A TIME-VARIANT NORM CONSTRAINED INTERPOLATION PROBLEM ARISING FROM RELAXED COMMUTANT LIFTING

A.E. FRAZHO, S. TER HORST, AND M.A. KAASHOEK

Abstract. A time-variant analogue of an interpolation problem equivalent to the relaxed commutant lifting problem is introduced and studied. In a somewhat less general form the problem already appears in the analysis of the set of all solutions to the three chain completion problem. The interpolants are upper triangular operator matrices of which the columns induce contractive operators. The set of all solutions of the problem is described explicitly. The results presented are time-variant analogues of the main theorems in [23].

0. Introduction

Time-variant versions of metric constrained interpolation problems and time-varying linear system theory have been intensively studied since the early 1990's; see the papers [1, 2, 7, 8, 13] and the books [14, 25, 20, 15] for a general overview and additional references. The connection with commutant lifting theory was made in [18], where a time-varying analogue of the commutant lifting theorem, known as the three chain completion theorem, was proved. An early version of a time-variant commutant lifting theorem appeared in Ball-Gohberg [6], which was later extended to the setting of nest algebras in [30] (see also [12]); the connection with the three chain theorem is explained in [5]. One of the recent developments in commutant lifting theory is the introduction of a relaxation of the commutant lifting setting in [21]. In the present paper we consider a time-variant norm constrained abstract interpolation problem, which in the time-invariant case is equivalent to the relaxed commutant lifting problem [23].

To state the interpolation problem considered in this paper we need some notation. Throughout $\mathcal{U}_k$ and $\mathcal{Y}_k$ are Hilbert spaces with k being an arbitrary integer, and the symbols $\mathbf{U}$ and $\mathbf{Y}$ stand for the Hilbert direct sums $\oplus_{k\in\mathbb{Z}}\mathcal{U}_k$ and $\oplus_{k\in\mathbb{Z}}\mathcal{Y}_k$, respectively. We shall consider operator matrices $H=[H_{j,\,k}]_{j,\,k\in\mathbb{Z}}$ of which the $(j,\,k)$-th entry $H_{j,\,k}$ is an operator from $\mathcal{U}_k$ into $\mathcal{Y}_j$. The set of all such operator matrices will be denoted by $\mathbb{M}(\mathbf{U},\mathbf{Y})$. By $\mathbb{UM}(\mathbf{U},\mathbf{Y})$ we denote the subset of $\mathbb{M}(\mathbf{U},\mathbf{Y})$ consisting of all $H=[H_{j,\,k}]_{j,\,k\in\mathbb{Z}}$ that are *upper triangular*, that is, $H_{j,\,k}=0$ for each $k<j$.

In the present paper we are particularly interested in those $H=[H_{j,\,k}]_{j,\,k\in\mathbb{Z}}$ in $\mathbb{UM}(\mathbf{U},\mathbf{Y})$ that have the additional property

$$\sum_{j=-\infty}^{k}\|H_{j,\,k}u_k\|^2\leq c_H\|u_k\|^2,\quad u_k\in\mathcal{U}_k\quad(k\in\mathbb{Z}),\tag{0.1}$$

where c_H is some constant depending on H only. The set of all such operator matrices is denoted by $\mathbb{UM}^2(\mathbf{U},\mathbf{Y})$. We say that H belongs to $\mathbb{UM}^2_{\text{ball}}(\mathbf{U},\mathbf{Y})$ whenever

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the constant c_H can be taken equal to one. Thus an upper triangular operator matrix H belongs to $\mathbb{UM}^2_{\text{ball}}(\mathbf{U},\mathbf{Y})$ if and only if for each $k\in\mathbb{Z}$ the k-th column of H induces a contractive operator from $\mathcal{U}_k$ into $\mathbf{Y}=\oplus_{k\in\mathbb{Z}}\mathcal{Y}_k$. The following is the main problem treated in this paper.

Problem 0.1. *Assume that for each $k\in\mathbb{Z}$ we have given a subspace $\mathcal{F}_k$ of $\mathcal{U}_k$ and a contraction*

$$\varnothing_k=\begin{bmatrix}\varnothing_{k,\,1}\\ \varnothing_{k,\,2}\end{bmatrix}:\mathcal{F}_k\to\begin{bmatrix}\mathcal{Y}_k\\ \mathcal{U}_{k-1}\end{bmatrix}. \tag{0.2}$$

Given this data, find all $H=\left[H_{j,\,k}\right]_{j,\,k\in\mathbb{Z}}$ in $\mathbb{UM}^2_{\text{ball}}(\mathbf{U},\mathbf{Y})$ such that for each $k\in\mathbb{Z}$ the following interpolation conditions hold:

$$H_{k,\,k}|_{\mathcal{F}_k}=\varnothing_{k,\,1},\quad H_{j,\,k}|_{\mathcal{F}_k}=H_{j,\,k-1}\varnothing_{k,\,2}\quad (j,k\in\mathbb{Z}, j<k). \tag{0.3}$$

In the time-invariant case, the spaces $\mathcal{F}_k=\mathcal{F}$, $\mathcal{U}_k=\mathcal{U}$, and $\mathcal{Y}_k=\mathcal{Y}$ and the contraction $\varnothing_k=\varnothing$ do not depend on k, and the operators $H_{j,k}$ depend only on the difference $j-k$. In this setting, the above problem reduces to the function theory problem considered in the first paragraph of [23]. To see this, note that in this time-invariant setting the operator matrix H can be identified with the $\mathcal{L}(\mathcal{U},\mathcal{Y})$-valued function F_H, analytic on the open unit disc $\mathbb{D}$, given by

$$F_H(l)=\sum_{\nu=0}^{\infty}l^{\nu}H_{-\nu}.$$

Moreover in this case the interpolation condition and the norm constraint in Problem 0.1 can be restated as

$$\varnothing_1+lF_H(l)\varnothing_2=F_H(l)|_{\mathcal{F}}\quad(l\in\mathbb{D})\quad\text{and}\quad\sum_{\nu=0}^{\infty}\|H_{-\nu}u\|^2\leq\|u\|^2\quad(u\in\mathcal{U}).$$

For a particular choice of the contractions $\varnothing_k$, Problem 0.1 appears in a natural way in the analysis of the set of all solutions to the three chain completion problem [18], [19]. Indeed, see Section 4 in [19] or Section XIV.3 in [20], where one can find Problem 0.1 with $\varnothing_k$ being an isometry for each $k\in\mathbb{Z}$.

To state our first main result some additional notation is needed. We use the symbol $\mathbb{UM}^{\infty}(\mathbf{U},\mathbf{Y})$ to denote the set of all double infinite upper triangular operator matrices H that induce bounded linear operators from the Hilbert space $\mathbf{U}=\oplus_{k\in\mathbb{Z}}\mathcal{U}_k$ into the Hilbert space $\mathbf{Y}=\oplus_{k\in\mathbb{Z}}\mathcal{Y}_k$. If this induced operator is a contraction, then we say that H belongs to $\mathbb{UM}^{\infty}_{\text{ball}}(\mathbf{U},\mathbf{Y})$. In particular, $\mathbb{UM}^{\infty}_{\text{ball}}(\mathbf{U},\mathbf{Y})\subset\mathbb{UM}^{2}_{\text{ball}}(\mathbf{U},\mathbf{Y})$. We write $\mathbb{UM}^{\infty}_{0}(\mathbf{U},\mathbf{Y})$ and $\mathbb{UM}^{\infty}_{\text{ball},0}(\mathbf{U},\mathbf{Y})$ for the sets of all strictly upper triangular operator matrices in $\mathbb{UM}^{\infty}(\mathbf{U},\mathbf{Y})$ and $\mathbb{UM}^{\infty}_{\text{ball}}(\mathbf{U},\mathbf{Y})$, respectively. Finally, when $\mathcal{U}_k=\mathcal{Y}_k$ for each $k\in\mathbb{Z}$, and hence $\mathbf{U}=\mathbf{Y}$, we shall always replace the argument $(\mathbf{U},\mathbf{Y})$ by $(\mathbf{U})$. Thus $\mathbb{M}(\mathbf{U})$ stands for $\mathbb{M}(\mathbf{U},\mathbf{U})$, and $\mathbb{UM}(\mathbf{U})$ stands for $\mathbb{M}(\mathbf{U},\mathbf{U})$, etc. We are now ready to state the first main result.

Theorem 0.1. *For each $k\in\mathbb{Z}$ let $\varnothing_k$ be the contraction given by* (0.2). *Choose*

$$Z^{(1)}=\left[(Z^{(1)}_{j,\,k})\right]_{j,\,k\in\mathbb{Z}}\in\mathbb{UM}^{\infty}(\mathbf{U},\mathbf{Y}),\ Z^{(2)}=\left[Z^{(2)}_{j,\,k}\right]_{j,\,k\in\mathbb{Z}}\in\mathbb{UM}^{\infty}_0(\mathbf{U}), \tag{0.4}$$

such that

$$\|Z^{(1)}u\|^2+\|Z^{(2)}u\|^2\leq\|u\|^2\quad(u\in\mathbf{U}) \tag{0.5}$$

and

$$Z^{(1)}_{j,\,k}|_{\mathcal{F}_k} = \begin{cases} \o_{k,\,1}, & \textit{if } j=k, \\ 0, & \textit{if } j\neq k, \end{cases} \qquad Z^{(2)}_{j,\,k}|_{\mathcal{F}_k} = \begin{cases} \o_{k,\,2}, & \textit{if } j=k-1, \\ 0, & \textit{if } j\neq k-1. \end{cases} \tag{0.6}$$

Then

$$H = Z^{(1)}(I_{\mathbf{U}} - Z^{(2)})^{-1} \tag{0.7}$$

is a well-defined upper triangular operator matrix, H belongs to $\mathbb{UM}^2_{\rm ball}(\mathbf{U},\mathbf{Y})$, and H is a solution of Problem *0.1.*

Let us explain why formula (0.7) makes sense. In general, for arbitrary infinite operator matrices the usual matrix product is not defined. The situation is different for upper triangular matrices. For instance, for $A \in \mathbb{UM}(\mathbf{U},\mathbf{Y})$ and $B \in \mathbb{UM}(\mathbf{U})$ the matrix product AB is well-defined and AB belongs to $\mathbb{UM}(\mathbf{U},\mathbf{Y})$. Furthermore, with the usual matrix product $\mathbb{UM}(\mathbf{U})$ is an algebra with the identity matrix $I_{\mathbf{U}}$ as a unit and an operator matrix $M = [M_{j,\,k}]_{j,\,k\in\mathbb{Z}} \in \mathbb{UM}(\mathbf{U})$ is invertible in $\mathbb{UM}(\mathbf{U})$ if and only if for each $j\in\mathbb{Z}$ the j-th diagonal entry $M_{j,\,j}$ is invertible as an operator on $\mathcal{U}_j$ (see Subsection 1.2 below for further details). From these remarks is clear that the operator $I_{\mathbf{U}} - Z^{(2)}$ in (0.7) is invertible in $\mathbb{UM}(\mathbf{U})$ and that the product in (0.7) is well-defined.

One can always find $Z^{(1)}$ and $Z^{(2)}$ satisfying the conditions (0.4), (0.5) and (0.6) in Theorem 0.1. For instance one can take

$$Z^{(1)}_{j,\,k} = \delta_{j,\,k}\o_{k,\,1}\Pi_{\mathcal{F}_k} \quad \text{and} \quad Z^{(2)}_{j,\,k} = \delta_{j,\,k-1}\o_{k,\,2}\Pi_{\mathcal{F}_k}. \tag{0.8}$$

Here $\delta_{j,\,k}$ is the Kronecker delta and the map $\Pi_{\mathcal{F}_j}$ is the orthogonal projection of $\mathcal{U}_j$ onto $\mathcal{F}_j$. The solution of Problem 0.1 corresponding to this choice of $Z^{(1)}$ and $Z^{(2)}$ is given by $\tilde{H} = [\tilde{H}_{j,k}]_{j,\,k\in\mathbb{Z}}$ with

$$\tilde{H}_{j,k} = \begin{cases} 0, & j>k, \\ \o_{k,\,1}\Pi_{\mathcal{F}_k}, & j=k, \\ (\o_{j,\,1}\Pi_{\mathcal{F}_j})(\o_{j+1,\,2}\Pi_{\mathcal{F}_{j+1}})\cdots(\o_{k-1,\,2}\Pi_{\mathcal{F}_{k-1}})(\o_{k,\,2}\Pi_{\mathcal{F}_k}), & j<k. \end{cases}$$

Thus Problem 0.1 is always solvable.

Our second main result shows, in particular, that the method of Theorem 0.1 gives all solutions to Problem 0.1, that is, given a solution H to Problem 0.1, there exists a pair of operator matrices $(Z^{(1)}, Z^{(2)})$ satisfying (0.4), (0.5) and (0.6) such that H is given by the formula (0.7). In general, such a pair $(Z^{(1)}, Z^{(2)})$ is not uniquely determined by H. This phenomenon already appears in the time-invariant case and can be illustrated by simple examples. For instance, assume all spaces $\mathcal{F}_k$ consist of the zero element only. In that case $H=0$ is in $\mathbb{UM}^2_{\rm ball}(\mathbf{U},\mathbf{Y})$ and is a solution, while (0.7) holds with $Z^{(1)}=0$ and with any $Z^{(2)}$ from $\mathbb{UM}^\infty_0(\mathbf{U})$.

Given a solution H to Problem 0.1, we shall describe the set of all pairs $(Z^{(1)}, Z^{(2)})$ satisfying (0.4), (0.5) and (0.6) stated in the above theorem such that H is given by (0.7). The precise result is given by Theorem 4.1 in Section 4. Here we only describe some of the main ingredients entering into the proof and present an abbreviated version of this theorem.

Let H be a solution to Problem 0.1, and let H_k be the k-th column of H. Recall that H_k defines a contraction from $\mathcal{U}_k$ into $\mathbf{Y}$. Let $D_{H_k} = (I_{\mathcal{U}_k} - (H_k)^*H_k)^{1/2}$ denote the corresponding defect operator, and let $\mathcal{D}_{H_k}$ be the corresponding defect space, i.e., $\mathcal{D}_{H_k}$ is the closure of the range of D_{H_k} in $\mathcal{U}_k$. Since (0.3) is satisfied, for each $f \in \mathcal{F}_k$ we have

$$\begin{aligned} \|D_{H_k} f\|^2 &= \|f\|^2 - \|H_k f\|^2 = \|f\|^2 - \|\o_{k,1} f\|^2 - \|H_{k-1}\o_{k,2} f\|^2 \\ &= \|f\|^2 - \|\o_{k,1} f\|^2 - \|\o_{k,2} f\|^2 + \|\o_{k,2} f\|^2 - \|H_{k-1}\o_{k,2} f\|^2 \\ &= \|D_{\o_k} f\|^2 + \|D_{H_{k-1}}\o_{k,2} f\|^2. \end{aligned}$$

Hence we can define a contraction $\o_{H_k}$ by

$$\o_{H_k} : \mathcal{F}_{H_k} := \overline{D_{H_k}\mathcal{F}_k} \to \mathcal{D}_{H_{k-1}}, \quad \o_{H_k} D_{H_k}|_{\mathcal{F}_k} = D_{H_{k-1}}\o_{k,2}. \tag{0.9}$$

Now put $\mathbf{D}_H = \oplus_{k\in\mathbb{Z}}\mathcal{D}_{H_k}$, and let $\mathbf{C}_{H,\o}$ be the set of all operator matrices $C = [C_{j,k}]_{j,k\in\mathbb{Z}}$ in $\mathbb{UM}^\infty_{\mathrm{ball},0}(\mathbf{D}_H)$ such that

$$C_{j,k}|_{\mathcal{F}_{H_k}} = \begin{cases} \o_{H_k}, & \text{if } j = k-1, \\ 0, & \text{if } j \neq k-1. \end{cases} \tag{0.10}$$

Thus

$$\begin{aligned} \mathbf{C}_{H,\o} = \{C \in \mathbb{UM}^\infty_{\mathrm{ball},0}(\mathbf{D}_H) \mid &\text{ the } (j,k)\text{-th entry } C_{j,k} \\ &\text{of } C \text{ satisfies (0.10) for each } j,k \in \mathbb{Z}\}. \end{aligned} \tag{0.11}$$

We can now state the abbreviated version of our second main result.

Theorem 0.2. *Let H be a solution to* Problem *0.1. Then there exists a pair of operator matrices $(Z^{(1)}, Z^{(2)})$ satisfying* (0.4), (0.5) *and* (0.6) *such that H is given by* (0.7). *Furthermore, the set of all such pairs $(Z^{(1)}, Z^{(2)})$ is in one-to-one correspondence with the set $\mathbf{C}_{H,\o}$.*

The full version of the above theorem (see Theorem 4.1 below) will also present necessary and sufficient conditions guaranteeing that the set $\mathbf{C}_{H,\o}$ consists of a single element only.

Let H be a solution to Problem 0.1. In the analysis of the set of all pairs $(Z^{(1)}, Z^{(2)})$ satisfying (0.4)–(0.7) the following problem enters in a natural way.

Problem 0.2. *Given $H \in \mathbb{UM}^2_{\mathrm{ball}}(\mathbf{U},\mathbf{Y})$, describe the set of operator matrices F in $\mathbb{UM}(\mathbf{U})$ satisfying*

$$F + F^* \geq H^*H + I_{\mathbf{U}} \quad \textit{and} \quad F_{j,j} = I_{\mathcal{U}_j} \quad (j \in \mathbb{Z}). \tag{0.12}$$

Note that the matrix product H^*H is well-defined because each column of H induces a contractive operator (see Subsection 1.4 for further details). The inequality sign in (0.12) means that the operator matrix $\frac{1}{2}(F+F^*) - H^*H - I_{\mathbf{U}}$ is non-negative (see Subsection 1.3 for the definition of this notion and further details). The fact that (0.12) appears in the analysis, follows from the observation that $F = (I_{\mathbf{U}} - Z^{(2)})^{-1}$ satisfies (0.12) whenever the pair $(Z^{(1)}, Z^{(2)})$ satisfies the conditions (0.4)–(0.7). This connection will be made more precise in Theorem 2.1. The solution to Problem 0.2 will be obtained as a corollary to Theorem 3.2.

For the time-invariant case Theorems 0.1 and Theorem 0.2 can be found in [23]. By using the reduction techniques developed in Chapter X of [20] (also [17]) Problem 0.1 can be transformed into a problem of the type considered in [23]. This transformation together with techniques from [20] can be used to present an alternative way to prove our main results. We shall not develop this approach in the present paper. Theorems 0.1 and Theorem 0.2 can be used to solve a time-variant analogue of the relaxed commutant lifting problem. We will describe this connection in the final section of the paper.

This paper consists of five sections not counting this introduction. The first section has a preliminary character. We introduce some additional notation and recall a number of elementary facts about operator matrices that will be used in the proofs. In Section 2 we outline a general approach to deal with Problem 0.1 and prove Theorem 0.1. Section 3 is divided into three subsections. In this section a time-variant analogue of the Cayley transform is used to relate operator matrices from $\mathbb{UM}^{\infty}_{\mathrm{ball},0}$ to positive real operator matrices from $\mathbb{UM}(\mathbf{U})$. We apply this result to solve Problem 0.2 and to parameterize the set of all its solutions. Theorem 0.2 is proved in Section 4; this section also presents the full version of Theorem 0.2 and its proof. Here we also discuss the problem of finding necessary and sufficient conditions for the existence of a unique solution to Problem 0.1. In the final section an example involving finite operator matrices will be presented and we discuss the connection with a time-variant analogue of the relaxed commutant lifting problem.

1. Preliminaries

In this section we bring together a number of elementary facts about operator matrices that will be used in the sequel. In what follows we assume the reader to be familiar with the notations introduced in the previous section.

1.1. **The set $\mathbb{M}(\mathbf{U},\mathbf{Y})$ and bounded operators.** The set $\mathbb{M}(\mathbf{U},\mathbf{Y})$ is a linear space with respect to the usual operation of matrix addition. Given $M = [M_{j,\,k}]_{j,\,k\in\mathbb{Z}}$ in $\mathbb{M}(\mathbf{U},\mathbf{Y})$ and $j \le k$ we write $\Delta_{j,\,k}(M)$ for the $\{j,k\}$-*finite section* of M, that is,

$$\Delta_{j,\,k}(M) = \begin{bmatrix} M_{j,\,j} & \cdots & M_{j,\,k} \\ \vdots & & \vdots \\ M_{k,\,j} & \cdots & M_{k,\,k} \end{bmatrix}. \tag{1.1}$$

Note that $\Delta_{j,\,k}(M)$ defines a bounded linear operator from $\mathcal{U}_j \oplus \cdots \oplus \mathcal{U}_k$ into $\mathcal{Y}_j \oplus \cdots \oplus \mathcal{Y}_k$.

In general, an operator matrix $M \in \mathbb{M}(\mathbf{U},\mathbf{Y})$ does not induce in a canonical way a bounded operator from $\mathbf{U} = \oplus_{k\in\mathbb{Z}}\,\mathcal{U}_k$ into the space $\mathbf{Y} = \oplus_{k\in\mathbb{Z}}\,\mathcal{Y}_k$. In order for this to happen it is necessary and sufficient that

$$\sup_{j\le k} \|\Delta_{j,\,k}(M)\| < \infty. \tag{1.2}$$

Furthermore, if (1.2) is satisfied, then the quantity in the left hand side of (1.2) is equal to the norm of $M = [M_{j,\,k}]_{j,\,k\in\mathbb{Z}}$ as an operator from $\mathbf{U}$ into $\mathbf{Y}$.

1.2. **Invertibility in the algebra $\mathbb{UM}(\mathbf{U})$.** Let $\mathbf{X} = \oplus_{k\in\mathbb{Z}}\,\mathcal{X}_k$, $\mathbf{U} = \oplus_{k\in\mathbb{Z}}\,\mathcal{U}_k$, and $\mathbf{Y} = \oplus_{k\in\mathbb{Z}}\,\mathcal{Y}_k$ be Hilbert space direct sums. If $B \in \mathbb{UM}(\mathbf{X},\mathbf{U})$ and $A \in \mathbb{UM}(\mathbf{U},\mathbf{Y})$, then the (block) matrix product AB is well-defined and $AB \in \mathbb{UM}(\mathbf{X},\mathbf{Y})$. Moreover, for $C \in \mathbb{UM}(\mathbf{X},\mathbf{Y})$ we have

$$AB = C \quad \Longleftrightarrow \quad \Delta_{j,\,k}(A)\Delta_{j,\,k}(B) = \Delta_{j,\,k}(C) \ (j \le k). \tag{1.3}$$

In particular, the set $\mathbb{UM}(\mathbf{U})$ is closed under the usual multiplication of matrices. In fact, from (1.3) we see that $\mathbb{UM}(\mathbf{U})$ is an algebra with the identity matrix $I_{\mathbf{U}}$ as a unit. From (1.3) it also follows that the operator matrix $M = [M_{i,\,j}]_{i,\,j\in\mathbb{Z}} \in \mathbb{UM}(\mathbf{U})$

is invertible in $\mathbb{UM}(\mathbf{U})$ if and only if for each $j \in \mathbb{Z}$ the j-th diagonal entry $M_{j,\,j}$ is an invertible operator on $\mathcal{U}_j$. In that case, we have

$$\Delta_{j,\,k}(M^{-1}) = \Delta_{j,\,k}(M)^{-1} \quad (j, k \in \mathbb{Z}, j \leq k). \tag{1.4}$$

In particular, the (j,j)-th entry of M^{-1} is equal to $M_{k,\,k}^{-1}$.

The above mentioned properties of $\mathbb{UM}(\mathbf{U})$ also follow from the fact that an operator matrix from $\mathbb{UM}(\mathbf{U})$ can be identified in the usual way with a linear transformation on the linear space $\mathbf{U}^+$. By definition, the space $\mathbf{U}^+$ consists of all double infinite one column matrices $\mathbf{u} = [u_j\,]_{j\in\mathbb{Z}}$, with $u_j \in \mathcal{U}_j$ for each $j \in \mathbb{Z}$, such that $u_\nu = 0$ for $\nu > \ell$, for some ℓ depending on $\mathbf{u}$.

1.3. **Hermitian and non-negative operator matrices.** An operator matrix $M \in \mathbb{M}(\mathbf{U})$, $M = [M_{j,\,k}]_{j,\,k\in\mathbb{Z}}$, is said to be *hermitian* if $M^* = M$, where M^* is the operator matrix $M^* = [(M_{k,\,j})^*]_{j,\,k\in\mathbb{Z}}$. The *real part* of $M \in \mathbb{M}(\mathbf{U})$ is the operator matrix $\operatorname{Re} M$ given by

$$\operatorname{Re} M = \frac{1}{2}(M + M^*). \tag{1.5}$$

Obviously, $\operatorname{Re} M$ is hermitian. We call $M \in \mathbb{M}(\mathbf{U})$ *non-negative* if for each $j,\, k \in \mathbb{Z}$, $j \leq k$, the finite section $\Delta_{j,\,k}(M)$ induces a non-negative operator on the Hilbert space direct sum $\mathcal{U}_j \oplus \cdots \oplus \mathcal{U}_k$. In that case M is hermitian. For operator matrices M and N in $\mathbb{M}(\mathbf{U})$ we say that M is *greater than or equal to* N, and write $M \geq N$, if the operator matrix $M - N$ is non-negative. Hence $M \geq 0$ means that M is non-negative. Finally, an operator matrix $M \in \mathbb{UM}(\mathbf{U})$ is said to be *positive real* whenever $\operatorname{Re} M$ is non-negative. Positive real operator matrices $M \in \mathbb{UM}(\mathbf{U})$ that induce bounded operators on $\mathbf{U} = \oplus_{k\in\mathbb{Z}}\,\mathcal{U}_k$ (i.e., $M \in \mathbb{UM}^\infty(\mathbf{U})$) have been extensively studied in [3, 4].

1.4. **The operator matrix H^*H.** Let $H \in \mathbb{UM}^2_{\rm ball}(\mathbf{U}, \mathbf{Y})$. Recall that for each $k \in \mathbb{Z}$ the k-th column H_k of H induces a contractive operator, also denoted by H_k, from $\mathcal{U}_k$ into $\mathbf{Y} = \oplus_{k\in\mathbb{Z}}\,\mathcal{Y}_k$. It follows that for each j and k in $\mathbb{Z}$ the product $(H_j)^*H_k$ is a well-defined contraction from $\mathcal{U}_k$ into $\mathcal{Y}_j$. We define H^*H to be the operator matrix in $\mathbb{M}(\mathbf{U})$ given by

$$H^*H = \left[\ (H_j)^*H_k\ \right]_{j,\,k\in\mathbb{Z}}. \tag{1.6}$$

Clearly, $H^*H \in \mathbb{M}(\mathbf{U})$ is hermitian. In fact, since for each $j,\, k \in \mathbb{Z}$, $j \leq k$,

$$\Delta_{j,\,k}(H^*H) = \begin{bmatrix} (H_j)^* \\ \vdots \\ (H_k)^* \end{bmatrix} \begin{bmatrix} H_j & \cdots & H_k \end{bmatrix},$$

the operator matrix H^*H is non-negative. Note that H^*H is the real part of the operator matrix $V \in \mathbb{UM}(\mathbf{U})$ be given by

$$V = \left[\ V_{j,k}\ \right]_{j,k\in\mathbb{Z}}, \quad V_{j,\,k} = \begin{cases} 2(H_j)^*H_k, & \text{for } j < k, \\ (H_j)^*H_j, & \text{for } j = k, \\ 0, & \text{for } j > k. \end{cases} \tag{1.7}$$

2. The set $\mathbb{UM}^2_{\text{ball}}(\mathbf{U},\mathbf{Y})$ and the proof of Theorem 0.1

The main result of this section (Theorem 2.1 below) shows how elements in $\mathbb{UM}^2_{\text{ball}}(\mathbf{U},\mathbf{Y})$ can be constructed from a pair of operators $Z^{(1)} \in \mathbb{UM}^\infty(\mathbf{U},\mathbf{Y})$ and $Z^{(2)} \in \mathbb{UM}^\infty_0(\mathbf{U})$ satisfying an additional norm constraint. This result together with Proposition 2.2 allows us to prove Theorem 0.1. Theorem 2.1 also shows the relevance of Problem 0.2 in the analysis of Problem 0.1.

Theorem 2.1. *Assume we have given operator matrices*

$$Z^{(1)} \in \mathbb{UM}^\infty(\mathbf{U},\mathbf{Y}), \quad Z^{(2)} \in \mathbb{UM}^\infty_0(\mathbf{U}) \tag{2.1}$$

satisfying the norm constraint

$$\|Z^{(1)}u\|^2 + \|Z^{(2)}u\|^2 \le \|u\|^2 \quad (u \in \mathbf{U}). \tag{2.2}$$

Then

$$H = Z^{(1)}(I_\mathbf{U} - Z^{(2)})^{-1} \quad \text{and} \quad F = (I_\mathbf{U} - Z^{(2)})^{-1} \tag{2.3}$$

are well-defined operator matrices,

$$H \in \mathbb{UM}^2_{\text{ball}}(\mathbf{U},\mathbf{Y}), \quad F \in \mathbb{UM}(\mathbf{U}) \tag{2.4}$$

and

$$F + F^* \ge H^*H + I_\mathbf{U}, \quad F_{j,\,j} = I_{\mathcal{U}_j} \quad (j \in \mathbb{Z}). \tag{2.5}$$

Conversely, if we have given H and F as in (2.4) *such that* (2.5) *holds, then F is an invertible element in $\mathbb{UM}(\mathbf{U})$, the operator matrices*

$$Z^{(1)} = HF^{-1} \quad \text{and} \quad Z^{(2)} = I_\mathbf{U} - F^{-1} \tag{2.6}$$

are well-defined and these operator matrices satisfy (2.1) *and* (2.2). *Moreover, the map $(Z^{(1)}, Z^{(2)}) \mapsto (H, F)$ defined by* (2.3) *is a one-to-one map from the set of all pairs $(Z^{(1)}, Z^{(2)})$ satisfying* (2.1) *and* (2.2) *onto the set of all pairs (H, F) satisfying* (2.4) *and* (2.5). *The inverse of this map is given by the map $(H, F) \mapsto (Z^{(1)}, Z^{(2)})$ defined by* (2.6).

Proof. We split the proof into four parts. In the first two parts $Z^{(1)}$ and $Z^{(2)}$ are given and satisfy (2.1) and (2.2), and we prove that H and F in (2.3) are well-defined and satisfy (2.4) and (2.5). In the third part we prove the reverse statement. In the final part we show that the maps $(Z^{(1)}, Z^{(2)}) \mapsto (H, F)$ and $(H, F) \mapsto (Z^{(1)}, Z^{(2)})$ in Theorem 2.1 are each others inverses.

Part 1. Assume that $Z^{(1)} \in \mathbb{UM}^\infty(\mathbf{U},\mathbf{Y})$ and $Z^{(2)} \in \mathbb{UM}^\infty_0(\mathbf{U})$ satisfy (2.2), and let H be given by the first part of (2.3). Our aim is to prove that $H \in \mathbb{UM}^2_{\text{ball}}(\mathbf{U},\mathbf{Y})$.

Fix $k, j \in \mathbb{Z}, j < k$, and $u_k \in \mathcal{U}_k$. Define

$$v = \begin{bmatrix} v_j \\ \vdots \\ v_k \end{bmatrix} = \Delta_{j,\,k}\left((I_\mathbf{U} - Z^{(2)})^{-1}\right)u, \quad \text{where } u = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ u_k \end{bmatrix} \in \oplus_{n=j}^k \mathcal{U}_n.$$

Note that $v \in \oplus_{n=j}^{k}\mathcal{U}_n$. Since $Z^{(2)}$ is strictly upper triangular, we have $v_k = u_k$. Observe that

$$\begin{aligned}\Delta_{j,\,k}(Z^{(2)})\Delta_{j,\,k}\left((I_{\mathbf{U}} - Z^{(2)})^{-1}\right) &= \Delta_{j,\,k}\left(Z^{(2)}(I_{\mathbf{U}} - Z^{(2)})^{-1}\right)\\ &= \Delta_{j,\,k}\left((I_{\mathbf{U}} - Z^{(2)})^{-1} - I_{\mathbf{U}}\right)\\ &= \Delta_{j,\,k}\left((I_{\mathbf{U}} - Z^{(2)})^{-1}\right) - I_{\oplus_{n=j}^{k}\mathcal{U}_n}\end{aligned}$$

and

$$\Delta_{j,\,k}(Z^{(1)})\Delta_{j,\,k}\left((I_{\mathbf{U}} - Z^{(2)})^{-1}\right) = \Delta_{j,\,k}\left(Z^{(1)}(I_{\mathbf{U}} - Z^{(2)})^{-1}\right) = \Delta_{j,\,k}(H).$$

Hence

$$\Delta_{j,\,k}(Z^{(2)})v = v - u =: \alpha \quad \text{and} \quad \Delta_{j,\,k}(Z^{(1)})v = \Delta_{j,\,k}(H)u =: \beta,$$

where, using $v_k = u_k$,

$$\alpha = \begin{bmatrix} v_j \\ \vdots \\ v_{k-1} \\ 0 \end{bmatrix} \quad \text{and} \quad \beta = \begin{bmatrix} H_{j,\,k}u_k \\ \vdots \\ H_{k-1,\,k}u_k \\ H_{k,\,k}u_k \end{bmatrix}.$$

The assumption (2.2) implies that

$$\begin{bmatrix} \Delta_{j,\,k}(Z^{(1)}) \\ \Delta_{j,\,k}(Z^{(2)}) \end{bmatrix}$$

is a contraction. Thus

$$\sum_{\nu=j}^{k-1}\|v_\nu\|^2 + \sum_{\nu=j}^{k}\|H_{\nu,\,k}u_k\|^2 = \|\alpha\|^2 + \|\beta\|^2 \leq \|v\|^2 = \sum_{\nu=j}^{k}\|v_\nu\|^2.$$

By comparing the first term in the left hand side of the above inequality with the term in the right side and using $v_k = u_k$, we see that $\sum_{\nu=j}^{k}\|H_{\nu,\,k}u_k\|^2$ is less than or equal to $\|u_k\|^2$. This holds for each $j \leq k$ and $u_k \in \mathcal{U}_k$, and therefore the operator defined by the k-th column of H is a contraction. Since $k \in \mathbb{Z}$ is arbitrary, $H \in \mathbb{UM}^2_{\text{ball}}(\mathbf{U}, \mathbf{Y})$.

Part 2. Under the same assumptions as in Part 1, let H and F be given by (2.3). Our aim is to prove that (2.5) holds. From the definition of F the right hand side of (2.5) is clear. Thus we have to prove the inequality in the left hand side of (2.5).

First assume that there exists an $N \geq 0$ such that $\mathcal{U}_k = \mathcal{Y}_k = \{0\}$ for $|k| > N$. In that case $\mathbb{M}(\mathbf{U}) = \mathcal{L}(\mathbf{U})$, $\mathbb{UM}(\mathbf{U}) = \mathbb{UM}^\infty(\mathbf{U})$ and $\mathbb{UM}(\mathbf{U}, \mathbf{Y}) = \mathbb{UM}^\infty(\mathbf{U}, \mathbf{Y})$. In particular, H and F are bounded operators. We have

$$\begin{aligned} I - Z^{(2)*}Z^{(2)} &= I - (I - I + Z^{(2)*})(I - I + Z^{(2)})\\ &= I - (I - F^{-*})(I - F^{-1})\\ &= F^{-*} + F^{-1} - F^{-*}F^{-1}.\end{aligned}$$

Thus

$$\begin{aligned} H^*H &= F^*Z^{(1)*}Z^{(1)}F \leq F^*(I - Z^{(2)*}Z^{(2)})F\\ &= F^*(F^{-*} + F^{-1} - F^{-*}F^{-1})F = F + F^* - I.\end{aligned}$$

So (2.5) holds in this case.

Now fix $j \leq k$. For $N \geq |j|, |k|$ set

$$\mathbf{U}_N = \oplus_{i=-N}^{N} \mathcal{U}_i, \quad \mathbf{Y}_N = \oplus_{i=-N}^{N} \mathcal{Y}_i,$$

and notice that

$$\begin{aligned} &\Delta_{-N,\,N}(F) = (I_{\mathbf{U}_N} - \Delta_{-N,\,N}(Z^{(2)}))^{-1} \quad \text{and} \\ &\Delta_{-N,\,N}(H) = \Delta_{-N,\,N}(Z^{(1)})(I_{\mathbf{U}_N} - \Delta_{-N,\,N}(Z^{(2)}))^{-1}. \end{aligned}$$

Next apply the result of the second paragraph of this part to

$$\Delta_{-N,\,N}(Z^{(1)}) \in \mathbb{UM}(\mathbf{U}_N, \mathbf{Y}_N) \quad \text{and} \quad \Delta_{-N,\,N}(Z^{(2)}) \in \mathbb{UM}_0(\mathbf{U}_N).$$

Using that $\Delta_{j,\,k}(\Delta_{-N,\,N}(M)) = \Delta_{j,\,k}(M)$ for each $M \in \mathbb{UM}(\mathbf{U})$, it follows that

$$\Delta_{j,\,k}(F) + \Delta_{j,\,k}(F^*) \geq \Delta_{j,\,k}(H^* Q_N H) + I, \tag{2.7}$$

where $Q_N \in \mathcal{L}(\mathbf{Y})$ is the orthogonal projection on $\mathbf{Y}_N$. Since

$$\Delta_{j,\,k}(H^* Q_N H) = \begin{bmatrix} (H_j)^* \\ \vdots \\ (H_k)^* \end{bmatrix} Q_n \begin{bmatrix} H_j & \cdots & H_k \end{bmatrix},$$

$\begin{bmatrix} H_j & \cdots & H_k \end{bmatrix}$ is a bounded linear operator, $Q_N \to I_{\mathbf{Y}}$ as $N \to \infty$ with convergence in the strong operator topology, and (2.7) holds for each $N \geq |j|, |k|$, it follows that

$$\Delta_{j,\,k}(F) + \Delta_{j,\,k}(F^*) \geq \Delta_{j,\,k}(H^* H) + I. \tag{2.8}$$

Thus (2.5) holds.

Part 3. In this part we assume that $H \in \mathbb{UM}^2_{\text{ball}}(\mathbf{U}, \mathbf{Y})$, $F \in \mathbb{UM}(\mathbf{U})$, and that condition (2.5) is fulfilled. We show that the operator matrices $Z^{(1)}$ and $Z^{(2)}$ defined by (2.6) are in $\mathbb{UM}^\infty(\mathbf{U}, \mathbf{Y})$ and $\mathbb{UM}^\infty_0(\mathbf{U})$, respectively, and that (2.2) is satisfied.

The second part of (2.5) implies that F is invertible in $\mathbb{UM}(\mathbf{U})$. Thus the operator matrices $Z^{(1)}$ and $Z^{(2)}$ in (2.6) are well defined. Moreover, for each $j \in \mathbb{Z}$ the j-th diagonal entry of F^{-1} is the identity operator $I_{\mathcal{U}_j}$, and thus the matrix $Z^{(2)}$ is strictly upper triangular. It remains to show that $Z^{(1)}$ and $Z^{(2)}$ satisfy (2.2), since this automatically implies that $Z^{(1)} \in \mathbb{UM}^\infty(\mathbf{U}, \mathbf{Y})$ and $Z^{(2)} \in \mathbb{UM}^\infty_0(\mathbf{U})$.

First note that it suffices to show that

$$\Delta_{j,\,k}(Z^{(1)})^* \Delta_{j,\,k}(Z^{(1)}) + \Delta_{j,\,k}(Z^{(2)})^* \Delta_{j,\,k}(Z^{(2)}) \leq \Delta_{j,\,k}(I_{\mathbf{U}}) \quad (j \leq k). \tag{2.9}$$

Indeed, if the above inequalities have been established, then we can use the results reviewed in the second part of Subsection 1.1 to derive (2.2). Fix $j \leq k$, and write Δ in place of $\Delta_{j,\,k}$. From $Z^{(2)} = I_{\mathbf{U}} - F^{-1}$, we see that $\Delta(Z^{(2)}) = \Delta(I_{\mathbf{U}}) - \Delta(F^{-1})$. Now use formula (1.4) to see that $\Delta(F^{-1}) = \Delta(F)^{-1}$. But then

$$\begin{aligned} \Delta(Z^{(2)})^* \Delta(Z^{(2)}) &= \\ &= \big(\Delta(I_{\mathbf{U}}) - \Delta(F)^{-*}\big)\big(\Delta(I_{\mathbf{U}}) - \Delta(F)^{-1}\big) \\ &= \Delta(I_{\mathbf{U}}) - \Delta(F)^{-1} - \Delta(F)^{-*} + \Delta(F)^{-*}\Delta(F)^{-1} \\ &= \Delta(F)^{-*}\big(\Delta(F)^*\Delta(F) - \Delta(F)^* - \Delta(F) + \Delta(I_{\mathbf{U}})\big)\Delta(F)^{-1}. \end{aligned}$$

Thus

$$\Delta(F)^* \left(\Delta(I_{\mathbf{U}}) - \Delta(Z^{(2)})^* \Delta(Z^{(2)})\right) \Delta(F) = \\ = \Delta(F)^* + \Delta(F) - \Delta(I_{\mathbf{U}}) \geq \Delta(H^*H).$$

Here we used that the first part of (2.5) implies (2.8).

Next we consider the equality $H = Z^{(1)}F$. Again using (1.3), with $\Delta = \Delta_{j,k}$, we have $\Delta(H) = \Delta(Z^{(1)})\Delta(F)$, and thus

$$\Delta(F)^*\Delta(Z^{(1)})^*\Delta(Z^{(1)})\Delta(F) = \Delta(H)^*\Delta(H).$$

By combining this with the result of the previous paragraph we see that

$$\Delta(F)^*\big(\Delta(I_{\mathbf{U}}) - \Delta(Z^{(1)})^*\Delta(Z^{(1)}) - \Delta(Z^{(2)})^*\Delta(Z^{(2)})\big)\Delta(F) \geq \\ \geq \Delta(H^*H) - \Delta(H)^*\Delta(H) \geq 0.$$

To see that the last inequality holds, let P denote the projection from $\mathbf{Y}$ onto $\mathcal{Y}_j \oplus \cdots \oplus \mathcal{Y}_k$ and observe that

$$\begin{aligned} \Delta(H^*H) &= \begin{bmatrix} (H_j)^*H_j & \cdots & (H_j)^*H_k \\ \vdots & & \vdots \\ (H_k)^*H_j & \cdots & (H_k)^*H_k \end{bmatrix} = \begin{bmatrix} (H_j)^* \\ \vdots \\ (H_k)^* \end{bmatrix} \begin{bmatrix} H_j & \cdots & H_k \end{bmatrix} \\ &\geq \begin{bmatrix} (H_j)^* \\ \vdots \\ (H_k)^* \end{bmatrix} P \begin{bmatrix} H_j & \cdots & H_k \end{bmatrix} = \Delta(H)^*\Delta(H). \end{aligned}$$

Since $\Delta(F)$ is invertible, we obtain (2.9).

Part 4. In case $Z^{(1)}$ and $Z^{(2)}$ satisfy the assumptions of Parts 1 and 2 and H and F are given by (2.3), it is clear that

$$HF^{-1} = Z^{(1)} \quad \text{and} \quad I - F^{-1} = Z^{(2)}.$$

If H and F satisfy the assumptions of Parts 3 and $Z^{(1)}$ and $Z^{(2)}$ are given by (2.6), then

$$(I - Z^{(2)})^{-1} = (I_{\mathbf{U}} - I_{\mathbf{U}} + F^{-1})^{-1} = F \quad \text{and} \quad Z^{(1)}(I_{\mathbf{U}} - Z^{(2)})^{-1} = HF^{-1}F = H.$$

Thus the maps $(Z^{(1)}, Z^{(2)}) \mapsto (H, F)$ and $(H, F) \mapsto (Z^{(1)}, Z^{(2)})$ in Theorem 2.1 are each others inverses. □

The next proposition is an addition to Theorem 2.1. It takes into account the interpolation condition on H in (0.3) and on the pair $(Z^{(1)}, Z^{(2)})$ in (0.6).

Proposition 2.2. *Let $Z^{(1)}$ and $Z^{(2)}$ be as in* (2.1) *and* (2.2)*, and define H and F by* (2.3)*. Then the pair $(Z^{(1)}, Z^{(2)})$ satisfies the interpolation conditions* (0.6) *if and only if H satisfies the interpolation condition* (0.3) *and*

$$F_{j,k}|_{\mathcal{F}_k} = F_{j,k-1}\o_{k,2} \quad (j, k \in \mathbb{Z}, j < k). \tag{2.10}$$

Proof. Let $Z^{(1)}$ and $Z^{(2)}$ satisfy (2.1) and (2.2), and let H and F be defined by (2.3).

We begin with a general remark about the interpolation conditions (0.3), (0.6), and (2.10). Recall that for each $k \in \mathbb{Z}$ the space $\mathcal{F}_k$ is a subspace of $\mathcal{U}_k$. In what follows τ_k is the canonical embedding of $\mathcal{F}_k$ into $\mathcal{U}_k$. Furthermore, $\mathbf{F}$ will denote the Hilbert direct sum $\oplus_{k\in\mathbb{Z}}\mathcal{F}_k$. Now let

$$E = [E_{j,\,k}]_{j,\,k\in\mathbb{Z}}, \quad \Omega^{(1)} = \left[\Omega^{(1)}_{j,\,k}\right]_{j,\,k\in\mathbb{Z}}, \quad \Omega^{(2)} = \left[\Omega^{(2)}_{j,\,k}\right]_{j,\,k\in\mathbb{Z}}, \tag{2.11}$$

be the operator matrices defined by

$$E_{j,\,k} = \begin{cases} \tau_k, & \text{if } j = k, \\ 0, & \text{if } j \neq k, \end{cases} \tag{2.12}$$

and

$$\Omega^{(1)}_{j,\,k} = \begin{cases} \emptyset_{k,\,1}, & \text{if } j = k, \\ 0, & \text{if } j \neq k, \end{cases} \qquad \Omega^{(2)}_{j,\,k} = \begin{cases} \emptyset_{k,\,2}, & \text{if } j = k-1, \\ 0, & \text{if } j \neq k-1. \end{cases} \tag{2.13}$$

Observe that both E and $\Omega^{(1)}$ are diagonal operator matrices, $E \in \mathbb{UM}^{\infty}_{\text{ball}}(\mathbf{F},\mathbf{U})$ and $\Omega^{(1)} \in \mathbb{UM}^{\infty}_{\text{ball}}(\mathbf{F},\mathbf{Y})$, while $\Omega^{(2)} \in \mathbb{UM}^{\infty}_{\text{ball},\,0}(\mathbf{F},\mathbf{U})$ is a shifted diagonal operator, that is, all the entries of $\Omega^{(2)}$ are zero except those in the first diagonal above the main diagonal. Using these operator matrices we can restate the interpolation conditions. In fact, we have

$$(0.3) \iff HE = \Omega^{(1)} + H\Omega^{(2)}, \tag{2.14}$$

$$(0.6) \iff Z^{(1)}E = \Omega^{(1)} \quad \text{and} \quad Z^{(2)}E = \Omega^{(2)}, \tag{2.15}$$

$$(2.10) \iff FE - E = F\Omega^{(2)}. \tag{2.16}$$

Now assume that the pair $(Z^{(1)}, Z^{(2)})$ satisfies the interpolation condition (0.6). Since H and F are given by (2.3), we have

$$H = Z^{(1)}(I_{\mathbf{U}} - Z^{(2)})^{-1} = Z^{(1)} + Z^{(1)}(I_{\mathbf{U}} - Z^{(2)})^{-1}Z^{(2)} = Z^{(1)} + HZ^{(2)},$$

$$F = (I_{\mathbf{U}} - Z^{(2)})^{-1} = I_{\mathbf{U}} + (I_{\mathbf{U}} - Z^{(2)})^{-1}Z^{(2)} = I_{\mathbf{U}} + FZ^{(2)}.$$

Hence, using (2.15), we see that

$$HE = Z^{(1)}E + HZ^{(2)}E = \Omega^{(1)} + H\Omega^{(2)}, \quad FE = E + FZ^{(2)}E = E + F\Omega^{(2)}.$$

But then we can use (2.14) to conclude that H satisfies (0.3), and we can use (2.16) to conclude that F satisfies (2.10).

Next assume that H and F satisfy the interpolation conditions (0.3) and (2.10), respectively. From (2.3) we see that

$$Z^{(1)} = HF^{-1} \quad \text{and} \quad Z^{(2)} = I - F^{-1}.$$

Note that (2.10) and (2.16) imply that $F^{-1}E = E - \Omega^{(2)}$. Hence

$$\begin{aligned} Z^{(1)}E &= HF^{-1}E = HE - H\Omega^{(2)} = \Omega^{(1)} \quad \text{by (2.14)}, \\ Z^{(2)}E &= E - F^{-1}E = \Omega^{(2)}. \end{aligned}$$

But then we can use the equivalence in (2.15) to conclude that the pair $(Z^{(1)}, Z^{(2)})$ satisfies the interpolation conditions (0.6) as desired. □

Proof of Theorem 0.1. Let $Z^{(1)}$ and $Z^{(2)}$ be a pair of operator matrices satisfying (0.4), (0.5) and (0.6). Define H by formula (0.7). Theorem 2.1 tells us that H is well-defined and $H \in \mathbb{UM}^{2}_{\text{ball}}(\mathbf{U},\mathbf{Y})$; see formula (2.4). Since $(Z^{(1)}, Z^{(2)})$ satisfies the interpolation conditions (0.6), we see from Proposition 2.2 that H satisfies (0.3). Thus H is a solution to Problem 0.1. □

3. Majorants of H^*H and the solution to Problem 0.2

In this section we solve Problem 0.2 and give a parametrization of the set of all its solutions. The first subsection, which has a preliminary character, deals with a time-variant version of the Cayley transform. The main result (Theorem 3.2 below) is presented in the second subsection, which is then used in the final subsection to solve Problem 0.2.

3.1. **The Cayley transform.** Let $C \in \mathbb{UM}^{\infty}_{\text{ball},0}(\mathbf{U})$. Then we know from Subsection 1.2 that $I_{\mathbf{U}} - C$ is invertible in the algebra $\mathbb{UM}(\mathbf{U})$. It follows that K given by

$$K = (I_{\mathbf{U}} + C)(I_{\mathbf{U}} - C)^{-1} \tag{3.1}$$

is a well-defined element of $\mathbb{UM}(\mathbf{U})$. We shall refer to K as the *Cayley transform* of C. The following proposition shows that K is positive real (see [3], page 94, for a related but somewhat less general result).

Proposition 3.1. *The map $C \mapsto K$ defined by* (3.1) *establishes a one-to-one correspondence between the set of all operator matrices C in $\mathbb{UM}^{\infty}_{\text{ball},0}(\mathbf{U})$ and the positive real $K \in \mathbb{UM}(\mathbf{U})$ satisfying $K_{j,\,j} = I_{\mathcal{U}_j}$ for all $j \in \mathbb{Z}$.*

Proof. Assume that K is defined by (3.1) for a $C \in \mathbb{UM}^{\infty}_{\text{ball},0}(\mathbf{D})$. Then

$$K = I_{\mathbf{U}} + 2C\,(I_{\mathbf{U}} - C)^{-1}.$$

Since $(I_{\mathbf{U}} - C)^{-1}$ is upper triangular and C is strictly upper triangular, it follows that $K_{j,\,j} = I_{\mathcal{U}_j}$ for each $j \in \mathbb{Z}$. Given an operator matrix M let $\Delta(M) = \Delta_{j,\,k}(M)$ denote the finite section of M for $j \leq k$. We have to prove (see Subsection 1.3) that $\Delta(\operatorname{Re} K)$ is non-negative. To do this note that

$$\begin{aligned}
2\Delta(\operatorname{Re} K) &= (I - \Delta(C^*))^{-1}(I + \Delta(C^*)) + (I + \Delta(C))(I - \Delta(C))^{-1} \\
&= (I - \Delta(C^*))^{-1}\left\{(I + \Delta(C^*))(I - \Delta(C)) + \right. \\
&\qquad \left. + (I - \Delta(C^*))(I + \Delta(C))\right\}(I - \Delta(C))^{-1} \\
&= 2(I - \Delta(C^*))^{-1}\left\{I - \Delta(C^*)\Delta(C)\right\}(I - \Delta(C))^{-1}.
\end{aligned}$$

In other words,

$$\Delta(\operatorname{Re} K) = (I - \Delta(C^*))^{-1}\left\{I - \Delta(C)^*\Delta(C)\right\}(I - \Delta(C))^{-1}. \tag{3.2}$$

Here $I = I_{\mathcal{U}_j \oplus \cdots \oplus \mathcal{U}_k}$. Because C is a contraction, $\Delta(C)$ is also a contraction. Hence all finite sections $\Delta(\operatorname{Re} K)$ of $\operatorname{Re} K$ are non-negative. Therefore K is positive real.

Conversely, for a positive real matrix K in $\mathbb{UM}(\mathbf{U})$ satisfying $\operatorname{diag} K_{j,\,j} = I_{\mathcal{U}_j}$ for $j \in \mathbb{Z}$, consider the operator matrix C defined by

$$C = (K - I_{\mathbf{U}})\,(K + I_{\mathbf{U}})^{-1}. \tag{3.3}$$

We know that $K + I_{\mathbf{U}} \in \mathbb{UM}(\mathbf{U})$ and for each $j \in \mathbb{Z}$ the j-th diagonal element of $K + I_{\mathbf{U}}$ is $2I_{\mathcal{U}_j}$. Hence $K + I$ is invertible in $\mathbb{UM}(\mathbf{U})$ (see Subsection 1.2). Moreover, $K - I$ is in $\mathbb{UM}_0(\mathbf{U})$. Thus C in (3.3) is a well defined operator matrix in $\mathbb{UM}_0(\mathbf{U})$. We claim that C is in $\mathbb{UM}^{\infty}_{\text{ball},0}(\mathbf{U})$. To see this let Δ and I be as in the previous paragraph. Using (1.3) and (1.4) we have

$$\Delta(C) = (\Delta(K) - I)\,(\Delta(K) + I)^{-1}.$$

Thus

$$\begin{aligned} I-\Delta(C)^*\Delta(C) &= I-(\Delta(K)^*+I)^{-1}(\Delta(K)^*-I)(\Delta(K)-I)(\Delta(K)+I)^{-1} \\ &= (\Delta(K)^*+I)^{-1}\left\{(\Delta(K)^*+I)(\Delta(K)+I)\right. \\ &\qquad \left.-(\Delta(K)^*-I)(\Delta(K)-I)\right\}(\Delta(K)+I)^{-1} \\ &= 2(\Delta(K)^*+I)^{-1}\left(\Delta(K)^*+\Delta(K)\right)(\Delta(K)+I)^{-1} \\ &= 4(\Delta(K)^*+I)^{-1}\Delta(\mathrm{Re}\,K)(\Delta(K)+I)^{-1}\geq 0. \end{aligned}$$

Hence any finite section of C is a contraction. Therefore C is a contraction, and thus in $\mathbb{UM}^{\infty}_{\mathrm{ball},0}(\mathbf{U})$.

Finally, one easily verifies that the maps $C\mapsto K$ given by (3.1) and $K\mapsto C$ given by (3.3) are each others inverses. Hence the map $C\mapsto K$ defined by (3.1) has the desired properties. □

If K in $\mathbb{UM}(\mathbf{U})$ is positive real with $K_{j,\,j}=I_{\mathcal{U}_j}$ for $j\in\mathbb{Z}$, then C defined by (3.3) will be called the *inverse Cayley transform* of K.

3.2. **Time-variant harmonic majorants of** H^*H**.** Let $H\in\mathbb{UM}^2_{\mathrm{ball}}(\mathbf{U},\mathbf{Y})$ be given, and consider the operator matrix H^*H (see Subsection 1.4). In the present subsection we describe the operator matrices $W\in\mathbb{UM}(\mathbf{U})$ satisfying

$$\mathrm{Re}\,W\geq H^*H \quad \text{and} \quad W_{j,\,j}=I_{\mathcal{U}_j} \quad (j\in\mathbb{Z}). \tag{3.4}$$

This description will be used in the next subsection to give the solution to Problem 0.2.

When $W\in\mathbb{UM}(\mathbf{U})$ satisfies the first identity in (3.4) we call $\mathrm{Re}\,W$ a *time-variant harmonic majorant* of H^*H. In that case, since H^*H is non-negative, W is automatically positive real. Time-variant harmonic majorants of H^*H do exist. In fact (see Subsection 1.4) the operator matrix V defined by (1.7) belongs to $\mathbb{UM}(\mathbf{U})$ and $\mathrm{Re}\,V=H^*H$. Thus H^*H is its own time-variant harmonic majorant.

To describe all $W\in\mathbb{UM}(\mathbf{U})$ satisfying (3.4) recall that $\mathbf{D}_H$ is the Hilbert space direct sum $\oplus_{k\in\mathbb{Z}}\mathcal{D}_{H_k}$, where H_k is the k-th column of H and $\mathcal{D}_{H_k}$ is the corresponding defect space. The latter space is well-defined because H_k defines a contraction from $\mathcal{U}_k$ in to $\mathbf{Y}$. We define ∇_H and Π_H to be the diagonal operator matrices in $\mathbb{UM}(\mathbf{D}_H)$ and $\mathbb{UM}(\mathbf{U},\mathbf{D}_H)$, respectively, given by

$$(\nabla_H)_{j,\,k}=\left\{\begin{array}{ll} D_{H_k} & \text{for } j=k,\\ 0 & \text{for } j\neq k.\end{array}\right. \quad \text{and} \quad (\Pi_H)_{j,\,k}=\left\{\begin{array}{ll} \Pi_{H_k} & \text{for } j=k,\\ 0 & \text{for } j\neq k.\end{array}\right. \tag{3.5}$$

In the definition of ∇_H we view D_{H_k} as an operator on $\mathcal{D}_{H_k}$, and in the definition of Π_H the operator Π_{H_k} is the orthogonal projection of $\mathcal{U}_k$ onto $\mathcal{D}_{H_k}$. We can now state the main result of this section.

Theorem 3.2. *Let $H\in\mathbb{UM}^2_{\mathrm{ball}}(\mathbf{U},\mathbf{Y})$, and let ∇_H and Π_H be the diagonal operator matrices given by* (3.5)*. Then all operator matrices $W\in\mathbb{UM}(\mathbf{U})$ satisfying* (3.4) *are determined by*

$$W=V+\Pi_H^*\nabla_H\,(I+C)\,(I-C)^{-1}\,\nabla_H\Pi_H, \tag{3.6}$$

where V in $\mathbb{UM}(\mathbf{U})$ is given by (1.7), and C is an arbitrary operator matrix in $\mathbb{UM}^{\infty}_{\mathrm{ball},0}(\mathbf{D}_H)$. Moreover, W and C in (3.4) *determine each other uniquely.*

Proof. Let $C\in\mathbb{UM}^{\infty}_{\mathrm{ball},0}(\mathbf{D}_H)$. Then W in (3.6) is equal to

$$W=V+\Pi_H^*\nabla_H^*K\nabla_H\Pi_H,$$

where K is the Cayley transform of C. Using $\operatorname{Re} K$ is non-negative this yields

$$\operatorname{Re} W = \operatorname{Re} V + \Pi_H^* \nabla_H^* \operatorname{Re} K \nabla_H \Pi_H \geq \operatorname{Re} V = H^*H,$$

and we have

$$W_{j,\,j} = V_{j,\,j} + D_{H_j} K_{j,\,j} D_{H_j} = H_j^* H_j + D_{H_j}^2 = I_{\mathcal{U}_j} \quad (j \in \mathbb{Z}).$$

So W satisfies (3.4).

To prove the converse implication, assume that $W \in \mathbb{UM}(\mathbf{U})$ satisfies (3.4). Since $\operatorname{Re} V = H^*H$, the first part of (3.4) implies that the real part of the operator matrix $\Lambda = W - V \in \mathbb{UM}(\mathbf{U})$ is non-negative. The second part of (3.4) gives $\Lambda_{j,\,j} = I_{\mathcal{U}_j} - H_j^* H_j = D_{H_j}^2$ for all $j \in \mathbb{Z}$. The fact that $\operatorname{Re} \Lambda$ is non-negative implies that for any $j < k$ the finite section $\Delta_{j,\,k}(\operatorname{Re} \Lambda)$ is a non-negative operator on $\oplus_{i=j}^k \mathcal{U}_i$. In particular, the two by two operator matrix

$$\begin{bmatrix} 2\Lambda_{j,\,j} & \Lambda_{j,\,k} \\ \Lambda_{j,\,k}^* & 2\Lambda_{k,\,k} \end{bmatrix} = \begin{bmatrix} 2D_{H_j}^2 & \Lambda_{j,\,k} \\ \Lambda_{j,\,k}^* & 2D_{H_k^2} \end{bmatrix}$$

is a non-negative operator on the Hilbert direct sum $\mathcal{U}_j \oplus \mathcal{U}_k$. Recall that an arbitrary operator matrix

$$\begin{bmatrix} A_1 & B^* \\ B & A_2 \end{bmatrix} \text{ acting on the Hilbert space direct sum} \begin{bmatrix} \mathcal{E}_1 \\ \mathcal{E}_2 \end{bmatrix}$$

is a non-negative operator if and only if A_1 and A_2 are non-negative and $B = A_2^{1/2} \Phi A_1^{1/2}$, where Φ is a contraction from $\overline{A_1 \mathcal{E}_1}$ into $\overline{A_2 \mathcal{E}_2}$. Moreover, in this case, B and Φ uniquely determine each other; see Theorem XVI.1.1 in [16] for further details. Thus there exists a unique operator $K_{j,\,k}$ from $\mathcal{D}_{H_j}$ into $\mathcal{D}_{H_k}$ such that $\Lambda_{k,\,j} = D_{H_k} K_{k,\,j} D_{H_j}$. Now set $K_{j,j} = I_{\mathcal{D}_{H_j}}$ and $K_{j,\,k} = 0$ for $j > k$, and put $K = \begin{bmatrix} K_{j,\,k} \end{bmatrix}_{j,k \in \mathbb{Z}}$. Then

$$K \in \mathbb{UM}(\mathbf{D}_H) \quad \text{and} \quad \Lambda = \Pi_H^* \nabla_H^* K \nabla_H \Pi_H.$$

Since the range of ∇_H is a dense set in $\mathbf{D}_H$ and Λ is positive real, it follows that K is positive real and that Λ and K determine each other uniquely. Let C be the inverse Cayley transform of K. Then $C \in \mathbb{UM}_{\text{ball},0}^\infty(\mathbf{D}_H)$ and W is given by (3.6). It also follows from the above that C and W determine each other uniquely. □

Corollary 3.3. *There is only one operator matrix $W \in \mathbb{UM}(\mathbf{U})$ satisfying* (3.4) *if and only if for each $k \in \mathbb{Z}$ the k-th column H_k of H defines an isometry from $\mathcal{U}_k$ into $\mathbf{Y}$. In this case $W = V$, where V is given by* (1.7)*, is the only operator matrix in $\mathbb{UM}(\mathbf{U})$ satisfying* (3.4).

Proof. The set $\mathbb{UM}_{\text{ball},0}^\infty(\mathbf{D}_H)$ consists of just one element if and only if $\mathbf{D}_H = \{0\}$, i.e., $\mathcal{D}_{H_j} = \{0\}$ for all j. The latter condition is equivalent to H_j being an isometry for each $j \in \mathbb{Z}$. □

3.3. **All solutions to Problem 0.2.** We now describe the solution to Problem 0.2. Fix a $H \in \mathbb{UM}_{\text{ball}}^2(\mathbf{U}, \mathbf{Y})$. Define $N \in \mathbb{UM}(\mathbf{U})$ by

$$N = \begin{bmatrix} N_{j,k} \end{bmatrix}_{j,k \in \mathbb{Z}}, \quad N_{j,\,k} = \begin{cases} (H_j)^* H_k, & \text{for } j \leq k, \\ 0, & \text{for } j > k, \end{cases} \tag{3.7}$$

Here, as before, H_k is the k-th column of H.

Theorem 3.4. *Let $H \in \mathbb{UM}^2_{\rm ball}(\mathbf{U},\mathbf{Y})$. Define $N \in \mathbb{UM}(\mathbf{U})$ by* (3.7)*, and let the matrices $\nabla_H \in \mathbb{UM}(\mathbf{D}_H)$ and $\Pi_H \in \mathbb{UM}(\mathbf{D}_H,\mathbf{U})$ be the diagonal operator matrices given by* (3.5)*. For each $C \in \mathbb{UM}^\infty_{\rm ball,0}(\mathbf{D}_H)$ put*

$$F = N + \Pi_H^* \nabla_H (I_{\mathbf{D}_H} - C)^{-1} \nabla_H \Pi_H. \tag{3.8}$$

Then $F \in \mathbb{UM}(\mathbf{U})$, and the map $C \mapsto F$ defined by (3.8) *is a one-to-one map from $\mathbb{UM}^\infty_{\rm ball,0}(\mathbf{D}_H)$ onto the set of all $F \in \mathbb{UM}(\mathbf{U})$ that satisfy* (0.12)*. In particular, $F \in \mathbb{UM}(\mathbf{U})$ satisfying* (0.12) *exist. Finally, there exists a unique $F \in \mathbb{UM}(\mathbf{U})$ satisfying* (0.12) *if and only if for each $k \in \mathbb{Z}$ the k-th column H_k of H defines an isometry from $\mathcal{U}_k$ into $\mathbf{Y}$. In this case $F = N$ is the only $F \in \mathbb{UM}(\mathbf{U})$ satisfying* (0.12)*.*

Proof. Assume W and F belong to $\mathbb{UM}(\mathbf{U})$ and determine each other uniquely via

$$W = 2F - I_{\mathbf{U}} \quad \text{and} \quad F = \frac{1}{2}(W + I). \tag{3.9}$$

Then $W_{j,\,j} = I_{\mathcal{U}_j}$ if and only if $F_{j,\,j} = I_{\mathcal{U}_j}$ for each $j \in \mathbb{Z}$. Moreover, $2\mathrm{Re}\, F = \mathrm{Re}\, W + I$. Hence F satisfies (2.5) (where $F + F^* = 2\mathrm{Re}\, F$) if and only if W satisfies (3.4).

To complete the proof it remains to show that the map $C \mapsto W$ given by (3.6) composed with the map $W \mapsto F$ in (3.9) gives the map $C \mapsto F$ in (3.8). Let $C \in \mathbb{UM}^\infty_{\rm ball,0}(\mathbf{D}_H)$, define W by (3.6) and F by (3.9). Notice that V in (1.7) and N in (3.7) are related via $V = 2N + \Pi_H^* \nabla_H^2 \Pi_H - I$. Thus

$$\begin{aligned}
W &= V + \Pi_H^* \nabla_H (I + C)(I - C)^{-1} \nabla_H \Pi_H \\
&= 2N + \Pi_H^* \nabla_H^2 \Pi_H + \Pi_H^* \nabla_H (I + C)(I - C)^{-1} \nabla_H \Pi_H - I \\
&= 2N + \Pi_H^* \nabla_H (I + C + I - C)(I - C)^{-1} \nabla_H \Pi_H - I \\
&= 2N + 2\Pi_H^* \nabla_H (I - C)^{-1} \nabla_H \Pi_H - I.
\end{aligned}$$

Hence

$$\begin{aligned}
F &= \frac{1}{2}(W + I) = \frac{1}{2}(2N + 2\Pi_H^* \nabla_H (I - C)^{-1} \nabla_H \Pi_H) \\
&= N + \Pi_H^* \nabla_H (I - C)^{-1} \nabla_H \Pi_H.
\end{aligned}$$

So F is given by (3.8). □

A state space example. Consider the state space system $\{A, B, E, D\}$, where A is an operator on $\mathcal{X}$ whose spectrum is contained in the open unit disc. The input space $\mathbf{U} = \oplus_{j=1}^n \mathcal{U}_j$ and the output space $\mathbf{Y} = \oplus_{j=1}^n \mathcal{Y}_j$. Furthermore, B is an operator mapping $\mathbf{U}$ into $\mathcal{X}$ and E is an operator mapping $\mathcal{X}$ into $\mathbf{Y}$, while $D \in \mathbb{UM}(\mathbf{U},\mathbf{Y})$ is a finite upper triangular operator matrix mapping $\mathbf{U} = \oplus_{j=1}^n \mathcal{U}_j$ into $\mathbf{Y} = \oplus_{j=1}^n \mathcal{Y}_j$.

Now, let H be the operator matrix (consisting of N doubly infinite columns) of which the k-th column $H_k = \mathrm{col}\{H_{j,\,k}\}_{j\in\mathbb{Z}}$, $k = 1, \ldots, n$, is given by

$$H_{j,\,k} = \begin{cases} EA^{-j-1}B_k & \text{when } j < 0, \\ D_{j,\,k} & \text{when } j = 1, \ldots, n, \\ 0 & \text{when } j > n. \end{cases}$$

Here $D_{j,\,k}$ is the $(j,\,k)$-th entry of the $n \times n$ operator matrix D, and B_k is the restriction of B to the k-th component $\mathcal{U}_k$ of $\mathbf{U} = \oplus_{j=1}^n \mathcal{U}_j$. For this H we consider the finite operator matrix version of Problem 0.2, that is, we seek all $F \in \mathbb{UM}(\mathbf{U})$ satisfying

$$F + F^* \geq H^*H + I_{\mathbf{U}} \quad \text{and} \quad F_{j,j} = I_{\mathcal{U}_j} \quad (j = 1, \ldots, n). \tag{3.10}$$

By setting $\mathcal{U}_j = \{0\}$ for $j < 0$ and $j > n$, we can identify $\mathbf{U}$ and $\oplus_{j\in\mathbb{Z}}\mathcal{U}_j$, and we may view H as an operator matrix in $\mathbb{M}(\mathbf{U}, \mathbf{E})$, where $\mathbf{E} = \oplus_{j\in\mathbb{Z}}\mathcal{E}_j$ with

$$\mathcal{E}_j = \begin{cases} \mathbf{Y} & \text{when } j < 0 \text{ or } j > n, \\ \mathcal{Y}_j & \text{when } j = 1, \ldots, n. \end{cases}$$

The fact that D is upper triangular implies that $H \in \mathbb{UM}(\mathbf{U}, \mathbf{E})$. Hence the results obtained above apply.

Since A has its spectrum in the open unit disc, we know that the Lyapunov equation $P = A^*PA + E^*E$ has a unique solution which is given by

$$P = \sum_{j=0}^{\infty} A^{*j} E^* E A^j.$$

Using this, we obtain $H^*H = D^*D + B^*PB$. It follows that for each $k = 1, \ldots, n$ the k-th column H_k of H induces a contraction from $\mathbf{U}$ into $\mathbf{E}$ if and only if $P_{\mathcal{U}_k}(D^*D + B^*PB)|_{\mathcal{U}_k}$ is a contraction, where $P_{\mathcal{U}_k}$ is the orthogonal projection onto $\mathcal{U}_k$. In other words, H is in $\mathbb{UM}^2_{\text{ball}}(\mathbf{U}, \mathbf{E})$ if and only if the block diagonal entries of $D^*D + B^*PB$ are contractions.

Now assume that H is in $\mathbb{UM}^2_{\text{ball}}(\mathbf{U}, \mathbf{E})$. Let N be the upper triangular part of $D^*D + B^*PB$; see (3.7). In this setting, D_{H_k} equals the positive square root of the operator $I - P_{\mathcal{U}_k}(D^*D + B^*PB)|_{\mathcal{U}_k}$, and ∇_H is the diagonal operator formed by $\{D_{H_k}\}_{k=1}^n$ on $\oplus_{k=1}^n \mathcal{D}_{H_k}$; see (3.5). Then the set of all operators $F \in \mathbb{UM}(\mathbf{U})$ satisfying (3.10) is determined by (3.8). Finally, $H = Z^{(1)}(I_{\mathbf{U}} - Z^{(2)})^{-1}$, where $Z^{(1)} = HF^{-1}$ and $Z^{(2)} = I - F^{-1}$. Moreover, the operator matrix $\begin{bmatrix} Z^{(1)*} & Z^{(2)*} \end{bmatrix}$ induces a contraction.

4. The full version of Theorem 0.2 and its proof

The following theorem is the full version of Theorem 0.2.

Theorem 4.1. *Let H be a solution to* Problem *0.1, and let $\mathbf{C}_{H,\varnothing}$ be the set of all operator matrices C in $\mathbb{UM}^\infty_{\text{ball},0}(\mathbf{D}_H)$ defined by* (0.11). *Fix $C \in \mathbf{C}_{H,\varnothing}$, and put*

$$F = N + \Pi_H^* \nabla_H (I_{\mathbf{D}_H} - C)^{-1} \nabla_H \Pi_H. \tag{4.1}$$

Here $N \in \mathbb{UM}(\mathbf{U})$ is defined by (3.7), *and $\nabla_H \in \mathbb{UM}(\mathbf{D}_H)$ and $\Pi_H \in \mathbb{UM}(\mathbf{D}_H, \mathbf{U})$ are the diagonal operator matrices given by* (3.5). *Then $F \in \mathbb{UM}(\mathbf{U})$, F satisfies* (0.12), *and*

$$F_{j,\,k}|_{\mathcal{F}_k} = F_{j,\,k-1}\varnothing_{k,2} \quad (j, k \in \mathbb{Z}, j < k). \tag{4.2}$$

Put

$$Z^{(1)} = HF^{-1} \quad \text{and} \quad Z^{(2)} = I_{\mathbf{U}} - F^{-1}. \tag{4.3}$$

Then the pair of operator matrices $(Z^{(1)}, Z^{(2)})$ satisfies (0.4), (0.5), (0.6), *and H is given by* (0.7). *Furthermore, the map $C \mapsto (Z^{(1)}, Z^{(2)})$ is a one-to one map from the set $\mathbf{C}_{H,\varnothing}$ onto the set of all pairs $(Z^{(1)}, Z^{(2)})$ satisfying* (0.4), (0.5), (0.6), *and such that H is given by* (0.7). *In particular, there exists a unique pair of operator matrices $(Z^{(1)}, Z^{(2)})$ satisfying* (0.4), (0.5) *and* (0.6) *such that H is given by* (0.7) *if and only if one of the following three conditions is satisfied:*

(1) $\mathcal{F}_{H_k} = \mathcal{D}_{H_k}$ *for each* $k \in \mathbb{Z}$;
(2) $\o_{H_k}$ *is a co-isometry for each* $k \in \mathbb{Z}$;
(3) *there exists an integer* $k \in \mathbb{Z}$ *such that* $\mathcal{F}_{H_j} = \mathcal{D}_{H_j}$ *for each* $j > k$ *and the operator* $\o_{H_j}$ *is a co-isometry for each* $j \le k$.

Let H be a solution to Problem 0.1. Recall that, in particular, this implies that H belongs to $\mathbb{UM}^2_{\rm ball}(\mathbf{U}, \mathbf{Y})$. Hence Theorem 3.4 applies to H. It will be convenient first to prove the following proposition.

Proposition 4.2. *Let* H *be a solution to Problem 0.1. Fix* C *in* $\mathbb{UM}^\infty_{\rm ball,0}(\mathbf{D}_H)$, *and let* $F = [F_{j,\,k}]_{j,k\in\mathbb{Z}}$ *in* $\mathbb{UM}(\mathbf{U})$ *by defined by* (4.1). *Then* F *satisfies* (4.2) *if and only if* C *belongs to the set* $\mathbf{C}_{H,\,\o}$ *defined by* (0.11).

Proof. Let H be a solution to Problem 0.1. We first show that

$$C \in \mathbf{C}_{H,\,\o} \quad \Longleftrightarrow \quad C\nabla_H \Pi_H E = \nabla_H \Pi_H \Omega^{(2)}. \tag{4.4}$$

Here E and $\Omega^{(2)}$ are the operator matrices in $\mathbb{UM}(\mathbf{F}, \mathbf{U})$ defined by (2.12) and (2.13). To prove (4.4) we use (0.9) and (0.10). From these formulas it follows that the operator matrix $C = [C_{j,\,k}]_{j,k\in\mathbb{Z}}$ in $\mathbb{UM}^\infty_{\rm ball,0}(\mathbf{D}_H)$ belongs to $\mathbf{C}_{H,\,\o}$ if and only if for each $j < k$ in $\mathbb{Z}$ and each $f_k \in \mathcal{F}_k$ we have

$$C_{j,\,k} D_{H_k} f_k = \begin{cases} D_{H_{k-1}} \o_{k,\,2} f_k & \text{for } j = k-1, \\ 0, & \text{for } j \neq k-1, \end{cases} \qquad f_k \in \mathcal{F}_k \quad (j < k \in \mathbb{Z}). \tag{4.5}$$

In the language of operator matrices (4.5) is equivalent to the right hand side of (4.4). Thus our claim follows.

By assumption H belongs to $\mathbb{UM}^2_{\rm ball}(\mathbf{U}, \mathbf{Y})$. Hence Theorem 3.4 applies to H. In particular, since (3.8), and (4.1) are the same identities, F is well-defined and belongs to $\mathbb{UM}(\mathbf{U})$. From (3.8) it follows that F is also given by the following formula:

$$F = I_{\mathbf{U}} + \tilde{N} + \Pi_H^* \nabla_H (I - C)^{-1} C \nabla_H \Pi_H, \tag{4.6}$$

where $\tilde{N} = [\tilde{N}_{i,\,j}] \in \mathbb{UM}(\mathbf{U})$ is the strictly upper triangular operator matrix given by

$$\tilde{N}_{i,\,j} = \begin{cases} (H_i)^* H_j, & \text{for } i < j, \\ 0, & \text{for } i \geq j. \end{cases} \tag{4.7}$$

We claim that

$$\tilde{N} E = N \Omega^{(2)}. \tag{4.8}$$

Note that both $\tilde{N}E$ and $N\Omega^{(2)}$ are strictly upper triangular operator matrices in $\mathbb{UM}(\mathbf{F}, \mathbf{U})$. Furthermore, for $j < k$ the (j, k)-th entry of $\tilde{N}E$ is equal to $(H_j)^* H_k \tau_k$, where τ_k is the canonical embedding of $\mathcal{F}_k$ into $\mathcal{U}_k$. Now observe, using the second part of (0.3), that for $j < k$ we have

$$\begin{aligned} (H_j)^* H_k \tau_k f_k &= \sum_{\nu=-\infty}^{j} H^*_{\nu,\,j} H_{\nu,\,k} f_k = \sum_{\nu=-\infty}^{j} H^*_{\nu,\,j} H_{\nu,\,k-1} \o_{k,\,2} f_k \\ &= (H_j)^* H_{k-1} \o_{k,\,2} f_k \quad (f_k \in \mathcal{F}_k). \end{aligned}$$

But $(H_j)^* H_{k-1} \o_{k,\,2}$ is precisely the (j, k)-th entry of $N\Omega^{(2)}$. Thus (4.8) is proved.

Next we use the two representations for F given by (4.1) and (4.6). We multiply (4.1) and (4.6) from the right by $\Omega^{(2)}$ and E, respectively, and subtract the resulting identities. Then, using (4.8), we obtain

$$F\Omega^{(2)} - FE + E = \Pi_H^* \nabla_H (I_{\mathbf{D}_H} - C)^{-1} (\nabla_H \Pi_H \Omega^{(2)} - C\nabla_H \Pi_H E).$$

Notice that $\nabla_H(I_{\mathbf{D}_H} - C)^{-1}$ acts as a one-to-one linear transformations on the linear space $\mathbf{D}_H^+$ (see the final paragraph of Subsection 1.2 for the definition of this space). Since Π_H^* is also one-to-one on vectors from $\mathbf{D}_H^+$, it follows that

$$F\Omega^{(2)} - FE + E = 0 \iff \nabla_H \Pi_H \Omega^{(2)} - C\nabla_H \Pi_H E = 0. \tag{4.9}$$

By comparing the left hand side of (4.9) with (2.16) and the right hand side of (4.9) with (4.4), we see that F satisfies (4.2) if and only if C belongs to $\mathbf{C}_{H,\,ø}$. □

Proof of Theorem 4.1. Let H be a solution to Problem 0.1. Let F be given by (4.1) with C from $\mathbf{C}_{H,\,ø}$. By assumption H satisfies (0.3). Proposition 4.2 tells us that F satisfies (2.10). Thus we can apply Proposition 2.2 to show that the pair $(Z^{(1)}, Z^{(2)})$ satisfies the interpolation condition (0.6). To see that the map $C \mapsto (Z^{(1)}, Z^{(2)})$ is a one-to one map from the set $\mathbf{C}_{H,\,ø}$ onto the set of all pairs $(Z^{(1)}, Z^{(2)})$ satisfying (0.4), (0.5), (0.6) and such that H is given by (0.7) it remains to apply Theorems 2.1 and 3.4.

In order to prove the claim in the final part of Theorem 4.1 note that there exists a unique pair of operator matrices $(Z^{(1)}, Z^{(2)})$ satisfying (0.4), (0.5) and (0.6) such that H is given by (0.7) if and only if the set $\mathbf{C}_{H,\,ø}$ is a singleton.

Let $C = \begin{bmatrix} C_{i,\,j} \end{bmatrix}_{i,j\in\mathbb{Z}} \in \mathbf{C}_{H,\,ø}$. Fix a $k \in \mathbb{Z}$. Then observe that $\mathcal{F}_{H_k} = \mathcal{D}_{H_k}$ implies that the k-th column of C is completely determined by

$$C_{i,\,k} = \begin{cases} ø_{H_k}, & \text{if } i = k-1, \\ 0, & \text{if } i \neq k-1. \end{cases}$$

Moreover, if $ø_{H_k}$ is a co-isometry, then we can use Corollary XXVII.5.3 in [24] to show that the $(k-1)$-th row of C is completely determined by

$$C_{k-1,\,j} = \begin{cases} ø_{H_k}\Pi_{\mathcal{F}_{H_k}}, & \text{if } j = k, \\ 0, & \text{if } j \neq k. \end{cases}$$

Here $\pi_{\mathcal{F}_{H_k}}$ denotes the orthogonal projection from $\mathcal{D}_{H_k}$ onto $\mathcal{F}_{H_k}$. From these two observations and the fact that C is strictly upper triangular it follows that the conditions (1), (2) and (3) are each sufficient for $\mathbf{C}_{H,\,ø}$ to be a singleton.

To see that these conditions are also necessary, assume that non of the conditions (1), (2) or (3) is satisfied, i.e., assume there exists a $k \in \mathbb{Z}$ with $\mathcal{F}_{H_k} \neq \mathcal{D}_{H_k}$ and such that $ø_{H_{k-1}}$ in not a co-isometry. In that case, set $\mathcal{G}_{H_k} = \mathcal{D}_{H_k} \ominus \mathcal{F}_{H_k}$ and let $D_{ø_{H_{k-1}}}$ and $\mathcal{D}_{ø_{H_{k-1}}}$ denote the defect operator and defect space of $ø_{H_{k-1}}$, respectively. Then both $\mathcal{G}_{H_k}$ and $\mathcal{D}_{ø_{H_{k-1}}}$ are not equal to $\{0\}$, and thus there exists a non-zero contraction N from $\mathcal{G}_{H_k}$ into $\mathcal{D}_{ø_{H_{k-1}}}$. Now define $C = \begin{bmatrix} C_{i,\,j} \end{bmatrix}_{i,j\in\mathbb{Z}} \in \mathbb{UM}^\infty_{\text{ball},0}(\mathbf{D}_H)$ by setting $C_{i,\,j} = 0$ in case $i \neq j-1$ and $(i,\,j) \neq (k-2, k)$, $C_{j-1,\,j} = ø_{H_j}\Pi_{\mathcal{F}_{H_j}}$ for each $j \in \mathbb{Z}$ and $C_{k-1,\,k} = D_{ø^*_{k-1}} N \Pi_{\mathcal{G}_{H_k}}$. One easily sees that C is in $\mathbf{C}_{H,\,ø}$. Moreover, this is not the only element of $\mathbf{C}_{H,\,ø}$, because $\mathbf{C}_{H,\,ø}$ always contains the operator matrix in $\mathbb{UM}^\infty_{\text{ball},0}(\mathbf{D}_H)$ that has zeros in all entries accept for the first upper diagonal on which $ø_{H_k}\Pi_{\mathcal{F}_{H_k}}$ is the entry in the $(k-1,k)$-th position. □

The arguments used to prove the claim in the final part of Theorem 4.1 can also be used to derive the following proposition.

Proposition 4.3. *There exists a unique pair of operator matrices* $(Z^{(1)}, Z^{(2)})$ *satisfying* (0.4), (0.5) *and* (0.6) *if and only if one of the following three conditions is satisfied:*

(1) $\mathcal{F}_k = \mathcal{U}_k$ *for each* $k \in \mathbb{Z}$*;*
(2) $\o_k$ *is a co-isometry for each* $k \in \mathbb{Z}$*;*
(3) *there exists a* $k \in \mathbb{Z}$ *such that* $\mathcal{F}_j = \mathcal{U}_j$ *for each* $j > k$ *and the operator* $\o_j$ *is a co-isometry for each* $j \leq k$.

In particular, if one of the conditions (1), (2) *or* (3) *is satisfied, then there exists a unique solution to Problem 0.1.*

Proof. Let $(Z^{(1)}, Z^{(2)})$ be a pair of operator matrices as in (0.4). Set $\mathbf{Z} = \oplus_{i\in\mathbb{Z}}(\mathcal{Y}_{i+1} \oplus \mathcal{U}_i)$, and define

$$\widetilde{Z} = \begin{bmatrix} \widetilde{Z}_{i,j} \end{bmatrix}_{i,j\in\mathbb{Z}} \in \mathbb{UM}_0^\infty(\mathbf{U}, \mathbf{Z}), \text{ where } \widetilde{Z}_{i,j} = \begin{bmatrix} Z^{(1)}_{i+1,j} \\ Z^{(2)}_{i,j} \end{bmatrix} : \mathcal{U}_j \to \begin{bmatrix} \mathcal{Y}_{i+1} \\ \mathcal{U}_i \end{bmatrix}.$$

Observe that (0.5) is equivalent to $\widetilde{Z} \in \mathbb{UM}^\infty_{\text{ball},0}(\mathbf{U}, \mathbf{Z})$, while (0.6) corresponds to

$$\widetilde{Z}_{j,k}|_{\mathcal{F}_k} = \begin{cases} \o_k, & \text{if } j = k-1, \\ 0, & \text{if } j \neq k-1. \end{cases} \tag{4.10}$$

Thus the pair $(Z^{(1)}, Z^{(2)})$ satisfies (0.5) and (0.6) if and only if $\widetilde{Z}$ is an element of

$$\mathbf{C}_{\o} = \{\widetilde{Z} \in \mathbb{UM}^\infty_{\text{ball},0}(\mathbf{U}, \mathbf{Z}) \mid \text{the } (i,j)\text{-th entry } \widetilde{Z}_{i,j} \text{ of } \widetilde{Z} \text{ satisfies (4.10) for each } i,j \in \mathbb{Z}\}.$$

The first statement now follows by translating the arguments in the proof of the last part of Theorem 4.1 to the present setting. The last statement of Proposition 4.3 follows immediately from the first part. □

The conditions listed in the above proposition are sufficient, but in general not necessary conditions for the existence of a unique solution. This is already the case in the time-invariant case; see [28, 27]. The problem to give necessary and sufficient conditions for the existence of a unique solution to Problem 0.1 remains open.

5. An example involving finite operator matrices and time-variant relaxed commutant lifting

In this section we present some examples of how our results can be applied. In each case the contractions $\o_k$, $k \in \mathbb{Z}$, are not given beforehand but are constructed from the given data. The first subsection deals with a 4×4 operator matrix problem. In the second subsection we introduce a time-variant analogue of the relaxed commutant lifting problem, and show how this time-variant problem can be solved by using Theorem 0.1.

5.1. **An example involving finite operator matrices.** When in Problem 0.1 all spaces $\mathcal{U}_k$ and $\mathcal{Y}_j$ are set to zero, with the exception of a finite numbers of k's and j's, finite operator matrix problems appear. We illustrate this with an example.

Consider the problem of finding all 4×4 operator matrices

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} \\ A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} \\ A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4} \end{bmatrix} : \begin{bmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \\ \mathcal{X}_3 \\ \mathcal{X}_4 \end{bmatrix} \to \begin{bmatrix} \mathcal{Y}_1 \\ \mathcal{Y}_2 \\ \mathcal{Y}_3 \\ \mathcal{Y}_4 \end{bmatrix} \tag{5.1}$$

such that

$$\| \begin{bmatrix} A_{1,k} & A_{1,k+1} \\ A_{2,k} & A_{2,k+1} \\ A_{3,k} & A_{3,k+1} \\ A_{4,k} & A_{4,k+1} \end{bmatrix} \| \leq 1, \quad k = 1, 2, 3. \tag{5.2}$$

The problem is always solvable (the zero matrix is a solution). Moreover all solutions can be obtained by repeatedly applying Parrott's lemma [29] (cf., Section IV.1 in [16] or Section XXVII.5 in [24]). Here we show that a more direct description of the set of all solutions can be obtained from Theorem 0.1. To do this we set $\mathcal{Y}_j = \{0\}$ for $j \neq 1, 2, 3, 4$ and put

$$\begin{aligned} &\mathcal{U}_k = \{0\}, \quad \mathcal{F}_k = \{0\} \quad (k \neq 4, 5, 6), \\ &\mathcal{U}_4 = \mathcal{X}_1 \oplus \mathcal{X}_2, \quad \mathcal{F}_4 = \{0\}, \\ &\mathcal{U}_5 = \mathcal{X}_2 \oplus \mathcal{X}_3, \quad \mathcal{F}_5 = \mathcal{X}_2 \oplus \{0\}, \quad \sigma_5(x_2 \oplus 0) = 0 \oplus x_2 \quad (x_2 \in \mathcal{X}_2), \\ &\mathcal{U}_6 = \mathcal{X}_3 \oplus \mathcal{X}_4, \quad \mathcal{F}_6 = \mathcal{X}_3 \oplus \{0\}, \quad \sigma_6(x_3 \oplus 0) = 0 \oplus x_3 \quad (x_2 \in \mathcal{X}_3). \end{aligned}$$

To formulate this as an upper triangular operator matrix problem, consider

$$\begin{aligned} H_{1,4} &= \begin{bmatrix} A_{1,1} & A_{1,2} \end{bmatrix}, \quad H_{1,5} = \begin{bmatrix} A_{1,2} & A_{1,3} \end{bmatrix}, \quad H_{1,6} = \begin{bmatrix} A_{1,3} & A_{1,4} \end{bmatrix} \\ H_{2,4} &= \begin{bmatrix} A_{2,1} & A_{1,2} \end{bmatrix}, \quad H_{2,5} = \begin{bmatrix} A_{2,2} & A_{2,3} \end{bmatrix}, \quad H_{2,6} = \begin{bmatrix} A_{2,3} & A_{2,4} \end{bmatrix} \\ H_{3,4} &= \begin{bmatrix} A_{3,1} & A_{3,2} \end{bmatrix}, \quad H_{3,5} = \begin{bmatrix} A_{3,2} & A_{3,3} \end{bmatrix}, \quad H_{3,6} = \begin{bmatrix} A_{3,3} & A_{3,4} \end{bmatrix} \\ H_{4,4} &= \begin{bmatrix} A_{1,1} & A_{1,2} \end{bmatrix}, \quad H_{4,5} = \begin{bmatrix} A_{1,2} & A_{1,3} \end{bmatrix}, \quad H_{4,6} = \begin{bmatrix} A_{1,3} & A_{1,4} \end{bmatrix} \end{aligned}$$

Then (5.1) and (5.2) are satisfied if and only if

$$\| \begin{bmatrix} H_{1,k} \\ H_{2,k} \\ H_{3,k} \\ H_{4,k} \end{bmatrix} \| \leq 1 \quad (k = 4, 5, 6),$$

$$\begin{bmatrix} H_{1,5} \\ H_{2,5} \\ H_{3,5} \\ H_{4,5} \end{bmatrix} |_{\mathcal{F}_5} = \begin{bmatrix} H_{1,4} \\ H_{2,4} \\ H_{3,4} \\ H_{4,4} \end{bmatrix} \sigma_5, \quad \text{and} \quad \begin{bmatrix} H_{1,6} \\ H_{2,6} \\ H_{3,6} \\ H_{4,6} \end{bmatrix} |_{\mathcal{F}_5} = \begin{bmatrix} H_{1,5} \\ H_{2,5} \\ H_{3,5} \\ H_{4,5} \end{bmatrix} \sigma_6.$$

Now consider our main problem with $\mathcal{Y}_j, \mathcal{U}_k = \{0\}$ and $\mathcal{F}_k$ as above. Furthermore, put

$$\emptyset_{5,1} = 0, \quad \emptyset_{6,1} = 0, \quad \text{and} \quad \emptyset_{5,2} = \sigma_5 \quad \emptyset_{6,2} = \sigma_6.$$

Since $\mathcal{F}_k = \{0\}$ for $k \neq 5, 6$, we don't have to consider the operators $\o_{k,1}$ and $\o_{k,1}$ for $k \neq 5, 6$. Note that

$$\begin{bmatrix} \o_{5,1} \\ \o_{6,1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \o_{5,2} \\ \o_{6,2} \end{bmatrix} \quad \text{are both contractions.}$$

It is now clear how the problem regarding the operator matrix A in (5.1) can be solved via Theorem 0.1.

5.2. **A time-variant relaxed commutant lifting problem.** In this subsection we present a time-varying version of the relaxed commutant lifting problem from [21] and explain the connection with Problem 0.1. We plan to come back to this time-variant relaxed commutant lifting problem in more detail in a future publication, where we will also discuss the relation with the time-invariant version, the three chain completion problem [18, 19] and its weighted versions [9].

A data set for the time-variant relaxed commutant lifting problem is a set $\Lambda = \{A_n, T'_n, U'_n, R_n, Q_n \mid n \in \mathbb{Z}\}$ consisting of Hilbert space operators with for each $k \in \mathbb{Z}$

$$T'_k : \mathcal{H}'_{k-1} \to \mathcal{H}'_k, \quad A_k : \mathcal{H}_k \to \mathcal{H}'_k, \quad R_k : \mathcal{H}_{0,k} \to \mathcal{H}_k, \quad Q_k : \mathcal{H}_{0,k-1} \to \mathcal{H}_k,$$

and such that T'_k and A_k are contractions and

$$T'_k A_{k-1} R_{k-1} = A_k Q_k, \quad R^*_{k-1} R_{k-1} \leq Q^*_k Q_k.$$

The operator U'_k is completely determined by the set of operators $\{T'_n \mid n \in \mathbb{Z}\}$ and can be seen as a time-varying analogue of the Sz.-Nagy-Schäffer isometric lifting of T'_k; cf., [10]. To define U'_k we set

$$\mathbf{D}'_n = \oplus_{i \leq n} \mathcal{D}_{T'_n} \quad \text{and} \quad \mathcal{K}'_n = \mathcal{H}_n \oplus \mathbf{D}'_n \quad (n \in \mathbb{Z}).$$

Then U'_k is the isometric operator mapping $\mathcal{K}'_{k-1}$ into $\mathcal{K}'_k$ given by

$$U'_k = \begin{bmatrix} T'_k & 0 \\ E_{\mathcal{D}_{T'_n}} D_{T'_n} & E_{\mathbf{D}'_{k-1}} \end{bmatrix} : \begin{bmatrix} \mathcal{H}'_{k-1} \\ \mathbf{D}'_{k-1} \end{bmatrix} \to \begin{bmatrix} \mathcal{H}'_k \\ \mathbf{D}'_k \end{bmatrix}. \tag{5.3}$$

Here $E_{\mathcal{D}_{T'_k}}$ and $E_{\mathbf{D}'_{k-1}}$ are the canonical embeddings of $\mathcal{D}_{T'_k}$ and $\mathbf{D}'_{k-1}$ into $\mathbf{D}'_k$, respectively. We then consider the following problem.

Problem 5.1. *Given the data set $\Lambda = \{A_n, T'_n, U'_n, R_n, Q_n \mid n \in \mathbb{Z}\}$, describe the sets of operators $\{B_n \mid n \in \mathbb{Z}\}$ with the property that for each $k \in \mathbb{Z}$ the operator B_k is a contraction from $\mathcal{H}_k$ into $\mathcal{K}'_k$ satisfying*

$$\Pi_{\mathcal{H}'_k} B_k = A_k \quad \textit{and} \quad U'_k B_{k-1} R_{k-1} = B_k Q_k. \tag{5.4}$$

Here $\Pi_{\mathcal{H}'_k}$ is the orthogonal projection from $\mathcal{K}'_k$ onto $\mathcal{H}'_k$.

After some translation and reduction steps it follows that the special case of Problem 5.1 with $\mathcal{H}_{0,k} = \mathcal{H}_k$ and $R_k = I_{\mathcal{H}_k}$ for each $k \in \mathbb{Z}$ is just the nonstationary commutant lifting problem considered in [10] (see also Section 3.5 in [11]).

With the data set Λ we associate a set of contractions $\{\o_n \mid n \in \mathbb{Z}\}$ of the form (0.2). For each $k \in \mathbb{Z}$, the contraction $\o_k$ is defined on the subspace $\mathcal{F}_k = \overline{D_{A_k} Q_k \mathcal{H}_{0,k-1}}$ of $\mathcal{D}_{A_k}$ and is given by

$$\o_k : \mathcal{F}_k \to \begin{bmatrix} \mathcal{D}_{T'_k} \\ \mathcal{D}_{A_{k-1}} \end{bmatrix}, \quad \o_k D_{A_k} Q_k = \begin{bmatrix} D_{T'_k} A_{k-1} R_{k-1} \\ D_{A_{k-1}} R_{k-1} \end{bmatrix}.$$

We refer to $\o_k$ as the *k-th underlying contraction of the data set* Λ. To see that $\o_k$ is in fact a contraction observe that for each $h \in \mathcal{H}_{0,k-1}$ we have

$$\begin{aligned} \|D_{A_k} Q_k h\|^2 &= \|Q_k h\|^2 - \|A_k Q_k h\|^2 \leq \|R_{k-1} h\|^2 - \|T_k' A_{k-1} R_{k-1} h\|^2 \\ &= \|R_{k-1} h\|^2 - \|A_{k-1} R_{k-1} h\|^2 + \|A_{k-1} R_{k-1} h\|^2 - \|T_k' A_{k-1} R_{k-1} h\|^2 \\ &= \|D_{A_{k-1}} R_{k-1} h\|^2 + \|D_{T_k'} A_{k-1} R_{k-1} h\|^2. \end{aligned}$$

It follows directly from this computation that $\o_k$ is contractive, and moreover, that $\o_k$ is an isometry if and only if $R_{k-1}^* R_{k-1} = Q_k^* Q_k$.

From a variation on Parrott's lemma [29] (cf., Section IV.1 in [16] or Section XXVII.5 in [24]) we obtain that an operator B_k from $\mathcal{H}_k$ into $\mathcal{K}_k'$ is a contraction that satisfies the first requirement of (5.4) if and only B_k is of the form

$$B_k = \begin{bmatrix} A_k \\ \Gamma_k D_{A_k} \end{bmatrix}, \quad \text{with } \Gamma_k : \mathcal{D}_{A_k} \to \mathbf{D}_k' \text{ a contraction.} \tag{5.5}$$

Moreover, the contraction Γ_k is uniquely determined by B_k. In this case, writing out $U_k' B_{k-1} R_{k-1}$ and $B_k Q_k$, it follows that the second requirement in (5.4) holds if and only if Γ_k and Γ_{k-1} satisfy

$$\Gamma_k|_{\mathcal{F}_k} = E_{\mathcal{D}_{T_k'}} \o_{k,1} + E_{\mathbf{D}_{k-1}'} \Gamma_{k-1} \o_{k,2}. \tag{5.6}$$

Here $\o_{k,1}$ is the first component of $\o_k$ mapping $\mathcal{F}_k$ into $\mathcal{D}_{T_k'}$ and $\o_{k,2}$ is the second component of $\o_k$ mapping $\mathcal{F}_k$ into $\mathcal{D}_{A_{k-1}}$. Thus, alternatively, we seek the sets of operators $\{\Gamma_n \mid n \in \mathbb{Z}\}$ such that for each $k \in \mathbb{Z}$ the operator Γ_k is a contraction from $\mathcal{D}_{A_k}$ into $\mathbf{D}_k'$ and (5.6) is satisfied.

Now set $\mathbf{D} = \oplus_{k\in\mathbb{Z}} \mathcal{D}_{A_k}$ and $\mathbf{D}' = \oplus_{k\in\mathbb{Z}} \mathcal{D}_{T_n'}$. With a set of contractions $\{B_n : \mathcal{H}_n' \to \mathcal{K}_n' \mid n \in \mathbb{Z}\}$ satisfying the first condition of (5.4) we associate an operator matrix $H \in \mathbb{UM}^2_{\mathrm{ball}}(\mathbf{D}, \mathbf{D}')$ in the following way.

Procedure 5.1. *Let $\{B_n : \mathcal{H}_n' \to \mathcal{K}_n' \mid n \in \mathbb{Z}\}$ be a set of contractions such that the first condition of* (5.4) *holds for each $k \in \mathbb{Z}$, and let Γ_k be the contraction from $\mathcal{D}_{A_k}$ into $\mathbf{D}_k'$ determined by* (5.5)*. For each $k \in \mathbb{Z}$ we view $\mathbf{D}_k'$ as a subspace of $\mathbf{D}'$ and write $\Pi_{\mathbf{D}_k'}$ for the orthogonal projection from $\mathbf{D}'$ onto $\mathbf{D}_k'$. Define H_k to be the contraction from $\mathcal{D}_{A_k}$ into $\mathbf{D}'$ given by $H_k = \Pi_{\mathbf{D}_k'}^* \Gamma_k$ and $H \in \mathbb{UM}^2_{\mathrm{ball}}(\mathbf{D}, \mathbf{D}')$ by*

$$H = \begin{bmatrix} \cdots & H_{-1} & H_0 & H_1 & \cdots \end{bmatrix}.$$

One can reverse this procedure in order to obtain a set of contractions $\{B_n \mid n \in \mathbb{Z}\}$ that satisfy (5.4) for each $k \in \mathbb{Z}$ from a operator matrix in $\mathbb{UM}^2_{\mathrm{ball}}(\mathbf{D}, \mathbf{D}')$.

Procedure 5.2. *Let $H \in \mathbb{UM}^2_{\mathrm{ball}}(\mathbf{D}, \mathbf{D}')$. For each $k \in \mathbb{Z}$ let H_k be the k-th column of H, i.e., $H_k = H|_{\mathcal{D}_{A_k}}$, set $\Gamma_k = \Pi_{\mathbf{D}_k'} H_k$ and define B_k by* (5.5)*.*

It is straightforward that Γ_k defined in Procedure 5.2 is a contraction, and thus that B_k is a contraction satisfying the first requirement of (5.4), and moreover, these procedures are each others inverse. Furthermore, the condition (5.6) on the contractions $\{\Gamma_n \mid n \in \mathbb{Z}\}$ translates to the columns in the operator matrix H obtained by Procedure 5.1 in the form

$$H_k|_{\mathcal{F}_k} = \tau_k' \o_{k,1} + H_{k-1} \o_{k,2}, \tag{5.7}$$

where τ_k' is the canonical embedding of $\mathcal{D}_{T_n'}$ into $\mathbf{D}'$, or, equivalently, in the form (0.3) with $\o_{k,1}$ and $\o_{k,2}$ as defined in the present subsection, $\mathcal{U}_k = \mathcal{D}_{A_k}$ and $\mathcal{Y}_k = \mathcal{D}_{T_k'}$ for each $k \in \mathbb{Z}$. The converse is also true. For an operator matrix $H \in$

$\mathbb{UM}^2_{\rm ball}(\mathbf{D},\mathbf{D}')$ such that the k-th column H_k of H satisfies (5.7), the contraction Γ_k obtained by Procedure 5.2 satisfies (5.6). In conclusion, we have the following result.

Theorem 5.1. *Let $\Lambda = \{A_n, T'_n, U'_n, R_n, Q_n \mid n \in \mathbb{Z}\}$ be a data set as described above. For any solution H of Problem 0.1 with $\o_k$ equal to the k-th underlying contraction of Λ for each $k \in \mathbb{Z}$, the set $\{B_n \mid n \in \mathbb{Z}\}$ obtained from Procedure 5.2 is a solution of Problem 5.1. Conversely, for any solution $\{B_n \mid n \in \mathbb{Z}\}$ of Problem 5.1, the operator matrix H obtained from Procedure 5.1 is a solution of Problem 0.1 in case $\o_k$ in (0.2) is the k-th underlying contraction of Λ for each $k \in \mathbb{Z}$.*

From Theorem 5.1 in combination with Theorems 0.1 and 0.2 it is clear how all solutions of Problem 5.1 can be described. Furthermore, in combination with the full version of the second main result, Theorem 4.1, a time-variant analogue of Theorem 1.2 in [22] (see also Theorem 1.2 in [26]) is obtained.

Not only can Problem 5.1 be seen as a special case of Problem 0.1, the converse is also true, as explained in the following theorem.

Theorem 5.2. *For each $k \in \mathbb{Z}$ let $\o_k$ be a contraction of the form (0.2) with $\mathcal{F}_k$ a subspace of $\mathcal{U}_k$. Set*

$$A_k = \begin{bmatrix} I_{\mathcal{Y}_{k+1}} & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{Y}_{k+1} \\ \mathcal{U}_k \end{bmatrix} \to \mathcal{Y}_{k+1}, \quad T'_k = 0 : \mathcal{Y}_k \to \mathcal{Y}_{k+1},$$

$$R_k = \begin{bmatrix} \o_{1,k+1} \\ \o_{2,k+1} \end{bmatrix} : \mathcal{F}_{k+1} \to \begin{bmatrix} \mathcal{Y}_{k+1} \\ \mathcal{Y}_k \end{bmatrix}, \quad Q_k = \begin{bmatrix} 0 \\ \Pi^*_{\mathcal{F}_k} \end{bmatrix} : \mathcal{F}_k \to \begin{bmatrix} \mathcal{Y}_{k+1} \\ \mathcal{Y}_k \end{bmatrix},$$

and define U'_k by (5.3) for each $k \in \mathbb{Z}$. Here $\Pi_{\mathcal{F}_k}$ denotes the orthogonal projection from $\mathcal{U}_k$ onto $\mathcal{F}_k$. Then $\Lambda = \{A_n, T'_n, U'_n, R_n, Q_n \mid n \in \mathbb{Z}\}$ is a data set for a time-variant relaxed commutant lifting problem, and $\o_k$ is the k-th underlying contraction of Λ for each $k \in \mathbb{Z}$.

Proof. Fix $k \in \mathbb{Z}$. Clearly, T'_k and A_k are contraction. Moreover, $T'_k A_{k-1} R_{k-1}$ and $A_n Q_n$ are both equal to the zero operator from $\mathcal{F}_k$ into $\mathcal{Y}_{k+1}$, and, since $R_{k-1} = \o_k$ is a contraction, we have $R^*_{k-1} R_{k-1} \le I_{\mathcal{F}_k} = Q^*_k Q_k$. It then follows that Λ is a data set for a time-variant relaxed commutant lifting problem. Next, observe that

$$D_{A_k} Q_k = \Pi_{\mathcal{F}_k}, \quad D_{T'_k} A_{k-1} R_{k-1} = \o_{1,k} \quad \text{and} \quad D_{A_{k-1}} R_{k-1} = \o_{2,k}.$$

This implies that the k-th underlying contraction of Λ is equal to $\o_k$. □

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Department of Aeronautics and Astronautics, Purdue University, West Lafayette, IN 47907, USA
E-mail address: frazho@ecn.purdue.edu

Department of Mathematics, Virginia Tech, Blacksburg VA 24061, USA
E-mail address: terhorst@vt.edu

Afdeling Wiskunde, Faculteit der Exacte Wetenschappen, Vrije Universiteit, De Boelelaan 1081a, 1081 HV Amsterdam, The Netherlands
E-mail address: ma.kaashoek@few.vu.nl