

INVOLUTIVE DISTRIBUTIONS OF OPERATOR-VALUED EVOLUTIONARY VECTOR FIELDS AND THEIR AFFINE GEOMETRY

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ABSTRACT. We generalize the notion of a Lie algebroid over infinite jet bundle by replacing the variational anchor with an N -tuple of differential operators whose images in the Lie algebra of evolutionary vector fields of the jet space are subject to collective commutation closure. The linear space of such operators becomes an algebra with bi-differential structural constants, of which we study the canonical structure. In particular, we show that these constants incorporate bi-differential analogues of Christoffel symbols.

Introduction. Lie algebroids [21] are an important and convenient construction that appears, e.g., in classical Poisson dynamics [2] or the theory of quantum Poisson manifolds [1, 22]. Essentially, Lie algebroids extend the tangent bundle TM over a smooth manifold M , retaining the information about the $C^\infty(M)$ -module structure for its sections. In the paper [10] we defined the Lie algebroids over the infinite jet spaces for mappings between smooth manifolds (e.g., from strings to space-time); the classical definition [21] is recovered by shrinking the source manifold to a point. A special case of Lie algebroids over spaces of finite jets for sections of the tangent bundle was first considered in [15]. Within the variational setup, the anchors become linear matrix differential operators that map sections which belong to horizontal modules [13] to the generating sections φ of evolutionary derivations ∂_φ on the jet space; by assumption, the images of such anchors are closed under commutation in the Lie algebra of evolutionary vector fields. The two main examples of variational anchors are the recursions with involutive images ([8]) and the Hamiltonian operators (see [19, 12, 13] and [8]) whose domains consist of the variational vectors and covectors, respectively.

In [8] we studied the *linear compatibility* of variational anchors, meaning that N operators with the common domain span the N -dimensional linear space \mathcal{A} such that each point $A_\lambda \in \mathcal{A}$ is itself an anchor with involutive image. For example, Poisson compatible Hamiltonian operators are linear compatible and *vice versa* (Hamiltonian operators are *Poisson compatible* if their linear combinations remain Hamiltonian). The

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linear compatibility¹ allows us to reduce the case of many operators A_1, \dots, A_N to one operator $A_\lambda = \sum \lambda_i \cdot A_i$ with the same properties.

In this paper we introduce a different notion of compatibility for the N operators. Strictly speaking, we consider the class of structures which is wider than the set of Lie algebroids over the jet spaces. Namely, we relax the assumption that each operator alone is a variational anchor, but, instead, we deal with N -tuples of total differential operators A_1, \dots, A_N whose images are subject to the collective commutation closure:

$$\left[\sum_{i=1}^N \text{im } A_i, \sum_{j=1}^N \text{im } A_j \right] \subseteq \sum_{k=1}^N \text{im } A_k.$$

This involutivity condition converts the linear space of operators to an algebra with bi-differential structural constants \mathbf{c}_{ij}^k , see (6) below. The Magri scheme [16] for the restrictions of compatible Hamiltonian operators to the hierarchy of Hamiltonians yields an example of such overlapping for $N = 2$ with $\mathbf{c}_{ij}^k \equiv 0$.

We study the standard decomposition of the structural constants \mathbf{c}_{ij}^k , which is similar to the previously known case (1) for $N = 1$ ([7, 8, 10]). From the bi-differential constants \mathbf{c}_{ij}^k we extract the components Γ_{ij}^k that act by total differential operators on both arguments at once. Our main result, Theorem 3, states that, under a change of coordinates in the domain, the symbols Γ_{ij}^k are transformed by a proper analogue (11) of the classical rule $\Gamma \mapsto g \Gamma g^{-1} + dg g^{-1}$ for the connection 1-forms Γ and reparametrizations g . We note that the bi-differential symbols Γ_{ij}^k are symmetric in lower indices if the common domain of the N operators A_i consists of the variational covectors and hence its elements acquire their own odd grading.²

This note is organized as follows. In section 1 we introduce the operators with collective commutation closure. For consistency, we recall here the cohomological formulation [11] of the Magri scheme which gives us an example. In section 2 we study the properties of the bi-differential constants that appear in such algebras of operators. The analogues of Christoffel symbols emerge here; as an example, we calculate them for the symmetry algebra of the Liouville equation.

1. COMPATIBLE DIFFERENTIAL OPERATORS

We begin with some notation; for a more detailed exposition of the geometry of integrable systems we refer to [19] and [4, 12, 14, 17]. In the sequel, the ground field is the field \mathbb{R} of real numbers and all mappings are C^∞ -smooth.

Let $\pi: E^{m+n} \xrightarrow{M^m} B^n$ be a vector bundle over an orientable n -dimensional manifold B^n and, similarly, let $\xi: N^{d+n} \rightarrow B^n$ be another linear bundle over B^n . Consider the bundle $\pi_\infty: J^\infty(\pi) \rightarrow B^n$ of infinite jets of sections for the bundle π and take

¹When the set of admissible linear combinations $\{\lambda\} \subsetneq \mathbb{R}^N$ has punctures near which the homomorphisms A_λ exhibit a nontrivial analytic behaviour, this concept reappears in the theory of continuous contractions of Lie algebras (see [18] and references therein).

²Throughout this paper we deal with a purely commutative setup, refraining from the treatment of super-manifolds. However, we emphasize that, on a super-manifold, the two notions of *parity* and *grading* (or *weight*) may be totally uncorrelated, see [22].

the pull-back $\pi_\infty^*(\xi): N^{d+n} \times_{B^n} J^\infty(\pi) \rightarrow J^\infty(\pi)$ of the bundle ξ along π_∞ . By definition, the $C^\infty(J^\infty(\pi))$ -module of sections $\Gamma(\pi_\infty^*(\xi)) = \Gamma(\xi) \otimes_{C^\infty(B^n)} C^\infty(J^\infty(\pi))$ is called *horizontal*, see [13] for further details.

For example, let $\xi := \pi$. Then the variational vectors $\varphi \in \Gamma(\pi_\infty^*(\pi))$ are the generating sections of evolutionary derivations ∂_φ on $J^\infty(\pi)$. For convenience, we shall use the shorthand notation $\varkappa(\pi) \equiv \Gamma(\pi_\infty^*(\pi))$ and $\Gamma\Omega(\xi_\pi) \equiv \Gamma(\pi_\infty^*(\xi))$.

Let us consider first the case $N = 1$ when there is only one total differential operator $A: \Gamma\Omega(\xi_\pi) \rightarrow \varkappa(\pi)$ with involutive image:

$$[\text{im } A, \text{im } A] \subseteq \text{im } A. \quad (1)$$

The operator A transfers the bracket in the Lie algebra $\mathfrak{g}(\pi) = (\varkappa(\pi), [,])$ to the Lie algebra structure $[\cdot, \cdot]_A$ on the quotient of its domain by the kernel. The standard decomposition of this bracket is [8, 10]

$$[\mathbf{p}, \mathbf{q}]_A = \partial_{A(\mathbf{p})}(\mathbf{q}) - \partial_{A(\mathbf{q})}(\mathbf{p}) + \{\{\mathbf{p}, \mathbf{q}\}\}_A. \quad \mathbf{p}, \mathbf{q} \in \Gamma\Omega(\xi_\pi). \quad (2)$$

The *linear compatibility* of operators (4), which means that their arbitrary linear combinations $A_\lambda = \sum_i \lambda_i \cdot A_i$ satisfy (1), reduces the case of $N \geq 2$ operators to the previous case with $N = 1$ as follows.

Theorem 1 ([8]). *The bracket $\{\{\cdot, \cdot\}\}_{A_\lambda}$ induced by the combination $A_\lambda = \sum_i \lambda_i \cdot A_i$ on the domain of linear compatible normal³ operators A_i is*

$$\{\{\cdot, \cdot\}\}_{\sum_{i=1}^N \lambda_i A_i} = \sum_{i=1}^N \lambda_i \cdot \{\{\cdot, \cdot\}\}_{A_i}.$$

The pairwise linear compatibility implies the collective linear compatibility of A_1, \dots, A_N .

Proof. Consider the commutator $[\sum_i \lambda_i A_i(\mathbf{p}), \sum_j \lambda_j A_j(\mathbf{q})]$, here $\mathbf{p}, \mathbf{q} \in \Gamma\Omega(\xi_\pi)$. On one hand, it is equal to

$$= \sum_{i \neq j} \lambda_i \lambda_j [A_i(\mathbf{p}), A_j(\mathbf{q})] + \sum_i \lambda_i^2 A_i (\partial_{A_i(\mathbf{p})}(\mathbf{q}) - \partial_{A_i(\mathbf{q})}(\mathbf{p}) + \{\{\mathbf{p}, \mathbf{q}\}\}_{A_i}). \quad (3)$$

On the other hand, the linear compatibility of A_i implies

$$= A_\lambda (\partial_{A_\lambda(\mathbf{p})}(\mathbf{q})) - A_\lambda (\partial_{A_\lambda(\mathbf{q})}(\mathbf{p})) + A_\lambda (\{\{\mathbf{p}, \mathbf{q}\}\}_{A_\lambda}).$$

The entire commutator is quadratic homogeneous in λ , whence the bracket $\{\{\cdot, \cdot\}\}_{A_\lambda}$ is linear in λ . From (3) we see that the individual brackets $\{\{\cdot, \cdot\}\}_{A_i}$ are contained in it. Therefore,

$$\{\{\mathbf{p}, \mathbf{q}\}\}_{A_\lambda} = \sum_\ell \lambda_\ell \cdot \{\{\mathbf{p}, \mathbf{q}\}\}_{A_\ell} + \sum_\ell \lambda_\ell \cdot \gamma_\ell(\mathbf{p}, \mathbf{q}),$$

where $\gamma_\ell: \Gamma\Omega(\xi_\pi) \times \Gamma\Omega(\xi_\pi) \rightarrow \Gamma\Omega(\xi_\pi)$.

³By definition, a total differential operator A is *normal* if $A \circ \nabla = 0$ implies $\nabla = 0$; in other words, it may be that $\ker A \neq 0$, but the kernel does not have any functional freedom for its elements, see [7].

We claim that all summands $\gamma_\ell(\cdot, \cdot)$, which do not depend on $\boldsymbol{\lambda}$ at all, vanish. Indeed, assume the converse. Let there be $\ell \in [1, \dots, N]$ such that $\gamma_\ell(\mathbf{p}, \mathbf{q}) \neq 0$; without loss of generality, suppose $\ell = 1$. Then set $\boldsymbol{\lambda} = (1, 0, \dots, 0)$, whence

$$\begin{aligned} \left[\sum_i \lambda_i A_i(\mathbf{p}), \sum_j \lambda_j A_j(\mathbf{q}) \right] &= \left[(\lambda_1 A_1)(\mathbf{p}), (\lambda_1 A_1)(\mathbf{q}) \right] = (\lambda_1 A_1)(\lambda_1 \gamma_1(\mathbf{p}, \mathbf{q})) \\ &+ (\lambda_1 A_1) \left(\partial_{(\lambda_1 A_1)(\mathbf{p})}(\mathbf{q}) - \partial_{(\lambda_1 A_1)(\mathbf{q})}(\mathbf{p}) + \lambda_1 \{ \{ \mathbf{p}, \mathbf{q} \} \}_{A_1} \right) = \lambda_1 A_1(\lambda_1 [\mathbf{p}, \mathbf{q}]_{A_1}). \end{aligned}$$

Consequently, $\gamma_\ell(\mathbf{p}, \mathbf{q}) \in \ker A_\ell$ for all \mathbf{p} and \mathbf{q} . Now we use the assumption that each operator A_ℓ is normal. This implies that $\gamma_\ell = 0$ for all ℓ , which concludes the proof. \square

Now we let $N > 1$ and consider N -tuples of linear total differential operators

$$A_1, \dots, A_N: \Gamma\Omega(\xi_\pi) \longrightarrow \mathfrak{X}(\pi), \quad (4)$$

whose images in the Lie algebra $\mathfrak{g}(\pi)$ of evolutionary vector fields on $J^\infty(\pi)$ are subject to collective commutation closure.

Definition 1. We say that $N \geq 2$ total differential operators (4) are *strong compatible* if the sum of their images is closed under commutation in the Lie algebra $\mathfrak{g}(\pi) = (\mathfrak{X}(\pi), [\cdot, \cdot])$ of evolutionary vector fields,

$$\left[\sum_i \text{im } A_i, \sum_j \text{im } A_j \right] \subseteq \sum_k \text{im } A_k, \quad 1 \leq i, j, k \leq N. \quad (5)$$

The involutivity (5) gives rise to the bi-differential operators $\mathbf{c}_{ij}^k: \Gamma\Omega(\xi_\pi) \times \Gamma\Omega(\xi_\pi) \rightarrow \Gamma\Omega(\xi_\pi)$ through

$$[A_i(\mathbf{p}), A_j(\mathbf{q})] = \sum_k A_k(\mathbf{c}_{ij}^k(\mathbf{p}, \mathbf{q})), \quad \mathbf{p}, \mathbf{q} \in \Gamma\Omega(\xi_\pi). \quad (6)$$

The structural constants \mathbf{c}_{ij}^k absorb the bi-differential action on \mathbf{p}, \mathbf{q} under the commutation in the images of the operators.

Remark 1. If $N = 1$ and there is a unique operator $A: \Gamma\Omega(\xi_\pi) \rightarrow \mathfrak{X}(\pi)$ satisfying (1), then we recover the definition of the variational anchor in the Lie algebroid over the infinite jet space $J^\infty(\pi)$, see [10]. By construction, $\mathbf{c}_{11}^1 \equiv [\cdot, \cdot]_{A_1}$ if $N = 1$. However, for $N > 1$ we obtain a wider class of structures because we do not assume that the image of each operator A_i alone is involutive, therefore it may well occur that $\mathbf{c}_{ii}^k \neq 0$ for some $k \neq i$.

The Magri scheme [16] for the restriction of two compatible Hamiltonian operators A_1, A_2 onto the commutative hierarchy of the descendants \mathcal{H}_i of the Casimirs \mathcal{H}_0 for A_1 gives us an example of (5) with $N = 2$ and $\mathbf{c}_{ij}^k \equiv 0$. Let us consider it in more detail; from now on, we standardly identify the Hamiltonian operators A with the variational Poisson bi-vectors \mathbf{A} , see [13]. We recall that the variational Schouten bracket $[[\cdot, \cdot]]$ of such bi-vectors satisfies the Jacobi identity

$$[[[\mathbf{A}_1, \mathbf{A}_2], \mathbf{A}_3] + [[\mathbf{A}_2, \mathbf{A}_3], \mathbf{A}_1] + [[\mathbf{A}_3, \mathbf{A}_1], \mathbf{A}_2]] = 0. \quad (7)$$

Hence the defining property $[[\mathbf{A}, \mathbf{A}]] = 0$ for a Poisson bi-vector \mathbf{A} implies that $d_A = [[\mathbf{A}, \cdot]]$ is a differential, giving rise to the Poisson cohomology H_A^k . Obviously, the Casimirs \mathcal{H}_0 such that $[[\mathbf{A}, \mathcal{H}_0]] = 0$ for a Poisson bi-vector \mathbf{A} constitute the group H_A^0 .

Theorem 2 ([11, 16]). *Suppose $[\mathbf{A}_1, \mathbf{A}_2] = 0$, $\mathcal{H}_0 \in H_{A_1}^0$ is a Casimir of \mathbf{A}_1 , and the first Poisson cohomology w.r.t. $d_{A_1} = [\mathbf{A}_1, \cdot]$ vanishes. Then for any $k > 0$ there is a Hamiltonian \mathcal{H}_k such that*

$$[\mathbf{A}_2, \mathcal{H}_{k-1}] = [\mathbf{A}_1, \mathcal{H}_k]. \quad (8)$$

Put $\varphi_k := A_1(\delta/\delta u(\mathcal{H}_k))$ such that $\partial_{\varphi_k} = [\mathbf{A}_1, \mathcal{H}_k]$. The Hamiltonians \mathcal{H}_i , $i \geq 0$, pairwise Poisson commute w.r.t. either A_1 or A_2 , the densities of \mathcal{H}_i are conserved on any equation $u_{t_k} = \varphi_k$, and the evolutionary derivations ∂_{φ_k} pairwise commute for all $k \geq 0$.

Standard proof of existence. The main homological equality (8) is established by induction on k . Starting with a Casimir \mathcal{H}_0 , we obtain

$$0 = [\mathbf{A}_2, 0] = [\mathbf{A}_2, [\mathbf{A}_1, \mathcal{H}_0]] = -[\mathbf{A}_1, [\mathbf{A}_2, \mathcal{H}_0]] \pmod{[\mathbf{A}_1, \mathbf{A}_2]} = 0,$$

using the Jacobi identity (7). The first Poisson cohomology $H_{A_1}^1 = 0$ is trivial by an assumption of the theorem, hence the closed element $[\mathbf{A}_2, \mathcal{H}_0]$ in the kernel of $[\mathbf{A}_1, \cdot]$ is exact: $[\mathbf{A}_2, \mathcal{H}_0] = [\mathbf{A}_1, \mathcal{H}_1]$ for some \mathcal{H}_1 . For $k \geq 1$, we have

$$[\mathbf{A}_1, [\mathbf{A}_2, \mathcal{H}_k]] = -[\mathbf{A}_2, [\mathbf{A}_1, \mathcal{H}_k]] = -[\mathbf{A}_2, [\mathbf{A}_2, \mathcal{H}_{k-1}]] = 0$$

using (7) and by $[\mathbf{A}_2, \mathbf{A}_2] = 0$. Consequently, by $H_{A_1}^1 = 0$ we have that $[\mathbf{A}_2, \mathcal{H}_k] = [\mathbf{A}_1, \mathcal{H}_{k+1}]$, and we thus proceed infinitely. \square

We see now that the inductive step — the existence of the $(k + 1)$ -st Hamiltonian functional in involution — is possible if and only if H_0 is a Casimir,⁴ and therefore the operators A_1 and A_2 are restricted onto the linear subspace which is spanned in the space of variational covectors by the Euler derivatives of the descendants of \mathcal{H}_0 , i.e., of the Hamiltonians of the hierarchy. We note that the image under A_2 of a generic section from the domain of operators A_1 and A_2 can not be resolved w.r.t. A_1 by (8). For example, the first and second Hamiltonian structures for the KdV equation, which equal, respectively, $A_1 = d/dx$ and $A_2 = -\frac{1}{2} \frac{d^3}{dx^3} + 2u \frac{d}{dx} + u_x$, are not strong compatible unless they are restricted onto some subspaces of their arguments. On the linear subspace of descendants of the Casimir $\int u dx$, we have $\text{im } A_2 \subset \text{im } A_1$ and, since the image of the Hamiltonian operator A_1 is involutive, we conclude that $[\text{im } A_1, \text{im } A_2] \subset \text{im } A_1$.

On the other hand, the strong compatibility of the restrictions of Poisson compatible operators A_1 and A_2 onto the hierarchy is valid since their images are commutative Lie algebras. Regarding the converse statement as a potential generator of multi-dimensional completely integrable systems, we formulate the open problem: Is the strong compatibility of Poisson compatible Hamiltonian operators achieved *only* for their restrictions onto the hierarchies of Hamiltonians in involution so that the bi-differential constants \mathbf{c}_{ij}^k necessarily vanish?

⁴The Magri scheme starts from any two Hamiltonians $\mathcal{H}_{k-1}, \mathcal{H}_k$ that satisfy (8), but we operate with maximal subspaces of the space of functionals such that the sequence $\{\mathcal{H}_k\}$ can not be extended with $k < 0$.

2. BI-DIFFERENTIAL CHRISTOFFEL SYMBOLS

Similarly to (2), we extract the total bi-differential parts of the structural constants \mathbf{c}_{ij}^k in (6) and obtain

$$\mathbf{c}_{ij}^k = \partial_{A_i(\mathbf{p})}(\mathbf{q}) \cdot \delta_j^k - \partial_{A_j(\mathbf{q})}(\mathbf{p}) \cdot \delta_i^k + \Gamma_{ij}^k(\mathbf{p}, \mathbf{q}), \quad \mathbf{p}, \mathbf{q} \in \Gamma\Omega(\xi_\pi), \quad (9)$$

where $\Gamma_{ij}^k \in \mathcal{C}\text{Diff}(\Gamma\Omega(\xi_\pi) \times \Gamma\Omega(\xi_\pi) \rightarrow \Gamma\Omega(\xi_\pi))$ and δ_i^k, δ_j^k are the Kronecker delta symbols. By definition, the three indices in Γ_{ij}^k match the respective operators A_i, A_j, A_k in (6). (The total number of the indices is much greater than three; moreover, the proper upper or lower location of the omitted indices depends on the (co)vector nature of the domain $\Gamma\Omega(\xi_\pi)$.) Obviously, the convention

$$\Gamma_{11}^1 = \{ \{ , \} \}_{A_1}$$

holds if $N = 1$. At the same time, for fixed i, j, k , the symbol Γ_{ij}^k remains a (class of) matrix differential operator in each of its two arguments $\mathbf{p}, \mathbf{q} \in \Gamma\Omega(\xi_\pi)$. The symbol Γ_{ij}^k represents a class of bi-differential operators because they are not uniquely defined. Indeed, they are gauged by the conditions

$$\sum_{k=1}^N A_k \left(\partial_{A_i(\mathbf{p})}(\mathbf{q}) \delta_j^k - \partial_{A_j(\mathbf{q})}(\mathbf{p}) \delta_i^k + \Gamma_{ij}^k(\mathbf{p}, \mathbf{q}) \right) = 0, \quad \mathbf{p}, \mathbf{q} \in \Gamma\Omega(\xi_\pi). \quad (10)$$

We let the r.h.s. of (10) be zero if the sum $\sum_\ell \text{im } A_\ell$ of the images is indecomposable, meaning that no nontrivial sections commute with all the others: $[A_k(\mathbf{p}), \sum_{\ell=1}^N \text{im } A_\ell] = 0$ implies that $\mathbf{p} \in \ker A_k$. For this it is sufficient that the sum of the images of A_ℓ in $\mathfrak{g}(\pi)$ is semi-simple and the Whitehead lemma holds for it [5]. Otherwise, the right-hand side of (10) belongs to the linear subspace of such nontrivial sections.

Example 1 (see [9, 10]). Consider the Liouville equation $\mathcal{E}_{\text{Liou}} = \{u_{xy} = \exp(2u)\}$. The differential generators of its conservation laws are $w = u_x^2 - u_{xx} \in \ker \frac{d}{dy} \Big|_{\mathcal{E}_{\text{Liou}}}$ and $\bar{w} = u_y^2 - u_{yy} \in \ker \frac{d}{dx} \Big|_{\mathcal{E}_{\text{Liou}}}$. The operators⁵ $\square = u_x + \frac{1}{2} \frac{d}{dx}$ and $\bar{\square} = u_y + \frac{1}{2} \frac{d}{dy}$ determine higher symmetries $\varphi, \bar{\varphi}$ of $\mathcal{E}_{\text{Liou}}$ by the formulas

$$\varphi = \square(p(x, [w])), \quad \bar{\varphi} = \bar{\square}(\bar{p}(y, [\bar{w}]))$$

for any variational covectors p, \bar{p} . The images of \square and $\bar{\square}$ are closed w.r.t. the commutation; for instance, the bracket (2) for \square contains $\{ \{ p, q \} \}_\square = \frac{d}{dx}(p) \cdot q - p \cdot \frac{d}{dx}(q)$, and similar for $\bar{\square}$. The two summands in the symmetry algebra $\text{sym } \mathcal{E}_{\text{Liou}} \simeq \text{im } \square \oplus \text{im } \bar{\square}$ commute between each other, $[\text{im } \square, \text{im } \bar{\square}] \doteq 0$ on $\mathcal{E}_{\text{Liou}}$. The operators $\square, \bar{\square}$ generate the bi-differential symbols

$$\begin{aligned} \Gamma_{\square\square}^\square &= \{ \{ , \} \}_\square = \frac{d}{dx} \otimes \mathbf{1} - \mathbf{1} \otimes \frac{d}{dx}, & \Gamma_{\bar{\square}\bar{\square}}^{\bar{\square}} &= \{ \{ , \} \}_{\bar{\square}} = \frac{d}{dy} \otimes \mathbf{1} - \mathbf{1} \otimes \frac{d}{dy}, \\ \Gamma_{\bar{\square}\square}^\square &= \frac{d}{dy} \otimes \mathbf{1}, & \Gamma_{\square\bar{\square}}^{\bar{\square}} &= -\mathbf{1} \otimes \frac{d}{dx}, & \Gamma_{\square\square}^\square &= -\mathbf{1} \otimes \frac{d}{dy}, & \Gamma_{\bar{\square}\bar{\square}}^{\bar{\square}} &= \frac{d}{dx} \otimes \mathbf{1}, \end{aligned}$$

where the notation is obvious. We note that $\Gamma_{\square\square}^\square(p, q) \doteq \Gamma_{\bar{\square}\bar{\square}}^{\bar{\square}}(p, q) \doteq \Gamma_{\square\bar{\square}}^\square(q, p) \doteq \Gamma_{\bar{\square}\square}^{\bar{\square}}(q, p) \doteq 0$ on $\mathcal{E}_{\text{Liou}}$ for any $p(x, [w])$ and $q(y, [\bar{w}])$.

⁵We denote the operators by \square and $\bar{\square}$, following the notation of [7, 9], see also references therein.

The matrix operators $\square, \bar{\square}$ are well-defined [7] for each 2D Toda chain $\mathcal{E}_{\text{Toda}}$ associated with a semi-simple complex Lie algebra. They exhibit the same properties as above.

Remark 2. The operators $\square, \bar{\square}$ yield the involutive distributions of evolutionary vector fields that are tangent to the *integral manifolds*, the 2D Toda differential equations. Generally, there is no Frobenius theorem for such distributions. Still, if the integral manifold exists and is an infinite prolongation of a differential equation $\mathcal{E} \subset J^\infty(\pi)$, then, by construction, this equation admits infinitely many symmetries of the form $\varphi = A_i(\mathbf{p})$ with free functional parameters $\mathbf{p} \in \Gamma\Omega(\xi_\pi)$. This property is close but not equivalent to the definition of Liouville-type systems (see [7, 9] and references therein).

The method by which we introduced the symbols Γ_{ij}^k suggests that, under reparametrizations g in the domain of the operators (4), they obey a proper analogue of the standard rule $\Gamma \mapsto g \Gamma g^{-1} + dg \cdot g^{-1}$ for the connection 1-forms Γ . This is indeed so.

Theorem 3 (Transformations of Γ_{ij}^k). *Let g be a reparametrization $\mathbf{p} \mapsto \tilde{\mathbf{p}} = g\mathbf{p}$, $\mathbf{q} \mapsto \tilde{\mathbf{q}} = g\mathbf{q}$ of sections $\mathbf{p}, \mathbf{q} \in \Gamma\Omega(\xi_\pi)$ in the domains⁶ of strong compatible operators (4). In this notation, the operators A_1, \dots, A_N are transformed by the formula $A_i \mapsto \tilde{A}_i = A_i \circ g^{-1}|_{w=w[\tilde{w}]}$. Then the bi-differential symbols $\Gamma_{ij}^k \in \mathcal{C}\text{Diff}(\Gamma\Omega(\xi_\pi) \times \Gamma\Omega(\xi_\pi) \rightarrow \Gamma\Omega(\xi_\pi))$ are transformed according to the rule*

$$\Gamma_{ij}^k(\mathbf{p}, \mathbf{q}) \mapsto \tilde{\Gamma}_{ij}^k(\tilde{\mathbf{p}}, \tilde{\mathbf{q}}) = (g \circ \Gamma_{ij}^k)(g^{-1}\tilde{\mathbf{p}}, g^{-1}\tilde{\mathbf{q}}) + \delta_i^{\tilde{k}} \cdot \partial_{\tilde{A}_j(\tilde{\mathbf{q}})}(g)(g^{-1}\tilde{\mathbf{p}}) - \delta_j^{\tilde{k}} \cdot \partial_{\tilde{A}_i(\tilde{\mathbf{p}})}(g)(g^{-1}\tilde{\mathbf{q}}). \quad (11)$$

Proof. Denote $A = A_i$ and $B = A_j$; without loss of generality we assume $i = 1$ and $j = 2$. Let us calculate the commutators of vector fields in the images of A and B using two systems of coordinates in the domain. We equate the commutators straightforwardly, because the fibre coordinates in the images of the operators are not touched at all. So, we have, originally,

$$[A(\mathbf{p}), B(\mathbf{q})] = B(\partial_{A(\mathbf{p})}(\mathbf{q})) - A(\partial_{B(\mathbf{q})}(\mathbf{p})) + A(\Gamma_{AB}^A(\mathbf{p}, \mathbf{q})) + B(\Gamma_{AB}^B(\mathbf{p}, \mathbf{q})) + \sum_{k=3}^N A_k(\Gamma_{AB}^k(\mathbf{p}, \mathbf{q})).$$

On the other hand, we substitute $\tilde{\mathbf{p}} = g\mathbf{p}$ and $\tilde{\mathbf{q}} = g\mathbf{q}$ in $[\tilde{A}(\tilde{\mathbf{p}}), \tilde{B}(\tilde{\mathbf{q}})]$, whence, by the Leibnitz rule, we obtain

$$\begin{aligned} [\tilde{A}(\tilde{\mathbf{p}}), \tilde{B}(\tilde{\mathbf{q}})] &= \tilde{B}(\partial_{\tilde{A}(\tilde{\mathbf{p}})}(g)(\mathbf{q})) + (\tilde{B} \circ g)(\partial_{\tilde{A}(\tilde{\mathbf{p}})}(\mathbf{q})) - \tilde{A}(\partial_{\tilde{B}(\tilde{\mathbf{q}})}(g)(\mathbf{p})) - (\tilde{A} \circ g)(\partial_{\tilde{B}(\tilde{\mathbf{q}})}(\mathbf{p})) \\ &\quad + (A \circ g^{-1})(\Gamma_{\tilde{A}\tilde{B}}^{\tilde{A}}(g\mathbf{p}, g\mathbf{q})) + (B \circ g^{-1})(\Gamma_{\tilde{A}\tilde{B}}^{\tilde{B}}(g\mathbf{p}, g\mathbf{q})) + \sum_{\tilde{k}=3}^N (A_{\tilde{k}} \circ g^{-1})(\Gamma_{\tilde{A}\tilde{B}}^{\tilde{k}}(g\mathbf{p}, g\mathbf{q})). \end{aligned}$$

⁶Under an invertible change $\tilde{w} = \tilde{w}[w]$ of fibre coordinates (see Example 1), the variational covectors are transformed by the inverse of the adjoint linearization $g = [(\ell_{\tilde{w}}^{(w)})^\dagger]^{-1}$, whereas for variational vectors, $g = \ell_{\tilde{w}}^{(w)}$ is the linearization.

Therefore,

$$\begin{aligned}\Gamma_{AB}^A(\mathbf{p}, \mathbf{q}) &= (g^{-1} \circ \Gamma_{\tilde{A}\tilde{B}}^{\tilde{A}})(g\mathbf{p}, g\mathbf{q}) - (g^{-1} \circ \partial_{B(\mathbf{q})}(g))(\mathbf{p}), \\ \Gamma_{AB}^B(\mathbf{p}, \mathbf{q}) &= (g^{-1} \circ \Gamma_{\tilde{A}\tilde{B}}^{\tilde{B}})(g\mathbf{p}, g\mathbf{q}) + (g^{-1} \circ \partial_{A(\mathbf{p})}(g))(\mathbf{q}), \\ \Gamma_{AB}^k(\mathbf{p}, \mathbf{q}) &= (g^{-1} \circ \Gamma_{\tilde{A}\tilde{B}}^k)(g\mathbf{p}, g\mathbf{q}) \quad \text{for } k \geq 3.\end{aligned}$$

Acting by g on these equalities and expressing $\mathbf{p} = g^{-1}\tilde{\mathbf{p}}$, $\mathbf{q} = g^{-1}\tilde{\mathbf{q}}$, we obtain (11) and conclude the proof. \square

Remark 3. Within the Hamiltonian formalism, it is very productive to postulate that the arguments of Hamiltonian operators, the variational covectors, are *odd*,⁷ see [22] and [13]. Indeed, in this particular situation they can be conveniently identified with Cartan 1-forms times the pull-back of the volume form $d \operatorname{vol}(B^n)$ for the base of the jet bundle. We preserve this *grading* for such domains of operators (when $N = 1$, we referred to such operators in [10] as variational anchors of *second kind*). If, moreover, π and ξ are super-bundles with Grassmann-valued sections, then the operators become bi-graded [22]. Their proper grading is -1 because their images in $\mathfrak{g}(\pi)$ have grading zero, but the \mathbb{Z}_2 -*parity*, if any, can be arbitrary.

Corollary 4. For strong compatible operators whose domain $\Gamma\Omega(\xi_\pi)$ consists of variational covectors, the grading of the arguments equals 1. Therefore, for any $i, j, k \in [1, \dots, N]$ and for any $\mathbf{p}, \mathbf{q} \in \Gamma\Omega(\xi_\pi)$ we have that

$$\Gamma_{ij}^k(\mathbf{p}, \mathbf{q}) = -\Gamma_{ji}^k(\mathbf{q}, \mathbf{p}) = (-1)^{|\mathbf{p}|_{\text{gr}} \cdot |\mathbf{q}|_{\text{gr}}} \cdot \Gamma_{ji}^k(\mathbf{q}, \mathbf{p}) \quad (12)$$

due to the skew-symmetry of the commutators in (5). Hence the symbols Γ_{ij}^k are symmetric in this case.

Proposition 5. If two normal operators A_i and A_j are simultaneously linear and strong compatible, then their ‘individual’ brackets Γ_{ii}^i and Γ_{jj}^j are

$$\{\{\mathbf{p}, \mathbf{q}\}\}_{A_i} = \Gamma_{ij}^j(\mathbf{p}, \mathbf{q}) + \Gamma_{ji}^j(\mathbf{p}, \mathbf{q}) \quad \text{and} \quad \{\{\mathbf{p}, \mathbf{q}\}\}_{A_j} = \Gamma_{ij}^i(\mathbf{p}, \mathbf{q}) + \Gamma_{ji}^i(\mathbf{p}, \mathbf{q})$$

for any $\mathbf{p}, \mathbf{q} \in \Gamma\Omega(\xi_\pi)$.

Proof. For brevity, denote $A = A_i$, $B = A_j$ and consider the linear combination $\mu A + \nu B$; by assumption, its image is closed under commutation. By Theorem 1, we have

$$\begin{aligned}(\mu A + \nu B)(\{\{\mathbf{p}, \mathbf{q}\}\}_{\mu A + \nu B}) &= \\ &= \mu^2 A(\{\{\mathbf{p}, \mathbf{q}\}\}_A) + \mu\nu \cdot A(\{\{\mathbf{p}, \mathbf{q}\}\}_B) + \mu\nu \cdot B(\{\{\mathbf{p}, \mathbf{q}\}\}_A) + \nu^2 B(\{\{\mathbf{p}, \mathbf{q}\}\}_A).\end{aligned}$$

On the other hand,

$$\begin{aligned}[(\mu A + \nu B)(\mathbf{p}), (\mu A + \nu B)(\mathbf{q})] &= \\ &= \mu^2 [A(\mathbf{p}), A(\mathbf{q})] + \mu\nu [A(\mathbf{p}), B(\mathbf{q})] - \mu\nu [A(\mathbf{q}), B(\mathbf{p})] + \nu^2 [B(\mathbf{p}), B(\mathbf{q})].\end{aligned}$$

Taking into account (9) and equating the coefficients of $\mu\nu$, we obtain

$$A(\{\{\mathbf{p}, \mathbf{q}\}\}_B) + B(\{\{\mathbf{p}, \mathbf{q}\}\}_A) = A(\Gamma_{AB}^A(\mathbf{p}, \mathbf{q})) + B(\Gamma_{AB}^B(\mathbf{p}, \mathbf{q})) - A(\Gamma_{AB}^A(\mathbf{q}, \mathbf{p})) - B(\Gamma_{AB}^B(\mathbf{q}, \mathbf{p})).$$

⁷Here we assume for simplicity that all fibre coordinates in π and ξ are permutable.

Using the formulas $\Gamma_{AB}^A(\mathbf{q}, \mathbf{p}) = -\Gamma_{BA}^A(\mathbf{p}, \mathbf{q})$ and $\Gamma_{AB}^B(\mathbf{q}, \mathbf{p}) = -\Gamma_{BA}^B(\mathbf{p}, \mathbf{q})$, see (12), we isolate the arguments of the operators and obtain the assertion. \square

CONCLUSION

For every \mathbb{k} -vector space V , the space of endomorphisms $\text{End}_{\mathbb{k}}(V)$ is a monoid with respect to the composition \circ . In this context, one can study relations between recursion operators. For instance, the structural relations for recursion operators of the Krichever–Novikov equations are described by hyperelliptic curves, see [3]. Likewise, we have the relation $R_1 \circ R_2 - R_2 \circ R_1 = R_1^2$ between two recursions for the dispersionless 3-component Boussinesq system, see [6]. Simultaneously, the space of endomorphisms carries the structure of a Lie algebra, which is given by the formula $[R_i, R_j] = R_i \circ R_j - R_j \circ R_i$ for every $R_i, R_j \in \text{End}_{\mathbb{k}}(V)$.

In this paper we proceed further and consider the class of structures on the linear spaces of total differential operators that, generally, do not in principle admit any associative composition. (The bracket of recursion operators that appears through (6) is different from the Richardson–Nijenhuis bracket [12], although we use similar geometric techniques.) The classification problem for such algebras of operators is completely open.

Discussion. We performed all the reasonings for local differential operators in a purely commutative setup; all the structures were defined on the empty jet spaces. A rigorous extension of these objects to \mathbb{Z}_2 -graded nonlocal operators on differential equations is a separate problem for future research. In addition, the use of difference operators subject to (5) can be a fruitful idea *au début* for discretization of integrable systems with free functional parameters in the symmetries (e.g., Toda-like difference systems [20]).

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