

Mahler Measure, Eisenstein Series and Dimers.

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Abstract

This note reveals a mysterious link between the partition function of certain dimer models on 2-dimensional tori and the L -function of their spectral curves. It also relates the partition function in certain families of dimer models to Eisenstein series.

Introduction

The *logarithmic Mahler measure* $\mathfrak{m}(F)$ and the *Mahler measure* $\mathbf{M}(F)$ of a Laurent polynomial $F(x, y)$ with complex coefficients are:

$$\mathfrak{m}(F) := \frac{1}{(2\pi i)^2} \oint \oint_{|x|=|y|=1} \log |F(x, y)| \frac{dx}{x} \frac{dy}{y}, \quad \mathbf{M}(F) := \exp(\mathfrak{m}(F)) . \quad (1)$$

Boyd [2] gives a survey of many Laurent polynomials for which $\mathfrak{m}(F)$ equals (numerically to many decimal places) a ‘simple’ non-zero rational number times the derivative at 0 of the L -function of the projective plane curve Z_F defined by the vanishing of F :

$$\mathfrak{m}(F) \cdot \mathbb{Q}^* = L'(Z_F, 0) \cdot \mathbb{Q}^* . \quad (2)$$

Tables 2, 5, 1 in [2] give the following families of cubic polynomials $\tilde{F}(X, Y, Z)$ for which (2) was checked numerically with $F(x, y) = (xy)^{-1} \tilde{F}(x, y, 1)$ and with integer values of the parameter s ,

$$X^2Y + XY^2 + X^2Z + XZ^2 + Y^2Z + YZ^2 - sXYZ \quad (3)$$

$$X^2Y + Y^2Z + Z^2X - sXYZ \quad (4)$$

$$X^2Y + XY^2 + XZ^2 + YZ^2 - sXYZ \quad (5)$$

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Deninger [3] and Rodriguez Villegas [8] showed that the experimentally observed relations (2) agree with predictions from the Bloch-Beilinson conjectures. Rodriguez Villegas [8] gave a proof of (2) for some values of s in Examples (3), (4), (5).

On the other hand, Kenyon, Okounkov and Sheffield gave a formula for the *partition function per fundamental domain* of a dimer model which is exactly the same as Formula (1) for the Mahler measure of the characteristic polynomial of that dimer model (see [5] Theorem 3.5). In this paper we show examples of dimer models with characteristic polynomials (3), (4), (5).

This suggests that there may be some mysterious link between the partition function of a dimer model and the L-function of its spectral curve.

Our dimer models come in families with the parameter s explicitly related to the weights in the dimer model. The results for the Mahler measures of the polynomials (3), (4), (5) presented in [9] now imply that, for $|s|$ sufficiently large, the partition functions in these families of dimer models are $Q_6^{-1}, Q_3^{-\frac{1}{3}}, Q_4^{-\frac{1}{2}}$ respectively, with

$$\begin{aligned} Q_6 &= \mathbf{q} \prod_{n \geq 1} (1 - \mathbf{q}^n)^{(-1)^{n-1} n \chi_{-3}(n)}, & Q_3 &= \mathbf{q} \prod_{n \geq 1} (1 - \mathbf{q}^n)^{9n \chi_{-3}(n)}, \\ Q_4 &= \mathbf{q} \prod_{n \geq 1} (1 - \mathbf{q}^n)^{4n \chi_{-4}(n)}, \end{aligned} \tag{6}$$

where $\chi_{-3}(n) = 0, 1, -1$ if $n \equiv 0, 1, 2 \pmod{3}$ and $\chi_{-4}(n) = 0, 1, 0, -1$ if $n \equiv 0, 1, 2, 3 \pmod{4}$ and where \mathbf{q} is explicitly related to s .

The Eisenstein series in the title are the logarithmic derivatives of the products in (6):

$$1 + \sum_{n \geq 1} \chi_{-3}(n) \frac{n^2 (-\mathbf{q})^n}{1 - \mathbf{q}^n}, \quad 1 - 9 \sum_{n \geq 1} \chi_{-3}(n) \frac{n^2 \mathbf{q}^n}{1 - \mathbf{q}^n}, \quad 1 - 4 \sum_{n \geq 1} \chi_{-4}(n) \frac{n^2 \mathbf{q}^n}{1 - \mathbf{q}^n}.$$

The products and Eisenstein series for Q_3 and Q_4 were also studied by Ramanujan (see [1]). We find the similarity with McMahon's function

$$M(\mathbf{q}) := \prod_{n \geq 1} (1 - \mathbf{q}^n)^{-n}$$

very intriguing, in particular because McMahon's function appears in the partition functions in [6, 7].

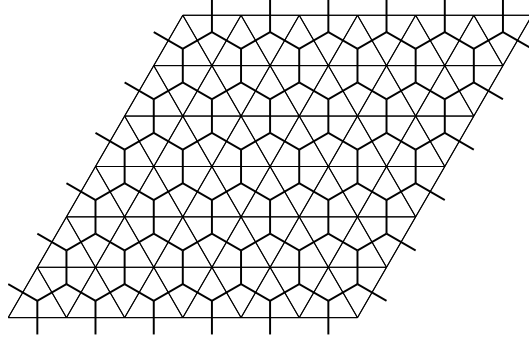


Figure 1:

1 Dimers

For general theory and background information on dimer models we refer to [4, 5, 6, 7]. We restrict ourselves to dimer models on the standard hexagonal graph Γ , which is the dual of the tessellation of the plane by regular triangles; see Figure 1

A *dimer configuration*, or perfect matching, on Γ is a collection \mathcal{M} of its edges such that each vertex of Γ is incident to exactly one edge from \mathcal{M} . Each edge of Γ intersects exactly one edge in the triangle tessellation and vice versa. Given a dimer configuration \mathcal{M} one can remove from the triangle tessellation all edges which intersect a Γ -edge belonging to \mathcal{M} . The two triangles adjacent to such an edge glue together to a rhombus and as a result one gets a rhombus tessellation of the plane (see Figure 5). Conversely every rhombus tessellation of the plane, in which each rhomb is the union of two adjacent triangles in the triangle tessellation, comes from a dimer configuration on Γ .

View the plane as the complex plane \mathbb{C} and let $\omega := e^{2\pi i/3}$. Then the vertices of the triangulation are precisely the elements of the lattice

$$\Lambda := \{a + b\omega \mid a, b \in \mathbb{Z}\}.$$

We first look at dimer configurations which are invariant under translations by vectors from the lattice 3Λ . One may also say that these are dimer configurations on the graph $\Gamma/3\Lambda$, embedded in the torus $\mathbb{C}/3\Lambda$. Figure 2 shows the fundamental domain and a labeling scheme for the vertices of the graph c.q. the triangles of the tessellation. It also shows the same adjacency

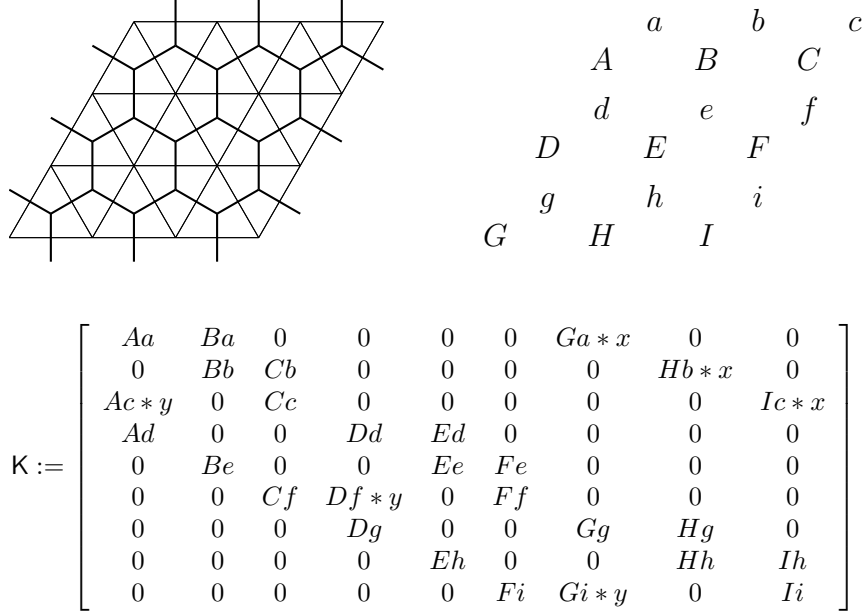


Figure 2: The matrix entries give an abstract, tautological weighting of the graph edges. The $*x$ (resp. $*y$) marks the graph edges for which the top-bottom (resp. left-right) sides of the fundamental parallelogram must be identified.

and labeling structure in the matrix K , which is the matrix of the *Kasteleijn operator* of the dimer model on $\Gamma/3\Lambda$. The determinant

$$P(x, y) := \det K \quad (7)$$

is called the *characteristic polynomial* and the zero locus of $P(x, y)$ in \mathbb{C}^{*2} is called the *spectral curve* of the dimer model (see [4] §2.1). A straightforward computation gives the characteristic polynomial shown in Figure 3.

Next we want to specify a 3Λ -invariant weight function W on the edges of Γ so that the characteristic polynomial of the dimer model with these weights is a constant times one of the polynomials (3), (4), (5).

So, the coefficients of certain monomials must be 0 and the coefficients of all other monomials different from XYZ must be equal and non-zero. Moreover, the ratio between the coefficient of XYZ and the other non-zero coefficient should be $-s$. Altogether these conditions constitute 9 equations on the weights. Which 9 equations exactly depends on the polynomial one wants to reconstruct.

$$\bar{P}(X, Y, Z) := Z^3 P(\frac{X}{Z}, \frac{Y}{Z}) =$$

[illegible]

Figure 3:

here $\begin{bmatrix} i & h & g & f & e & d & c & b & a \\ F & E & D & C & B & A & I & H & G \end{bmatrix}$ denotes the product $Fi * Eh * Dg * Cf * Be * Ad * Ic * Hb * Ga$ of the weights. It also shows the dimer configuration as a map from ∇ - to Δ -triangles.

The graph $\Gamma/3\Lambda$ has 27 edges and 18 vertices. If one multiplies the weights of the three edges incident to a vertex by the same non-zero number k , the whole characteristic polynomial gets multiplied by k . So there is a torus \mathbb{C}^{*18} acting on the solutions of the 9 equations, but the 1-dimensional subtorus given by multiplication factors k for the nine Δ -vertices and k^{-1} for the nine ∇ -vertices ($k \in \mathbb{C}^*$) acts trivially. So, for fixed s , the solution space of the 9 equations modulo the action of \mathbb{C}^{*18} has dimension at least $1 = 27 - 9 - 17$.

Without further delving into general solutions we now exhibit just three examples from a larger collection we found by trial and error. Correctness of these examples can easily be checked by direct inspection of the general characteristic polynomial $\tilde{P}(X, Y, Z)$ in Figure 3.

As a common structure in these examples there are only four edge weights $(0, 1, w, m)$ and s is a function of m . The parameter w confirms that for fixed s the solution space modulo \mathbb{C}^{*18} has dimension ≥ 1 . Edges with weight 0 can not occur in a dimer configuration. Thus if the dimer configuration is presented as a rhombus tessellation the 0-weight edges of Γ turn into edges of the tessellation which *must be present*; it is like boundary conditions. This is shown for Example 2 in Figure 4. Any rhombus tessellation which respects the boundary conditions is allowed. Figure 5 shows the 3Λ -periodic dimer configurations for Example 2.

Example 1 *Setting $W(Hg) = W(Dd) = W(Eh) = 0$, $W(Hb) = W(Ee) = W(Df) = m$, $W(Ii) = W(Ic) = W(Gi) = W(Aa) = W(Ac) = W(Ga) = 1$ and all remaining weights = w yields a characteristic polynomial of type (3):*

$$mw^6(X^2Y + XY^2 + X^2Z + XZ^2 + Y^2Z + YZ^2) - (4 + 3m + 3m^2 + m^3)w^6XYZ$$

$$\text{with } s = (4 + 3m + 3m^2 + m^3)m^{-1}.$$

Example 2 *Setting $W(Hg) = W(Dd) = W(Eh) = W(Be) = W(Cb) = W(Ff) = 0$, $W(Hb) = W(Ee) = W(Df) = m$, $W(Ii) = W(Ic) = W(Gi) = W(Aa) = W(Ac) = W(Ga) = 1$ and all remaining weights = w yields a characteristic polynomial of type (4):*

$$mw^6(X^2Z + XY^2 + YZ^2) - (2 + m^3)w^6XYZ$$

$$\text{with } s = (2 + m^3)m^{-1}.$$

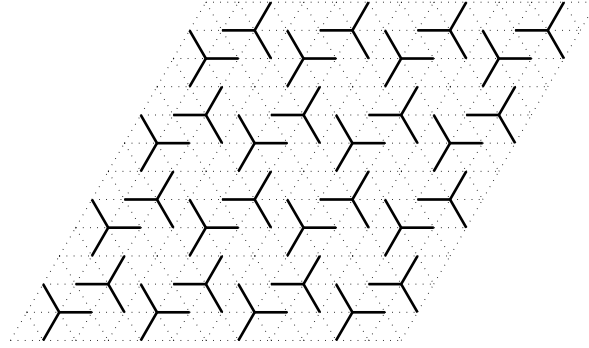


Figure 4:

Example 3 *Setting $W(Aa) = W(Bb) = W(Cf) = W(Gi) = W(Fe) = 0$, $W(Cb) = W(Ff) = W(Be) = 1$, $W(Cc) = W(Fi) = W(Ba) = m$ and all remaining weights $= w$ yields a characteristic polynomial of type (5):*

$$mw^6(X^2Y + XY^2 + XZ^2 + YZ^2) - (2 + m^2 + m^3)w^6XYZ$$

with $s = (2 + m^2 + m^3)m^{-1}$.

Remark: Among the examples in [5] there is a dimer model on the square-octagon graph with characteristic polynomial of type (5): $x + x^{-1} + y + y^{-1} - s$.

As in [5] §3.2, we now fix a 3Λ -invariant \mathbb{R} -valued weight function W on the edges of Γ and consider for a positive integer n the dimer configurations on (Γ, W) which are invariant under translations by vectors in the lattice $3n\Lambda$. So, the preceding discussion and examples concern the case $n = 1$. Theorem 3.3 of [5] expresses the characteristic polynomial $P_n(x, y)$ of the dimer model on $(\Gamma/3n\Lambda, W)$ in terms of the characteristic polynomial $P(x, y)$ of the dimer model on $(\Gamma/3\Lambda, W)$:

$$P_n(x, y) = \prod_{u^n=x} \prod_{v^n=y} P(u, v).$$

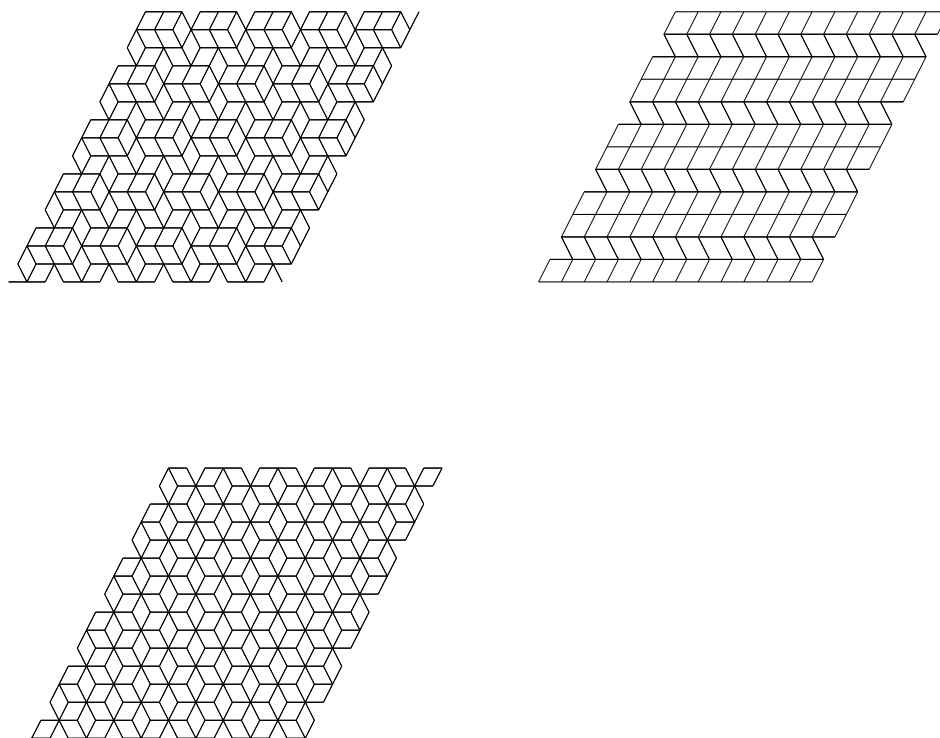


Figure 5: 180° degree rotation of the top left picture and $\pm 60^\circ$ degree rotations of the top right picture give also legitimate tessellations.

Corollary 3.4 of [5] uses this to compute the *partition function* $Z(\Gamma/3n\Lambda, W)$ for the dimer model on $(\Gamma/3n\Lambda, W)$:

$$Z(\Gamma/3n\Lambda, W) = \frac{1}{2}(-Z_n^{(0,0)} + Z_n^{(0,1)} + Z_n^{(1,0)} + Z_n^{(1,1)})$$

$$\text{where} \quad Z_n^{(a,b)} = \prod_{u^n=(-1)^a} \prod_{v^n=(-1)^b} P(u, v). \quad (8)$$

Finally, [5] Thm. 3.5 computes the *partition function per fundamental domain* Z for periodic dimer models on (Γ, W) with period lattice $3n\Lambda$, $n \in \mathbb{N}$:

$$\begin{aligned} \log Z &:= \lim_{n \rightarrow \infty} \frac{1}{n^2} \log Z(\Gamma/3n\Lambda, W) \\ &= \frac{1}{(2\pi i)^2} \oint \oint_{|x|=|y|=1} \log |P(x, y)| \frac{dx}{x} \frac{dy}{y} =: \mathbf{m}(P). \end{aligned} \quad (9)$$

Note: $|P(x, y)| = P(x, y)$ in (9), because the polynomial $P(x, y)$ has real coefficients and (8) contains with every factor also its complex conjugate.

Corollary 1 *The partition function per fundamental domain is equal to the Mahler measure of the characteristic polynomial:*

$$Z = \mathbf{M}(P).$$

■

We now combine the above examples with Examples # 6, # 3, # 4 of [9].

Example 1 (continued) *According to [9] Example # 6 the Mahler measure of the polynomial*

$$(X + Y)(Y + Z)(Z + X) - tXYZ$$

with $|t|$ sufficiently large, is \mathbf{Q}_6^{-1} where

$$\mathbf{Q}_6 = \mathbf{q} \prod_{n \geq 1} (1 - \mathbf{q}^n)^{(-1)^{n-1} n \chi_{-3}(n)} \quad \text{and} \quad t = \frac{\eta(\mathbf{q}^2)^3 \eta(\mathbf{q}^3)^9}{\eta(\mathbf{q})^3 \eta(\mathbf{q}^6)^9}$$

and $\chi_{-3}(n) = 0, 1, -1$ if $n \equiv 0, 1, 2 \pmod{3}$. Thus, if we rescale the dimer model in Example 1 by multiplying all edge-weights by $(mw^6)^{-\frac{1}{9}}$, the partition function per fundamental domain for the rescaled dimer model is \mathbf{Q}_6^{-1} with

$$t - 2 = s = (4 + 3m + 3m^2 + m^3)m^{-1}.$$

Example 2 (continued) According to [9] Example # 3 the Mahler measure of the polynomial

$$X^2Y + Y^2Z + Z^2X - tXYZ$$

with $|t|$ sufficiently large, is $\mathbf{Q}_3^{-\frac{1}{3}}$ where

$$\mathbf{Q}_3 = \mathbf{q} \prod_{n \geq 1} (1 - \mathbf{q}^n)^{9n\chi_{-3}(n)} \quad \text{and} \quad t^3 = 27 + \frac{\eta(\mathbf{q})^{12}}{\eta(\mathbf{q}^3)^{12}}.$$

Thus, if we rescale the dimer model in Example 2 by multiplying all edge-weights by $(mw^6)^{-\frac{1}{9}}$, the partition function per fundamental domain for the rescaled dimer model is $\mathbf{Q}_3^{-\frac{1}{3}}$ with

$$t = s = (2 + m^3)m^{-1}.$$

Example 3 (continued) According to [9] Example # 4 the Mahler measure of the polynomial

$$(X + Y)(XY + Z^2) - tXYZ$$

with $|t|$ sufficiently large, is $\mathbf{Q}_4^{-\frac{1}{2}}$ where

$$\mathbf{Q}_4 = \mathbf{q} \prod_{n \geq 1} (1 - \mathbf{q}^n)^{4n\chi_{-4}(n)} \quad \text{and} \quad t = \frac{\eta(\mathbf{q}^2)^{12}}{\eta(\mathbf{q})^4 \eta(\mathbf{q}^4)^8}$$

and $\chi_{-4}(n) = 0, 1, 0, -1$ if $n \equiv 0, 1, 2, 3 \pmod{4}$. Thus, if we rescale the dimer model in Example 3 by multiplying all edge-weights by $(mw^6)^{-\frac{1}{9}}$, the partition function per fundamental domain for the rescaled dimer model is $\mathbf{Q}_4^{-\frac{1}{2}}$ with

$$t = s = (2 + m^2 + m^3)m^{-1}.$$

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