Algebraic Cycles, Branes and De Rham-Witt

Jan Stienstra

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Witt vectors, Foliations and absolute De Rham cohomology

Algebraic Cycles: web of intersecting subvarieties of a variety.

Branes: boundary conditions for open strings in a manifold.

X smooth, projective algebraic variety over alg. closed field k.

$Z^d(X)$:= free abelian group on the set of irreducible subvarieties of codimension d in X.

$B^d(X)$:= subgroup of $Z^d(X)$ generated by cycles $\mathrm{div}(f)$, the divisor of zeros and poles of a rational function f on a codimension $d-1$ subvariety of X.

codimension d **algebraic cycle** on X is an element of $Z^d(X)$.

rational equivalence:

$$\alpha, \beta \in Z^d(X): \qquad \alpha \sim \beta \quad \Leftrightarrow \quad \alpha - \beta \in B^d(X)$$

d**-th Chow group** of X

$$\mathrm{CH}^d(X) := Z^d(X) \,/\, B^d(X)$$

Chow ring of X

$$\mathrm{CH}^\bullet(X) = \bigoplus_{d \geq 0} \mathrm{CH}^d(X)$$

Product on $\mathrm{CH}^\bullet(X)$ given by intersecting algebraic cycles.

Do not use set theoretic unions and intersections!
Multiplicities are subtle!

$K_0(X)$:= **Grothendieck group** of the category of **locally free $\mathcal{O}_X$-modules of finite type**.

$K_0(X)$ is a **λ-ring**, with λ^i-operation coming from the i^{th} exterior power operation on locally free sheaves.

$Gr_\bullet K_0(X)$:= graded ring associated with the **γ-filtration** on $K_0(X)$.

$K_0(\mathrm{coh}(X))$:= **Grothendieck group** of the category of **coherent sheaves** on X.

$Gr_\bullet K_0(\mathrm{coh}(X))$:= graded group associated with filtration on $K_0(\mathrm{coh}(X))$ **by codimension of support** of coherent sheaf on X.

For smooth X the natural map is an isomorphism:

$$K_0(X) \xrightarrow{\simeq} K_0(\mathrm{coh}(X))$$

Grothendieck-Riemann-Roch:
For X smooth projective over a field there are natural homomorphisms of graded rings, which are isomorphisms modulo torsion

$$Gr_\bullet K_0(X) \quad \xrightarrow{\simeq} \quad Gr_\bullet K_0(\mathrm{coh}(X)) \quad \xleftarrow{\simeq} \quad \mathrm{CH}^\bullet(X)$$

Grothendieck's Chern character gives an isomorphism for every d

$$Gr_d K_0(X) \otimes \mathbb{Q} \quad \xleftarrow{\simeq} \quad \{\xi \in K_0(X) \otimes \mathbb{Q} \mid \psi_n(\xi) \;=\; n^d \, \xi \,,\; \forall n\}$$

here ψ_n is the n-th **Adams operation**.

Remark: Quillen has defined higher algebraic K-groups $K_i(X)$ $(i \geq 0)$ of the category of locally free sheaves of finite type on X. These also carry an action by Adams operations. The corresponding eigenspaces are the **motivic cohomology groups** of X. Bloch has defined the **higher Chow groups** of X.
Upon tensoring with $\mathbb{Q}$ the higher Chow groups coincide with the motivic cohomology groups.

Quillen's **higher algebraic K-theory** of exact categories.

For commutative ring R with unit $K_d(R)$ is the d^{th} algebraic K-group of the category of finitely generated projective R-modules.

For scheme X let $\mathcal{K}_{d,X}$ denote the Zariski sheaf associated with the pre-sheaf

$$\text{affine open } U = \operatorname{Spec}(R) \quad \mapsto \quad K_d(R)$$

Theorem:	$\mathrm{CH}^d(X) \quad \simeq \quad H^d(X, \mathcal{K}_{d,X})$

For $d = 1$ this is in fact the classical formula

$$\operatorname{Pic}(X) \quad \simeq \quad H^1(X, \mathcal{O}_X^*)$$

For $d = 2$ this result is due to Bloch.
For general d it was conjectured by Gersten and proven by Quillen.

Question (Spencer Bloch, 1970's):

Determine, for smooth projective X, the structure of the functor

$$\{\textit{augmented Artinian local } k\textit{-algebras}\} \longrightarrow \{\textit{Abelian groups}\}$$
$$A \quad \mapsto \quad H^m(X, \mathcal{K}_{n, X\times A/X})$$

Here $\mathcal{K}_{n,X\times A/X}$ is the Zariski sheaf associated with the pre-sheaf

$$\text{affine open } U = \operatorname{Spec}(R) \quad \mapsto \quad \ker(K_n(R \otimes_k A) \to K_n(R))$$

Classical theory tells that for $m = n = 1$ this functor is pro-representable by the formal group of the Picard variety of X.

Artin and Mazur have considered the case $n = 1$ and arbitrary m. They gave conditions for when this functor is pro-representable by a formal group.

e.g. $n = 1$, $m = 2$ gives the *formal Brauer group* of X.

In 1976 Bloch proposed the case $m = n = 2$ to me as a research problem for my PhD thesis.
One PhD thesis and three post-doc jobs later I formulated the answer as follows (published in Crelle 355 (1985)):

Theorem. Let X be a smooth projective variety over a perfect field k of characteristic $p > 0$.
Then there is a functorial homomorphism for every m and n:

$$\left(H^m(X, W\Omega^\bullet_X) \otimes_{\mathcal{R}} \widehat{W}\Omega_A^{n-1-\bullet}\right) \quad \longrightarrow \quad H^m(X, \mathcal{K}^s_{n,X\times A/X})$$

of which the kernel and the cokernel are both essentially 0.

Main ingredients in the proof of the theorem are

- detailed information about the De Rham-Witt complex given in the works of Bloch, Deligne, Illusie and Raynaud
- an alternative construction of the De Rham-Witt complex based on the K-theory of categories of finitely generated projective modules equiped with an endomorphism.

Theorem. Let X be a smooth projective variety over a perfect field k of characteristic $p > 0$.
Then there is a functorial homomorphism for every m and n:

$$\Big(H^m(X, W\Omega_X^\bullet) \otimes_{\mathcal{R}} \widehat{W}\Omega_A^{n-1-\bullet}\Big) \quad \longrightarrow \quad H^m(X, \mathcal{K}^s_{n,X\times A/X})$$

of which the kernel and the cokernel are both essentially 0.

- $\mathcal{K}^s_{n,X\times A/X}$ is the **symbol part** of $\mathcal{K}_{n,X\times A/X}$, i.e. the image under multiplication in K-theory of

  $$(1 + \mathcal{O}_X \otimes_k \mathsf{m}_A) \otimes_{\mathbb{Z}} \mathcal{O}^*_{X\times A} \otimes_{\mathbb{Z}} \ldots \otimes_{\mathbb{Z}} \mathcal{O}^*_{X\times A} \qquad ;$$

  m_A is the maximal ideal of A.
- $\mathcal{R} = W(k)[F, V, d]$ is the **Raynaud ring**;
  $W(k)$ the p-typical **Witt vectors** of k;
  F is **Frobenius**, V **Verschiebung**,
  d the derivation in the **De Rham-Witt complex** of X:

  $$W\mathcal{O}_X \xrightarrow{d} W\Omega^1_X \xrightarrow{d} \ldots \xrightarrow{d} W\Omega^i_X \xrightarrow{d} \ldots$$
- $H^m(X, W\Omega_X^\bullet) \ :=$

$$H^m(X, W\mathcal{O}_X) \xrightarrow{d} H^m(X, W\Omega^1_X) \xrightarrow{d} \ldots \xrightarrow{d} H^m(X, W\Omega^i_X) \xrightarrow{d} \ldots$$

  viewed as an $\mathcal{R}$-module.
- $\widehat{W}\Omega_A^\bullet$ is the **formal De Rham-Witt complex** of A.
- We call a functor $G : \{augm.Art.loc.k\text{-}algs\} \rightarrow \{Ab.grps\}$ **essentially zero** if for every A there is a surjection $A' \twoheadrightarrow A$ inducing the zero map $G(A') \rightarrow G(A)$.

The alternative construction of the De Rham-Witt complex **works for any commutative unital ring R, without assumptions about the characteristic or about regularity.**

$\mathrm{End}(R)$ denotes the exact category whose

- objects: pairs (M, α) consisting of a finitely generated projective R-module M and an R-linear endomorphism α of M
- morphisms: $(M, \alpha) \to (M', \alpha')$ is an R-linear map $f : M \to M'$ s.t. $f \circ \alpha = \alpha' \circ f$.
- short exact sequences: underlying sequence of R-modules is exact

$K_i(\mathrm{End}(R))$, for $i = 0, 1, 2, \ldots$, denote Quillen's K-groups of the exact category $\mathrm{End}(R)$.

$$K_*(\mathrm{End}(R)) \;=\; \bigoplus_{i\geq 0} K_i(\mathrm{End}(R))$$

is a **graded commutative ring** with product induced by tensor product

$$(M,\alpha)\otimes(M',\alpha') \;=\; (M\otimes M', \alpha\otimes\alpha')$$

There is a **Frobenius** operator F_n for every $n \geq 1$ induced by

$$(M,\alpha) \quad\mapsto\quad (M,\alpha^n)$$

There is a **Verschiebung** operator V_n for every $n \geq 1$ induced by

$$(M,\alpha) \quad\mapsto\quad (M^{\oplus n}, \begin{pmatrix} 0 & 0 & \dots & 0 & \alpha \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & 0 & \ddots & 0 & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix})$$

There is also a map

$$d : K_i(\mathrm{End}(R)) \;\to\; K_{i+1}(\mathrm{End}(R))\,,$$

constructed as follows.

Consider the polynomial ring in one variable $\mathbb{Z}[u]$ and the bi-exact functor

$$\begin{array}{rcl} \mathrm{End}(R) \times \mathrm{End}(\mathbb{Z}[u]) & \longrightarrow & \mathrm{End}(R) \\ (M,\alpha)\,, \quad (N,\beta) & \mapsto & (M \otimes_{\mathbb{Z}[u]} N\,,\ 1 \otimes \beta) \end{array}$$

here M is considered as a $\mathbb{Z}[u]$-module: $um = \alpha m$, $\forall m \in M$.

This induces maps

$$K_i(\mathrm{End}(R)) \otimes_{\mathbb{Z}} K_j(\mathrm{End}(\mathbb{Z}[u])) \longrightarrow K_{i+j}(\mathrm{End}(R))\,.$$

The map

$$d : K_i(\mathrm{End}(R)) \to K_{i+1}(\mathrm{End}(R))\,,$$

comes from a particular element in $K_1(\mathrm{End}(\mathbb{Z}[u]))$.

Remark. The above construction of the map d is a special case of a general construction of functorial operations on $K_*(\mathrm{End}(R))$.

Relations:

$$V_1 = F_1 = \mathsf{1}$$

$$F_n F_m = F_{nm}\ , \qquad V_n V_m = V_{nm} \qquad (\forall n, m)$$

$$F_n V_n = n\mathsf{1} \qquad (\forall n)$$

$$V_p F_p = p\mathsf{1} \qquad \text{if } p \text{ prime and } pR = 0$$

$$F_n V_m \;=\; V_m F_n \qquad \text{if} \quad (n,m) = 1$$

$$V_n\, d \;=\; n\, d\, V_n\ , \qquad d\, F_n \;=\; n\, F_n\, d \qquad (\forall n)$$

$$F_n\, d\, V_n \;=\; d \quad \text{if } n \text{ odd}\,, \qquad F_2\, d\, V_2 \;=\; d \quad \text{if } 2R = 0$$

$$2\, d^2 \;=\; 0; \qquad d^2 \;=\; 0 \quad \text{if } 2R = 0$$

$$F_n(\mathsf{a}\, \mathsf{b}) \;=\; (F_n\, \mathsf{a})\, (F_n\, \mathsf{b}) \qquad (\forall n)$$

$$V_n(\mathsf{a}\, F_n \mathsf{b}) \;=\; (V_n \mathsf{a})\, \mathsf{b} \qquad (\forall n)$$

$$d(\mathsf{a}\, \mathsf{b}) \;=\; (d\mathsf{a})\, \mathsf{b} \,+\, (-1)^i \mathsf{a}\, (d\mathsf{b})$$

for all $\mathsf{a} \in K_i(\mathrm{End}(R))$ and $\mathsf{b} \in K_j(\mathrm{End}(R))$

Exact functors

$$\begin{aligned}
\mathrm{End}(R) &\longrightarrow P(R)\,, & (M,\alpha) &\mapsto M \\
P(R) &\longrightarrow \mathrm{End}(R)\,, & M &\mapsto (M,0)
\end{aligned}$$

where $P(R)$ = category of fin. gen. projective R-modules.

These induce direct sum decomposition:

$$K_i(\mathrm{End}(R)) \;=\; K_i(R) \oplus \widetilde{K}_i(\mathrm{End}(R))$$

with

$$\widetilde{K}_i(\mathrm{End}(R)) \;:=\; \ker(K_i(\mathrm{End}(R)) \to K_i(R))$$

Theorem(Almkvist, Grayson)
Let $\mathbb{W}(R)$ denote the ring of **big Witt vectors** of R.
Then the map

$$\begin{aligned}
\widetilde{K}_0(\mathrm{End}(R)) &\longrightarrow (1+tR[[t]])^{\times} \;=\; \mathbb{W}(R) \\
[M,\alpha]-[M,0] &\mapsto \det(1-t\alpha)^{-1}
\end{aligned}$$

is an injective homomorphism of rings, which commutes with the Frobenius and Verschiebung operators.

Moreover the **Teichmüller lifting** is given by

$$\begin{array}{ccccc}
R & \to & \widetilde{K}_0(\mathrm{End}(R)) & \to & \mathbb{W}(R) \\
x & \mapsto & [R,x]-[R,0] & \mapsto & (1-xt)^{-1}
\end{array}$$

The inverse map from the image of $\widetilde{K}_0(\mathrm{End}(R))$ in $\mathbb{W}(R)$ back to $\widetilde{K}_0(\mathrm{End}(R))$ is given by

$$\left(1-\sum_{j=1}^{n} r_j t^j\right)^{-1} \mapsto \left[R^{\oplus n}, \begin{pmatrix} 0 & 0 & \dots & 0 & r_n \\ 1 & 0 & \dots & 0 & r_{n-1} \\ 0 & 1 & \dots & 0 & r_{n-2} \\ \vdots & 0 & \ddots & 0 & \vdots \\ 0 & 0 & \dots & 1 & r_1 \end{pmatrix}\right] - [R^{\oplus n}, 0]$$

We will describe a decreasing filtration by homogeneous ideals

$$\mathrm{Fil}_n \widetilde{K}_*(\mathrm{End}(R)) , \qquad n = 1, 2, 3, \dots$$

such that the Frobenius maps F_m, Verschiebung maps V_m and the derivation d extend to the completion

$$\widetilde{K}_*(\mathrm{End}(R))^\wedge \ := \ \lim_{\leftarrow n} \widetilde{K}_*(\mathrm{End}(R))/\mathrm{Fil}_n \widetilde{K}_*(\mathrm{End}(R))$$

and such that

$$\widetilde{K}_0(\mathrm{End}(R))^\wedge \ = \ \mathbb{W}(R) \, .$$

For a commutative unital ring A let $\mathrm{Nil}(A)$ denote the full exact subcategory of $\mathrm{End}(A)$ with objects those (M, α) for which α is **nilpotent**.

The **fundamental theorem of K-theory** gives an isomorphism for every $i \geq 0$:

$$\mathrm{NK}_{i+1}(A) \simeq \widetilde{K}_i(\mathrm{Nil}(A))$$

where

$$\begin{aligned} \mathrm{NK}_{i+1}(A) &:= \mathrm{coker}(K_{i+1}(A) \to K_{i+1}(A[u])) \\ \widetilde{K}_i(\mathrm{Nil}(A)) &:= \ker(K_i(\mathrm{Nil}(A)) \to K_i(A)) \end{aligned}$$

for the maps induced by the inclusion $A \hookrightarrow A[u]$ and the forgetful functor $\mathrm{Nil}(A) \to P(A)$, respectively.

Let $s_0\,,\ s_1\ :\ K_{i+1}(A[u]) \to K_{i+1}(A)$ denote the homorphisms induced by the substitutions $u \mapsto 0$ and $u \mapsto 1$, respectively. Then $s_1 - s_0$ gives a well defined homorphism

$$s_1 - s_0\ :\ \mathrm{NK}_{i+1}(A) \longrightarrow K_{i+1}(A)\,.$$

By composing the latter homomorphism with the isomorphism from the fundamental theorem we obtain, for every $i \geq 0$, a homomorphism

$$\int_0^1\ :\ \widetilde{K}_i(\mathrm{Nil}(A)) \longrightarrow K_{i+1}(A)\,.$$

The bi-exact functor

$$\begin{array}{ccccc} \mathrm{End}(R) & \times & \mathrm{Nil}(A) & \longrightarrow & \mathrm{Nil}(R\otimes_{\mathbb{Z}} A) \\ (M,\alpha)\ , & & (N,\beta) & \mapsto & (M\otimes_{\mathbb{Z}} N\ ,\ \alpha\otimes\beta) \end{array}$$

induces

$$\begin{array}{ccccc} \widetilde{K}_i(\mathrm{End}(R)) & \otimes_{\mathbb{Z}} & \widetilde{K}_j(\mathrm{Nil}(A)) & \longrightarrow & \widetilde{K}_{i+j}(\mathrm{Nil}(R\otimes_{\mathbb{Z}} A)) \\ \mathsf{a}\ , & & \mathsf{b} & \mapsto & \mathsf{ab} \end{array}$$

and by composition with $\int_0^1$:

$$\begin{array}{ccccc} \widetilde{K}_i(\mathrm{End}(R)) & \otimes_{\mathbb{Z}} & \widetilde{K}_j(\mathrm{Nil}(A)) & \longrightarrow & K_{i+j+1}(R\otimes_{\mathbb{Z}} A) \\ \mathsf{a}\ , & & \mathsf{b} & \mapsto & \displaystyle\int_0^1 \mathsf{ab} \end{array}$$

Remark: The above pairing $(\mathsf{a},\mathsf{b}) \mapsto \int_0^1 \mathsf{ab}$ underlies the map

$$\Big(H^m(X, W\Omega^{\bullet}_X)\otimes_{\mathcal{R}} \widehat{W}\Omega_A^{n-1-\bullet}\Big) \quad\longrightarrow\quad H^m(X, \mathcal{K}^s_{n,X\times A/X})$$

in my theorem.

$\mathrm{Nil}(A)$ carries an increasing filtration by full exact subcategories:

$$\mathrm{Fil}^n \mathrm{Nil}(A) \ := \ \{ \ (M, \alpha) \mid \alpha^n = 0 \ \} \, .$$

This yields an increasing filtration on the K-groups:

$$\mathrm{Fil}^n \widetilde{K}_j(\mathrm{Nil}(A)) \ := \ \mathrm{image}(\widetilde{K}_j(\mathrm{Fil}^n \mathrm{Nil}(A)) \to \widetilde{K}_j(\mathrm{Nil}(A)))$$

Now define

$$\mathrm{Fil}_n \widetilde{K}_i(\mathrm{End}(R)) \ :=$$

$$\left\{ \mathsf{a} \in \widetilde{K}_i(\mathrm{End}(R)) \ \middle| \ \begin{array}{l} \forall A\,,\ \forall j\,,\ \forall \mathsf{b} \in \widetilde{K}_j(\mathrm{Fil}^n \mathrm{Nil}(A)) \\ \int_0^1 \mathsf{ab} \ = \ 0 \text{ in } K_{i+j+1}(R \otimes_{\mathbb{Z}} A) \end{array} \right\}$$

$$\widetilde{K}_i(\mathrm{End}(R))^{\wedge} \ := \ \lim_{\leftarrow n} \widetilde{K}_i(\mathrm{End}(R)) / \mathrm{Fil}_n \widetilde{K}_i(\mathrm{End}(R))$$

We define the **generalized De Rham-Witt complex** of R as the completion of the smallest subring of $\widetilde{K}_*(\mathrm{End}(R))^\wedge$ which contains $\mathbb{W}(R)$ and is closed under the operations d, F_n, V_n.

The **generalized formal De Rham-Witt complex** of a commutative unital ring A is similarly defined from $\widetilde{K}_*(\mathrm{Nil}(A))$ and the multiplicative group

$$\widehat{\mathbb{W}}(A) \;:=\; (1 + u\,A^{\mathrm{nil}}[u])^\times \;\subset\; \mathrm{NK}_1(A) \;=\; \widetilde{K}_0(\mathrm{Nil}(A))$$

of polynomials with constant term 1 and all other coefficients nilpotent in A. No need for completion.

If R and A are rings of prime characteristic p one can split off the De Rham-Witt complex $W\Omega^\bullet_R$ of R and the formal De Rham-Witt complex $\widehat{W}\Omega^\bullet_A$ of A from the above generalized forms by means of the idempotent operator (μ is the Möbius function):

$$\sum_{n,\, p\nmid n} \frac{\mu(n)}{n} V_n F_n \;=\; \prod_{\ell \text{ prime}\neq p} \left(1 - \frac{1}{\ell} V_\ell F_\ell\right)$$

In physics **branes** are boundary conditions for open strings in a target manifold $\mathcal{M}$.

One speaks of **D-branes** (D for Dirichlet) if the boundary conditions specify the position of the endpoints of the string.

One speaks of **D-branes of B-type** if the boundary conditions require the string to end on a *holomorphic submanifold* in the target manifold $\mathcal{M}$.

Chien-Hao Liu and Shing-Tung Yau are working on a project to find structures in algebraic geometry to model D-branes of B-type.

see e.g. their paper
D-branes and Azumaya noncommutative geometry: From Polchinski to Grothendieck

Two key ideas:

- Polchinski: stacks of branes ($\approx$ branes with multiplicities) should be described by matrix valued functions on $\mathcal{M}$.
- Grothendieck: instead of substructures of $\mathcal{M}$ use structures that map into $\mathcal{M}$.

Liu and Yau implement these ideas in what they call an **Azumaya scheme with fundamental module**.

The visible part of such a structure consists of a scheme X and a locally free $\mathcal{O}_X$-module of finite type $\mathcal{E}$.

The (as yet) invisible part of such a structure can only be probed through the set of morphisms from it to other schemes Y (acting as target spaces like $\mathcal{M}$).

Liu and Yau define these sets of morphisms as

$$\mathrm{Mor}((X,\mathcal{E}),Y) \;:=\; \left\{ \begin{array}{l} \text{coherent } \mathcal{O}_{X\times Y}\text{-modules } \widetilde{\mathcal{E}} \text{ on} \\ X \times Y \text{ with } p_{1*}\widetilde{\mathcal{E}} \,=\, \mathcal{E} \end{array} \right\}$$

They remark that it may suffice to "probe" the structure of the Azumaya scheme with fundamental module by taking Y from a fixed (well chosen) collection of "basic spaces".

They also say (parenthetically)
"Furthermore, from .... , one expects that one finally has to consider everything in the derived(-category) sense."

Comments and Questions.

- The assignment

  $$Y \mapsto \mathrm{Mor}((X, \mathcal{E}), Y)$$

  is a *covariant* functor from the category of schemes to the category of sets.

  One should compare this with standard constructions in moduli problems where one uses a *contravariant* functor:

  $$F : \textit{schemes} \to \textit{sets}$$

  and shows that there is a scheme Z such that

  $$F(Y) \;=\; \mathrm{Mor}(Y, Z) \qquad \text{for all schemes } Y\,.$$

- What does it mean:
  has to consider everything in the derived(-category) sense?

  Are X and Y fixed and can only $\widetilde{\mathcal{E}}$ vary?

  For fixed X and Y the $\widetilde{\mathcal{E}}$'s form an exact category:

  $$\mathrm{Brane}(X;Y) \;:=\; \left\{ \begin{array}{l} \text{coherent } \mathcal{O}_{X\times Y}\text{-modules } \widetilde{\mathcal{E}} \text{ on } X \times Y \text{ s.t.} \\ p_{1*}\widetilde{\mathcal{E}} \text{ is locally free of finite type on } X \end{array} \right\}$$

  Should one then take the corresponding derived category?

  Or can one take its K-theory $K_*(\mathrm{Brane}(X;Y))$ instead?

  Should one then vary Y and consider for fixed X the covariant functor

  $$Y \;\mapsto\; K_*(\mathrm{Brane}(X;Y))$$

  on the category of schemes?

Liu-Yau in affine context:

$$\begin{aligned} X &= \mathrm{Spec}(R) \\ Y &= \mathrm{Spec}(A) \\ \mathcal{E} &= \text{finitely generated projective } R\text{-module } M \end{aligned}$$

Then

$$\mathrm{Mor}((X,\mathcal{E}),Y) \;=\; \mathrm{Hom}_{\mathrm{rings}}(A,\mathrm{End}_R(M))\,.$$

For fixed (R, M) the question becomes:

Study the *contravariant* functor:

$$\begin{aligned} \{\textit{commutative unital rings}\} &\longrightarrow \{\textit{sets}\} \\ A &\mapsto \mathrm{Hom}_{\mathrm{rings}}(A,\mathrm{End}_R(M)) \end{aligned}$$

For fixed R and A we have the exact category

$$\mathrm{End}(R;A) \;:=\; \left\{ \text{pairs } (M,\varphi) \;\middle|\; \begin{array}{l} M \text{ fin. gen. proj. } R\text{-mod.} \\ \varphi : A \to \mathrm{End}_R(M) \text{ ring hom.} \end{array} \right\}$$

Example:

$$\mathrm{End}(R;\mathbb{Z}[u]) \;=\; \mathrm{End}(R)$$

So there is a link with De Rham-Witt when Azumaya schemes are "probed" by morphisms to the affine line.

A ring homomorphism $f : A' \to A$ induces an exact functor

$$\mathrm{End}(R;A) \;\longrightarrow\; \mathrm{End}(R;A')\;, \qquad (M,\varphi) \;\mapsto\; (M,\varphi\circ f)$$

This makes the construction *contravariantly* functorial in A.

Recall the question Spencer Bloch asked in 1970's:

Determine, for smooth projective X, the structure of the functor

$$\{\textit{augmented Artinian local } k\textit{-algebras}\} \longrightarrow \{\textit{Abelian groups}\}$$
$$A \quad \mapsto \quad H^m(X, \mathcal{K}_{n, X\times A/X})$$

and notice behind the façade of K-theory, sheaves and cohomology on the affine level the exact category

$$\{\text{fin. gen. proj. } R \otimes_k A\text{-modules}\}$$

Being an augmented Artinian local algebra over the field k the ring A is a finite dimensional k-vector space.
Therefore every fin. gen. proj. $R \otimes_k A$-module M is also

a fin. gen. proj. R-module with A-module structure given by a ring homomorphism $\varphi : A \to \mathrm{End}_R(M)$

A ring homomorphism $f : A' \to A$ induces an exact functor

$$\{\text{f.g.pr. } R \otimes_k A'\text{-mod.}\} \xrightarrow{-\otimes_{R\otimes_k A'}(R\otimes_k A)} \{\text{f.g.pr. } R \otimes_k A\text{-mod.}\}$$

This makes the construction *covariantly* functorial in A.

As Liu and Yau remarked, it may suffice to "probe" the structure of the Azumaya scheme with fundamental module by taking Y from a fixed (well chosen) collection of "basic spaces".

Do toric varieties constitute a good collection of "basic spaces" for that purpose?

Since toric varieties are locally modeled on commutative semigroups this would mean in the affine context :

For a commutative unital ring R and and a commutative semigroup S look at the exact category of representations of S in finitely generated projective R-modules:

$$\mathrm{Rep}(R;S) \;:=\; \left\{ \text{pairs } (M,\varphi) \;\middle|\; \begin{array}{l} M \text{ fin. gen. proj. } R\text{-mod.} \\ \varphi : S \to \mathrm{End}_R(M) \text{ s.-grp hom.} \end{array} \right\}$$

Restricting to representations by nilpotent endomorphisms should probably be the same as working with Artinian local algebras.

Question:
Working with Artinian local algebras is a kind of analysis.
Are there other sensible ways to bring in (non-Archimedian) analysis?

c.f. Payne's work on analytification and tropicalization using homomorphisms from commutative semi-groups into the semiring $(\mathbb{R}_{\geq 0}\,,\ \max\,,\ \cdot)$.