}(\mathbb{L}, \mathbb{Z})$. By the general theory of toric varieties [9] this lattice-fan pair gives rise to a toric variety. We are going to describe the construction of the toric variety for the case of $\mathbb{L}_{\mathbb{Z}}^{\vee}$ and the secondary fan. Before starting we must point out that [9] works with a fan of *closed cones*, while the cones in our Definition 3 of the secondary fan are *not closed*; see also Formula (44). This difference, however, only affects a few minor subtleties in the formulation at intermediate stages. The monoids (55) and therefore also the resulting toric varieties are the same as in [9].

We denote the pairing between $\mathbb{L}_{\mathbb{R}} := \mathbb{L} \otimes_{\mathbb{Z}} \mathbb{R}$ and $\mathbb{L}_{\mathbb{R}}^{\vee}$ by $\langle \cdot, \cdot \rangle$. For each cone \mathcal{C} in the secondary fan (see (44)) one considers the affine scheme $\mathcal{U}_{\mathcal{C}} := \text{Spec } \mathbb{Z}[\mathbb{L}_{\mathcal{C}}]$ associated with the monoid ring $\mathbb{Z}[\mathbb{L}_{\mathcal{C}}]$ of the monoid¹

$$\mathbb{L}_{\mathcal{C}} := \{ \ell \in \mathbb{L} \mid \langle \omega, \ell \rangle \geq 0 \text{ for all } \omega \in \mathcal{C} \}. \quad (55)$$

In down-to-earth terms, a complex point of $\mathcal{U}_{\mathcal{C}}$ is just a homomorphism from the additive monoid $\mathbb{L}_{\mathcal{C}}$ to the multiplicative monoid \mathbb{C} , sending $0 \in \mathbb{L}_{\mathcal{C}}$ to $1 \in \mathbb{C}$.

For cones \mathcal{C} and \mathcal{C}' in the secondary fan such that \mathcal{C}' is contained in the closure $\overline{\mathcal{C}}$ of \mathcal{C} , there are inclusions

$$\mathcal{C}' \subset \overline{\mathcal{C}}, \quad \mathbb{L}_{\mathcal{C}'} \supset \mathbb{L}_{\mathcal{C}}, \quad \mathbb{Z}[\mathbb{L}_{\mathcal{C}'}] \supset \mathbb{Z}[\mathbb{L}_{\mathcal{C}}], \quad \mathcal{U}_{\mathcal{C}'} \subset \mathcal{U}_{\mathcal{C}};$$

¹Alternative terminology: monoid = semi-group

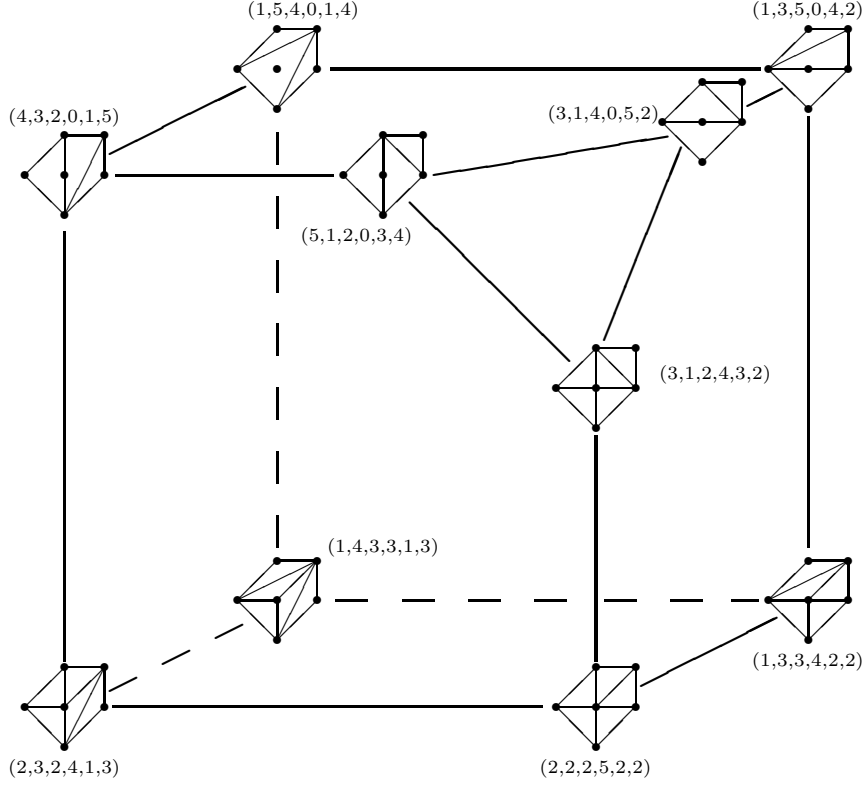


FIGURE 7. The secondary polytope and all regular triangulations for \mathcal{A} as in (50).

more precisely, the following lemma shows that the inclusion $\mathcal{U}_{\mathcal{C}'} \hookrightarrow \mathcal{U}_{\mathcal{C}}$ is an open immersion associated with the inversion of an element in the ring $\mathbb{Z}[\mathbb{L}_{\mathcal{C}}]$.

Lemma 5. *In the above situation $\mathbb{L}_{\mathcal{C}'} = \mathbb{L}_{\mathcal{C}} + \mathbb{Z}\lambda$ for some $\lambda \in \mathbb{L}_{\mathcal{C}}$.*

Proof. If $\mathcal{C}' = \mathcal{C}$, the result is trivial. So assume $\mathcal{C}' \neq \mathcal{C}$. Since \mathcal{C} is a rational polyhedral cone it is spanned by finitely many $\omega_1, \dots, \omega_p \in \mathbb{L}_{\mathbb{Z}}^{\vee}$, i.e. every point in \mathcal{C} is a linear combination with non-negative real coefficients of $\omega_1, \dots, \omega_p$. Moreover since $\mathcal{C}' \subset \overline{\mathcal{C}}$ there is a $\lambda \in \mathbb{L}_{\mathcal{C}}$ such that $\langle \omega', \lambda \rangle = 0$ for all $\omega' \in \mathcal{C}'$ and $\langle \omega, \lambda \rangle > 0$ for all $\omega \in \mathcal{C}$. Take $\mu \in \mathbb{R}_{>0}$ such that for $j = 1, \dots, p$ we have $\langle \omega_j, \lambda \rangle > \mu$ if $\omega_j \notin \mathcal{C}'$. For every $\underline{\ell} \in \mathbb{L}_{\mathcal{C}'}$ and every non-negative integer $r > -\frac{1}{\mu} \min_j \langle \omega_j, \underline{\ell} \rangle$, one now easily checks that $\langle \omega, \underline{\ell} + r\lambda \rangle \geq 0$ for every $\omega \in \mathcal{C}$, and hence $\underline{\ell} + r\lambda \in \mathbb{L}_{\mathcal{C}}$. \square

Definition 7. *The toric variety associated with the secondary fan is the scheme that results from glueing the affine schemes $\mathcal{U}_{\mathcal{C}}$, where \mathcal{C} ranges over all cones in the secondary fan, using the open immersions $\mathcal{U}_{\mathcal{C}'} \hookrightarrow \mathcal{U}_{\mathcal{C}}$ for $\mathcal{C}' \subset \overline{\mathcal{C}}$. We denote this toric variety by $\mathcal{V}_{\mathcal{A}}$.*

For every cone \mathcal{C} of the secondary fan the monoid $\mathbb{L}_{\mathcal{C}}$ splits as a disjoint union $\mathbb{L}_{\mathcal{C}} = \mathbb{L}_{\mathcal{C}}^0 \amalg \mathbb{L}_{\mathcal{C}}^+$ where $\mathbb{L}_{\mathcal{C}}^0$ (resp. $\mathbb{L}_{\mathcal{C}}^+$) is the set of elements which do (resp. do not) have an inverse in the additive monoid $\mathbb{L}_{\mathcal{C}}$. One easily checks that

$$\mathbb{L}_{\mathcal{C}}^0 := \{ \underline{\ell} \in \mathbb{L} \mid \langle \omega, \underline{\ell} \rangle = 0 \quad \forall \omega \in \mathcal{C} \}, \quad \mathbb{L}_{\mathcal{C}}^+ := \{ \underline{\ell} \in \mathbb{L} \mid \langle \omega, \underline{\ell} \rangle > 0 \quad \forall \omega \in \mathcal{C} \}.$$

If $\mathcal{C} = \{0\}$, then $\mathbb{L}_{\mathcal{C}}^0 = \mathbb{L}_{\mathcal{C}}$ and $\mathbb{L}_{\mathcal{C}}^+ = \emptyset$. If $\mathcal{C} \neq \{0\}$, the elements of $\mathbb{L}_{\mathcal{C}}^+$ generate a proper ideal $I_{\mathcal{C}}$ in the ring $\mathbb{Z}[\mathbb{L}_{\mathcal{C}}]$ and one has in $\mathcal{U}_{\mathcal{C}}$ the closed subscheme

$$\mathcal{B}_{\mathcal{C}} := \text{Spec } \mathbb{Z}[\mathbb{L}_{\mathcal{C}}] / I_{\mathcal{C}}.$$

Let us see what this amounts to for the complex points of $\mathcal{U}_{\mathcal{C}}$, viewed as homomorphisms from the additive monoid $\mathbb{L}_{\mathcal{C}}$ into the multiplicative monoid \mathbb{C} . Each such homomorphism has to send invertible elements to invertible elements, i.e. $\mathbb{L}_{\mathcal{C}}^0$ into \mathbb{C}^* . If $\mathcal{C} = \{0\}$, the set of complex points of $\mathcal{U}_{\mathcal{C}}$ can therefore be identified with the set (in fact, d -dimensional torus) of group homomorphisms from \mathbb{L} into \mathbb{C}^* :

$$\mathcal{U}_{\{0\}}(\mathbb{C}) = \text{Hom}(\mathbb{L}, \mathbb{C}^*) = \mathbb{L}_{\mathbb{Z}}^{\vee} \otimes_{\mathbb{Z}} \mathbb{C}^*. \quad (56)$$

If $\mathcal{C} \neq \{0\}$, the complex points of $\mathcal{B}_{\mathcal{C}}$ are those monoid homomorphisms that send all elements of $\mathbb{L}_{\mathcal{C}}^+$ to $\{0\}$. So, the set of complex points of $\mathcal{B}_{\mathcal{C}}$ can be identified with the set of group homomorphisms from $\mathbb{L}_{\mathcal{C}}^0$ into \mathbb{C}^* :

$$\mathcal{B}_{\mathcal{C}}(\mathbb{C}) = \text{Hom}(\mathbb{L}_{\mathcal{C}}^0, \mathbb{C}^*).$$

This is a torus of dimension $d - \dim \mathcal{C}$.

If \mathcal{C} is a maximal cone, then $\mathbb{L}_{\mathcal{C}}^0 = \{0\}$ and $\mathcal{B}_{\mathcal{C}}(\mathbb{C})$ is only one point, which we denote as $\mathbf{p}_{\mathcal{C}}$. For every positive real number $r < 1$ the homomorphisms $\mathbb{L}_{\mathcal{C}} \rightarrow \mathbb{C}$ mapping $\mathbb{L}_{\mathcal{C}}^+$ into the disc of radius r centred at 0 in \mathbb{C} form an open neighborhood of $\mathbf{p}_{\mathcal{C}}$, which we will also call the disc of radius r about $\mathbf{p}_{\mathcal{C}}$ in $\mathcal{U}_{\mathcal{C}}(\mathbb{C})$.

Example. For Gauss's hypergeometric structures $\mathbb{L} = \mathbb{Z}(1, 1, -1, -1) \subset \mathbb{R}^4$. So, $\mathbb{L}_{\mathbb{R}}^{\vee} \simeq \mathbb{R}$ and the secondary fan has two maximal cones: $\mathbb{R}_{>0}$ and $\mathbb{R}_{<0}$. One can easily see that the associated toric variety is the projective line \mathbb{P}^1 (see [9] p.6).

Example. For Appell's F_4 the secondary fan is shown in Figure 4. One can easily see that the associated toric variety is the projective plane \mathbb{P}^2 (see [9] p.6-7).

Example. For Appell's F_1 the secondary fan is shown in Figure 3. One can easily see that the associated toric variety is the projective plane \mathbb{P}^2 with three points blown up (see [9]).

5.2. Convergence of Fourier Γ -series and the secondary fan

We want to use the toric variety associated with the secondary fan to put the domains of convergence of the Fourier Γ -series (37) in the proper perspective. Let us write $\widehat{\mathbf{w}} \in \mathbb{L}_{\mathbb{C}}^{\vee}$ for the image of $\mathbf{w} \in \mathbb{C}^N$ under the natural projection $\mathbb{C}^N \longrightarrow \mathbb{L}_{\mathbb{C}}^{\vee} := \text{Hom}(\mathbb{L}, \mathbb{C})$ (linear dual; cf. (40)). Then $\mathbf{w} \cdot \underline{\ell} = \langle \widehat{\mathbf{w}}, \underline{\ell} \rangle$ for all $\underline{\ell} \in \mathbb{L}$ and (37) can be rewritten as

$$\Psi_{\mathbb{L}, \underline{\gamma}}(\mathbf{w}) = e^{2\pi i \mathbf{w} \cdot \underline{\gamma}} \sum_{\underline{\ell} \in \mathbb{L}} \frac{e^{2\pi i \langle \widehat{\mathbf{w}}, \underline{\ell} \rangle}}{\prod_{j=1}^N \Gamma(\gamma_j + \ell_j + 1)}. \quad (57)$$

A vector $\widehat{\mathbf{w}} \in \mathbb{L}_{\mathbb{C}}^{\vee}$ defines a homomorphism

$$\mathbb{L} \rightarrow \mathbb{C}^*, \quad \underline{\ell} \mapsto e^{2\pi i \langle \widehat{\mathbf{w}}, \underline{\ell} \rangle}$$

and, hence, a complex point of the toric variety $\mathcal{V}_{\mathcal{A}}$. This point lies in the disc of radius $r < 1$ about the special point $\mathbf{p}_{\mathcal{C}}$ corresponding to a maximal cone \mathcal{C} of the secondary fan if and only if $\langle \widehat{\mathbf{w}}, \underline{\ell} \rangle > -\frac{\log r}{2\pi}$ for every non-zero $\underline{\ell} \in \mathbb{L}_{\mathcal{C}}$; this means that $\widehat{\mathbf{w}}$ should lie ‘sufficiently far’ inside the cone \mathcal{C} .

Recall that a maximal cone \mathcal{C} of the secondary fan corresponds to a regular triangulation of the polytope $\Delta_{\mathcal{A}}$. The index sets of the vertices of the maximal simplices in this triangulation constitute a list $T_{\mathcal{C}}$ of subsets of $\{1, \dots, N\}$ with $N - d$ elements, and according to (44)

$$\mathcal{C} = \bigcap_{J \in T_{\mathcal{C}}} (\text{positive span of } \{\mathbf{b}_j\}_{j \notin J}). \quad (58)$$

Now note that for $\underline{\ell} = (\ell_1, \dots, \ell_N) \in \mathbb{L}$ almost tautologically $\ell_j = \langle \underline{\ell}, \mathbf{b}_j \rangle$. This shows that for \mathbb{L}_J as defined in (38)

$$\mathbb{L}_J \subset \mathbb{L}_{\mathcal{C}} \quad \text{for every } J \in T_{\mathcal{C}}. \quad (59)$$

The above arguments together with those in Section 3.5 show:

Proposition 7. *Let \mathcal{C} be a maximal cone of the secondary fan. Let $J \in T_{\mathcal{C}}$. Let $\underline{\gamma} = (\gamma_1, \dots, \gamma_N) \in \mathbb{C}^N$ be such that $\gamma_j \in \mathbb{Z}_{\leq 0}$ for $j \notin J$. Then there is a positive real constant $r < 1$ (depending on $\underline{\gamma}$) such that the Fourier Γ -series $\Psi_{\mathbb{L}, \underline{\gamma}}(\mathbf{w})$ in (57) converges for every $\mathbf{w} \in \mathbb{C}^N$ for which $\widehat{\mathbf{w}}$ defines a point in the disc of radius r about the special point $\mathbf{p}_{\mathcal{C}}$ in the toric variety $\mathcal{V}_{\mathcal{A}}$. \square*

5.3. Solutions of GKZ differential equations and the secondary fan

Let us look for local solutions to the GKZ differential equations (19)-(20) associated with an N -element subset $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_N\} \subset \mathbb{Z}^{k+1}$ and a vector $\mathbf{c} \in \mathbb{C}^{k+1}$. Let \mathcal{C} be a maximal cone in the secondary fan of \mathcal{A} . According to Proposition 7, every vector $\underline{\gamma} = (\gamma_1, \dots, \gamma_N) \in \mathbb{C}^N$ which satisfies

$$\gamma_1 \mathbf{a}_1 + \dots + \gamma_N \mathbf{a}_N = \mathbf{c}, \quad (60)$$

$$\exists J \in T_{\mathcal{C}} \quad \text{such that} \quad \gamma_j \in \mathbb{Z}_{\leq 0} \quad \text{for } j \notin J, \quad (61)$$

yields a Fourier Γ -series $\Psi_{\mathbb{L}, \underline{\gamma}}(\mathbf{w})$ converging for every $\mathbf{w} \in \mathbb{C}^N$ for which $\widehat{\mathbf{w}}$ defines a point in a sufficiently small disc about the point $\mathbf{p}_{\mathcal{C}}$ in $\mathcal{V}_{\mathcal{A}}$. According to Section 3.6 the corresponding Γ -series satisfies the GKZ differential equations (19)-(20) for \mathcal{A} and \mathbf{c} . If $\underline{\gamma} \equiv \underline{\gamma}' \pmod{\mathbb{L}}$, then the two Fourier Γ -series are equal. Lemma 6 will imply that the number of \mathbb{L} -congruence classes of solutions to (60)-(61) is finite and, hence, *the Fourier Γ -series we obtain in this way have a common domain of convergence.*

Remark. Because of the factor $e^{2\pi i \mathbf{w} \cdot \underline{\gamma}}$ in (57) the Fourier Γ -series $\Psi_{\mathbb{L}, \underline{\gamma}}(\mathbf{w})$ will in general not descend to a function on some disc about $\mathbf{p}_{\mathcal{C}}$ in $\mathcal{V}_{\mathcal{A}}$. On the other hand, if $\underline{\gamma}$ and $\underline{\gamma}'$ both satisfy (60)-(61), then $\underline{\gamma} - \underline{\gamma}' \in \mathbb{L}_{\mathcal{C}}$ and $\mathbf{w} \cdot (\underline{\gamma} - \underline{\gamma}') = \langle \widehat{\mathbf{w}}, \underline{\gamma} - \underline{\gamma}' \rangle$ for every $\mathbf{w} \in \mathbb{C}^N$. This means that the quotient $\Psi_{\mathbb{L}, \underline{\gamma}}(\mathbf{w}) \Psi_{\mathbb{L}, \underline{\gamma}'}^{-1}(\mathbf{w})$ does descend to a function on some disc about $\mathbf{p}_{\mathcal{C}}$ in $\mathcal{V}_{\mathcal{A}}$.

Lemma 6. *Fix $\mathbf{c} \in \mathbb{C}^{k+1}$ and a $k+1$ -element set $J \subset \{1, \dots, N\}$ such that the vectors \mathbf{a}_j with $j \in J$ are linearly independent. Then the number of classes modulo \mathbb{L} of vectors $\underline{\gamma} = (\gamma_1, \dots, \gamma_N) \in \mathbb{C}^N$ which satisfy Equation (60) and $\gamma_j \in \mathbb{Z}$ for $j \in J' := \{1, \dots, N\} \setminus J$, is equal to $|\det((\mathbf{a}_j)_{j \in J})|$.*

Proof. Since the vectors \mathbf{a}_j with $j \in J$ are linearly independent, the equation $\sum_{j=1}^N \gamma_j \mathbf{a}_j = \mathbf{c}$ can be solved in parametric form with the components γ_j for $j \in J'$ as free parameters. Every solution is the sum of one particular solution of the inhomogeneous system (e.g. the solution with $\gamma_j = 0$ for $j \in J'$) and a solution of the homogeneous system. So it suffices to determine the number of \mathbb{L} -equivalence classes of solutions of the equation $\sum_{j=1}^N \gamma_j \mathbf{a}_j = \mathbf{0}$ with $\gamma_j \in \mathbb{Z}$ for $j \in J'$. The solutions themselves lie in $\mathbb{L} \otimes \mathbb{Q}$.

Take any $d \times N$ -matrix \mathbf{B} whose rows form a \mathbb{Z} -basis of \mathbb{L} . This amounts to choosing an isomorphism $\mathbb{L} \simeq \mathbb{Z}^d$. Let $\mathbf{B}_{J'}$ (resp. \mathbf{B}_J) denote the submatrix of \mathbf{B} formed by the columns with index in J' (resp. in J). As in the proof of Lemma 2 one sees that the matrix $\mathbf{B}_{J'}$ is invertible over \mathbb{Q} and that the set of solutions of $\sum_{j=1}^N \gamma_j \mathbf{a}_j = \mathbf{0}$ with $\gamma_j \in \mathbb{Z}$ for $j \in J'$ is $\mathbb{Z}^d(\mathbf{B}_{J'})^{-1} \subset \mathbb{Q}^d \simeq \mathbb{L} \otimes \mathbb{Q}$; the notation $\mathbb{Z}^d(\mathbf{B}_{J'})^{-1}$ refers to the fact that here \mathbb{Z}^d consists of row vectors. The number of classes modulo \mathbb{L} of such solutions is therefore

$$\# \left(\mathbb{Z}^d(\mathbf{B}_{J'})^{-1} / \mathbb{Z}^d \right) = \# \left(\mathbb{Z}^d / \mathbb{Z}^d \mathbf{B}_{J'} \right) = |\det(\mathbf{B}_{J'})|.$$

Thus we must prove:

$$|\det(\mathbf{B}_{J'})| = |\det((\mathbf{a}_j)_{j \in J})|. \quad (62)$$

Proof of (62): Let \mathbf{A} (resp. \mathbf{A}_J resp. $\mathbf{A}_{J'}$) denote the matrix with columns \mathbf{a}_j with $j \in \{1, \dots, N\}$ (resp. $j \in J$ resp. $j \in J'$). Then $\mathbf{A}\mathbf{B}^t = \mathbf{0}$ and hence

$$\mathbf{A}_J^{-1} \mathbf{A}_{J'} = -(\mathbf{B}_{J'}^{-1} \mathbf{B}_J)^t. \quad (63)$$

As in Cramer's rule one sees that the matrix entries on the left hand side of (63) are all of the form $\pm(\det \mathbf{A}_J)^{-1}(\det \mathbf{A}_I)$ with $I \subset \{1, \dots, N\}$ such that $\#I = k+1$

and $\sharp(I \cap J) = k$. The corresponding matrix entries on the right hand side of (63) are $\pm(\det \mathbf{B}_{J'})^{-1}(\det \mathbf{B}_{I'})$ with $I' := \{1, \dots, N\} \setminus I$. Thus we see

$$|\det \mathbf{A}_J|^{-1} |\det \mathbf{A}_I| = |\det \mathbf{B}_{J'}|^{-1} |\det \mathbf{B}_{I'}|,$$

first for every $I \subset \{1, \dots, N\}$ such that $\sharp I = k+1$ and $\sharp(I \cap J) = k$ and then, by induction, for every $k+1$ -element subset $I \subset \{1, \dots, N\}$. Consequently there are coprime positive integers a, b such that

$$b |\det \mathbf{A}_I| = a |\det \mathbf{B}_{I'}| \quad (64)$$

for every $k+1$ -element subset $I \subset \{1, \dots, N\}$. Now recall that the columns $\mathbf{a}_1, \dots, \mathbf{a}_N$ of \mathbf{A} generate \mathbb{Z}^{k+1} . This implies that the greatest common divisor of the numbers $\det \mathbf{A}_I$ is 1. So $a = 1$ in (64). On the other hand, the rows of \mathbf{B} form a \mathbb{Z} -basis of \mathbb{L} . Therefore for every prime number p the rows of the matrix $\mathbf{B} \bmod p\mathbb{Z}$ are linearly independent over the field $\mathbb{Z}/p\mathbb{Z}$ and at least one of the numbers $\det \mathbf{B}_{I'}$ must be not divisible by p . This shows $b = 1$ in (64) and finishes the proof of Formula (62). \square

Lemma 7. *Let \mathcal{C} be a maximal cone of the secondary fan. Let $\underline{\gamma}^1, \dots, \underline{\gamma}^p$ be solutions to the equations (60)-(61) such that $\underline{\gamma}^i \not\equiv \underline{\gamma}^j \bmod \mathbb{L}$ for $i \neq j$. Then the Fourier Γ -series $\Psi_{\mathbb{L}, \underline{\gamma}^1}(\mathbf{w}), \dots, \Psi_{\mathbb{L}, \underline{\gamma}^p}(\mathbf{w})$ are linearly independent over \mathbb{C} .*

Proof. Fix a positive real constant $r < 1$ such that all the given Fourier Γ -series converge for every $\mathbf{w} \in \mathbb{C}^N$ for which $\widehat{\mathbf{w}}$ defines a point in the disc of radius r about the special point $\mathbf{p}_{\mathcal{C}}$ in the toric variety $\mathcal{V}_{\mathcal{A}}$ (cf. Proposition 7). Next choose $\mathbf{w} \in \mathbb{C}^N$ such that $\widehat{\mathbf{w}}$ defines a point in that disc and such that no two of the numbers $\Im \mathbf{w} \cdot (\underline{\gamma}^j + \underline{\ell})$ with $1 \leq j \leq p$ and $\underline{\ell} \in \mathbb{L}_{\mathcal{C}}$ such that the $\underline{\ell}$ -th term in the Fourier Γ -series $\Psi_{\mathbb{L}, \underline{\gamma}^j}(\mathbf{w})$ is not 0, are equal. For this \mathbf{w} the set of those numbers $\Im \mathbf{w} \cdot (\underline{\gamma}^j + \underline{\ell})$ assumes its minimum for a unique pair, say $(\underline{\gamma}^m, \underline{\ell}^m)$. This implies

$$\lim_{t \rightarrow \infty} e^{-2\pi i t \mathbf{w} \cdot (\underline{\gamma}^m + \underline{\ell}^m)} \Psi_{\mathbb{L}, \underline{\gamma}^j}(t\mathbf{w}) = 0 \quad \text{if } j \neq m, \quad \text{resp.} \quad \neq 0 \quad \text{if } j = m.$$

The linear independence claimed in the lemma now follows immediately. \square

It follows from Lemma 6 that the number of \mathbb{L} -congruence classes of solutions to the Equations (60)-(61) is less than or equal to

$$\sum_{J \in T_{\mathcal{C}}} |\det ((\mathbf{a}_j)_{j \in J})| = \text{volume } \Delta_{\mathcal{A}}.$$

Definition 8. *Let \mathcal{C} be a maximal cone of the secondary fan of $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_N\}$ and let $\mathbf{c} \in \mathbb{C}^{k+1}$. One says that \mathbf{c} is \mathcal{C} -resonant if the number of \mathbb{L} -congruence classes of solutions to the equations (60)-(61) is less than $\text{volume } \Delta_{\mathcal{A}}$.*

This means that \mathbf{c} is \mathcal{C} -resonant if and only if there is a $\underline{\gamma} = (\gamma_1, \dots, \gamma_N) \in \mathbb{C}^N$ which satisfies $\gamma_1 \mathbf{a}_1 + \dots + \gamma_N \mathbf{a}_N = \mathbf{c}$, and for which there are two different sets J_1 and J_2 on the list $T_{\mathcal{C}}$ such that $\gamma_j \in \mathbb{Z}$ for $j \in \{1, \dots, N\} \setminus (J_1 \cap J_2)$.

Corollary 2. *If \mathbf{c} is not \mathcal{C} -resonant, the Fourier Γ -series $\Psi_{\mathbb{L}, \underline{\gamma}}(\mathbf{w})$ associated with solutions $\underline{\gamma}$ of the equations (60)-(61) are linearly independent and span a space of local solutions of the GKZ differential equations (19)-(20) of dimension equal to volume $\Delta_{\mathcal{A}}$. According to the discussion in Section 2.7 this is then the full space of local solutions if (for instance) the polytope $\Delta_{\mathcal{A}}$ admits a unimodular triangulation. \square*

6. Extreme resonance in GKZ systems

In this section \mathcal{C} is a maximal cone of the secondary fan of $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_N\}$ for which the corresponding regular triangulation of $\Delta_{\mathcal{A}}$ is unimodular, i.e.

$$|\det((\mathbf{a}_j)_{j \in J})| = 1 \quad \text{for every } J \in T_{\mathcal{C}}.$$

This means that for every $J \in T_{\mathcal{C}}$ the set $\{\mathbf{a}_j\}_{j \in J}$ is a \mathbb{Z} -basis of \mathbb{Z}^{k+1} . Consequently, for every $\mathbf{c} \in \mathbb{Z}^{k+1}$ all solutions $\underline{\gamma} = (\gamma_1, \dots, \gamma_N)$ of the equations (60)-(61) lie in \mathbb{Z}^N and are therefore congruent modulo \mathbb{L} . So a vector $\mathbf{c} \in \mathbb{Z}^{k+1}$ is \mathcal{C} -resonant, in an extreme way: all Fourier Γ -series coming from solutions of (60)-(61) are equal! *In this section we will demonstrate how one can obtain, locally near the point $\mathbf{p}_{\mathcal{C}}$ on $\mathcal{V}_{\mathcal{A}}$, more solutions of the GKZ differential equations (19)-(20) from an ‘infinitesimal deformation’ of this Fourier Γ -series.*

Definition 9. *For \mathcal{A} and \mathcal{C} as above we define the ring*

$$\mathcal{R}_{\mathcal{A}, \mathcal{C}} := \mathbb{Z}[E_1, \dots, E_N] / (\mathcal{I}_{\mathcal{A}} + \mathcal{I}_{\mathcal{C}}) \quad (65)$$

where $\mathbb{Z}[E_1, \dots, E_N]$ is just the polynomial ring over \mathbb{Z} in N variables, $\mathcal{I}_{\mathcal{A}}$ is the ideal generated by the linear forms which are the components of the vector

$$E_1 \mathbf{a}_1 + \dots + E_N \mathbf{a}_N, \quad (66)$$

and $\mathcal{I}_{\mathcal{C}}$ is the ideal generated by the monomials

$$E_{i_1} \cdots E_{i_s} \quad \text{with } \{i_1, \dots, i_s\} \not\subset J \quad \text{for all } J \in T_{\mathcal{C}}. \quad (67)$$

We write ε_j for the image of E_j in $\mathcal{R}_{\mathcal{A}, \mathcal{C}}$.

So in $\mathcal{R}_{\mathcal{A}, \mathcal{C}}$ we have the relations

$$\varepsilon_1 \mathbf{a}_1 + \dots + \varepsilon_N \mathbf{a}_N = 0, \quad (68)$$

$$\varepsilon_{i_1} \cdots \varepsilon_{i_s} = 0 \quad \text{if } \{i_1, \dots, i_s\} \not\subset J \quad \text{for all } J \in T_{\mathcal{C}}. \quad (69)$$

Relation (68) means that the vector $\underline{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_N) \in \mathcal{R}_{\mathcal{A}, \mathcal{C}}^N$ lies in $\mathbb{L} \otimes_{\mathbb{Z}} \mathcal{R}_{\mathcal{A}, \mathcal{C}}$.

Remark. The ideal $\mathcal{I}_{\mathcal{C}}$ is well-known in combinatorial algebra [21], where it is called the *Stanley-Reisner ideal* of the triangulation $T_{\mathcal{C}}$. The ring $\mathbb{Z}[E_1, \dots, E_N] / \mathcal{I}_{\mathcal{C}}$ is called the *Stanley-Reisner ring*.

The following facts about the ring $\mathcal{R}_{\mathcal{A}, \mathcal{C}}$ are proven in [22] §2.

- Proposition 8.** 1. $\mathcal{R}_{\mathcal{A},\mathcal{C}}$ is a free \mathbb{Z} -module of rank equal to volume $\Delta_{\mathcal{A}}$.
 2. $\mathcal{R}_{\mathcal{A},\mathcal{C}}$ is a graded ring and each ε_j has degree 1.
 3. Denoting the homogeneous part of degree i in $\mathcal{R}_{\mathcal{A},\mathcal{C}}$ by $\mathcal{R}_{\mathcal{A},\mathcal{C}}^{(i)}$ one has isomorphisms (see also §4.1)

$$\mathcal{R}_{\mathcal{A},\mathcal{C}}^{(0)} = \mathbb{Z}, \quad \mathcal{R}_{\mathcal{A},\mathcal{C}}^{(1)} \simeq \mathbb{L}_{\mathbb{Z}}^{\vee}, \quad \varepsilon_j \mapsto \mathbf{b}_j. \quad (70)$$

4. The Poincaré series of the graded ring $\mathcal{R}_{\mathcal{A},\mathcal{C}}$ is

$$\sum_{i \geq 0} \left(\text{rank } \mathcal{R}_{\mathcal{A},\mathcal{C}}^{(i)} \right) T^i = \sum_{m=0}^{k+1} S_{\mathcal{C},m} T^m (1-T)^{k+1-m}, \quad (71)$$

where $S_{\mathcal{C},0} = 1$ and $S_{\mathcal{C},m}$, for $m \geq 1$, is the number of simplices with m vertices in the triangulation of $\Delta_{\mathcal{A}}$ corresponding with \mathcal{C} . In particular $\mathcal{R}_{\mathcal{A},\mathcal{C}}^{(i)} = 0$ for $i \geq k+1$ and the elements $\varepsilon_1, \dots, \varepsilon_N$ are nilpotent. \square

For the examples at the end of Sections 4.3 and 4.1 (see also Figures 1, 3, 4) we find:

Example. For $\mathbb{L} = \mathbb{Z}(-2, 1, 1) \subset \mathbb{Z}^3$ there is only one unimodular triangulation, namely $T_{\mathcal{C}} = \{\{1, 2\}, \{1, 3\}\}$. One easily checks that in this case

$$\mathcal{R}_{\mathcal{A},\mathcal{C}} = \mathbb{Z} \oplus \mathbb{Z}\varepsilon, \quad \varepsilon^2 = 0, \quad (\varepsilon_1, \varepsilon_2, \varepsilon_3) = (-2\varepsilon, \varepsilon, \varepsilon).$$

Example. For Gauss $\mathbb{L} = \mathbb{Z}(1, 1, -1, -1) \subset \mathbb{Z}^4$ and there are two unimodular triangulations, which both lead to

$$\mathcal{R}_{\mathcal{A},\mathcal{C}} = \mathbb{Z} \oplus \mathbb{Z}\varepsilon, \quad \varepsilon^2 = 0.$$

For one triangulation $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ is $(-\varepsilon, -\varepsilon, \varepsilon, \varepsilon)$, for the other $(\varepsilon, \varepsilon, -\varepsilon, -\varepsilon)$.

Example: For $\mathbb{L} = \mathbb{Z}(-3, 1, 1, 1) \subset \mathbb{Z}^4$ there is only one unimodular triangulation, namely $T_{\mathcal{C}} = \{\{1, 3, 4\}, \{1, 2, 4\}, \{1, 2, 3\}\}$. One easily checks that in this case

$$\mathcal{R}_{\mathcal{A},\mathcal{C}} = \mathbb{Z} \oplus \mathbb{Z}\varepsilon \oplus \mathbb{Z}\varepsilon^2, \quad \varepsilon^3 = 0, \quad (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = (-3\varepsilon, \varepsilon, \varepsilon, \varepsilon).$$

Example. For Appell's F_1 $\mathbb{L} = \mathbb{Z}(1, -1, 0, -1, 1, 0) \oplus \mathbb{Z}(1, 0, -1, -1, 0, 1)$. There are six unimodular triangulations (see Figure 3). One can check that for the triangulation $T_{\mathcal{C}} = \{\{3, 4, 5, 6\}, \{1, 2, 3, 4\}, \{2, 3, 4, 5\}\}$ the relations (66)-(67) yield

$$\begin{aligned} (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6) &= \varepsilon(1, -1, 0, -1, 1, 0) + \delta(1, 0, -1, -1, 0, 1), \\ \varepsilon_1 \varepsilon_5 &= \varepsilon_1 \varepsilon_6 = \varepsilon_2 \varepsilon_6 = 0, \end{aligned}$$

and hence:

$$\mathcal{R}_{\mathcal{A},\mathcal{C}} = \mathbb{Z} \oplus \mathbb{Z}\varepsilon \oplus \mathbb{Z}\delta, \quad \varepsilon^2 = \delta^2 = \varepsilon\delta = 0.$$

Example. For Appell's F_4 $\mathbb{L} = \mathbb{Z}(1, -1, 1, -1, 0, 0) \oplus \mathbb{Z}(1, 0, 1, 0, 0, -1, -1)$. There are three unimodular triangulations (see Figure 4). One can check that for the triangulation $T_C = \{\{1, 3, 4, 6\}, \{1, 3, 4, 5\}, \{1, 2, 3, 5\}, \{1, 2, 3, 6\}\}$ the relations (66)-(67) yield

$$\begin{aligned} (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6) &= \varepsilon(1, -1, 1, -1, 0, 0) + \delta(1, 0, 1, 0, 0, -1, -1), \\ \varepsilon_2 \varepsilon_4 &= \varepsilon_5 \varepsilon_6 = 0, \end{aligned}$$

and hence:

$$\mathcal{R}_{\mathcal{A},C} = \mathbb{Z} \oplus \mathbb{Z}\varepsilon \oplus \mathbb{Z}\delta \oplus \mathbb{Z}\varepsilon\delta, \quad \varepsilon^2 = \delta^2 = 0.$$

For $z \in \mathbb{C}$ and nilpotent ε one can define $\frac{1}{\Gamma(z+\varepsilon)}$ as an element of $\mathbb{C}[\varepsilon]$ by using the Taylor expansion of the function $\frac{1}{\Gamma}$ at z :

$$\frac{1}{\Gamma(z+\varepsilon)} := \frac{1}{\Gamma(z)} + \varepsilon \left(\frac{1}{\Gamma}\right)'(z) + \frac{\varepsilon^2}{2} \left(\frac{1}{\Gamma}\right)''(z) + \frac{\varepsilon^3}{3!} \left(\frac{1}{\Gamma}\right)'''(z) + \dots$$

One defines similarly $\Gamma(1+\varepsilon)$. Thus for $z \in \mathbb{C}$ and nilpotent ε also $\frac{\Gamma(1+\varepsilon)}{\Gamma(z+1+\varepsilon)}$ has been defined. From (30) one sees that for $m \in \mathbb{Z}$:

$$\frac{\Gamma(1+\varepsilon)}{\Gamma(m+1+\varepsilon)} = \begin{cases} \frac{1}{(1+\varepsilon)(2+\varepsilon)\cdots(m+\varepsilon)} & \text{if } m > 0 \\ 1 & \text{if } m = 0 \\ \varepsilon(\varepsilon-1)(\varepsilon-2)\cdots(\varepsilon+m+1) & \text{if } m < 0 \end{cases} \quad (72)$$

Finally, for $z \in \mathbb{C}$, $u \in \mathbb{C}^*$ (with a choice of a branch of $\log u$) and nilpotent ε one has naturally

$$e^{\varepsilon z} := \sum_{m \geq 0} \frac{1}{m!} \varepsilon^m z^m, \quad u^\varepsilon := e^{\varepsilon \log u}.$$

We are ready to present our deformation of the (Fourier) Γ -series:

Definition 10. For $\underline{\gamma} = (\gamma_1, \dots, \gamma_N) \in \mathbb{Z}^N$ and $\underline{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_N) \in \mathcal{R}_{\mathcal{A},C}^N$ we define

$$\Psi_{\mathbb{L}, \underline{\gamma}, \underline{\varepsilon}}(\mathbf{w}) := \sum_{\underline{\ell} \in \mathbb{L}} \prod_{j=1}^N \frac{\Gamma(1+\varepsilon_j)}{\Gamma(\gamma_j + \ell_j + 1 + \varepsilon_j)} e^{2\pi i \mathbf{w} \cdot (\underline{\gamma} + \underline{\ell} + \underline{\varepsilon})}, \quad (73)$$

$$\Phi_{\mathbb{L}, \underline{\gamma}, \underline{\varepsilon}}(\mathbf{u}) := \sum_{\underline{\ell} \in \mathbb{L}} \prod_{j=1}^N \frac{\Gamma(1+\varepsilon_j)}{\Gamma(\gamma_j + \ell_j + 1 + \varepsilon_j)} u_j^{\gamma_j + \ell_j + \varepsilon_j}. \quad (74)$$

Remark. From the point of view of deforming $\underline{\gamma}$ it seems more natural to consider

$$\Psi_{\mathbb{L}, \underline{\gamma} + \underline{\varepsilon}}(\mathbf{w}) := \sum_{\underline{\ell} \in \mathbb{L}} \prod_{j=1}^N \frac{1}{\Gamma(\gamma_j + \ell_j + 1 + \varepsilon_j)} e^{2\pi i \mathbf{w} \cdot (\underline{\gamma} + \underline{\ell} + \underline{\varepsilon})}, \quad (75)$$

$$\Phi_{\mathbb{L}, \underline{\gamma} + \underline{\varepsilon}}(\mathbf{u}) := \sum_{\underline{\ell} \in \mathbb{L}} \prod_{j=1}^N \frac{u_j^{\gamma_j + \ell_j + \varepsilon_j}}{\Gamma(\gamma_j + \ell_j + 1 + \varepsilon_j)}; \quad (76)$$

i.e.

$$\Psi_{\mathbb{L}, \underline{\gamma} + \underline{\varepsilon}}(\mathbf{w}) = \frac{\Psi_{\mathbb{L}, \underline{\gamma}, \underline{\varepsilon}}(\mathbf{w})}{\prod_{j=1}^N \Gamma(1 + \varepsilon_j)}, \quad \Phi_{\mathbb{L}, \underline{\gamma} + \underline{\varepsilon}}(\mathbf{u}) = \frac{\Phi_{\mathbb{L}, \underline{\gamma}, \underline{\varepsilon}}(\mathbf{u})}{\prod_{j=1}^N \Gamma(1 + \varepsilon_j)}.$$

Indeed, expanding these functions in coordinates with respect to a basis of $\mathcal{R}_{\mathcal{A}, \mathcal{C}}$ is for (75) and (76) essentially just Taylor expansion, if one views the expressions as functions of $\underline{\gamma}$, while the interpretation as (multi-valued) local solutions of GKZ differential equations with values in $\mathcal{R}_{\mathcal{A}, \mathcal{C}} \otimes \mathbb{C}$ (see below) are equally true for (75)-(76) in place of (73)-(74). We prefer, however, the latter because their coordinates are series with rational coefficients, whereas the coefficients of the coordinate series of the former involve interesting, but mysterious non-rational numbers like the Euler-Masceroni constant and values of Riemann's zeta-function. We can be slightly more informative about the coefficients in (75)-(76): there is the well-known formula for the Γ -function due to Gauss

$$\Gamma(s) = \lim_{n \rightarrow \infty} \left[\frac{n! n^s}{s(s+1) \cdots (s+n)} \right],$$

from which one easily derives the expansion

$$\log \Gamma(1+s) = -\gamma s + \sum_{m=2}^{\infty} (-1)^m \zeta(m) \frac{s^m}{m}$$

where γ denotes the Euler-Masceroni constant and ζ is Riemann's zeta-function. By exponentiating and re-expanding one finds the Taylor expansion for $\Gamma(1+s)$ and then eventually the expansion of $\left[\prod_{j=1}^N \Gamma(1 + \varepsilon_j) \right]^{-1}$.

Lemma 8. *There are finitely many $\underline{\ell}^{(1)}, \dots, \underline{\ell}^{(r)} \in \mathbb{L}_{\mathcal{C}}$ (with $\mathbb{L}_{\mathcal{C}}$ as in (55)) such that the series $\Psi_{\mathbb{L}, \underline{\gamma}, \underline{\varepsilon}}(\mathbf{w})$ and $\Phi_{\mathbb{L}, \underline{\gamma}, \underline{\varepsilon}}(\mathbf{u})$ involve only terms with*

$$\underline{\ell} \in \bigcup_{i=1}^r (-\underline{\ell}^{(i)} + \mathbb{L}_{\mathcal{C}}).$$

In particular for $\underline{\gamma} = 0$ the series involve only terms with $\underline{\ell} \in \mathbb{L}_{\mathcal{C}}$.

Proof. It follows immediately from (72) and (67) that for the terms which appear with non-zero coefficient, the set $\{j \mid \gamma_j + \ell_j < 0\}$ is contained in some J on the list $T_{\mathcal{C}}$. Suppose $\{j \mid \gamma_j + \ell_j < 0\} \subset J \in T_{\mathcal{C}}$. Then $\gamma_j + \ell_j \geq 0$ for every $j \in J' := \{1, \dots, N\} \setminus J$. The vector $\sum_{j \in J'} \max(0, \gamma_j) \mathbf{a}_j$ is a \mathbb{Z} -linear combination of the

vectors \mathbf{a}_i with $i \in J$, because the triangulation is unimodular. Such a relation is an element of \mathbb{L} . Thus one sees that \mathbb{L} contains an element $\underline{\ell}^J = (\ell_1^J, \dots, \ell_N^J)$ with $\ell_j^J = \max(0, \gamma_j)$ for all $j \in J'$. So $\ell_j^J + \ell_j \geq 0$ for every $j \in J'$. In the notation introduced in (38) this can be written as $\underline{\ell}^J \in \mathbb{L}_J$ and $\underline{\ell}^J + \underline{\ell} \in \mathbb{L}_J$. The lemma now follows from (59). \square

Partial sums (with finitely many terms) of the series (73) resp. (74) can be evaluated as elements in the ring $\mathcal{R}_{\mathcal{A}, \mathcal{C}} \otimes \mathbb{C}$ and be written in coordinates with respect to a \mathbb{Z} -basis of the finite rank \mathbb{Z} -module $\mathcal{R}_{\mathcal{A}, \mathcal{C}}$. These coordinates are again partial sums of series. In [22] §3 one finds estimates on the growth of the coefficients of these series and on a common domain of convergence. Thus $\Psi_{\mathbb{L}, \underline{\gamma}, \underline{\varepsilon}}(\mathbf{w})$ and $\Phi_{\mathbb{L}, \underline{\gamma}, \underline{\varepsilon}}(\mathbf{u})$ are functions with values in $\mathcal{R}_{\mathcal{A}, \mathcal{C}} \otimes \mathbb{C}$. The function $\Psi_{\mathbb{L}, \underline{\gamma}, \underline{\varepsilon}}(\mathbf{w})$ is defined for $\mathbf{w} \in \mathbb{C}^N$ with $\Im \widehat{\mathbf{w}}$ ‘sufficiently far’ inside the cone \mathcal{C} (cf. §5.2). Because of the appearance of logarithms $\Phi_{\mathbb{L}, \underline{\gamma}, \underline{\varepsilon}}(\mathbf{u})$ is actually a multi-valued function, defined on some open disc about 0 in $\mathbb{C}^{\mathcal{A}}$ with the divisor $u_1 \cdots u_N = 0$ removed. The multi-valuedness is easily described using the relation $u_j = e^{2\pi i w_j}$ which matches w_j with a choice of $\log u_j$. A different choice adds an integer to w_j . Now note that for $\mathbf{m} \in \mathbb{Z}^N$

$$\Psi_{\mathbb{L}, \underline{\gamma}, \underline{\varepsilon}}(\mathbf{w} + \mathbf{m}) = e^{2\pi i \mathbf{m} \cdot \underline{\varepsilon}} \Psi_{\mathbb{L}, \underline{\gamma}, \underline{\varepsilon}}(\mathbf{w}).$$

This formula can also be read as a precise expression for local monodromy. Since $\{\mathbf{m} \cdot \underline{\varepsilon} \mid \mathbf{m} \in \mathbb{Z}^N\} = \mathcal{R}_{\mathcal{A}, \mathcal{C}}^{(1)}$, we can summarize our analysis of the multi-valuedness of $\Phi_{\mathbb{L}, \underline{\gamma}, \underline{\varepsilon}}(\mathbf{u})$ as follows:

Proposition 9. $\Phi_{\mathbb{L}, \underline{\gamma}, \underline{\varepsilon}}(\mathbf{u})$ is a multi-valued function with values in $\mathcal{R}_{\mathcal{A}, \mathcal{C}} \otimes \mathbb{C}$. Different branches of this function are related by multiplication with an element $e^{2\pi i \omega}$ with $\omega \in \mathcal{R}_{\mathcal{A}, \mathcal{C}}^{(1)}$. \square

The same arguments as those used in Section 3.6 show immediately

Proposition 10. The $\mathcal{R}_{\mathcal{A}, \mathcal{C}} \otimes \mathbb{C}$ -valued function $\Phi_{\mathbb{L}, \underline{\gamma}, \underline{\varepsilon}}(\mathbf{u})$ satisfies the GKZ system of differential equations (19)-(20) for \mathcal{A} and $\mathbf{c} = \sum_{j=1}^N \gamma_j \mathbf{a}_j$.

The \mathbb{C} -valued functions which arise as coordinates of $\Phi_{\mathbb{L}, \underline{\gamma}, \underline{\varepsilon}}(\mathbf{u})$ with respect to a basis of $\mathcal{R}_{\mathcal{A}, \mathcal{C}}$ satisfy the same GKZ system of differential equations. \square

Example. For $\mathbb{L} = \mathbb{Z}(-3, 1, 1, 1)$ and \mathcal{C} the cone corresponding to the unimodular triangulation $T_{\mathcal{C}} = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}\}$

$$\mathcal{R}_{\mathcal{A}, \mathcal{C}} = \mathbb{Z} \oplus \mathbb{Z}\varepsilon \oplus \mathbb{Z}\varepsilon^2, \quad \varepsilon^3 = 0, \quad (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = (-3\varepsilon, \varepsilon, \varepsilon, \varepsilon).$$

For $\underline{0} = (0, 0, 0, 0)$ one then finds, using (72) and setting $z = u_1^{-3}u_2u_3u_4$,

$$\begin{aligned}
\Phi_{\mathbb{L}, \underline{0}, \varepsilon}(\mathbf{u}) &= \sum_{m \in \mathbb{Z}} \frac{\Gamma(1-3\varepsilon)}{\Gamma(1-3m-3\varepsilon)} \left(\frac{\Gamma(1+\varepsilon)}{\Gamma(1+m+\varepsilon)} \right)^3 u_1^{-3m-3\varepsilon} u_2^{m+\varepsilon} u_3^{m+\varepsilon} u_4^{m+\varepsilon} \\
&= z^\varepsilon \left(1 + \sum_{m \geq 1} \frac{(-3\varepsilon)(-3\varepsilon-1) \cdots (-3\varepsilon-3m+1)}{((1+\varepsilon) \cdots (m+\varepsilon))^3} z^m \right) \\
&= \left(1 + \varepsilon \log z + \frac{\varepsilon^2}{2} \log^2 z \right) \left(1 + \varepsilon G_1(z) + \varepsilon^2 G_2(z) \right) \\
&= 1 + (\log z + G_1(z))\varepsilon + \left(\frac{1}{2} \log^2 z + G_1(z) \log z + G_2(z) \right) \varepsilon^2
\end{aligned}$$

with

$$\begin{aligned}
G_1(z) &= 3 \sum_{m \geq 1} (-1)^m \frac{(3m-1)!}{(m!)^3} z^m \\
G_2(z) &= 9 \sum_{m \geq 1} (-1)^m \frac{(3m-1)!}{(m!)^3} \left(\sum_{j=m+1}^{3m-1} \frac{1}{j} \right) z^m.
\end{aligned}$$

Similarly, for $\underline{\gamma} = (-1, 0, 0, 0)$ we obtain

$$\begin{aligned}
\Phi_{\mathbb{L}, \underline{\gamma}, \varepsilon}(\mathbf{u}) &= \sum_{m \in \mathbb{Z}} \frac{\Gamma(1-3\varepsilon)}{\Gamma(-3m-3\varepsilon)} \left(\frac{\Gamma(1+\varepsilon)}{\Gamma(m+1+\varepsilon)} \right)^3 u_1^{-1-3m-3\varepsilon} u_2^{m+\varepsilon} u_3^{m+\varepsilon} u_4^{m+\varepsilon} \\
&= u_1^{-1} \sum_{m \geq 0} \frac{(-3\varepsilon)(-3\varepsilon-1) \cdots (-3\varepsilon-3m)}{((1+\varepsilon) \cdots (m+\varepsilon))^3} (u_1^{-3}u_2u_3u_4)^{m+\varepsilon} \\
&= u_1^{-1} \left(1 + \varepsilon \log z + \frac{\varepsilon^2}{2} \log^2 z \right) \left(\varepsilon F_1(z) + \varepsilon^2 F_2(z) \right) \\
&= u_1^{-1} F_1(z) \varepsilon + u_1^{-1} (F_1(z) \log z + F_2(z)) \varepsilon^2
\end{aligned}$$

with

$$\begin{aligned}
F_1(z) &= -3 \sum_{m \geq 0} (-1)^m \frac{(3m)!}{(m!)^3} z^m, \\
F_2(z) &= -9 \sum_{m \geq 1} (-1)^m \frac{(3m)!}{(m!)^3} \left(\sum_{j=m+1}^{3m} \frac{1}{j} \right) z^m.
\end{aligned}$$

Note that in agreement with Proposition 1

$$\Phi_{\mathbb{L}, \underline{\gamma}, \varepsilon}(\mathbf{u}) = \frac{\partial}{\partial u_1} \Phi_{\mathbb{L}, \underline{0}, \varepsilon}(\mathbf{u}) = -3u_1^{-1} z \frac{\partial}{\partial z} \Phi_{\mathbb{L}, \underline{0}, \varepsilon}(\mathbf{u}).$$

The components of $\Phi_{\mathbb{L}, \underline{0}, \varepsilon}(\mathbf{u})$ are three linearly independent solutions of the GKZ system of differential equations with $\mathbf{c} = 0$, whereas the components of $\Phi_{\mathbb{L}, \underline{\gamma}, \varepsilon}(\mathbf{u})$ yield only two linearly independent solutions of the GKZ system for $\mathbf{c} = -\mathbf{a}_1$.

Since in this case volume $\Delta_{\mathcal{A}} = 3$ we find enough solutions for $\mathbf{c} = 0$, but not enough for $\mathbf{c} = -\mathbf{a}_1$ (see Section 2.7).

The phenomenon observed at the end of the previous example – namely that our method yields enough solutions if $\mathbf{c} = 0$, but misses solutions if $\mathbf{c} \neq 0$ – occurs quite generally. Below, in Theorem 3, we quote [22] Theorem 5 and also recall some conclusions (e.g. Proposition 9) found earlier in the present notes:

Theorem 3. *Let \mathcal{C} be a maximal cone of the secondary fan of \mathcal{A} for which the corresponding regular triangulation of $\Delta_{\mathcal{A}}$ is unimodular. Let $\underline{0} = (0, \dots, 0)$. Then the coordinates of the $\mathcal{R}_{\mathcal{A}, \mathcal{C}} \otimes \mathbb{C}$ -valued function $\Phi_{\mathbb{L}, \underline{0}, \underline{\varepsilon}}(\mathbf{u})$ with respect to a basis of the free \mathbb{Z} -module $\mathcal{R}_{\mathcal{A}, \mathcal{C}}$ constitute a basis for the local solution space of the GKZ system of differential equations (19)-(20) for \mathcal{A} and $\mathbf{c} = 0$. These multi-valued functions are invariant under the action (1) of the torus \mathbb{T}^{k+1} and descend therefore to multi-valued functions on a disc minus a divisor centered at the point $\mathbf{p}_{\mathcal{C}}$ in the toric variety $\mathcal{V}_{\mathcal{A}}$. The multi-valuedness of these functions is given by multiplying $\Phi_{\mathbb{L}, \underline{0}, \underline{\varepsilon}}(\mathbf{u})$ with elements in the group $\{e^{2\pi i \omega} \mid \omega \in \mathcal{R}_{\mathcal{A}, \mathcal{C}}^{(1)}\}$. \square*

Remark. In [4] Anne de Boo carefully re-examined the preceding method and improved it by also taking $\underline{\gamma}$ into account. In this way he obtained full local solution spaces for GKZ systems of differential equations for many more instances of the triangulation of $\Delta_{\mathcal{A}}$ and of the parameter \mathbf{c} .

Very recently Borisov and Horja [5] found a way to obtain enough solutions for any $\mathbf{c} \in \mathbb{Z}^{k+1}$ and any triangulation. Their method is close in spirit to the method in this Section 6. We recommend [5] for further reading on this aspect of GKZ hypergeometric structures.

7. GKZ for Lauricella's F_D

Since Lauricella's F_D also plays an important role in other lectures in this School, we put details of the GKZ theory for Lauricella's F_D together in this section.

7.1. Series, \mathbb{L} , \mathcal{A} and the primary polytope $\Delta_{\mathcal{A}}$

Recall that in Section 3.2.4 we found, starting from the power series expansion of Lauricella's F_D in $k-1$ variables

$$F_D(a, \mathbf{b}, c | \mathbf{z}) := \sum_{\mathbf{m}} \frac{(a)_{|\mathbf{m}|} (\mathbf{b})_{\mathbf{m}}}{(c)_{|\mathbf{m}|} \mathbf{m}!} \mathbf{z}^{\mathbf{m}},$$

that the lattice \mathbb{L} is generated by the rows of the following $(k-1) \times (2k)$ -matrix

$$\begin{pmatrix} 1 & -1 & 0 & \dots & 0 & -1 & 1 & 0 & \dots & 0 \\ 1 & 0 & -1 & \ddots & \vdots & -1 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots & \vdots & \ddots & \ddots & 0 \\ 1 & 0 & \dots & 0 & -1 & -1 & 0 & \dots & 0 & 1 \end{pmatrix}. \quad (77)$$

So for \mathcal{A} we can take the set of columns of the $(k+1) \times (2k)$ -matrix

$$\begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots & 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}. \quad (78)$$

This notation is consistent with the main part of this text: \mathcal{A} is a subset of \mathbb{Z}^{k+1} ; moreover $N = 2k$ and $d = \text{rank } \mathbb{L} = k - 1$.

The primary polytope $\Delta_{\mathcal{A}}$ is the direct product of a $(k-1)$ -simplex and a 1-simplex and, for $k = 3$, looks like the prism in Figure 1. The vectors $\mathbf{a}_1, \dots, \mathbf{a}_k$ are in the bottom face of the prism; $\mathbf{a}_{k+1}, \dots, \mathbf{a}_{2k}$ are in the top face. The numbering is such that the difference vectors $\mathbf{a}_{k+j} - \mathbf{a}_j$, for $j = 1, \dots, k$ are all equal.

7.2. Integrals and differential equations for F_D

In [18] Lauricella's F_D in variables z_0, \dots, z_n is introduced via the integrals

$$F_{\alpha}(z_0, \dots, z_n) := \int_{\alpha} (z_0 - \zeta)^{-\mu_0} \dots (z_n - \zeta)^{-\mu_n} d\zeta \quad (79)$$

over suitable intervals α , with endpoints in $\{z_0, \dots, z_n, \infty\}$. Note that because of the translation invariance property

$$F_{\alpha}(z_0 + a, \dots, z_n + a) = F_{\alpha}(z_0, \dots, z_n) \quad (80)$$

the integral (79) is in fact a function of just n variables: $z_1 - z_0, \dots, z_n - z_0$.

GKZ theory can deal efficiently with (multiplicative) torus actions on the variables, but it can not accommodate for translation invariance like (80). So we eliminate the translation invariance during the passage to GKZ and consider the integrals (with the same μ_0, \dots, μ_n)

$$I_{\sigma}(u_1, \dots, u_{2n}) = \int_{\sigma} (u_1 + u_{n+1}\xi)^{-\mu_1} \dots (u_n + u_{2n}\xi)^{-\mu_n} \xi^{-\mu_0} d\xi, \quad (81)$$

which are of the type considered in Section 2.4.

The GKZ differential equations satisfied by these integrals can be found with the methods used in Section 2.3. For instance, for $j = 1, \dots, n$

$$\begin{aligned} \frac{\partial I_{\sigma}}{\partial u_j} &= -\mu_j \int_{\sigma} (u_1 + u_{n+1}\zeta)^{-\mu_1} \dots (u_n + u_{2n}\zeta)^{-\mu_n} \xi^{-\mu_0} \frac{d\zeta}{u_j + u_{j+n}\zeta} \\ \frac{\partial I_{\sigma}}{\partial u_{j+n}} &= -\mu_j \int_{\sigma} (u_1 + u_{n+1}\zeta)^{-\mu_1} \dots (u_n + u_{2n}\zeta)^{-\mu_n} \xi^{-\mu_0} \frac{\zeta d\zeta}{u_j + u_{j+n}\zeta} \end{aligned}$$

and, hence, for $i, j = 1, \dots, n$

$$\frac{\partial^2 I_{\sigma}}{\partial u_i \partial u_{j+n}} = \frac{\partial^2 I_{\sigma}}{\partial u_j \partial u_{i+n}},$$

i.e. I_{σ} satisfies the differential equations (19) with \mathbb{L} as in (77) and $k = n$.

Similarly, for $s \in \mathbb{C}$ close to 1, we have

$$\begin{aligned} I_\sigma(u_1, \dots, u_n, su_{n+1}, \dots, su_{2n}) &= s^{\mu_0-1} I_\sigma(u_1, \dots, u_{2n}), \\ I_\sigma(u_1, \dots, u_{j-1}, su_j, u_{j+1}, \dots, u_{j+n-1}, su_{j+n}, u_{j+n+1}, \dots, u_{2n}) &= \\ &= s^{-\mu_j} I_\sigma(u_1, \dots, u_{2n}). \end{aligned}$$

This leads to the differential equations (20) with $k = n$, \mathcal{A} as in (78) and $\mathbf{c} = (\mu_0 - 1, -\mu_1, \dots, -\mu_n)^t$.

As we have seen in Section 3.2.4 the power series $F_D(a, \mathbf{b}, c | \mathbf{z})$ is, up to a constant factor, the Γ -series associated with the above \mathbb{L} and with $\underline{\gamma} = (\gamma_1, \dots, \gamma_N) = (c - 1, -b_1, \dots, -b_{k-1}, -a, 0, \dots, 0)$. The parameter \mathbf{c} in the GKZ differential equations (20) is therefore

$$\mathbf{c} = \sum_{j=1}^{2k} \gamma_j \mathbf{a}_j = (-a, c - a - 1, -b_1, \dots, -b_{k-1})^t =: (c_0, c_1, c_2, \dots, c_k)^t.$$

The system of differential equations (20) can now be written as

$$\frac{\partial \Phi}{\partial u_{j+k}} = -u_{j+k}^{-1} \left(u_j \frac{\partial \Phi}{\partial u_j} - c_j \Phi \right) \quad \text{for } j = 1, \dots, k, \quad (82)$$

$$u_1 \frac{\partial \Phi}{\partial u_1} + \dots + u_k \frac{\partial \Phi}{\partial u_k} = (-c_0 + c_1 + \dots + c_k) \Phi. \quad (83)$$

The system (19) is equivalent with the following $\frac{1}{2}k(k-1)$ differential equations

$$\frac{\partial^2 \Phi}{\partial u_i \partial u_{j+k}} = \frac{\partial^2 \Phi}{\partial u_j \partial u_{i+k}} \quad \text{for } 1 \leq i < j \leq k. \quad (84)$$

Next we substitute (82) into (84) and set

$$u_j = z_j \quad \text{if } 1 \leq j \leq k, \quad u_j = 1 \quad \text{if } k+1 \leq j \leq 2k.$$

The result is the system of $\frac{1}{2}k(k-1)$ differential equations

$$(z_i - z_j) \frac{\partial^2 \Phi}{\partial z_i \partial z_j} = c_i \frac{\partial \Phi}{\partial z_j} - c_j \frac{\partial \Phi}{\partial z_i} \quad \text{for } 1 \leq i < j \leq k. \quad (85)$$

The above substitution turns (83) into

$$z_1 \frac{\partial \Phi}{\partial z_1} + \dots + z_k \frac{\partial \Phi}{\partial z_k} = (-c_0 + c_1 + \dots + c_k) \Phi. \quad (86)$$

The system of differential equations (85)-(86) is then equivalent with the GKZ system (19)-(20) for Lauricella's F_D . The Equations (85) appear in this form also in [18], and (86) appears in loc.cit. in an 'integrated' form:

$$\Phi(e^t z_1, \dots, e^t z_k) = e^{(-c_0 + c_1 + \dots + c_k)t} \Phi(z_1, \dots, z_k).$$

The match with [18] becomes exact, if one eliminates in loc. cit. the translation invariance by setting $z_0 = 0$ (like we did in passing from (79) to (81)).

7.3. Triangulations of $\Delta_{\mathcal{A}}$, secondary polytope and fan for F_D

Consider a triangulation \mathcal{T} of the prism $\Delta_{\mathcal{A}}$ by k -dimensional simplices with vertices in the set \mathcal{A} . Then the bottom $(k-1)$ -simplex $[\mathbf{a}_1, \dots, \mathbf{a}_k]$ must be a face of exactly one k -simplex in the triangulation, say σ_1 . Let \mathbf{a}_{k+s_1} be the vertex of σ_1 opposite to the face $[\mathbf{a}_1, \dots, \mathbf{a}_k]$. So $1 \leq s_1 \leq k$. The face of σ_1 opposite to the vertex s_1 has vertices \mathbf{a}_{k+s_1} and \mathbf{a}_i with $1 \leq i \leq k$, $i \neq s_1$. This must be a face of exactly one other k -dimensional simplex in the triangulation, say σ_2 . Let \mathbf{a}_{k+s_2} be the remaining vertex of σ_2 . So $1 \leq s_2 \leq k$ and $s_2 \neq s_1$. The face of σ_2 opposite to the vertex \mathbf{a}_{s_2} has vertices \mathbf{a}_{k+s_1} , \mathbf{a}_{k+s_2} and \mathbf{a}_i with $1 \leq i \leq k$, $i \neq s_1, s_2$. This must be a face of exactly one other k -dimensional simplex, say σ_3 . Let \mathbf{a}_{k+s_3} be the remaining vertex of σ_3 . So $1 \leq s_3 \leq k$ and $s_3 \neq s_1, s_2$. And so on. Thus the triangulation \mathcal{T} of $\Delta_{\mathcal{A}}$ determines a permutation τ of $\{1, \dots, k\}$ with $\tau(i) = s_i$.

There is an obvious converse to this procedure associating to a permutation τ of $\{1, 2, 3, \dots, k\}$ the triangulation with maximal simplices $\sigma_1^{(\tau)}, \dots, \sigma_k^{(\tau)}$ where

$$\sigma_j^{(\tau)} := \text{convex hull} (\{\mathbf{a}_{\tau(i)} \mid j \leq i \leq k\} \cup \{\mathbf{a}_{k+\tau(i)} \mid 1 \leq i \leq j\}). \quad (87)$$

These triangulations are unimodular; i.e. all k -simplices have volume 1. So when constructing the secondary polytope one only has to count for every triangulation how many simplices come together in the points $\mathbf{a}_1, \dots, \mathbf{a}_N$. With the above formula for the simplex $\sigma_j^{(\tau)}$ one easily finds that the vector associated with the permutation τ is $(\tau^{-1}(1), \dots, \tau^{-1}(k), k+1-\tau^{-1}(1), \dots, k+1-\tau^{-1}(k))$.

The secondary polytope is the convex hull of these points as τ runs through all permutations of $\{1, 2, 3, \dots, k\}$. By translating over the vector corresponding to the identity permutation the secondary polytope moves to the convex hull of the points $(\tau^{-1}(1)-1, \dots, \tau^{-1}(k)-k, 1-\tau^{-1}(1), \dots, k-\tau^{-1}(k))$ in the space $\mathbb{L}_{\mathbb{R}} := \mathbb{L} \otimes \mathbb{R}$.

Example/Exercise. The reader is invited to determine with the above algorithm the permutations corresponding to the maximal cones of the secondary fan of Appell's F_1 (= Lauricella's F_D with $k=3$) shown in Figure 3.

Recall from Section 4.1 that the secondary fan is a partition of the real vector space $\mathbb{L}_{\mathbb{R}}^{\vee} = \text{Hom}(\mathbb{L}, \mathbb{R})$ into rational cones, all with their apex in 0. Corollary 1 and Formula (44) describe these cones. The vectors $\mathbf{b}_1, \dots, \mathbf{b}_N$ are the images of the standard basis vectors of \mathbb{R}^N under the natural surjection $\mathbb{R}^N \longrightarrow \mathbb{L}_{\mathbb{R}}^{\vee}$. In the present situation we choose the rows of the matrix (77) as a basis for $\mathbb{L}_{\mathbb{R}} = \mathbb{L} \otimes \mathbb{R}$. On $\mathbb{L}_{\mathbb{R}}^{\vee}$ we use coordinates with respect to the dual basis. The columns of (77) then represent the vectors $\mathbf{b}_1, \dots, \mathbf{b}_N$ in these coordinates.

Now consider a vector $\mathbf{t} = (t_2, \dots, t_k)$ in $\mathbb{L}_{\mathbb{R}}^{\vee}$. Put $t_1 = 0$. Then \mathbf{t} defines a partial ordering $<_{\mathbf{t}}$ on the set $\{1, 2, \dots, k\}$ by

$$i <_{\mathbf{t}} j \quad \Leftrightarrow \quad t_i < t_j.$$

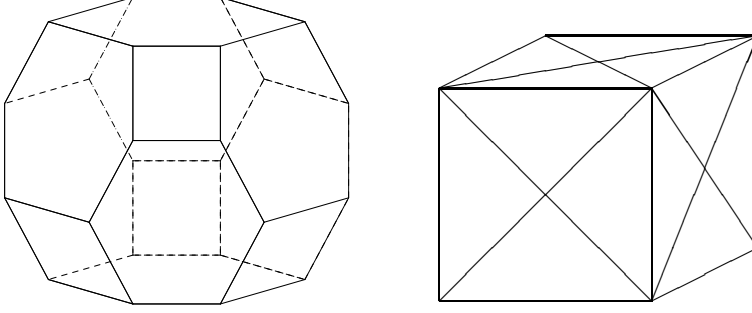


FIGURE 8. Secondary polytope (left) and Secondary fan (right) for F_D with $k = 4$. All cones in the fan have their apex at the centre of the cube. Shown are the intersections of the cones with some faces of the cube. The reader is invited to label the maximal cones with the permutations of $\{1, 2, 3, 4\}$.

The indexing and ordering is such that for $h = 1, \dots, k$

$$\mathbf{t} = \sum_{i >_t h} (t_i - t_h) \mathbf{b}_{i+k} + \sum_{i <_t h} (t_h - t_i) \mathbf{b}_i. \quad (88)$$

One also easily checks that these are the only expressions for \mathbf{t} as positive linear combination of a linearly independent subset of $\{\mathbf{b}_1, \dots, \mathbf{b}_N\}$. Corollary 1 and Formula (44) now tell exactly in which cone of the secondary fan \mathbf{t} lies. In particular, \mathbf{t} lies in the interior of a maximal cone if and only if no two of the numbers t_1, t_2, \dots, t_k are equal. In that case, the ordering $<_t$ is a total ordering, or what amounts to the same a permutation of $\{1, 2, \dots, k\}$. More precisely, if we associate with \mathbf{t} the permutation τ defined by

$$\tau(1) <_t \tau(2) <_t \dots <_t \tau(k-1) <_t \tau(k),$$

then the index set which effectively appears in (88) is the complement of the index set in (87) with $h = \tau(j)$.

Thus we have shown:

Corollary 3. *The maximal cones in the secondary fan for F_D are the connected components of the complement in \mathbb{R}^{k-1} of the union of hyperplanes*

$$\bigcup_{1 \leq i < j \leq k} H_{ij}, \quad \text{with equation for } H_{ij}: \quad t_i = t_j,$$

where on the right t_2, t_3, \dots, t_k are coordinates on \mathbb{R}^{k-1} and $t_1 = 0$. □

Remarks. Most GKZ systems do not have a secondary fan which is cut out by a hyperplane arrangement. F_D is something special.

Exercise. In Looijenga's lectures [18] the natural domain of definition for the *Schwarz map* is $\mathbb{P}(V_n^\circ)$. The notations are: $\mathbb{P}(V_n^\circ)$ is $(\mathbb{C}^{n+1})^\circ$ modulo the natural \mathbb{C}^* -action with weights $(1, 1, \dots, 1)$ and modulo translations over $\mathbb{C}(1, 1, \dots, 1)$,

$$(\mathbb{C}^{n+1})^\circ := \mathbb{C}^{n+1} \setminus \bigcup_{i < j} \{\text{hyperplane with equation } z_i = z_j\}.$$

How does $\mathbb{P}(V_n^\circ)$ relate to the toric variety \mathcal{V}_A ?

8. A glimpse of Mirror Symmetry

8.1. GKZ data from Calabi-Yau varieties

One of the manifestations of the Mirror Symmetry phenomenon is a relation between two families of 3-dimensional Calabi-Yau varieties, matching complex geometry on one family with symplectic geometry on the other.

In the language of complex geometry a smooth Calabi-Yau variety is a compact smooth Kähler manifold X with trivial canonical bundle, i.e. $\Omega_X^{\dim X} \simeq \mathcal{O}_X$, which also satisfies $H^0(X, \Omega_X^i) = 0$ for $0 < i < \dim X$. Note: not all definitions in the literature require this second condition. Moreover, there are definitions which allow certain types of singularities.

A Calabi-Yau variety of dimension 1 is an *elliptic curve*. A Calabi-Yau variety of dimension 2 is a *K3 surface*. A Calabi-Yau variety of dimension 3 is usually called a *Calabi-Yau threefold*. Standard examples of Calabi-Yau varieties, all given as complete intersections in a product of projective spaces, are shown in the second column of Table 1. From the homogeneous degrees of the defining equations and the coordinates of the ambient projective space one builds a lattice \mathbb{L} for use in GKZ context. This is shown in the third column of Table 1. The lattice \mathbb{L} comes naturally with an embedding into some \mathbb{Z}^N and the quotient $\mathbb{M} := \mathbb{Z}^N / \mathbb{L}$ is torsion free, isomorphic to \mathbb{Z}^{k+1} , $k+1 = N - d$. As in Section 4 we let $\mathbf{a}_1, \dots, \mathbf{a}_N \in \mathbb{M}$ denote the images of the standard basis vectors of \mathbb{Z}^N and $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_N\}$. Using Corollary 1 and Formula (44) one checks that in these examples the positive span of the last d columns of matrix \mathbf{B} is a maximal cone \mathcal{C} in the secondary fan of \mathbb{L} . Using (62) one checks that the triangulation of $\Delta_{\mathcal{A}}$ corresponding to \mathcal{C} is unimodular. Next one computes the ring $\mathcal{R}_{\mathcal{A}, \mathcal{C}}$ in Definition 65 and one finds that it is (isomorphic to) the cohomology ring of the ambient space in the second column of Table 1:

$$\mathcal{R}_{\mathcal{A}, \mathcal{C}} = \begin{cases} \mathbb{Z}[\varepsilon] / (\varepsilon^{r+1}) & \text{if the ambient space is } \mathbb{P}^r \\ \mathbb{Z}[\delta_1, \dots, \delta_r] / (\delta_1^2, \dots, \delta_r^2) & \text{if the ambient space is } (\mathbb{P}^1)^r \\ \mathbb{Z}[\delta_1, \delta_2] / (\delta_1^3, \delta_2^3) & \text{if the ambient space is } (\mathbb{P}^2)^2 \end{cases}$$

dim.	Calabi-Yau variety	B
1	cubic curve in \mathbb{P}^2	$(-3, 1, 1, 1)$
1	\cap two quadrics in \mathbb{P}^3	$(-2, -2, 1, 1, 1, 1)$
1	curve of degree $(2, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$	$\begin{pmatrix} -2 & 1 & 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$
1	\cap two surf. deg. $(1, 1, 1)$ in $(\mathbb{P}^1)^3$	$\begin{pmatrix} -1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$
2	quartic surface in \mathbb{P}^3	$(-4, 1, 1, 1, 1)$
2	\cap quadric and cubic in \mathbb{P}^4	$(-2, -3, 1, 1, 1, 1, 1)$
2	\cap three quadrics in \mathbb{P}^5	$(-2, -2, -2, 1, 1, 1, 1, 1, 1)$
2	surface of deg. $(2, 2, 2)$ in $(\mathbb{P}^1)^3$	$\begin{pmatrix} -2 & 1 & 0 & 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$
3	quintic hypersurface in \mathbb{P}^4	$(-5, 1, 1, 1, 1, 1)$
3	\cap two cubics in \mathbb{P}^5	$(-3, -3, 1, 1, 1, 1, 1, 1)$
3	3-fold of deg. $(3, 3)$ in $\mathbb{P}^2 \times \mathbb{P}^2$	$\begin{pmatrix} -3 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$
3	\cap four quadrics in \mathbb{P}^7	$(-2, -2, -2, -2, 1, 1, 1, 1, 1, 1, 1, 1)$
3	3-fold of deg. $(2, 2, 2, 2)$ in $(\mathbb{P}^1)^4$	$\begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$
	\cap means ‘intersection of’.	$\mathbb{L} = \mathbb{Z}$ -span of rows of B

TABLE 1. Standard examples of Calabi-Yau varieties.

Choosing a \mathbb{Z} -basis for \mathbb{M} we write $\mathbf{a}_j = (a_{1j}, \dots, a_{(k+1)j})$. With \mathcal{A} one now associates the *Laurent polynomial* in the variables x_1, \dots, x_{k+1} with undetermined coefficients $u_j = \mathbf{u}_{\mathbf{a}_j}$ (cf.(11)):

$$P_{\mathcal{A}}(\mathbf{x}) = P_{\mathcal{A}}(x_1, \dots, x_{k+1}) = \sum_{\mathbf{a} \in \mathcal{A}} \mathbf{u}_{\mathbf{a}} \mathbf{x}^{\mathbf{a}} = \sum_{j=1}^N u_j \prod_{i=1}^{k+1} x_i^{a_{ij}}. \quad (89)$$

Since for each \mathbf{a}_j the coordinates sum to 1, this Laurent polynomial is homogeneous: $P_{\mathcal{A}}(t\mathbf{x}) = tP_{\mathcal{A}}(\mathbf{x})$ for every $t \in \mathbb{C}^*$. As the coefficients \mathbf{u} vary the zero loci of $P_{\mathcal{A}}(\mathbf{x})$ sweep out a family of hypersurfaces in $(\mathbb{C}^*)^{k+1} / \mathbb{C}^* = (\mathbb{C}^*)^k$. Both $(\mathbb{C}^*)^k$ and the hypersurfaces can be suitably compactified. This family of compactified hypersurfaces is then the *mirror in the sense of [3] of the family of Calabi-Yau varieties* in the second column in Table 1. The members of this mirror family are Calabi-Yau varieties if the original Calabi-Yau varieties have codimension 1 in the ambient space. In case of codimension > 1 the mirror family consists of *generalized Calabi-Yau varieties* in the sense of [3].

We will now discuss details for the first three examples of Calabi-Yau threefolds. Other examples can be treated in the same way.

8.2. The quintic in \mathbb{P}^4

This is the original example with which Mirror Symmetry entered the mathematical arena; see [7]. The matrix \mathbf{B} in Table 1 is of the form $\mathbf{B} = (\tilde{\mathbf{B}} \mathbb{I}_d)$, where \mathbb{I}_d is the $d \times d$ -identity matrix. The matrix $\mathbf{A} = (\mathbb{I}_{N-d} - \tilde{\mathbf{B}}^t)$ then satisfies $\mathbf{B}\mathbf{A}^t = 0$ and its columns generate \mathbb{Z}^{k+1} . We apply row operations (i.e. a basis transformation in \mathbb{Z}^{k+1}) so that the Laurent polynomial $P_{\mathcal{A}}(\mathbf{x})$ in (89) assumes a pleasant form:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

We let $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_N\}$ denote the columns of the right-hand matrix. Then

$$P_{\mathcal{A}}(\mathbf{x}) = x_1 (u_1 + u_2 x_2 + u_3 x_3 + u_4 x_4 + u_5 x_5 + u_6 (x_2 x_3 x_4 x_5)^{-1}) .$$

Remark. The Laurent polynomial $P_{\mathcal{A}}(\mathbf{x})$ can be dehomogenized by setting $x_1 = 1$ and subsequently be homogenized to the degree 5 polynomial in 5 variables

$$\begin{aligned} \tilde{P}_{\mathcal{A}}(\mathbf{X}) = & u_1 X_1 X_2 X_3 X_4 X_5 + u_2 X_2^2 X_3 X_4 X_5 + u_3 X_2 X_3^2 X_4 X_5 + \\ & + u_4 X_2 X_3 X_4^2 X_5 + u_5 X_2 X_3 X_4 X_5^2 + u_6 X_1^5 . \end{aligned}$$

The polynomial $\tilde{P}_{\mathcal{A}}(\mathbf{X})$ defines a family of special quintic hypersurfaces in \mathbb{P}^4 , which is the mirror of the family of general quintic hypersurfaces in \mathbb{P}^4 . Traditionally the mirror family is presented as a quotient of the hypersurface

$$Z_1^5 + Z_2^5 + Z_3^5 + Z_4^5 + Z_5^5 - 5\psi Z_1 Z_2 Z_3 Z_4 Z_5 = 0$$

by a specific action of the group $(\mathbb{Z}/5\mathbb{Z})^3$ (see [7] §2). The hypergeometric integrals and series constructed from the periods of this ‘Fermat-like quintic’ are however the same as those coming from $\tilde{P}_{\mathcal{A}}(\mathbf{X})$.

The periods of the mirror Calabi-Yau hypersurfaces are given by integrals

$$I_{\sigma}^{-}(\mathbf{u}) := \frac{1}{(2\pi i)^4} \int_{\sigma} P_{\mathcal{A}}(1, x_2, x_3, x_4, x_5)^{-1} \frac{dx_2}{x_2} \frac{dx_3}{x_3} \frac{dx_4}{x_4} \frac{dx_5}{x_5} . \quad (90)$$

As shown in Section 2.3, these integrals viewed as functions of u_1, \dots, u_6 , satisfy the GKZ system of differential equations (19)-(20) with \mathcal{A} as above and $\mathbf{c} = -\mathbf{a}_1$.

If the numbers $|u_j u_1^{-1}|$ for $2 \leq j \leq 6$ are sufficiently small, the domain of integration for one of the above period integrals $I_{\sigma}^{-}(\mathbf{u})$ can be taken to be

$\sigma = \{|x_2| = |x_3| = |x_4| = |x_5| = 1\}$. Using geometric series, the binomial and residue theorems, one obtains for this period integral the series expansion:

$$I_{\sigma}^{-}(\mathbf{u}) = u_1^{-1} \sum_{n \geq 0} (-1)^n \frac{(5n)!}{(n!)^5} z^n \quad \text{with} \quad z = u_1^{-5} u_2 u_3 u_4 u_5 u_6. \quad (91)$$

Now look at the series $\Phi_{\mathbb{L}, \underline{\gamma}, \underline{\varepsilon}}(\mathbf{u})$ for $\mathbb{L} = \mathbb{Z}(-5, 1, 1, 1, 1, 1)$ and $\underline{\gamma} = (-1, 0, 0, 0, 0, 0)$ defined in (74). In this case $\underline{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6) = (-5\varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon)$. The triangulation is given by the list $T_{\mathcal{C}}$ consisting of the five sets one gets by deleting from $\{1, 2, 3, 4, 5, 6\}$ one number > 1 . The minimal set not contained in a set on the list $T_{\mathcal{C}}$ is $\{2, 3, 4, 5, 6\}$. Thus we see

$$\mathcal{R}_{\mathcal{A}, \mathcal{C}} = \mathbb{Z} \oplus \mathbb{Z}\varepsilon \oplus \mathbb{Z}\varepsilon^2 \oplus \mathbb{Z}\varepsilon^3 \oplus \mathbb{Z}\varepsilon^4, \quad \varepsilon^5 = 0,$$

and, with Pochhammer symbol notation (30) and $z = u_1^{-5} u_2 u_3 u_4 u_5 u_6$,

$$\Phi_{\mathbb{L}, \underline{\gamma}, \underline{\varepsilon}}(\mathbf{u}) = -5\varepsilon u_1^{-1} \sum_{n \geq 0} (-1)^n \frac{(1 + 5\varepsilon)_{5n}}{((1 + \varepsilon)_n)^5} z^{n+\varepsilon}. \quad (92)$$

So, the function $\Phi_{\mathbb{L}, \underline{\gamma}, \underline{\varepsilon}}(\mathbf{u})$ takes values in the vector space $\varepsilon \mathcal{R}_{\mathcal{A}, \mathcal{C}} \otimes \mathbb{C}$. Now let $\text{Ann}(\varepsilon) = \{x \in \mathcal{R}_{\mathcal{A}, \mathcal{C}} \mid x\varepsilon = 0\}$ denote the annihilator ideal of ε and let $\overline{\mathcal{R}}_{\mathcal{A}, \mathcal{C}} := \mathcal{R}_{\mathcal{A}, \mathcal{C}} / \text{Ann}(\varepsilon)$. Then as a vector space $\varepsilon \mathcal{R}_{\mathcal{A}, \mathcal{C}} \otimes \mathbb{C}$ is isomorphic to $\overline{\mathcal{R}}_{\mathcal{A}, \mathcal{C}} \otimes \mathbb{C}$. The latter has, however, the advantage of being a ring. Let $\overline{\varepsilon}$ denote the class of ε in $\overline{\mathcal{R}}_{\mathcal{A}, \mathcal{C}}$. Then

$$\overline{\mathcal{R}}_{\mathcal{A}, \mathcal{C}} = \mathbb{Z} \oplus \mathbb{Z}\overline{\varepsilon} \oplus \mathbb{Z}\overline{\varepsilon}^2 \oplus \mathbb{Z}\overline{\varepsilon}^3, \quad \overline{\varepsilon}^4 = 0.$$

Moreover we can write

$$\Phi_{\mathbb{L}, \underline{\gamma}, \underline{\varepsilon}}(\mathbf{u}) = -5u_1^{-1} z^{\overline{\varepsilon}} \sum_{n \geq 0} (-1)^n \frac{(1 + 5\overline{\varepsilon})_{5n}}{((1 + \overline{\varepsilon})_n)^5} z^n, \quad (93)$$

and view it as a function with values in the ring $\overline{\mathcal{R}}_{\mathcal{A}, \mathcal{C}} \otimes \mathbb{C}$. We expand this function with respect to the basis $\{1, \overline{\varepsilon}, \overline{\varepsilon}^2, \overline{\varepsilon}^3\}$:

$$\Phi_{\mathbb{L}, \underline{\gamma}, \underline{\varepsilon}}(\mathbf{u}) = \Phi_0(\mathbf{u}) + \Phi_1(\mathbf{u})\overline{\varepsilon} + \Phi_2(\mathbf{u})\overline{\varepsilon}^2 + \Phi_3(\mathbf{u})\overline{\varepsilon}^3. \quad (94)$$

By-passing all motivations, justifications and interpretations from string theory and Hodge theory (see, however, Section 8.5 and [8] p. 263) we define the *canonical coordinate*

$$q := -\exp\left(\frac{\Phi_1(\mathbf{u})}{\Phi_0(\mathbf{u})}\right) \quad (95)$$

and the *prepotential*

$$\mathcal{F}(q) = \frac{5}{2} \left(\frac{\Phi_1(\mathbf{u})}{\Phi_0(\mathbf{u})} \frac{\Phi_2(\mathbf{u})}{\Phi_0(\mathbf{u})} - \frac{\Phi_3(\mathbf{u})}{\Phi_0(\mathbf{u})} \right). \quad (96)$$

Note that these are functions of z , because they are constructed from quotients of solutions to the same GKZ system. In fact $q = -z + O(z^2)$ and we can invert this relation so as to get z as a function $z(q)$ of q . We then want to view the prepotential

N_1	=	2875
N_2	=	609250
N_3	=	317206375
N_4	=	242467530000
N_5	=	229305888887625
N_6	=	248249742118022000
N_7	=	295091050570845659250
N_8	=	375632160937476603550000
N_9	=	503840510416985243645106250

TABLE 2. The numbers N_j for the quintic threefold.

as a function of q . The recipe for extracting results about the enumerative geometry of the general quintic threefold is to take

$$\mathcal{F}(q) = \frac{5}{6} \log^3 q + \sum_{j \geq 1} N_j \operatorname{Li}_3(q^j), \quad (97)$$

where Li_3 is the *trilogarithm function* $\operatorname{Li}_3(x) := \sum_{n \geq 1} \frac{x^n}{n^3}$. Then one of the miracles of mirror symmetry is that all numbers N_j are positive integers and that in fact N_j equals the number of rational curves of degree j on a general quintic threefold [7, 8]. The first few of these N_j are shown in Table 2.

Actual computations proceed as follows: compute F_0, \dots, f_3 from

$$\sum_{n \geq 0} (-1)^n \frac{(1 + 5\bar{\varepsilon})_{5n}}{((1 + \bar{\varepsilon})_n)^5} z^n = F_0(z) + F_1(z)\bar{\varepsilon} + F_2(z)\bar{\varepsilon}^2 + F_3(z)\bar{\varepsilon}^3$$

and $f_i(z) := \frac{F_i(z)}{F_0(z)}$. That implies

$$\Phi_{\mathbb{L}, \underline{\gamma}, \underline{\varepsilon}}(\mathbf{u}) = \Phi_0(\mathbf{u}) z^{\bar{\varepsilon}} (1 + f_1(z)\bar{\varepsilon} + f_2(z)\bar{\varepsilon}^2 + f_3(z)\bar{\varepsilon}^3) \quad (98)$$

Comparing the expansions of $\log \Phi_{\mathbb{L}, \underline{\gamma}, \underline{\varepsilon}}(\mathbf{u})$ which result from (94) and (98), i.e.

$$\begin{aligned} \log \Phi_{\mathbb{L}, \underline{\gamma}, \underline{\varepsilon}}(\mathbf{u}) &= \log(\Phi_0(z)) + \frac{\Phi_1(\mathbf{u})}{\Phi_0(\mathbf{u})} \bar{\varepsilon} + \left(-\frac{1}{2} \left[\frac{\Phi_1(\mathbf{u})}{\Phi_0(\mathbf{u})} \right]^2 + \frac{\Phi_2(\mathbf{u})}{\Phi_0(\mathbf{u})} \right) \bar{\varepsilon}^2 \\ &\quad + \left(\frac{1}{3} \left[\frac{\Phi_1(\mathbf{u})}{\Phi_0(\mathbf{u})} \right]^3 - \frac{\Phi_1(\mathbf{u})}{\Phi_0(\mathbf{u})} \frac{\Phi_2(\mathbf{u})}{\Phi_0(\mathbf{u})} + \frac{\Phi_3(\mathbf{u})}{\Phi_0(\mathbf{u})} \right) \bar{\varepsilon}^3 \\ &= \log(\Phi_0(z)) + (\log z + f_1(z)) \bar{\varepsilon} + \left(-\frac{1}{2} f_1(z)^2 + f_2(z) \right) \bar{\varepsilon}^2 \\ &\quad + \left(\frac{1}{3} f_1(z)^3 - f_1(z) f_2(z) + f_3(z) \right) \bar{\varepsilon}^3, \end{aligned}$$

we see that q and $\mathcal{F}(q)$ can easily be computed from the already known f_1, f_2, f_3 :

$$\begin{aligned} q &= -z \exp(f_1(z)), \\ \mathcal{F}(q) &= \frac{5}{2} \left(\frac{1}{3} \log^3(-q) - \left(\frac{1}{3} f_1(z(q))^3 - f_1(z(q)) f_2(z(q)) + f_3(z(q)) \right) \right). \end{aligned}$$

8.3. The intersection of two cubics in \mathbb{P}^5

This is one of the examples discussed in [17]. Here we treat it as a highly resonant GKZ system. As in the case of the quintic the matrix \mathbf{B} in Table 1 is of the form $\mathbf{B} = (\tilde{\mathbf{B}} \mathbb{I}_d)$ and we apply row operations to the matrix $(\mathbb{I}_{N-d} - \tilde{\mathbf{B}}^t)$ so that the Laurent polynomial $P_{\mathcal{A}}(\mathbf{x})$ in (89) assumes a pleasant form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \end{pmatrix}$$

We let $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_N\}$ denote the columns of the right-hand matrix. Then

$$P_{\mathcal{A}}(\mathbf{x}) = x_1 P_{\mathcal{A},1}(x_3, x_4, x_5) + x_2 P_{\mathcal{A},2}(x_5, x_6, x_7)$$

with

$$\begin{aligned} P_{\mathcal{A},1}(x_3, x_4, x_5) &= u_1 + u_3 x_3 + u_4 x_4 + u_8 (x_3 x_4 x_5)^{-1} \\ P_{\mathcal{A},2}(x_5, x_6, x_7) &= u_2 + u_5 x_5 (x_6 x_7)^{-1} + u_6 x_6 + u_7 x_7. \end{aligned}$$

This way of combining the two Laurent polynomials in five variables, $P_{\mathcal{A},1}$ and $P_{\mathcal{A},2}$, to one Laurent polynomial in seven variables $P_{\mathcal{A}}$ is known as *Cayley's trick* (see [12, 13, 14]). The two polynomials $P_{\mathcal{A},1}$ and $P_{\mathcal{A},2}$, suitably homogenized, define a family of Calabi-Yau complete intersection threefolds in $\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$. The corresponding period integrals are (cf. Section 2.4 and [12, 14])

$$I_{\sigma}^{-}(\mathbf{u}) := \frac{1}{(2\pi i)^5} \int_{\sigma} P_{\mathcal{A},1}(x_3, x_4, x_5)^{-1} P_{\mathcal{A},2}(x_5, x_6, x_7)^{-1} \frac{dx_3}{x_3} \frac{dx_4}{x_4} \frac{dx_5}{x_5} \frac{dx_6}{x_6} \frac{dx_7}{x_7}. \quad (99)$$

One can show as in Sections 2.3 and 2.4 that these integrals viewed as functions of u_1, \dots, u_8 , satisfy the GKZ system of differential equations (19)-(20) with \mathcal{A} as above and $\mathbf{c} = -\mathbf{a}_1 - \mathbf{a}_2$.

If the numbers $|u_3 u_1^{-1}|$, $|u_4 u_1^{-1}|$, $|u_8 u_1^{-1}|$ and $|u_5 u_2^{-1}|$, $|u_6 u_2^{-1}|$, $|u_7 u_2^{-1}|$ are sufficiently small, the domain of integration for one of the above period integrals $I_{\sigma}^{-}(\mathbf{u})$ can be taken to be $\sigma = \{|x_3| = |x_4| = |x_5| = |x_6| = |x_7| = 1\}$. This period integral admits the series expansion, with $z = u_1^{-3} u_2^{-3} u_3 u_4 u_5 u_6 u_7 u_8$,

$$I_{\sigma}^{-}(\mathbf{u}) = u_1^{-1} u_2^{-1} \sum_{n \geq 0} \left(\frac{(3n)!}{(n!)^3} \right)^2 z^n. \quad (100)$$

The series $\Phi_{\mathbb{L}, \underline{\gamma}, \underline{\varepsilon}}(\mathbf{u})$ for $\mathbb{L} = \mathbb{Z}(-3, -3, 1, 1, 1, 1, 1, 1)$, $\underline{\gamma} = (-1, -1, 0, 0, 0, 0, 0, 0)$ and $\underline{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_8) = (-3\varepsilon, -3\varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon)$ reads

$$\Phi_{\mathbb{L}, \underline{\gamma}, \underline{\varepsilon}}(\mathbf{u}) = 9\varepsilon^2 u_1^{-1} u_2^{-1} \sum_{n \geq 0} \left(\frac{(1 + 3\varepsilon)_{3n}}{((1 + \varepsilon)_n)^3} \right)^2 z^{n+\varepsilon}, \quad (101)$$

N_1	$=$	1053
N_2	$=$	52812
N_3	$=$	6424326
N_4	$=$	1139448384
N_5	$=$	249787892583
N_6	$=$	62660964509532
N_7	$=$	17256453900822009
N_8	$=$	5088842568426162960
N_9	$=$	1581250717976557887945

TABLE 3. The numbers N_j for the intersection of two cubics in \mathbb{P}^5 .

and is evaluated in $\mathcal{R}_{\mathcal{A},C} = \mathbb{Z} \oplus \mathbb{Z}\varepsilon \oplus \mathbb{Z}\varepsilon^2 \oplus \mathbb{Z}\varepsilon^3 \oplus \mathbb{Z}\varepsilon^4 \oplus \mathbb{Z}\varepsilon^5$, $\varepsilon^6 = 0$. The function $\Phi_{\mathbb{L},\underline{\gamma},\underline{\varepsilon}}(\mathbf{u})$ actually takes values in the vector space $\varepsilon^2 \mathcal{R}_{\mathcal{A},C} \otimes \mathbb{C}$. As in the case of the quintic, we replace this space by the isomorphic space $\overline{\mathcal{R}}_{\mathcal{A},C} \otimes \mathbb{C}$, where $\text{Ann}(\varepsilon^2) := \{x \in \mathcal{R}_{\mathcal{A},C} \mid x\varepsilon^2 = 0\}$ and $\overline{\mathcal{R}}_{\mathcal{A},C} := \mathcal{R}_{\mathcal{A},C} / \text{Ann}(\varepsilon^2)$. Let $\overline{\varepsilon}$ denote the class of ε in $\overline{\mathcal{R}}_{\mathcal{A},C}$. Then

$$\overline{\mathcal{R}}_{\mathcal{A},C} = \mathbb{Z} \oplus \mathbb{Z}\overline{\varepsilon} \oplus \mathbb{Z}\overline{\varepsilon}^2 \oplus \mathbb{Z}\overline{\varepsilon}^3, \quad \overline{\varepsilon}^4 = 0.$$

Proceeding as in the case of the quintic we write

$$\begin{aligned} \Phi_{\mathbb{L},\underline{\gamma},\underline{\varepsilon}}(\mathbf{u}) &= 9u_1^{-1}u_2^{-1} \sum_{n \geq 0} \left(\frac{(1 + 3\overline{\varepsilon})_{3n}}{((1 + \overline{\varepsilon})_n)^3} \right)^2 z^{n+\overline{\varepsilon}} \\ &= \Phi_0(\mathbf{u}) + \Phi_1(\mathbf{u})\overline{\varepsilon} + \Phi_2(\mathbf{u})\overline{\varepsilon}^2 + \Phi_3(\mathbf{u})\overline{\varepsilon}^3 \\ &= \Phi_0(\mathbf{u})z^{\overline{\varepsilon}} (1 + f_1(z)\overline{\varepsilon} + f_2(z)\overline{\varepsilon}^2 + f_3(z)\overline{\varepsilon}^3), \end{aligned}$$

and almost exactly as for the quintic we extract from $\log \Phi_{\mathbb{L},\underline{\gamma},\underline{\varepsilon}}(\mathbf{u})$ a *canonical coordinate* and a *prepotential*:

$$q := z \exp(f_1(z)) = z + O(z^2), \quad (102)$$

$$\mathcal{F}(q) := \frac{9}{2} \left(\frac{1}{3} \log^3 q - \left(\frac{1}{3} f_1(z(q))^3 - f_1(z(q))f_2(z(q)) + f_3(z(q)) \right) \right). \quad (103)$$

Finally we compute the numbers N_j from the expansion

$$\mathcal{F}(q) = \frac{3}{2} \log^3 q + \sum_{j \geq 1} N_j \text{Li}_3(q^j). \quad (104)$$

The first few of the numbers N_j are shown in Table 3 and agree with those in [17].

8.4. The hypersurface of degree (3, 3) in $\mathbb{P}^2 \times \mathbb{P}^2$

Again the matrix \mathbf{B} in Table 1 is of the form $\mathbf{B} = (\tilde{\mathbf{B}} \mathbb{I}_d)$, and we apply row operations to the matrix $(\mathbb{I}_{N-d} - \tilde{\mathbf{B}}^t)$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 3 & 3 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}$$

We let $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_N\}$ denote the columns of the right-hand matrix. Then

$$P_{\mathcal{A}}(\mathbf{x}) = x_1 (u_1 + u_2 x_2 + u_3 x_3 + u_4 x_4 + u_5 x_5 + u_6 (x_2 x_4)^{-1} + u_7 (x_3 x_5)^{-1}) .$$

The periods of the mirror Calabi-Yau hypersurfaces are given by integrals

$$I_{\sigma}^{-}(\mathbf{u}) := \frac{1}{(2\pi i)^4} \int_{\sigma} P_{\mathcal{A}}(1, x_2, x_3, x_4, x_5)^{-1} \frac{dx_2}{x_2} \frac{dx_3}{x_3} \frac{dx_4}{x_4} \frac{dx_5}{x_5} ; \quad (105)$$

As shown in Section 2.3 these integrals viewed as functions of u_1, \dots, u_7 , satisfy the GKZ system of differential equations (19)-(20) with \mathcal{A} as above and $\mathbf{c} = -\mathbf{a}_1$.

If the numbers $|u_j u_1^{-1}|$ for $2 \leq j \leq 7$ are sufficiently small, the domain of integration for one of the above period integrals $I_{\sigma}^{-}(\mathbf{u})$ can be taken to be $\sigma = \{|x_2| = |x_3| = |x_4| = |x_5| = 1\}$. This integral admits the expansion:

$$I_{\sigma}^{-}(\mathbf{u}) = u_1^{-1} \sum_{n_1, n_2 \geq 0} (-1)^{n_1+n_2} \frac{(3n_1+3n_2)!}{(n_1!)^3 (n_2!)^3} z_1^{n_1} z_2^{n_2} \quad (106)$$

with $z_1 = u_1^{-3} u_2 u_4 u_6$ and $z_2 = u_1^{-3} u_3 u_5 u_7$. Now look at the series $\Phi_{\mathbb{L}, \underline{\gamma}, \underline{\varepsilon}}(\mathbf{u})$ for $\mathbb{L} = \mathbb{Z}(-3, 1, 0, 1, 0, 1, 0) \oplus \mathbb{Z}(-3, 0, 1, 0, 1, 0, 1)$, $\underline{\gamma} = (-1, 0, 0, 0, 0, 0, 0)$ and $\underline{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7) = \delta_1(-3, 1, 0, 1, 0, 1, 0) + \delta_2(-3, 0, 1, 0, 1, 0, 1)$. Using Corollary 1 and the matrix \mathbf{B} from Table 1 for this example one easily checks that the triangulation is given by the list $T_{\mathcal{C}}$ consisting of the nine sets one gets by deleting from $\{1, 2, 3, 4, 5, 6, 7\}$ one even and one odd number > 1 . The minimal sets not contained in a set on the list $T_{\mathcal{C}}$ are $\{2, 4, 6\}$ and $\{3, 5, 7\}$. Thus we see

$$\begin{aligned} \mathcal{R}_{\mathcal{A}, \mathcal{C}} &= \mathbb{Z} \oplus \mathbb{Z} \delta_1 \oplus \mathbb{Z} \delta_2 \oplus \mathbb{Z} \delta_1^2 \oplus \mathbb{Z} \delta_1 \delta_2 \oplus \mathbb{Z} \delta_2^2 \oplus \mathbb{Z} \delta_1^2 \delta_2 \oplus \mathbb{Z} \delta_1 \delta_2^2 \oplus \mathbb{Z} \delta_1^2 \delta_2^2, \\ \delta_1^3 &= \delta_2^3 = 0. \end{aligned}$$

Thus, with $z_1 = u_1^{-3} u_2 u_4 u_6$ and $z_2 = u_1^{-3} u_3 u_5 u_7$,

$$\Phi_{\mathbb{L}, \underline{\gamma}, \underline{\varepsilon}}(\mathbf{u}) = -3(\delta_1 + \delta_2) u_1^{-1} \sum_{n_1, n_2 \geq 0} (-1)^{n_1+n_2} \frac{(1+3\delta_1+3\delta_2)_{3n_1+3n_2}}{((1+\delta_1)_{n_1}(1+\delta_2)_{n_2})^3} z_1^{n_1+\delta_1} z_2^{n_2+\delta_2} .$$

The function $\Phi_{\mathbb{L}, \underline{\gamma}, \underline{\varepsilon}}(\mathbf{u})$ takes values in the vector space $(\delta_1 + \delta_2) \mathcal{R}_{\mathcal{A}, \mathcal{C}} \otimes \mathbb{C}$. As in the previous cases, we replace this space by the isomorphic one $\overline{\mathcal{R}}_{\mathcal{A}, \mathcal{C}} \otimes \mathbb{C}$, where

$\text{Ann}(\delta_1 + \delta_2) := \{x \in \mathcal{R}_{\mathcal{A},C} \mid x(\delta_1 + \delta_2) = 0\}$ and $\overline{\mathcal{R}}_{\mathcal{A},C} := \mathcal{R}_{\mathcal{A},C} / \text{Ann}(\delta_1 + \delta_2)$. Let $\overline{\delta}_1$ and $\overline{\delta}_2$ denote the classes of δ_1 and δ_2 , respectively, in $\overline{\mathcal{R}}_{\mathcal{A},C}$. Then

$$\begin{aligned} \overline{\mathcal{R}}_{\mathcal{A},C} &= \mathbb{Z} \oplus \mathbb{Z}\overline{\delta}_1 \oplus \mathbb{Z}\overline{\delta}_2 \oplus \mathbb{Z}\overline{\delta}_2^2 \oplus \mathbb{Z}\overline{\delta}_1^2 \oplus \mathbb{Z}\overline{\delta}_1^2\overline{\delta}_2, \\ \overline{\delta}_1\overline{\delta}_2 &= \overline{\delta}_1^2 + \overline{\delta}_2^2, \quad \overline{\delta}_1^2\overline{\delta}_2 = \overline{\delta}_1\overline{\delta}_2^2, \quad \overline{\delta}_1^3 = \overline{\delta}_2^3 = \overline{\delta}_1^2\overline{\delta}_2^2 = 0. \end{aligned}$$

Proceeding as in the previous examples we write

$$\begin{aligned} \Phi_{\mathbb{L},\underline{\gamma},\underline{\varepsilon}}(\mathbf{u}) &= -3u_1^{-1} \sum_{n_1, n_2 \geq 0} (-1)^{n_1+n_2} \frac{(1+3\overline{\delta}_1+3\overline{\delta}_2)^{3n_1+3n_2}}{((1+\overline{\delta}_1)_{n_1}(1+\overline{\delta}_2)_{n_2})^3} z_1^{n_1+\overline{\delta}_1} z_2^{n_2+\overline{\delta}_2} \\ &= \Phi_0(\mathbf{u}) + \Phi_{1,1}(\mathbf{u})\overline{\delta}_1 + \Phi_{1,2}(\mathbf{u})\overline{\delta}_2 + \Phi_{2,1}(\mathbf{u})\overline{\delta}_2^2 + \Phi_{2,2}(\mathbf{u})\overline{\delta}_1^2 + \Phi_3(\mathbf{u})\overline{\delta}_1^2\overline{\delta}_2 \\ &= \Phi_0(\mathbf{u})z_1^{\overline{\delta}_1}z_2^{\overline{\delta}_2} \left(1 + f_{1,1}(\mathbf{z})\overline{\delta}_1 + f_{1,2}(\mathbf{z})\overline{\delta}_2 + f_{2,1}(\mathbf{z})\overline{\delta}_2^2 + f_{2,2}(\mathbf{z})\overline{\delta}_1^2 + f_3(\mathbf{z})\overline{\delta}_1^2\overline{\delta}_2\right). \end{aligned}$$

Here $\mathbf{z} = (z_1, z_2)$. From the $\overline{\delta}_1$ and $\overline{\delta}_2$ components we construct two *canonical coordinates* (cf. (111))

$$q_1 := -z_1 \exp(f_{1,1}(\mathbf{z})), \quad q_2 := -z_2 \exp(f_{1,2}(\mathbf{z})). \quad (107)$$

We view z_1, z_2 as functions of q_1, q_2 via the inverse of relation (107). The *prepotential* in this case is (cf. (114))

$$\mathcal{F}(q) = \frac{3}{2} \left(\frac{\Phi_{1,1}(\mathbf{u})}{\Phi_0(\mathbf{u})} \frac{\Phi_{2,1}(\mathbf{u})}{\Phi_0(\mathbf{u})} + \frac{\Phi_{1,2}(\mathbf{u})}{\Phi_0(\mathbf{u})} \frac{\Phi_{2,2}(\mathbf{u})}{\Phi_0(\mathbf{u})} - \frac{\Phi_3(\mathbf{u})}{\Phi_0(\mathbf{u})} \right). \quad (108)$$

The $-$ -signs in (107) and the factor 3 in (108) are needed to match the calculations below with the results in [16] Appendix B2.

We expand $\log \Phi_{\mathbb{L},\underline{\gamma},\underline{\varepsilon}}(\mathbf{u})$ on the basis $\{1, \overline{\delta}_1, \overline{\delta}_2, \overline{\delta}_2^2, \overline{\delta}_1^2, \overline{\delta}_1^2\overline{\delta}_2\}$ of $\overline{\mathcal{R}}_{\mathcal{A},C}$. The $\overline{\delta}_1^2\overline{\delta}_2$ -coordinate is on the one hand

$$\log(-q_1) \log(-q_2) \log(q_1 q_2) - \frac{2}{3} \mathcal{F}(q)$$

and on the other hand it is

$$f_{1,1}(\mathbf{z})^2 f_{1,2}(\mathbf{z}) + f_{1,1}(\mathbf{z}) f_{1,2}(\mathbf{z})^2 - f_{1,1}(\mathbf{z}) f_{2,1}(\mathbf{z}) - f_{1,2}(\mathbf{z}) f_{2,2}(\mathbf{z}) + f_3(\mathbf{z}).$$

Computing the coefficients N_{j_1, j_2} in the expansion

$$\mathcal{F}(q_1, q_2) = \frac{3}{2} \log(-q_1) \log(-q_2) \log(q_1 q_2) + \sum_{j_1, j_2 \geq 0, j_1+j_2 > 0} N_{j_1, j_2} \text{Li}_3(q_1^{j_1} q_2^{j_2})$$

is now somewhat more involved than in the previous examples. We leave it as an **exercise in Mathematica, Maple or PARI programming**.

A table of the numbers N_{j_1, j_2} for this example appears in [16] Appendix B2 under the name $X_{(3|3)}(1, 1, 1|1, 1, 1)$. In [16] one finds many more 2-parameter models.

8.5. The Schwarz map for some extended GKZ systems

In this section we briefly discuss how the $\overline{\mathcal{R}}_{\mathcal{A},C} \otimes \mathbb{C}$ -valued function $\Phi_{\mathbb{L},\underline{\gamma},\underline{\varepsilon}}(\mathbf{u})$ which we met in the preceding examples, can be viewed as a Schwarz map and what are some special features of the image.

First note that, since $\dim \overline{\mathcal{R}}_{\mathcal{A},C} \otimes \mathbb{C} < \dim \mathcal{R}_{\mathcal{A},C} \otimes \mathbb{C} = \text{volume } \Delta_{\mathcal{A}}$, the components of $\Phi_{\mathbb{L},\underline{\gamma},\underline{\varepsilon}}(\mathbf{u})$ with respect to some basis of $\overline{\mathcal{R}}_{\mathcal{A},C} \otimes \mathbb{C}$ can not suffice as a basis for the solution space of the GKZ system. They do however constitute a basis for the solution space of some extension of the GKZ system (see [16]). So, strictly speaking we are not talking about the Schwarz map for the GKZ system, but for an extension thereof. Since we do explicitly have all these basis solutions for the extended system, we need not care about this system itself.

In the examples, coming from (families of) Calabi-Yau threefolds, the ring $\overline{\mathcal{R}}_{\mathcal{A},C}$ is graded and splits in homogeneous pieces,

$$\overline{\mathcal{R}}_{\mathcal{A},C} = \overline{\mathcal{R}}_{\mathcal{A},C}^{(0)} \oplus \overline{\mathcal{R}}_{\mathcal{A},C}^{(1)} \oplus \overline{\mathcal{R}}_{\mathcal{A},C}^{(2)} \oplus \overline{\mathcal{R}}_{\mathcal{A},C}^{(3)},$$

with degrees 0, 1, 2, 3 and ranks 1, d , d , 1, respectively; recall $d = \text{rank } \mathbb{L}$. We fix a basis for $\overline{\mathcal{R}}_{\mathcal{A},C}$ by fixing bases for the homogeneous pieces

$$\overline{\mathcal{R}}_{\mathcal{A},C}^{(0)} : e_0 = 1, \quad \overline{\mathcal{R}}_{\mathcal{A},C}^{(1)} : e_{1,1}, \dots, e_{1,d}, \quad \overline{\mathcal{R}}_{\mathcal{A},C}^{(2)} : e_{2,1}, \dots, e_{2,d}, \quad \overline{\mathcal{R}}_{\mathcal{A},C}^{(3)} : e_3,$$

and expand $\Phi_{\mathbb{L},\underline{\gamma},\underline{\varepsilon}}(\mathbf{u})$ with respect to this basis

$$\Phi_{\mathbb{L},\underline{\gamma},\underline{\varepsilon}}(\mathbf{u}) = \Phi_0(\mathbf{u})e_0 + \sum_{i=1}^d \Phi_{1,i}(\mathbf{u})e_{1,i} + \sum_{i=1}^d \Phi_{2,i}(\mathbf{u})e_{2,i} + \Phi_3(\mathbf{u})e_3. \quad (109)$$

The Schwarz map lands in the projective space

$$\mathbb{P}(\overline{\mathcal{R}}_{\mathcal{A},C} \otimes \mathbb{C})$$

and $\Phi_0(\mathbf{u}), \dots, \Phi_3(\mathbf{u})$ are homogeneous coordinates for the image points. Since these functions are solutions of the same GKZ system their quotients, and hence also the Schwarz map, are defined on some open set in $\mathcal{V}_{\mathcal{A}}$ near the special point \mathbf{p}_C corresponding to the maximal cone C in the secondary fan. The map is multi-valued and we do fully control the local monodromy.

The image of the Schwarz map has dimension equal to $\dim \mathcal{V}_{\mathcal{A}} = \text{rank } \mathbb{L} = d$, whereas the projective space $\mathbb{P}(\overline{\mathcal{R}}_{\mathcal{A},C} \otimes \mathbb{C})$ has dimension $1 + 2d$. For the description of the image of the Schwarz map we want to profit from the description of the moduli of Calabi-Yau threefolds by Bryant and Griffiths [6]. In the theory of moduli of Calabi-Yau threefolds one writes the holomorphic 3-form in coordinates with respect to a basis of the third cohomology space given by topological 3-cycles. These coordinates are the period integrals of the 3-form. We know $\Phi_0(\mathbf{u})$ explicitly as a period integral (see (90), (99), (105)), but we still need an argument for the other coordinates in (109) to be periods. Such an argument may be that inspection of the local monodromy shows that the extended GKZ system of differential equations satisfied by the known period integral $\Phi_0(\mathbf{u})$ is irreducible, for then all periods must be linear combinations of $\Phi_0(\mathbf{u}), \dots, \Phi_3(\mathbf{u})$.

Having matched (109) with the coordinates (= periods) of the holomorphic 3-form with respect to a basis of topological 3-cycles, we must check that the basis e_0, \dots, e_3 satisfies the requirements for application of the Bryant-Griffiths theory, i.e. we need to know that with respect to the alternating bilinear form \langle, \rangle on the third cohomology space of the Calabi-Yau threefold

$$\langle e_0, e_3 \rangle = -\langle e_3, e_0 \rangle = -\langle e_{1,i}, e_{2,i} \rangle = \langle e_{2,i}, e_{1,i} \rangle = 1 \quad \text{for } i = 1, \dots, d, \quad (110)$$

and all other $\langle e_r, e_s \rangle = 0$. In an example at the end of this section we show how to derive (110) from the explicitly known local monodromy and logarithmic pieces of $\Phi_{\mathbb{L}, \gamma, \underline{\varepsilon}}(\mathbf{u})$.

We are now all set for applying [6]. First define the *canonical coordinates*

$$q_i := \exp \left(\frac{\Phi_{1,i}(\mathbf{u})}{\Phi_0(\mathbf{u})} \right) \quad \text{for } i = 1, \dots, d. \quad (111)$$

The derivations $q_i \frac{\partial}{\partial q_i}$ act on the cohomology spaces of the Calabi-Yau threefolds in the family. Griffiths' transversality and the Riemann bilinear relations imply

$$\left\langle \frac{\Phi_{\mathbb{L}, \gamma, \underline{\varepsilon}}(\mathbf{u})}{\Phi_0(\mathbf{u})}, q_i \frac{\partial}{\partial q_i} \left(\frac{\Phi_{\mathbb{L}, \gamma, \underline{\varepsilon}}(\mathbf{u})}{\Phi_0(\mathbf{u})} \right) \right\rangle = 0. \quad (112)$$

Write $\varphi_3 := \frac{\Phi_3(\mathbf{u})}{\Phi_0(\mathbf{u})}$ and $\varphi_{a,j} := \frac{\Phi_{a,j}(\mathbf{u})}{\Phi_0(\mathbf{u})}$ for $a = 1, 2, j = 1, \dots, d$. These are (multivalued) functions of q_1, \dots, q_d , and in fact $\varphi_{1,j} = \log q_j$. Then the left hand side of (112) evaluates to

$$q_i \frac{\partial \varphi_3}{\partial q_i} - \sum_{j=1}^d \left(\varphi_{1,j} q_i \frac{\partial \varphi_{2,j}}{\partial q_i} \right) + \varphi_{2,i} = q_i \frac{\partial}{\partial q_i} \left(\varphi_3 - \sum_{j=1}^d \varphi_{1,j} \varphi_{2,j} \right) + 2 \varphi_{2,i}.$$

According to (112) this equals 0 and thus

$$\varphi_{2,i} = q_i \frac{\partial \mathcal{F}}{\partial q_i} \quad (113)$$

where

$$\mathcal{F} := \frac{1}{2} \left(-\varphi_3 + \sum_{j=1}^d \varphi_{1,j} \varphi_{2,j} \right) \quad (114)$$

is the so-called *prepotential*.

Example. Thus we recover the canonical coordinate (95) for the quintic in \mathbb{P}^4 up to a $-$ -sign and the prepotential (96) up to a factor 5 (which is the degree of the quintic). And similarly for the intersection of two cubics in \mathbb{P}^3 and the hypersurface of degree (3, 3) in $\mathbb{P}^2 \times \mathbb{P}^2$. The factors 'sign' and 'degree' are needed to match the results of our calculations with the tables of enumerative data in the literature. Moreover, if the wrong sign is used, the numbers N_{j_1, \dots, j_d} are often even not integers.

Conclusion. *The above discussion shows that the canonical coordinates and the prepotential act like a parametrization for the image of the Schwarz map: the image points have coordinates $(1, t_1, \dots, t_{2d+1})$ with*

$$\begin{aligned} t_j &= \log q_j & \text{for } j = 1, \dots, d, \\ t_{d+j} &= q_j \frac{\partial \mathcal{F}}{\partial q_j} & \text{for } j = 1, \dots, d, \\ t_{2d+1} &= -2\mathcal{F} + \sum_{j=1}^d t_j t_{d+j}. \end{aligned}$$

Remark. On the graded ring $\overline{\mathcal{R}}_{\mathcal{A},C}$ there is an involution \cdot^* given for homogeneous elements by $x^* = (-1)^{\deg x} x$. We fix the linear map $\tau : \overline{\mathcal{R}}_{\mathcal{A},C} \xrightarrow{\text{project}} \overline{\mathcal{R}}_{\mathcal{A},C}^{(3)} \xrightarrow{\simeq} \mathbb{Z}$. Then, in the examples of Sections 8.2, 8.3, 8.4 the alternating bilinear form defined by the ordered basis and the relations (110), is in fact

$$\langle x, y \rangle = \tau(x^* y).$$

Moreover, in those examples the trick of expanding $\log \Phi_{\mathbb{L}, \gamma, \varepsilon}(\mathbf{u})$ in two ways showed that the prepotential is a polynomial of degree 3 in $\log q_1, \dots, \log q_d$ plus a power series in q_1, \dots, q_d .

Example. Let us check that (110) holds for the ordered basis

$$\{e_0, e_{1,1}, e_{1,2}, e_{2,1}, e_{2,2}, e_3\} := \{1, \overline{\delta}_1, \overline{\delta}_2, \overline{\delta}_2^2, \overline{\delta}_1^2, \overline{\delta}_1^2 \overline{\delta}_2\}$$

in the example of Section 8.4. The alternating bilinear form (on the third cohomology space in a family of Calabi-Yau threefolds) is invariant under the local monodromy operators, which in this case are given by multiplication by $\exp(2\pi i \overline{\delta}_1)$ and $\exp(2\pi i \overline{\delta}_2)$. This means that the matrices for multiplication with $\overline{\delta}_1$ and $\overline{\delta}_2$ and the Gramm matrix of the alternating bilinear form, everything with respect to the above basis, must satisfy $\text{Gramm}^t = -\text{Gramm}$ and

$$\text{mat}(\overline{\delta}_a) \cdot \text{Gramm} = -\text{Gramm} \cdot \text{mat}(\overline{\delta}_a)^t, \quad a = 1, 2. \quad (115)$$

One easily checks

$$\text{mat}(\overline{\delta}_1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad \text{mat}(\overline{\delta}_2) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The general anti-symmetric matrix solution to (115) has, up to a non-zero scalar factor, the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & x \\ 0 & 0 & 1 & 0 & 0 & y \\ -1 & 0 & 0 & -x & -y & 0 \end{pmatrix}.$$

When we evaluate (112) using this Gramm matrix, we find, for $i = 1, 2$,

$$\begin{aligned} & q_i \frac{\partial \varphi_3}{\partial q_i} - \varphi_{1,1} q_i \frac{\partial \varphi_{2,1}}{\partial q_i} - \varphi_{1,2} q_i \frac{\partial \varphi_{2,2}}{\partial q_i} + \varphi_{2,i} + \\ & + x \left(\varphi_{2,1} q_i \frac{\partial \varphi_3}{\partial q_i} - \varphi_{3,q_i} \frac{\partial \varphi_{2,1}}{\partial q_i} \right) + y \left(\varphi_{2,2} q_i \frac{\partial \varphi_3}{\partial q_i} - \varphi_{3,q_i} \frac{\partial \varphi_{2,2}}{\partial q_i} \right) = 0. \end{aligned} \quad (116)$$

We want to estimate the growth of the various terms in this expression by looking at the logarithmic pieces. So, recall that in this example

$$\Phi_{\mathbb{L}, \gamma, \varepsilon}(\mathbf{u}) = \Phi_0(\mathbf{u}) z_1^{\bar{\delta}_1} z_2^{\bar{\delta}_2} \times (\text{power series in } z_1, z_2)$$

and

$$\begin{aligned} z_1^{\bar{\delta}_1} z_2^{\bar{\delta}_2} &= (1 + (\log z_1) \bar{\delta}_1 + \frac{1}{2} (\log z_1)^2 \bar{\delta}_1^2) (1 + (\log z_2) \bar{\delta}_2 + \frac{1}{2} (\log z_2)^2 \bar{\delta}_2^2) \\ &= 1 + (\log z_1) \bar{\delta}_1 + (\log z_2) \bar{\delta}_2 + \left(\frac{1}{2} (\log z_2)^2 + (\log z_1) (\log z_2) \right) \bar{\delta}_2^2 \\ &\quad + \left(\frac{1}{2} (\log z_1)^2 + (\log z_1) (\log z_2) \right) \bar{\delta}_1^2 + \frac{1}{2} (\log z_1) (\log z_2) (\log z_1 z_2) \bar{\delta}_1^2 \bar{\delta}_2^2. \end{aligned}$$

Moreover, up to addition of power series, $\log q_1 \asymp \log z_1$ and $\log q_2 \asymp \log z_2$. Thus we see that the highest order logarithmic contributions are

$$\begin{aligned} -\varphi_{3,q_1} \frac{\partial \varphi_{2,1}}{\partial q_1} + \varphi_{2,1} q_1 \frac{\partial \varphi_3}{\partial q_1} &\asymp -\frac{1}{2} ((\log q_1)^2 (\log q_2) + (\log q_1) (\log q_2)^2) (\log q_2) \\ &\quad + \left(\frac{1}{2} (\log q_2)^2 + (\log q_1) (\log q_2) \right) ((\log q_1) (\log q_2) + \frac{1}{2} (\log q_2)^2) \\ &= \frac{1}{2} (\log q_1)^2 (\log q_2)^2 + \frac{1}{2} (\log q_1) (\log q_2)^3 + \frac{1}{4} (\log q_2)^4 \end{aligned}$$

and

$$\begin{aligned} -\varphi_{3,q_1} \frac{\partial \varphi_{2,2}}{\partial q_1} + \varphi_{2,2} q_1 \frac{\partial \varphi_3}{\partial q_1} &\asymp -\frac{1}{2} ((\log q_1)^2 (\log q_2) + (\log q_1) (\log q_2)^2) (\log q_1 + \log q_2) \\ &\quad + \left(\frac{1}{2} (\log q_1)^2 + (\log q_1) (\log q_2) \right) ((\log q_1) (\log q_2) + \frac{1}{2} (\log q_2)^2) \\ &= \frac{1}{4} (\log q_1)^2 (\log q_2)^2 \end{aligned}$$

So if we consider (116) for $i = 1$ and $q_2 \downarrow 0$ the dominant $\frac{1}{4} (\log q_2)^4$ term forces $x = 0$. Having $x = 0$ we consider (116) for $i = 1$ and $q_1 = q_2 \downarrow 0$. Once again there is a dominant $\frac{1}{4} (\log q_2)^4$, forcing $y = 0$.

This finishes the proof for the fact that in the example of Section 8.4 the ordered basis $\{1, \bar{\delta}_1, \bar{\delta}_2, \bar{\delta}_2^2, \bar{\delta}_1^2, \bar{\delta}_1^2 \bar{\delta}_2\}$ satisfies (110).

Exercise. Apply the techniques of the preceding example and show that the basis $\{1, \bar{\varepsilon}, \bar{\varepsilon}^2, \bar{\varepsilon}^3\}$ in Sections 8.2 and 8.3 satisfies (110).

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