

MAPPING CLASS GROUPS AND MODULI SPACES OF CURVES

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1. INTRODUCTION

It is classical that there is a very strong relation between the topology of \mathcal{M}_g , the moduli space of smooth projective curves of genus g , and the structure of the mapping class group Γ_g , the group of homotopy classes of orientation preserving diffeomorphisms of a compact orientable surface of genus g . The geometry of \mathcal{M}_g , the topology of \mathcal{M}_g , and the structure of Γ_g are all intimately related. Until recently, the principal tools for studying these topics were Teichmüller theory (complex analysis and hyperbolic geometry), algebraic geometry, and geometric topology. Recently, a fourth cornerstone has been added, and that is physics which enters through the theories of quantum gravity and conformal field theory. Already these new ideas have had a remarkable impact on the subject through the ideas of Witten and the work of Kontsevich. In this article, we survey some recent developments in the understanding of moduli spaces. Some of these are classical (do not use physical ideas), while others are modern. One message we would like to convey is that algebraic geometers, topologists, and physicists who work on moduli spaces of curves may have a lot to learn from each other.

Having said this, we should immediately point out that, partly due to our own limitations, there are important developments that we have not included in this survey. Our most notable omission is the arithmetic aspect of the theory, much of which originates in Grothendieck's fundamental works [19], [20]. We direct readers

1991 *Mathematics Subject Classification*. Primary 14-02, 14H10, 32G15; Secondary 20F34.
First author supported in part by grants from the National Science Foundation.

to the volume [21] and to the recent papers of Ihara, Nakamura and Oda for other recent developments (see Nakamura's survey [71] for references). Other topics we have not covered include conformal field theory and recent work of Ivanov [38] and Ivanov and McCarthy [39] on homomorphisms from mapping class groups and arithmetic groups to mapping class groups. Of particular importance is Ivanov's version of Margulis rigidity for mapping class groups [38] which he obtains using some recent fundamental work of Kaimanovich and Masur [47] on the ergodic theory of Teichmüller space.

We shall denote the moduli space of n pointed smooth projective curves of genus g by \mathcal{M}_g^n . Knudsen, Mumford and Deligne constructed a canonical compactification $\overline{\mathcal{M}}_g^n$ of it. It is the moduli space of stable n pointed projective curves of genus g . It is a projective variety with only finite quotient singularities. Perhaps the most important developments of the decade concern the Chow rings¹ of \mathcal{M}_g^n and $\overline{\mathcal{M}}_g^n$.

The first Chern class of the relative cotangent bundle of the universal curve associated to the i th point is a class $\overline{\tau}_i$ in $\text{CH}^1(\overline{\mathcal{M}}_g^n)$. One can consider monomials in the $\overline{\tau}_i$'s of polynomial degree equal to the dimension of some $\overline{\mathcal{M}}_g^n$. For such a monomial, one can take the degree of the monomial as a zero cycle on $\overline{\mathcal{M}}_g^n$ to obtain a rational number. These can be assembled into a generating function. Witten conjectured that this formal power series satisfies a system of partial differential operators. Kontsevich proved this using topological arguments, and thereby provided inductive formulas for these intersection numbers. These developments are surveyed in Section 6.

For each positive integer i , Mumford defined a *tautological class* $\overline{\kappa}_i$ in $\text{CH}^i(\overline{\mathcal{M}}_g)$. The restrictions κ_i of these classes to $\text{CH}^\bullet(\mathcal{M}_g)$ generate a subalgebra of $\text{CH}^\bullet(\mathcal{M}_g)$ which is called the *tautological algebra* of \mathcal{M}_g . Faber has conjectured that this ring has the structure of the (p, p) part of the cohomology ring of a smooth complex projective variety of complex dimension $g - 2$. That is, it satisfies Poincaré duality and has the "Hard Lefschetz Property" with respect to κ_1 . Considerable evidence now exists for this conjecture, much of which is presented in Section 5. Other developments on the Chow ring, such as explicit computations in low genus, are also surveyed there.

In the early 80s, Harer proved that the cohomology in a given degree of \mathcal{M}_g is independent of the genus once the genus is sufficiently large relative to the degree. These stable cohomology groups form a graded commutative algebra which is known to be free. The tautological classes κ_i freely generate a polynomial algebra inside the stable cohomology ring. Mumford and others have conjectured that the stable cohomology of \mathcal{M}_g is generated by the κ_i 's. Some progress has been made towards this conjecture which we survey throughout the paper. In Section 4 we consider the stabilization maps from an algebro-geometric point of view, and in Section 10 we survey Kontsevich's methods for constructing classes in the cohomology of the \mathcal{M}_g^n .

We have also tried to advertise the fecund work of Dennis Johnson on the Torelli groups. The Torelli group T_g is the subgroup of the mapping class group Γ_g consisting of those diffeomorphism classes that act trivially on the homology of the reference surface. This mysterious group, in some sense, measures the difference

¹All Chow rings and cohomology groups in this paper are with \mathbb{Q} coefficients except when explicit coefficients are used.

between curves and abelian varieties and appears to play a subtle role in the geometry of \mathcal{M}_g . Johnson proved that T_g is finitely generated when $g \geq 3$ and computed its first integral homology group. These computations have direct geometric applications, especially when combined with M. Saito's work in Hodge theory — for example, they restrict the normal functions defined over \mathcal{M}_g and its standard level covers. From this, one can give a computation of the Picard group of the generic curve with a level l structure. Johnson's work and its applications is surveyed in Section 7.

Since Γ_g is the orbifold fundamental group of \mathcal{M}_g , an algebraic variety, one should be able to apply Hodge theory and Galois theory to study its structure. In Section 9 we survey recent work on applications of Hodge theory to understanding the structure of the Torelli groups, mainly via Malcev completion. In Section 10 we combine this Hodge theory with recent results of Kawazumi and Morita to show that the cohomology of \mathcal{M}_g constructed by Kontsevich using graph cohomology are, after stabilization, polynomials in the κ_i 's. Thus Hodge theory provides some evidence for Mumford's conjecture that the stable cohomology of the mapping class group is generated by the κ_i 's.

Some of the results we discuss have not yet appeared in the literature, at least not in the form in which we present them. Rather than mention all such results, we simply mention a few instances where we believe our presentation to be novel: the correspondences in Section 4.1, the rôle of the fundamental normal function for orbifold fundamental groups in Section 8.6, Theorem 9.11 and the contents of Section 10.3.

Notation and Conventions. All varieties will be defined over \mathbb{C} unless explicitly stated to the contrary. Unless explicit coefficients are used, all (co)homology groups are with rational coefficients. We will often abbreviate *mixed Hodge structure* by MHS. The sub- or superscript *pr* on a (co)homology group will denote the primitive part in both the context of the Hard Lefschetz Theorem and in the context of Hopf algebras.

Acknowledgements. We would like to thank Carel Faber for his comments on part of an earlier version of this paper and Shigeyuki Morita for explaining to us some of his recent work. We also appreciate the useful comments by a referee. We gratefully acknowledge support by the AMS that enabled us to attend this conference.

2. MAPPING CLASS GROUPS

Fix a compact connected oriented reference surface S_g of genus g , and a sequence of distinct points (x_0, x_1, x_2, \dots) in S_g . Let us write S_g^n for the open surface $S - \{x_1, \dots, x_n\}$ and π_g^n for its fundamental group $\pi_1(S_g^n, x_0)$. This group admits a presentation with generators $\alpha_{\pm 1}, \dots, \alpha_{\pm g}, \beta_1, \dots, \beta_n$ and relation

$$(\alpha_1, \alpha_{-1}) \cdots (\alpha_g, \alpha_{-g}) = \beta_1 \cdots \beta_n,$$

where (x, y) denotes the commutator of x and y .² The generators are represented by loops that do not meet outside the base point; β_i is represented by a loop that follows an arc to a point close to x_i , makes a simple loop around x_i , and returns to the base point along the same arc.

Let $\text{Diff}^+(S)_r^n$ denote the group of orientation preserving diffeomorphisms of S that fix the x_i for $i = 1, \dots, n+r$, and are the identity on $T_{x_i}S$ for $i = n+1, \dots, n+r$.

²For $n = 0$ the righthand side is to be interpreted as the unit element.

Although not really necessary at this stage, it is convenient to assume that $2g - 2 + n + 2r > 0$. In other words, we do not consider the cases where (g, n, r) is $(0, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(0, 2, 0)$ or $(1, 0, 0)$. We will keep this assumption throughout the paper.

The *mapping class group* $\Gamma_{g,r}^n$ is defined to be the group of connected components of this group:

$$\Gamma_{g,r}^n = \pi_0 \text{Diff}^+(S)_r^n.$$

We omit the decorations n and r when they are zero. The mapping class group Γ_g^n acts on π_g^n by outer automorphisms. A theorem that goes back to Baer (1928) and Nielsen (1927) [72] identifies Γ_g , via this representation, with the subgroup of $\text{Out}(\pi_g)$ (of index two) that acts trivially on $H_2(\pi_g) \cong H_2(S_g)$.

When $n \geq 1$ we can consider the diagonal action of $\text{Aut}(\pi_g^n)$ on $(\pi_g^n)^n$. Clearly, $\text{Out}(\pi_g^n)$ acts on the set of orbits of π_g^n (which acts by inner automorphisms on each component) in $(\pi_g^n)^n$. Now Γ_g^n can be identified with the group of outer automorphisms of π_g^n that preserve the image of $(\beta_1, \dots, \beta_n)$ in $\pi_g^n \setminus (\pi_g^n)^n$. If we choose x_n as a base point, then a corresponding assertion holds: Γ_g^n can be identified with a subgroup of $\text{Aut}(\pi_1(S_g^{n-1}, x_n))$ that is characterized in a similar way. The evident homomorphism $\Gamma_g^n \rightarrow \Gamma_g^{n-1}$ is surjective and its kernel can be identified with $\pi_1(S_g^{n-1}, x_n)$ (acting by inner automorphisms). Ivanov and McCarthy [39] recently showed that the resulting exact sequence cannot be split.

2.1. Generators and basic properties. Although a lot is known about these groups they are still poorly understood. Let us quickly review some of their basic properties. Dehn proved in [10] that the mapping class groups are generated by the ‘twists’ that are now named after him: if α is a simple (unoriented) loop on S_g^{n+r} , then parameterize a regular neighborhood of α in S_g^{n+r} by the cylinder $[0, 1] \times S^1$ (preserving orientations) and define an automorphism of S_g that on this neighborhood is given by $(t, z) \mapsto (t, e^{2\pi i t} z)$ and is the identity elsewhere. The isotopy class of this automorphism only depends on the isotopy class of α and is called the *Dehn twist* along α . (Perhaps we should add that α is, in turn, already determined by its free homotopy class, in other words, by the associated conjugacy class in π_g^n .) The corresponding element of $\Gamma_{g,r}^n$ is the identity precisely when α bounds a disk in $S_g - \{x_{n+1}, \dots, x_{n+r}\}$ which meets $\{x_1, \dots, x_n\}$ in at most one point. Several people have found a finite presentation for the mapping class groups. One with few generators was given by Waynryb [79]. From this presentation one sees that the mapping class groups considered here are perfect when $g \geq 3$ (a result due to Powell [75] in the undecorated case).

There is an obvious homomorphism $\Gamma_{g,n} \rightarrow \Gamma_g^n$. It is easy to see that it is surjective and that the kernel is generated by the Dehn twists around the points x_1, \dots, x_n . These Dehn twists generate a free abelian central subgroup of $\Gamma_{g,n}$ of rank n . Now recall that a central extension of a discrete group G by \mathbb{Z} determines an extension class in $H^2(G; \mathbb{Z})$; it has a geometric interpretation as a first Chern class. In the present case we have n such classes $\tau_i \in H^2(\Gamma_g^n; \mathbb{Z})$, $i = 1, \dots, n$.

Conversely, each subgroup of $H^2(G; \mathbb{Z})$ determines a central extension of G by that subgroup. Harer proved that $H^2(\Gamma_{g,r}; \mathbb{Z})$ is infinite cyclic if $g \geq 3$ [27], so that there is a corresponding central extension

$$0 \rightarrow \mathbb{Z} \rightarrow \tilde{\Gamma}_{g,r} \rightarrow \Gamma_{g,r} \rightarrow 1.$$

Since $H_1(\Gamma_{g,r}; \mathbb{Z})$ vanishes, this central extension is perfect (and universal). A nice presentation of it was recently given by Gervais [16]. The (imperfect) central extension by $\frac{1}{12}\mathbb{Z}$ containing this extension appears in the theory of conformal blocks; it has a simple geometric description which we will give in Section 3.

2.2. Stable cohomology. The mapping class groups $\Gamma_{g,r}^n$ turn up in a connected sum construction that we describe next. It is convenient to do this in a somewhat abstract setting. Suppose we are given a closed, oriented (but not necessarily connected) surface S , a finite subset $Y \subset S$, and a fixed point free involution ι of Y . Assume that ι has been lifted to an orientation reversing linear involution $\tilde{\iota}$ on the spaces of rays $\text{Ray}(TS|Y)$. The *real oriented blow up* $S_Y \rightarrow S$ is a surface with boundary canonically isomorphic to $\text{Ray}(TS|Y)$. So $\tilde{\iota}$ defines an orientation reversing involution of this boundary. Welding the boundary components of S_Y by means of this involution produces a closed surface $S(\tilde{\iota})$. Some care is needed to give it a differentiable structure inducing the given one on S_Y . Although there is no unique way to do this, all natural choices lie in the same isotopy class. If S happens to have a complex structure, then each choice of a real ray L in $T_p S \otimes_{\mathbb{C}} T_{\iota(p)} S$ determines a lift of ι over the pair $\{p, \iota(p)\}$: if l is a ray in $T_p S$, then $\tilde{\iota}(l)$ is determined uniquely by the condition $l \otimes_{\mathbb{C}} \tilde{\iota}(l) = L$.

If $S(\tilde{\iota})$ is connected, then each finite subset X of $S - Y$ determines a natural homomorphism from the mapping class group which is perhaps best denoted by $\Gamma(S)_Y^X$ (a product of groups of the type $\Gamma_{g,r}^n$) to the mapping class group $\Gamma(S(\tilde{\iota}))^X$. The image of this homomorphism is simply the stabilizer of the simple loops indexed by Y/ι that are images of boundary components of S_Y . Its kernel is a free abelian group whose generators can be labeled by a system of representatives R of ι orbits in Y . Indeed, for each element y of R , take the composite of the Dehn twist around y and the inverse of the Dehn twist around $\iota(y)$. These maps appear in the stability theorems and are at the root of the recent operad theoretic approaches to the study of the cohomology of mapping class groups.

Theorem 2.1 (Stability theorem, Harer [29]). *There exists a positive constant c with the following property. If $S(\tilde{\iota})$ is connected and S' is a connected component of S and X a finite subset of $S' \setminus Y$, then the homomorphism*

$$\Gamma(S')_{Y \cap S'}^X \rightarrow \Gamma(S(\tilde{\iota}))^X$$

induces an isomorphism on integral cohomology in degree $\leq c \cdot \text{genus}(S')$.

The constant c appearing in this theorem was $1/3$ in Harer's original paper. It was later improved to $1/2$ by Ivanov in [35]. Most recently, Harer [31] has showed that we can take c to be about $2/3$ and that this is the minimal possible value. There is also a version for twisted coefficients, due to Ivanov [36].

Harer's theorem says essentially that the k th cohomology group of $\Gamma_{g,r}^n$ depends only on n , provided that g is large enough. These stable cohomology groups are the cohomology of a single group, namely the group Γ_{∞}^n of compactly supported mapping classes of a surface S_{∞} of infinite genus (with one end, say) that fix a given set of n distinct points.

Among the homomorphisms defined above are maps $\Gamma_{g,1} \times \Gamma_{g',1} \rightarrow \Gamma_{g+g'}$. These stabilize and define homomorphisms of \mathbb{Q} algebras

$$\mu : H^{\bullet}(\Gamma_{\infty}) \rightarrow H^{\bullet}(\Gamma_{\infty}) \otimes H^{\bullet}(\Gamma_{\infty}).$$

This defines a coproduct on $H^\bullet(\Gamma_\infty)$. Together with the cup product, this gives $H^\bullet(\Gamma_\infty)$ the structure of a connected graded-bicommutative Hopf algebra. The classification of such Hopf algebras implies that $H^\bullet(\Gamma_\infty)$ is free as a graded algebra and is generated by its set of primitive elements

$$H_{\text{pr}}^\bullet(\Gamma_\infty) := \{x \in H^+(\Gamma_\infty) : \mu(x) = x \otimes 1 + 1 \otimes x\}.$$

For each $i > 0$, Mumford [69] and Morita [63] independently found a class κ_i in $H_{\text{pr}}^{2i}(\Gamma_\infty)$ (we shall recall the definition in Section 4) and Miller [61] and Morita [63] independently showed that each κ_i is nonzero. So the κ_i 's generate a polynomial subalgebra of the stable cohomology. Mumford conjectured that they span all of $H_{\text{pr}}^\bullet(\Gamma_\infty)$. This has been verified by Harer in a series of papers [27], [30], [32] in degrees ≤ 4 .³

The first Chern class $\tau_i \in H^2(\Gamma_\infty^n; \mathbb{Z})$ stabilizes also and we may think of it as an element of $H^\bullet(\Gamma_\infty^n; \mathbb{Z})$ ($i = 1, \dots, n$). The forgetful map $\Gamma_\infty^n \rightarrow \Gamma_\infty$ gives $H^\bullet(\Gamma_\infty^n)$ the structure of a module over this Hopf algebra. From the stability theorem one can deduce:

Theorem 2.2 (Looijenga [58]). *The algebra $H^\bullet(\Gamma_\infty^n; \mathbb{Z})$ is freely generated by the classes τ_1, \dots, τ_n as a graded-commutative $H^\bullet(\Gamma_\infty; \mathbb{Z})$ algebra.*

3. MODULI SPACES

A conformal structure and an orientation on S_g determine a complex structure on S_g . The *Teichmüller space* $\mathcal{X}_{g,r}^n$ is the space of conformal structures on S_g (with some reasonable topology) up to isotopies that fix $\{x_1, \dots, x_{n+r}\}$ pointwise and act trivially on the tangent spaces $T_x S$ for $i = n+1, \dots, n+r$. It is, in a natural way, a complex manifold of dimension $3g - 3 + n + 2r$. As a real manifold it is diffeomorphic to a cell. The group $\Gamma_{g,r}^n$ acts naturally on it. This action is properly discontinuous and a subgroup of finite index acts freely.

If Γ is any subgroup of $\Gamma_{g,r}^n$ that acts freely, then the orbit space $\Gamma \backslash \mathcal{X}_{g,r}^n$ is a classifying space for Γ and so its singular integral cohomology coincides with $H^\bullet(\Gamma; \mathbb{Z})$. This is even true with twisted coefficients: if V is a Γ module, then the trivial sheaf over $\mathcal{X}_{g,r}^n$ with fiber V comes with an obvious (diagonal) action of Γ . Passing to Γ orbits yields a locally constant sheaf \mathbb{V} on $\Gamma \backslash \mathcal{X}_{g,r}^n$. The cohomology of this sheaf equals $H^\bullet(\Gamma; V)$. For an arbitrary subgroup Γ of $\Gamma_{g,r}^n$ these statements still hold as long as we take our coefficients to be \mathbb{Q} vector spaces (but \mathbb{V} need no longer be locally constant). For $\Gamma = \Gamma_{g,r}^n$, we denote the orbit space by $\mathcal{M}_{g,r}^n$.

The space $\mathcal{M}_{g,r}^n$ is, in a natural way, a normal analytic space and the obvious forgetful maps such as $\mathcal{M}_{g,r}^n \rightarrow \mathcal{M}_g^n$ are analytic. An interpretation as a coarse moduli space makes it possible to lift this analytic structure to the algebraic category. To see this, we first choose a nonzero vector in each tangent space $T_x S_g$. Each triple $(C; x, v)$, where C is a connected nonsingular complex projective curve C of genus g , x an injective map $x : \{1, \dots, n+r\} \rightarrow C$, and v a nowhere zero section of TC over $\{n+1, \dots, n+r\}$, determines an element of $\mathcal{M}_{g,r}^n$. This point depends only on the isomorphism class of (C, x, v) with respect to the obvious notion of isomorphism. Since each conformal structure on S gives S the structure of a nonsingular complex projective curve, $\mathcal{M}_{g,r}^n$ can be identified with the space of isomorphism classes of such triples. From the work of Knudsen, Mumford and Deligne, we know that \mathcal{M}_g^n is, in a natural way, a quasi-projective

³He also tells us that he has checked that there are no stable primitive classes in degree 5.

orbifold. Recall that they also constructed a projective completion $\overline{\mathcal{M}}_g^n$ of \mathcal{M}_g^n , the *Deligne-Mumford completion* [12], that also admits the interpretation of a coarse moduli space. Its points parameterize the connected stable n pointed curves (C, x) of arithmetic genus g , where we now allow C to have ordinary double points, but still require x to map to the smooth part of C and the automorphism group of (C, x) to be finite. The *Deligne-Mumford boundary* $\overline{\mathcal{M}}_g^n - \mathcal{M}_g^n$ is a normal crossing divisor in the orbifold sense. There is a projective morphism $\overline{\mathcal{M}}_g^{n+1} \rightarrow \overline{\mathcal{M}}_g^n$, defined by forgetting the last point. It comes with n sections x_1, \dots, x_n . The fibers of this morphism are stable n pointed curves (modulo finite automorphism groups) and the morphism can be regarded as the universal stable n pointed curve (in an orbifold sense). Let ω denote the relative dualizing sheaf of this morphism, considered as a line bundle in the orbifold sense. We can then think of $\mathcal{M}_{g,r}^n$ as the set of $(v_{n+1}, \dots, v_{n+r})$ in the total space of $x_{n+1}^* \omega \oplus \dots \oplus x_{n+r}^* \omega$ restricted to \mathcal{M}_g^{n+r} that have each component nonzero. So $\mathcal{M}_{g,r}^n$ is also quasi-projective.

Each finite quotient group G of Γ_g^n determines, in an obvious way, a Galois cover $\mathcal{M}_g^n[G] \rightarrow \mathcal{M}_g^n$. The Deligne-Mumford completion $\overline{\mathcal{M}}_g^n[G]$ of this cover is, by definition, the normalization of $\overline{\mathcal{M}}_g^n$ in $\mathcal{M}_g^n[G]$.

Theorem 3.1 (Looijenga [55]). *There exists a finite group G such that $\overline{\mathcal{M}}_g[G]$ is smooth with a normal crossing divisor as Deligne-Mumford boundary.*

This has been extended by De Jong and Pikaart [74] to arbitrary characteristic, and by Boggi and Pikaart (independently) to the n -pointed case. (They show that it also can be arranged that each irreducible component of the Deligne-Mumford boundary of $\overline{\mathcal{M}}_g^n[G]$ is smooth.) This makes it relatively easy to define the Chow algebra of $\overline{\mathcal{M}}_g^n$: if $\overline{\mathcal{M}}_g^n[G]$ is smooth, then define $\mathrm{CH}^\bullet(\overline{\mathcal{M}}_g^n)$ to be the G invariant part of $\mathrm{CH}^\bullet(\overline{\mathcal{M}}_g^n[G])$ (we take algebraic cycles modulo rational equivalence with coefficients in \mathbb{Q}). It is easy to see that this is independent of the choice of G .

The central extension of Γ_g by \mathbb{Z} ($g \geq 3$) discussed in Section 2.1 takes the geometric form of a complex line bundle over Teichmüller space with Γ_g action and hence yields an orbifold line bundle over \mathcal{M}_g . Its twelfth tensor power has a concrete description: it is the determinant bundle of the direct image of the relative dualizing sheaf of $\mathcal{M}_g^1 \rightarrow \mathcal{M}_g$ (this is a rank g vector bundle). The orbifold fundamental group of the associated \mathbb{C}^\times bundle is just the central extension of Γ_g by $\frac{1}{12}\mathbb{Z}$ mentioned in Section 2.1.

Since Γ_g is perfect when $g \geq 3$, we have $H^1(\Gamma_g) = 0$. Ivanov has asked the following question:

Question 3.2 (Ivanov). Is it true that $H^1(\Gamma)$ vanishes for all finite index subgroups Γ of Γ_g , at least when g is sufficiently large?

This would imply that the Picard group of each finite unramified cover of \mathcal{M}_g (in the orbifold sense) is finitely generated. The answer to Ivanov's questions is affirmative, for example, for subgroups of finite index of Γ_g , $g \geq 3$, that contain the Torelli group — see (7.4).

4. ALGEBRO-GEOMETRIC STABILITY

The Deligne-Mumford completion $\overline{\mathcal{M}}_g^n$ comes with a natural stratification into orbifolds, with each stratum parameterizing stable n pointed curves of a fixed topological type T . Denote this stratum by $\mathcal{M}(T)$. It has codimension equal to the number of singular points of T . The normalization of the topological type T is an oriented closed surface S that comes with n distinct numbered points $X = \{x_1, \dots, x_n\}$ and a finite subset Y of $S - X$ with a fixed point free involution ι , so that T is recovered by identifying the points of Y according to ι . These topological data define a moduli space $\mathcal{M}(S)^{X \cup Y}$ of the same type (we hope that the notation is self-explanatory) and there is a natural morphism $\mathcal{M}(S)^{X \cup Y} \rightarrow \mathcal{M}(S/\iota)^X$ that is a Galois cover of orbifolds. This morphism extends to a finite surjective morphism from the Deligne-Mumford completion $\overline{\mathcal{M}}(S)^{X \cup Y}$ to the closure of $\mathcal{M}(T)$ in $\overline{\mathcal{M}}_g^n$. The resulting morphism $\overline{\mathcal{M}}(S)^{X \cup Y} \rightarrow \overline{\mathcal{M}}_g^n$ has only self-intersections of normal crossing type and so carries a normal bundle in the orbifold sense. This normal bundle is a direct sum of line bundles with one summand for each ι orbit $\{p, p'\}$, namely $p^*\omega^{-1} \otimes p'^*\omega^{-1}$. (To see this, notice that the restriction of the universal curve to $\overline{\mathcal{M}}(T)$ has a quadratic singularity along the locus defined by the pair $\{p, p'\}$. Associating to a local defining equation its hessian determines a natural isomorphism between $p^*\omega^{-1} \otimes p'^*\omega^{-1}$ and the normal bundle of a divisor in the Deligne-Mumford boundary passing through $\overline{\mathcal{M}}(T)$.) We now see before us an algebro-geometric incarnation of the map that appears in the stability theorem: the set of normal vectors that point towards the interior \mathcal{M}_g^n is the restriction to $\mathcal{M}(S)$ of the total space of the direct sum of \mathbb{C}^\times bundles in this normal bundle. So $\mathcal{M}(S)_Y^X$ maps to the latter space, and although we do not have a morphism $\mathcal{M}(S)_Y^X \rightarrow \mathcal{M}_g^n$, the map on cohomology behaves as if there were. In particular, the map $H^\bullet(\mathcal{M}_g^n) \rightarrow H^\bullet(\mathcal{M}(S)_Y^X)$ is a MHS morphism. So the stability theorem implies:

Theorem 4.1 (Algebro-geometric stability). *Suppose that the finite set X is contained in a connected component S' of S of genus g' , so that $\mathcal{M}(S')_{Y \cap S'}^X$ appears as a factor of $\mathcal{M}(S)_Y^X$. Choose points in the remaining factors so that we have an inclusion of $\mathcal{M}(S')_{Y \cap S'}^X$ in $\mathcal{M}(S)_Y^X$. Then for $k \leq cg'$ the composite map*

$$H^k(\mathcal{M}_g^n) \rightarrow H^k(\mathcal{M}(S)_Y^X) \rightarrow H^k(\mathcal{M}(S')_{Y \cap S'}^X)$$

is an isomorphism and so is the map

$$H^k(\mathcal{M}_{g'}^n) \rightarrow H^k(\mathcal{M}(S')_{Y \cap S'}^X)$$

induced by the forgetful morphism $\mathcal{M}(S')_{Y \cap S'}^X \rightarrow \mathcal{M}(S')^X \cong \mathcal{M}_{g'}^n$. These maps are also MHS morphisms.

So the stable rational cohomology $H^\bullet(\Gamma_\infty^n)$ comes with a natural MHS. A geometric consequence of this result is that each stable rational cohomology class of \mathcal{M}_g^n (that is, a class whose degree is in the stability range) extends across the open part of the blow up of $\overline{\mathcal{M}}(T)$ parameterizing the normal directions pointing towards the interior. Pikaart showed that these partial extensions can be made to come from a single extension to $\overline{\mathcal{M}}_g^n$, at least if g is large compared with k . But then it is not hard to show that if this is possible for large g , then it is possible in the stable range and so the conclusion is:

Theorem 4.2 (Pikaart [73]). *The restriction map $H^k(\overline{\mathcal{M}}_g^n) \rightarrow H^k(\mathcal{M}_g^n)$ is surjective in the stable range. Consequently, the MHS on $H^k(\Gamma_\infty^n)$ is pure of weight k .*

Mumford's Conjecture, if known, would imply this result, and so Pikaart's Theorem is evidence for the truth of this conjecture.

We illustrate this theorem with the known stable classes. We have seen in the previous section that $\overline{\mathcal{M}}_g^n$ comes with n orbifold line bundles $x_i^*\omega$, $i = 1, \dots, n$. Let $\overline{\tau}_{n,i}$ denote the first Chern class of this line bundle, regarded as an element of $\text{CH}^1(\overline{\mathcal{M}}_g^n)$. The restriction of this class to $\text{CH}^1(\mathcal{M}_g^n)$ is a pull-back of the restriction of $\overline{\tau}_{n-1,i}$ to \mathcal{M}_g^{n-1} (when $n \geq 1$) and so we denote that restriction simply by τ_i . The underlying cohomology class of τ_i in $H^{2i}(\mathcal{M}_g^n) \cong H^{2i}(\Gamma_g^n)$ is what we denoted earlier by that symbol, in particular, it is stable.

For the definition of the tautological classes of \mathcal{M}_g^n , we shall not use Mumford's original definition, but a modification proposed by Arbarello-Cornalba. This might begin with the observation that the "functor" which associates to an $(n+1)$ -pointed stable genus g curve $(C; x_1, \dots, x_n, x)$ the cotangent space T_x^*C defines an orbifold line bundle over $\overline{\mathcal{M}}_g^{n+1}$. It is not quite the same as the relative dualizing sheaf ω of the forgetful map $\overline{\mathcal{M}}_g^{n+1} \rightarrow \overline{\mathcal{M}}_g^n$: a little computation shows that it is in fact $\omega(\sum_{i=1}^n (x_i))$. This is perhaps a more natural bundle to consider than ω . In any case, we denote the direct image of $c_1(\omega(\sum_{i=1}^n (x_i))^{i+1} \in \text{CH}^{i+1}(\overline{\mathcal{M}}_g^{n+1}))$ under the projection $\overline{\mathcal{M}}_g^{n+1} \rightarrow \overline{\mathcal{M}}_g^n$ by $\overline{\kappa}_{n,i} \in \text{CH}^i(\overline{\mathcal{M}}_g^n)$ and its restriction to \mathcal{M}_g^n by $\kappa_{n,i}$. The cohomology class underlying $\kappa_{n,i}$ can be regarded as an element of $H^{2i}(\Gamma_g^n)$ (of Hodge bidegree (i, i)). These cohomology classes stabilize and, for $n = 0$, they define the nonzero primitive elements of degree $2i$ alluded to in 2.2.

We regard (for $k = 0, 1, \dots, n$) $\text{CH}^\bullet(\overline{\mathcal{M}}_g^n)$ as a $\text{CH}^\bullet(\overline{\mathcal{M}}_g^k)$ -algebra via the obvious forgetful morphism, and view the classes $\overline{\kappa}_{k,i}$ as elements of $\text{CH}^\bullet(\overline{\mathcal{M}}_g^n)$ when appropriate. The class $\overline{\kappa}_{n,i}$ is then not equal to $\overline{\kappa}_{n-1,i}$, but according to formula (1.10) of [4] we have:

$$\overline{\kappa}_{n,i} = \overline{\kappa}_{n-1,i} + (\overline{\tau}_{n,n})^i.$$

As Arbarello-Cornalba explain, the classes $\overline{\kappa}_{n,i}$ possess a nice property not enjoyed by Mumford's classes. First recall that every stratum of $\overline{\mathcal{M}}_g^n$ is the image of a finite map $\overline{\mathcal{M}}(S)^{X \cup Y} \rightarrow \overline{\mathcal{M}}_g^n$ and that $\overline{\mathcal{M}}(S)^{X \cup Y}$ is a product of varieties of the type $\overline{\mathcal{M}}_{g_\alpha}^{n_\alpha}$. The pull-back of $\overline{\kappa}_{n,i}$ along this map is the sum of the classes $\overline{\kappa}_{n_\alpha,i}$ (pulled back along the projection $\overline{\mathcal{M}}(S)^{X \cup Y} \rightarrow \overline{\mathcal{M}}_{g_\alpha}^{n_\alpha}$). Carel Faber pointed out to us that a similar property is enjoyed by the divisor class of the Deligne-Mumford boundary, but we know of no other examples. Since this behaviour is reminiscent of that of a primitive element in a Hopf algebra under the coproduct, we ask:

Question 4.3. What other collections $\left\{ \mu_{g,n} \in \text{CH}^k(\mathcal{M}_g^n) \right\}_{g,n}$ have this property?

4.1. Correspondences between moduli spaces. There is an altogether different way to relate the cohomology of the moduli spaces \mathcal{M}_g^n for different values of g . This involves certain Hecke type correspondences. For simplicity we shall restrict ourselves to the undecorated case $n = 0$. We return to the reference surface S_g and suppose that we are given a subgroup π of π_g of finite index d , say. (For what follows only its conjugacy class will matter.) This subgroup determines an unramified

finite covering $\tilde{S} \rightarrow S$ of closed oriented surfaces. The genus \tilde{g} of \tilde{S} is then equal to $d(g-1)+1$. Consider the group of pairs $(\tilde{h}, h) \in \text{Diff}^+(\tilde{S}) \times \text{Diff}^+(S)$ such that \tilde{h} is a lift of h . Let $\Gamma_g(\pi)$ be its group of connected components. The projection $\Gamma_g(\pi) \rightarrow \Gamma_g$ has as kernel the group of covering transformations of $\tilde{S} \rightarrow S$ (so is finite) and its image consists of the outer automorphisms of π_g that come from an automorphism which preserves the subgroup π (so is of finite index, e , say). There is a corresponding finite covering of moduli spaces $p_1 : \mathcal{M}_g(\pi) \rightarrow \mathcal{M}_g$, where $\mathcal{M}_g(\pi)$ is simply the coarse moduli space of finite unramified coverings of nonsingular complex projective curves $\tilde{C} \rightarrow C$ topologically equivalent to $\tilde{S} \rightarrow S$. There is also a finite map $p_2 : \mathcal{M}_g(\pi) \rightarrow \mathcal{M}_{\tilde{g}}$. Together they define a one-to-finite correspondence $p_2 p_1^{-1}$ from \mathcal{M}_g to $\mathcal{M}_{\tilde{g}}$. This extends over the Deligne-Mumford compactifications: if $p_1 : \overline{\mathcal{M}}_g(\pi) \rightarrow \overline{\mathcal{M}}_g$ denotes the normalization of $\overline{\mathcal{M}}_g$ in $\mathcal{M}_g(\pi)$, then p_2 extends to a finite morphism $p_2 : \overline{\mathcal{M}}_g(\pi) \rightarrow \overline{\mathcal{M}}_{\tilde{g}}$. We have an induced map

$$T_\pi := \epsilon^{-1} p_{1*} p_2^* : \text{CH}^\bullet(\overline{\mathcal{M}}_{\tilde{g}}) \rightarrow \text{CH}^\bullet(\overline{\mathcal{M}}_g)$$

and likewise on cohomology. A computation shows that any monomial in the tautological classes is an ‘‘eigen class’’ for such correspondences:

Proposition 4.4. *The map T_π sends $\overline{\kappa}_{i_1} \overline{\kappa}_{i_2} \cdots \overline{\kappa}_{i_r}$ to $d^r \overline{\kappa}_{i_1} \overline{\kappa}_{i_2} \cdots \overline{\kappa}_{i_r}$.*

This proposition suggests the consideration, for given positive integers r and s , of sequences of classes $(x_g \in \text{CH}^s(\overline{\mathcal{M}}_g))_{g \geq 2}$ of fixed degree that have the property that $T_\pi(x_g) = d^r x_{(d-1)g+1}$ for each index d subgroup π of π_g .

Question 4.5. Is for such a system the image of x_g in $H^\bullet(\mathcal{M}_g)$ stable? Is it in fact a polynomial of degree r in primitive stable classes?

An affirmative answer would give us a notion of stability for the Chow groups of the moduli spaces \mathcal{M}_g .

5. CHOW ALGEBRAS AND THE TAUTOLOGICAL CLASSES

We have already encountered some of the basic classes on $\overline{\mathcal{M}}_g^n$: the first Chern classes $\overline{\tau}_i \in \text{CH}^1(\overline{\mathcal{M}}_g^n)$ ($i = 1, \dots, n$) and the tautological classes $\overline{\kappa}_i$ ($i = 1, 2, \dots$). More such classes come from the boundary: if $\prod_i \overline{\mathcal{M}}_{g_i}^{n_i} \rightarrow \overline{\mathcal{M}}_g^n$ is a Galois covering of a stratum of the boundary as in Section 4, then we can add to these the push-forwards along this map of the exterior products of the corresponding classes on the factors. Let us call the subalgebra of $\text{CH}^\bullet(\overline{\mathcal{M}}_g^n)$ generated by all these classes the *tautological subalgebra* and denote it by $\mathcal{R}^\bullet(\overline{\mathcal{M}}_g^n)$. The image of this algebra in $\text{CH}^\bullet(\mathcal{M}_g^n)$ is denoted by $\mathcal{R}^\bullet(\mathcal{M}_g^n)$; it is generated by the classes $\kappa_{n,i}$ ($i = 1, 2, \dots$) and τ_i ($i = 1, \dots, n$). It is possible that these classes generate the rational Chow ring of $\overline{\mathcal{M}}_g^n$ modulo homological equivalence, but this is of course unknown. In any case, these subalgebras are preserved under pull-back and push-forward along the natural maps that we have met so far.

The first computations were done by Mumford [69] who found a presentation of $\text{CH}^\bullet(\overline{\mathcal{M}}_2)$. Subsequently Faber [13] calculated $\text{CH}^\bullet(\overline{\mathcal{M}}_2^1)$, $\text{CH}^\bullet(\overline{\mathcal{M}}_3)$ and obtained partial results on $\text{CH}^\bullet(\overline{\mathcal{M}}_4)$. In all these cases the tautological algebra is the whole Chow algebra.

This is also the case for $\overline{\mathcal{M}}_0^n$, whose Chow algebra was computed by Keel. This is a very remarkable algebra which appears in other contexts. Because of this, we describe it explicitly. We first introduce notation for the divisor classes on

$\overline{\mathcal{M}}_0^n$. The boundary divisor $\overline{\mathcal{M}}_0^n - \mathcal{M}_0^n$ parameterizes all singular stable n pointed rational curves. Its components correspond to the topological types of n pointed stable rational curves with exactly one singular point. Such curves have exactly two irreducible components. By collecting the points x_i lying on the same component, we obtain a partition P of $\{1, \dots, n\}$ into two subsets. The stability property implies that both members of P have at least two elements. We denote the corresponding class in $\text{CH}^1(\overline{\mathcal{M}}_0^n)$ by $D(P)$.

Theorem 5.1 (Keel [49]). *The Chow algebra $\text{CH}^\bullet(\overline{\mathcal{M}}_0^n)$ coincides with $H^\bullet(\overline{\mathcal{M}}_0^n)$ and, as a \mathbb{Q} algebra, is generated by the $D(P)$'s subject to the following relations:*

- (i) *If $\{i, j, k\}$ are distinct integers in $\{1, \dots, n\}$, then the sum of the $D(P)$'s for which P separates i from $\{j, k\}$ is independent of j and k (and equals $\overline{\tau}_i$).*
- (ii) *$D(P) \cdot D(P') = 0$ if P and P' are independent in the sense that the partition they generate has four nonempty members.*

The relations (ii) are geometrically obvious since the divisors $D(P)$ and $D(P')$ do not meet if P and P' are independent. The additive relations (i) are not difficult to see either: if C is a stable n pointed rational curve, then a moment of thought shows that there is a unique morphism $z : C \rightarrow \mathbb{P}^1$ that is an isomorphism on one irreducible component, constant on the other irreducible components, and is such that $z(x_i) = 1$, $z(x_j) = 0$ and $z(x_k) = \infty$. The differential $z^{-1}dz$ restricted to x_i defines a section of $x_i^* \omega$. The image of $z^{-1}dz$ in $T_{x_i}^* C$ vanishes precisely when z collapses the irreducible component containing x_i .

In [59], Manin derives a formula for the Poincaré polynomial of $\overline{\mathcal{M}}_0^n$. Such a formula was independently found by Getzler [17] who also obtained the \mathcal{S}_n equivariant Poincaré polynomial of $H^\bullet(\overline{\mathcal{M}}_0^n)$. That is, he determined the character of the \mathcal{S}_n representations $H^k(\overline{\mathcal{M}}_0^n)$, $k \geq 0$. Kaufmann [53] recently gave a formula for the intersection number of classes of strata of complementary dimension.

We now turn to the Chow and cohomology algebras of the moduli spaces \mathcal{M}_g . First we list some results about the Chow algebras.

$$\begin{aligned} \text{CH}^\bullet(\mathcal{M}_1^n) &= \mathbb{Q} \text{ for } n = 1, 2 \text{ (folklore)} \\ \text{CH}^\bullet(\mathcal{M}_2) &= \mathbb{Q} \text{ (folklore),} \\ \text{CH}^\bullet(\mathcal{M}_2^1) &= \mathbb{Q}[\tau]/(\tau^2) \text{ (Mumford [69]),} \\ \text{CH}^\bullet(\mathcal{M}_3) &= \mathbb{Q}[\kappa_1]/(\kappa_1^2) \text{ (Faber[13]),} \\ \text{CH}^\bullet(\mathcal{M}_3^1) &= \mathbb{Q}[\kappa_1, \tau]/(\kappa_1^2, 4\tau^2 - \tau\kappa_1) \text{ (Faber[13]),} \\ \text{CH}^\bullet(\mathcal{M}_4) &= \mathbb{Q}[\kappa_1]/(\kappa_1^3) \text{ (Faber [13]),} \\ \text{CH}^\bullet(\mathcal{M}_5) &= \mathbb{Q}[\kappa_1]/(\kappa_1^4) \text{ (Izadi [40] combined with Faber [14]).} \end{aligned}$$

The reason that such computations can be made is that, when g and n are both small, the moduli space \mathcal{M}_g^n has a concrete description. For example, when $g = 2$, each curve is hyperelliptic and therefore given by configuration of 6 points on the projective line. In the case $g = 3$ a nonhyperelliptic curve is realized by its canonical system as a quartic curve in \mathbb{P}^2 . The double cover of the projective plane along this curve is a Del Pezzo surface of degree 2, i.e., is obtained by blowing up 7 points in the plane in general position. General curves of genus 4 and 5 can be described as complete intersections of multidegrees $(2, 3)$ (in \mathbb{P}^3) and $(2, 2, 2)$ (in \mathbb{P}^4), respectively.

5.1. The tautological algebra of \mathcal{M}_g and Faber's Conjecture. On the basis of numerous calculations, Faber, around 1993, made the following conjecture.

Conjecture 5.2 (Faber[1]). *The tautological algebra $\mathcal{R}^\bullet(\mathcal{M}_g)$ is a graded Frobenius algebra with socle in degree $g-2$. That is, $\dim \mathcal{R}^{g-2}(\mathcal{M}_g) = 1$, and the intersection product defines a nondegenerate bilinear form $\mathcal{R}^i(\mathcal{M}_g) \times \mathcal{R}^{g-2-i}(\mathcal{M}_g) \rightarrow \mathcal{R}^{g-2}(\mathcal{M}_g)$ ($i = 0, \dots, g-2$). Moreover, κ_1 has the Lefschetz property in $\mathcal{R}^\bullet(\mathcal{M}_g)$ in the sense that multiplication by $(\kappa_1)^{g-2-2i}$ maps $\mathcal{R}^i(\mathcal{M}_g)$ isomorphically onto $\mathcal{R}^{g-2-i}(\mathcal{M}_g)$ for $0 \leq i \leq (g-2)/2$.*

Since the conjecture was made, evidence for it has been growing. For example:

Theorem 5.3 (Looijenga [57]). *The algebra $\mathcal{R}^\bullet(\mathcal{M}_g)$ is trivial in degree $> g-2$ and $\mathcal{R}^{g-2}(\mathcal{M}_g)$ is generated by the class of the hyperelliptic locus (a closed irreducible variety of codimension $g-2$).*

In particular $\kappa_1^{g-1} = 0$. Since κ_1 is ample on \mathcal{M}_g , we recover a theorem of Diaz [11] which asserts that every complete subvariety of \mathcal{M}_g must be of $\dim \leq g-2$.

Actually, in [57] a stronger result is proven, which, among other things, implies that $\mathcal{R}^k(\mathcal{M}_g^n) = 0$ for $k > g-2+n$. An induction argument then shows that $\mathcal{R}^{3g-3+n}(\overline{\mathcal{M}}_g^n)$ is spanned by the classes of the zero dimensional strata. But zero dimensional strata can be connected by one dimensional strata and the one dimensional strata are all rational. This shows that $\mathcal{R}^{3g-3+n}(\overline{\mathcal{M}}_g^n) \cong \mathbb{Q}$.⁴

Faber recently proved that the tautological class κ_{g-2} is nonzero. To describe his result, we find it convenient to introduce a compactly supported version of the tautological algebra: let $\mathcal{R}_c^\bullet(\mathcal{M}_g^n)$ be defined as the set of elements in $\mathcal{R}^\bullet(\overline{\mathcal{M}}_g^n)$ that restrict trivially to the Deligne-Mumford boundary. This is a graded ideal in $\mathcal{R}^\bullet(\overline{\mathcal{M}}_g^n)$ and the intersection product defines a map

$$\mathcal{R}^\bullet(\mathcal{M}_g^n) \times \mathcal{R}_c^\bullet(\mathcal{M}_g^n) \rightarrow \mathcal{R}_c^\bullet(\mathcal{M}_g^n)$$

that makes $\mathcal{R}_c^\bullet(\mathcal{M}_g^n)$ a $\mathcal{R}^\bullet(\mathcal{M}_g^n)$ -module. Notice that every complete subvariety of \mathcal{M}_g of codimension d whose class is in $\mathcal{R}^\bullet(\overline{\mathcal{M}}_g^n)$ defines a nonzero element of $\mathcal{R}_c^d(\mathcal{M}_g^n)$ (but it is by no means clear that such elements span $\mathcal{R}_c^d(\mathcal{M}_g^n)$). A somewhat stronger form of the first part of Faber's Conjecture is:

Conjecture 5.4. *The intersection pairings*

$$\mathcal{R}^k(\mathcal{M}_g) \times \mathcal{R}_c^{3g-3-k}(\mathcal{M}_g) \rightarrow \mathcal{R}_c^{3g-3}(\mathcal{M}_g) \cong \mathbb{Q}, \quad k = 0, 1, 2, \dots$$

are perfect (Poincaré duality) and $\mathcal{R}_c^\bullet(\mathcal{M}_g)$ is a free $\mathcal{R}^\bullet(\mathcal{M}_g)$ module of rank one.

Faber [14] finds a compactly supported class I_g in $\mathcal{R}_c^{2g-1}(\mathcal{M}_g)$ with $\kappa_{g-2} \cdot I_g \neq 0$. So $\mathcal{R}_c^\bullet(\mathcal{M}_g)$ should be the ideal generated by this element. Faber verified his conjecture for genera ≤ 15 by writing down many relations in $\mathcal{R}^\bullet(\mathcal{M}_g)$ (this evidently gives an upper bound) and using the nonvanishing of κ_{g-2} (this gives a surprisingly strong lower bound).

A refined form of Conjecture 5.4 (which we shall not state here) also takes care of the Lefschetz property.

Question 5.5. Does the tautological ring of $\overline{\mathcal{M}}_g^n$ satisfy Poincaré duality? Does it have the Lefschetz property with respect to $\overline{\kappa}_{n,1}$? (It is known that $\overline{\kappa}_{n,1}$ is ample [8].)

⁴(December 2000) Faber and Pandharipande have observed that this argument is incomplete; a correct argument has been recently found by Graber and Vakil (math.AG/0011100).

5.2. Cohomology of some moduli spaces. As may be expected, even less is known about the cohomology algebras. Here is an incomplete list of special results. In genus 0 we have that the Chow algebra of $\overline{\mathcal{M}}_0^n$ maps isomorphically onto its rational cohomology algebra. The cohomology of \mathcal{M}_0^n is easily computed if we start out from the observation that this space is the projective arrangement of type A_{n-2} . It then follows for instance, that its cohomology in degree p is of type (p, p) . There are similar descriptions of the moduli spaces of n -pointed hyperelliptic curves of genus g when $n = 0, 1, 2$ that involve arrangements of type A or D . These again should enable us to determine their rational cohomology ring, but it seems that this hasn't been done yet. In the same spirit arrangements of various types (among them E_6 and E_7) were used in [54] to prove that

$$H^\bullet(\mathcal{M}_3) = \text{CH}^\bullet(\mathcal{M}_3) + \mathbb{Q}u,$$

where u is a class of degree 6 of Hodge bidegree $(6, 6)$ and

$$H^\bullet(\mathcal{M}_3^1) = \text{CH}^\bullet(\mathcal{M}_3^1) + \mathbb{Q}u + \mathbb{Q}u\tau + \mathbb{Q}u\kappa_1 + \mathbb{Q}v,$$

where v is a class of degree 7 and of Hodge bidegree $(6, 6)$.

Question 5.6. The image of the tautological algebra in $H^{2p}(\mathcal{M}_g^n)$ consists of classes of type (p, p) . Are all such classes of this form?

A version of the Hodge conjecture asserts that the rational classes in degree $2p$ of type (p, p) are in the image of the Chow algebra, so modulo this conjecture we are asking whether every Chow class on \mathcal{M}_g^n is homologically equivalent to a tautological class.

6. THE RIBBON GRAPH PICTURE

Around 1981 Thurston, Mumford and Harer observed that partial completions of the Teichmüller spaces \mathcal{X}_g^n with $n > 0$ possess two natural Γ_g^n equivariant triangulations. One is based on the hyperbolic geometry of S_g^n (Thurston) and the other based on the singular euclidean geometry of S_g^n (Mumford, Harer). The last approach was actually a direct, but very powerful application of work that Jenkins and Strebel had done 10–20 years earlier. It is this approach that we shall explain.

The basic notion is that of a *ribbon graph*. This is a finite graph⁵ G together with a cyclic order on the set of oriented edges⁶ emanating from each vertex. As we shall see, there is a canonical construction of a surface that contains G and of which G is a deformation retract. This construction should explain the name. We first give a somewhat more abstract characterization of ribbon graphs which is very useful in some applications.

Let $X(G)$ be the set of oriented edges of G . Let σ_1 be the involution of $X(G)$ that reverses the orientation of each edge. The set $X_1(G)$ of σ_1 orbits can be identified with the set of edges of G . The cyclic orderings define another permutation σ_0 of $X(G)$ as follows. Each oriented edge e has an initial vertex $\text{in}(e)$ and a terminal vertex $\text{term}(e)$. Define $\sigma_0(e)$ to be the successor of e with respect to the given cyclic order on the set of oriented edges that have $\text{in}(e)$ as their initial vertex. The set of orbits $X_0(G)$ of σ_0 can be identified with the set of vertices of G . Put $\sigma_\infty := (\sigma_1\sigma_0)^{-1} = \sigma_0^{-1}\sigma_1$. Call an orbit of this permutation a *boundary cycle*.

⁵For us a graph is a cell complex of pure dimension one; its zero cells are called *vertices* and its one cells *edges*. So it has no isolated vertices.

⁶An oriented edge of a graph is an edge together with an orientation of it.

(Draw a picture to see why.) The set of boundary cycles will be denoted $X_\infty(G)$. These data form a complete invariant of G , for we can reverse the construction and associate to a finite nonempty set X endowed with a fixed point free involution σ_1 and a permutation σ_0 of X , a ribbon graph $G(X, \sigma_0, \sigma_1)$ whose oriented edges are indexed by X and such that σ_0 and σ_1 are the permutations defined above.

For every oriented edge e of G we form the one point compactification Δ_e of the half strip $e \times [0, \infty)$; this is just a 2-simplex, parameterized in an unusual way. We make identifications along the boundaries of these simplices with the help of σ_0 and $\sigma_\infty: e \times \{0\}$ is identified with $\sigma_1 e \times \{0\}$ and $\{\text{term}(e)\} \times [0, \infty)$ with $\{\text{in}(\sigma_\infty e)\} \times [0, \infty)$ (in either case, the identification map is essentially the identity). This is easily seen to be a compact, triangulated surface $S(G)$ that contains G as a subcomplex. Its vertex set can be identified with the disjoint union of the vertex set of G (so $X_0(G)$) and $X_\infty(G)$. We call vertices of the latter type *cusps*. Notice that G is a deformation retract of $S(G) - X_\infty(G)$ and that the surface is canonically oriented if we insist that the cyclic orderings of the edges emanating from each vertex are induced by the orientation. Let us say that the ribbon graph G is *n-pointed* if we are given an injection $y: \{1, \dots, n\} \hookrightarrow X_\infty(G) \cup X_0(G)$ whose image contains $X_\infty(G)$ and the vertices of valency ≤ 2 .

Suppose that we are given a *metric* l on G . That is, a function that assigns to every (unoriented) edge of X a positive real number. Give $[0, \infty)$ the standard metric and every half strip $e \times [0, \infty)$ the product metric. This defines (at least locally) a metric on $S(G)$. This metric is euclidean except possibly at the vertices. However, it is not difficult to show that the underlying conformal structure extends across all the vertices of $S(G)$ so that we end up with a compact Riemann surface $C(G, l)$. Notice that each cusp has a ‘‘circumference’’ — this is the length of the associated boundary cycle. It is clear that we get the same complex structure if l is replaced by a positive multiple of it and so we may just as well assume that the total length of G is 1. With this convention, the sum of the circumferences of the cusps is 2. The work of Jenkins and Strebel shows that all compact Riemann surfaces arise in this way:

Theorem 6.1 (Strebel [78]). *Let $(C; x: \{1, \dots, n\} \hookrightarrow C)$ be an n -pointed connected Riemann surface (so that the complement of the image of x has negative Euler characteristic as usual) and let c_1, \dots, c_n be nonnegative real numbers, not all zero. Then there exists an n -pointed metrized ribbon graph (G, y, l) , with $y(i)$ a cusp of G of circumference c_i when $c_i > 0$ and a vertex of G otherwise, such that $(C(G, l), y)$ and (C, x) are isomorphic as n -pointed Riemann surfaces. Moreover, (G, y, l) is unique up to the obvious notion of isomorphism.*

The results of Strebel also include a continuity property: a continuous variation of the complex structure on C corresponds to a continuous variation of (G, y, l) in a sense that we make precise. Denote by \mathcal{RG}_g^n the set of isomorphism classes of n -pointed ribbon graphs (G, y) that are marked in the sense that we are given an isotopy class of homeomorphisms $h: S_g \rightarrow S(G)$ with $h(x_i) = y(i)$, $i = 1, \dots, n$. On this set Γ_g^n acts, and it is easy to see that the number of orbits of markings is finite.

Suppose that $(G, y, [h])$ represents an element of \mathcal{RG}_g^n . Denote the geometric realization of the abstract simplex on the set $X_1(G)$ by $\Delta(G)$. Notice that the metrics l on G that give G unit length are parameterized by the interior of $\Delta(G)$. The circumferences of the cusps add up to two, so half the circumferences are the

barycentric coordinates of a simplicial projection $\lambda : \Delta(G) \rightarrow \Delta^{n-1}$. Let s be an edge of G that is not a loop and does not connect two vertices in the image of y . Then collapsing that edge yields a member $(G/s, y/s, [h]/s)$ of \mathcal{RG}_g^n . We can regard $\Delta(G/s)$ as a face of $\Delta(G)$. Making these identifications produces a simplicial complex which we will denote by $\widehat{\mathbb{X}}_g^n$. It comes with a simplicial map $\lambda : \widehat{\mathbb{X}}_g^n \rightarrow \Delta^{n-1}$. We have a simplicial action of Γ_g^n on $\widehat{\mathbb{X}}_g^n$ which preserves the fibers of λ . The union of relative interiors of simplices of $\widehat{\mathbb{X}}_g^n$ indexed by the elements of \mathcal{RG}_g^n is an open subset \mathbb{X}_g^n of $\widehat{\mathbb{X}}_g^n$. The results of Strebel can be strengthened to:

Proposition 6.2 (cf. [56]). *The above construction defines a Γ_g^n equivariant homeomorphism of \mathbb{X}_g^n onto $\mathcal{X}_g^n \times \Delta^{n-1}$.*

Now consider the quotient space

$$\widehat{\mathbb{M}}_g^n := \Gamma_g^n \backslash \widehat{\mathbb{X}}_g^n.$$

This is a finite simplicial orbicomplex that is equipped with a simplicial map $\lambda : \widehat{\mathbb{M}}_g^n \rightarrow \Delta^{n-1}$. We regard this complex as a compactification of its open subset $\mathbb{M}_g^n := \Gamma_g^n \backslash \mathbb{X}_g^n$. According to the above theorem, the latter is canonically homeomorphic with $\mathcal{M}_g^n \times \Delta^{n-1}$. This raises the question of how this compactification compares to that of Deligne-Mumford. The answer is essentially due to Kontsevich:

Theorem 6.3 (Kontsevich [50], see also [56]). *The simplicial orbicomplex $\widehat{\mathbb{M}}_g^n$ is a quotient space of $\overline{\mathcal{M}}_g^n \times \Delta^{n-1}$. Moreover, the part of $\widehat{\mathbb{M}}_g^n$ where $\lambda_i > 0$ carries an oriented piecewise linear circle bundle (in the orbifold sense) whose pull-back to $\overline{\mathcal{M}}_g^n \times \{\lambda \in \Delta^{n-1} | \lambda_i > 0\}$ is the oriented circle bundle coming from the standard line bundle $\overline{\tau}_i$. In particular, the part of $\widehat{\mathbb{M}}_g^n$ lying over the interior of Δ^{n-1} carries the tautological cohomology classes underlying $\overline{\tau}_{n,i}$, $i = 1, \dots, n$.*

The defining equivalence relation on $\overline{\mathcal{M}}_g^n \times \Delta^{n-1}$ is a little subtle and we refer to [56] for details regarding both statement and proof.⁷

This compactification of $\mathcal{M}_g^n \times \Delta^{n-1}$ plays a crucial rôle in Kontsevich's proof of the Witten conjectures. There are however earlier applications. These include Harer's stability theorem we met before, the computation of the Euler characteristic of \mathcal{M}_g^n , and the proof that Γ_g^n is a virtual duality group of dimension $4g - 4 + n$. We shall not explain the relation with stability here, but we will briefly touch on the other applications.

There is also a remarkable arithmetic aspect of ribbon graphs that is presently under intense investigation, but which we merely mention in passing. This is the observation, made by Grothendieck in a research proposal [20], that for a metrized ribbon graph (G, l) all of whose edges have equal length, the corresponding Riemann surface $C(G, l)$ is, in a canonical way, a ramified covering of the Riemann sphere \mathbb{P}^1 with ramification locus contained in $\{0, 1, \infty\}$. The graph G appears here as the preimage of the interval $[0, 1]$, its vertex set as the preimage of 0 and the set of cusps as the preimage of ∞ . The preimage of 1 consists of the midpoints of the edges. At these points we have simple ramification. A covering of this type is naturally an algebraic curve defined over some number field. Conversely, every connected covering of the Riemann sphere of this type arises in this manner. The

⁷The space used by Kontsevich is not quite $\widehat{\mathbb{M}}_g^n$, but basically the part lying over the interior of Δ^{n-1} times a half line. In this case the circumference map λ has image $(0, \infty)^n$.

absolute Galois group of \mathbb{Q} acts on the collection of isomorphism types of such coverings, and thus also on each finite set \mathcal{RG}_g^n . It is very difficult to come to grips with this action. For more information we refer to the collection [21] and to Grothendieck's manuscripts [19] and [20]. Since these metrized ribbon graphs represent the barycenters of the simplices of \mathbb{M}_g^n , one can also think of this as an action of the absolute Galois group on the simplices of \mathbb{M}_g^n , but the significance of this is not clear to us.

6.1. Virtual duality and virtual Euler characteristic. We first make some observations about simplicial complexes. Let K be a simplicial complex, L a subcomplex. Set $U := K - L$. Then U admits a canonical deformation retraction onto the union of the closed simplices of the barycentric subdivision of K that lie in U . This is a subcomplex, called the *spine* of U , whose k -simplices correspond to strictly increasing chains $\sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_k$ of simplices of K not in L . Further, if Γ is a group of automorphisms of K that preserves L , and if

- (i) U is contractible,
- (ii) a subgroup of Γ of finite index acts freely on U ,

then $\Gamma \backslash U$ is a simplicial 'orbicomplex' that is also a virtual classifying space for Γ .⁸ It has $\Gamma \backslash \text{spine}(U)$ as deformation retract, and so the dimension of this spine is an upper bound for the virtual homological dimension of Γ .

We apply this in the situation where K is the preimage of the first vertex of Δ^{n-1} in $\widehat{\mathbb{X}}_g^n$ under λ and $U = K \cap \mathbb{X}_g^n$. Notice that $U \cong \mathcal{X}_g^n$. The simplices meeting U are indexed by the elements of \mathcal{RG}_g^n with a single boundary cycle. A simple calculation shows that when $g \geq 1$, the number of edges of such a graph is at most $6g - 5 + 2n$ and at least $2g - 1 + n$. For $g = 0$ these numbers are $2n - 5$, resp. $n - 2$. So the spine of U has dimension $\leq 4g - 4 + n$, resp. $n - 3$. One can verify that this is, in fact, an equality. It follows that U admits a subcomplex of this dimension as an equivariant deformation retract. Hence:

Theorem 6.4 (Harer [28]). *If $n \geq 1$, then for every level structure, the moduli space $\mathcal{M}_g^n[G]$ contains a subcomplex of dimension $4g - 4 + n$ (when $g > 0$) or $n - 3$ (when $g = 0$) as a deformation retract.*

From this he deduces a similar result for the case when $n = 0$: $\mathcal{M}_g[G]$ has the homotopy type of complex of dimension $4g - 5$.

Problem 6.5. Is there a Lefschetz type of proof of this fact? For instance, the Lefschetz property would follow if one can find an orbifold stratification of \mathcal{M}_g^n with all strata affine subvarieties of codimension $\leq g$ ($n \geq 1$) or $\leq g - 1$ ($n = 0$). That would also show that the cohomological dimension of \mathcal{M}_g^n for quasicohereant sheaves is $\leq g - 1$ ($n \geq 1$) or $\leq g - 2$ ($n = 0$).

Let us return to the general situation considered earlier and suppose, in addition, that

- (iii) $\Gamma \backslash K$ is a finite complex, and
- (iv) U is a simplicial manifold of dimension d , say.

These conditions are satisfied in the case at hand. It is then natural (and standard) to assign to each simplex of K the weight that is the reciprocal of the order of its

⁸This means there is a normal subgroup $\Gamma_1 \subset \Gamma$ of finite index such that a Γ/Γ_1 -cover of this space classifies Γ_1 .

Γ stabilizer. This weighting is constant on orbits. Wall's Euler characteristic of Γ is simply the usual alternating sum of the number of Γ orbits of simplices not in L , except that each is counted with its weight. Equivalently, it is the orbifold Euler characteristic of the quotient $\Gamma \backslash K$.

In the present case, a ribbon graph G defining a member of \mathcal{RG}_g^n gives a contribution $|\text{Aut}(G)|^{-1}(-1)^{|X_1(G)|}$ to the virtual Euler characteristic. The computation of the resulting sum is a combinatorial problem that was first solved by Harer and Zagier. Kontsevich [50] later gave a shorter proof. The answer is:

Theorem 6.6 (Harer-Zagier [34]). *The orbifold Euler characteristic of \mathcal{M}_g^n equals*

$$(-1)^{n-1} \frac{(2g+n-3)!}{(2g-2)!} \zeta(1-2g).$$

Here ζ denotes the Riemann zeta function.

Harer and Zagier also find formulae for the actual Euler characteristics of \mathcal{M}_g^1 and \mathcal{M}_g . These are often negative so that there must be lot of cohomology in odd degrees.

For the discussion of virtual duality we go back to the general situation and assume that beyond the four conditions already imposed we have:

- (v) L has the homotopy type of a bouquet of r -spheres.

Then the theory of Bieri-Eckmann can be invoked in a virtual setting: if we set $D := \tilde{H}_r(L; \mathbb{Z})$ and regard D as a Γ module in an obvious way, then $H_{d-r-1}(\Gamma; D)$ is of rank one and for any Γ -module V with rational coefficients the cap products

$$\cap : H^k(\Gamma; V) \otimes H_{d-r-1}(\Gamma; D) \rightarrow H_{d-r-1-k}(\Gamma; V \otimes D), \quad k = 0, 1, 2, \dots$$

are isomorphisms. One calls D the *Steinberg module* of Γ .

Harer [28] proves that in the present case hypothesis (v) is satisfied: L is a subcomplex of dimension $2g - 3 - n$, resp. $n - 4$ which is $(2g - 4 - n)$ -connected, resp. $(n - 5)$ -connected when $g > 0$, resp. $g = 0$.

We shall call the corresponding orbifold local system \mathbb{D} over \mathcal{M}_g^n the *Steinberg sheaf*. The homology group $H_{4g-4+n}(\mathcal{M}_g^n; \mathbb{D})$ is of rank one. For every orbifold local system \mathbb{V} of rational vector spaces on \mathcal{M}_g^n , cap product with a generator of this homology group defines isomorphisms

$$H^k(\mathcal{M}_g^n; \mathbb{V}) \xrightarrow{\sim} H_{4g-4+n-k}(\mathcal{M}_g^n; \mathbb{V} \otimes \mathbb{D}), \quad k = 0, 1, 2, \dots$$

when $g > 0$ and $n > 0$ (and similar isomorphisms in the remaining cases). In particular, taking \mathbb{V} to be \mathbb{Q} , we see that $H_\bullet(\mathcal{M}_g^n; \mathbb{D})$ has a canonical MHS. This suggests that \mathbb{D} has some Hodge theoretic significance. Unfortunately it is not of finite rank, yet we wonder:

Question 6.7. Is the Steinberg sheaf motivic? In particular, does it have natural completions that carry (compatible) Hodge and étale structures?

6.2. Intersection numbers on the Deligne-Mumford completion. The intersection numbers in question are those defined by monomials in the $\bar{\tau}_i$'s. To be precise, define for every such monomial $\bar{\tau}_1^{d_1} \dots \bar{\tau}_n^{d_n}$ (with all $d_i \geq 0$) the intersection number $\int_{\mathcal{M}_g^n} \bar{\tau}_1^{d_1} \dots \bar{\tau}_n^{d_n}$ where g is chosen in such a way that this has a possibility of being nonzero: $3g - 3 + n = d_1 + \dots + d_n$. A physics interpretation suggests that

we should combine these numbers into the generating function

$$\sum_{n=1}^{\infty} \frac{1}{n!} \sum_{g > 1 - \frac{1}{2}n} \sum_{d_1 + \dots + d_n = 3g - 3 + n} t_{d_1} \dots t_{d_n} \int_{\mathcal{M}_g^n} \overline{\tau}_1^{d_1} \dots \overline{\tau}_n^{d_n}.$$

Now pass to a new set of variables T_1, T_3, T_5, \dots by setting

$$t_i = 1.3.5 \dots (2i+1) T_{2i+1}.$$

The resulting expansion $F(T_1, T_3, T_5, \dots)$ encodes all these intersection numbers.

Witten [80] conjectured two other characterizations of this function, both of which allow computation of its coefficients. These were proved by Kontsevich in his celebrated paper [50]. Perhaps the most useful characterization is the one which says that F is killed by a Lie algebra of differential operators isomorphic to the Lie algebra of polynomial vector fields in one variable. This Lie algebra comes with a basis $(L_k)_{k > -1}$ corresponding to the vector fields $(z^k \partial / \partial z)_{k > -1}$ and Witten verified the identities $L_k(F) = 0$ for $k = -1, 0$ within the realm of algebraic geometry. However no such proof is known for $k \geq 1$. Kontsevich's strategy is to represent the classes $\overline{\tau}_1^{d_1} \dots \overline{\tau}_n^{d_n}$ by piecewise differential forms on the ribbon graph model that can actually be integrated. This allows him to convert the intersection numbers into weighted sums over ribbon graphs. This leads to a new characterization of the generating function that is more manageable. Still a great deal of ingenuity is needed to complete the proof of Witten's Conjecture.

7. TORELLI GROUPS AND MODULI

In the early 80s, Dennis Johnson published a series of pioneering papers [42, 43, 44] on the Torelli groups. Although this work is in geometric topology, it has several interesting applications to algebraic geometry. Here we review some of his work.

First a remark on notation. In the remainder of the paper we will write V_g for the symplectic vector space $H_1(S_g)$ and $Sp_g(\mathbb{Z})$ for the group $\text{Aut}(H_1(S_g, \mathbb{Z}), \langle \cdot, \cdot \rangle)$; this does not really clash with standard notation, since a choice of a symplectic basis of $H_1(S_g; \mathbb{Z})$ identifies this with the standard integral symplectic group of genus g . Likewise, Sp_g will stand for the algebraic \mathbb{Q} -group defined by the symplectic transformations of V_g ; so its group of \mathbb{Q} -points, $Sp_g(\mathbb{Q})$, is just the group of symplectic automorphisms of V_g .

The mapping class group $\Gamma_{g,r}^n$ acts on the homology of the reference surface S_g . Since each of its elements preserves the orientation of S_g , we have a homomorphism

$$(1) \quad \Gamma_{g,r}^n \rightarrow Sp_g(\mathbb{Z}).$$

which is surjective. The *Torelli group* $T_{g,r}^n$ is defined to be its kernel⁹ so that we have an extension

$$1 \rightarrow T_{g,r}^n \rightarrow \Gamma_{g,r}^n \rightarrow Sp_g(\mathbb{Z}) \rightarrow 1.$$

The homology groups of $T_{g,r}^n$ are therefore $Sp_g(\mathbb{Z})$ modules.

The simplest kind of element of $T_{g,r}^n$ is a Dehn twist along a simple loop in S_g^{n+r} that separates S into two connected components. We call such a loop a *separating simple loop*. Another type of element of $T_{g,r}^n$ is determined by a *separating pair of simple loops*. This is a pair of two disjoint nonisotopic loops α_1, α_2 on S_g^{n+r} that together separate S into two connected components. The Dehn twist along

⁹Note that there is no general agreement on the definition of $T_{g,r}^n$ when $r+n > 1$.

α_1 composed with the inverse of the Dehn twist along α_2 is in $T_{g,r}^n$. The first of Johnson's results is:

Theorem 7.1 (Johnson [42, 43, 44]). *When $g \geq 3$, $T_{g,r}^n$ is generated by elements associated to a finite number of separating simple loops and a finite number of separating pairs of simple loops. If $[S_g] \in \wedge^2 H_1(S_g; \mathbb{Z})$ corresponds to the fundamental class of S_g , then there are natural $Sp_g(\mathbb{Z})$ equivariant surjective homomorphisms*

$$\tau_g^1 : T_g^1 \rightarrow \wedge^3 H_1(S_g; \mathbb{Z}) \text{ and } \tau_g : T_g \rightarrow \wedge^3 H_1(S_g; \mathbb{Z}) / ([S_g] \wedge H_1(S_g; \mathbb{Z})).$$

In both cases, the kernel of τ is the subgroup generated by the elements associated to simple separating loops. Finally, the kernels of the induced homomorphisms

$$H_1(T_g^1; \mathbb{Z}) \rightarrow \wedge^3 H_1(S_g; \mathbb{Z}) \text{ and } H_1(T_g; \mathbb{Z}) \rightarrow \wedge^3 H_1(S_g; \mathbb{Z}) / ([S_g] \wedge H_1(S_g; \mathbb{Z}))$$

are both 2-torsion.

Johnson also finds an explicit description of this 2-torsion. We will give it in a moment, but first we want to point out an algebro-geometric consequence of this theorem. Let $\widetilde{\mathcal{M}}_g \subset \overline{\mathcal{M}}_g$ be the complement of the irreducible divisor whose generic point parametrizes irreducible singular stable curves, and let $\widetilde{\mathcal{M}}_g^1$ be its preimage in $\overline{\mathcal{M}}_g^1$.

Corollary 7.2. *When $g \geq 3$, the orbifold fundamental group of $\widetilde{\mathcal{M}}_g$ (resp. $\widetilde{\mathcal{M}}_g^1$) is isomorphic to an extension of $Sp_g(\mathbb{Z})$ by $\wedge^3 H_1(S_g; \mathbb{Z}) / ([S_g] \wedge H_1(S_g; \mathbb{Z}))$ (resp. $\wedge^3 H_1(S_g; \mathbb{Z})$).*

Johnson's theorem shows that the $Sp_g(\mathbb{Z})$ action on $H_1(T_{g,r}^n)$ is the restriction of a representation of the algebraic group Sp_g . We shall see shortly the importance of this property. Let $\lambda_1, \lambda_2, \dots, \lambda_g$ be a fundamental set of weights of Sp_g so that λ_j corresponds to the j th fundamental representation of Sp_g . This last representation can be realized as the natural Sp_g action on the primitive part of $\wedge^j V_g$.

The next result follows from Johnson's Theorem by standard arguments.

Corollary 7.3. *For each $g \geq 3$, there is a natural $Sp_g(\mathbb{Z})$ equivariant isomorphism*

$$\tau_{g,r}^n : H^1(T_{g,r}^n) \xrightarrow{\sim} V(\lambda_3) \oplus V(\lambda_1)^{\oplus(r+n)}.$$

A theorem of Ragnathan [76] implies that when $g \geq 2$, the first cohomology of each finite index subgroup of $Sp_g(\mathbb{Z})$ with coefficients in a rational representation of $Sp_g(\mathbb{Q})$ vanishes. So Johnson's computation also gives:

Corollary 7.4. *If $g \geq 3$, then every finite index subgroup of $\Gamma_{g,r}^n$ that contains $T_{g,r}^n$ has zero first Betti number.*

The situation is very different when $g < 3$. The Torelli groups T_1 and T_1^1 are trivial, while Geoff Mess [60] proved that when $g = 2$, T_2 is a countably generated free group. He also computed $H_1(T_2; \mathbb{Z})$. It is the $Sp_2(\mathbb{Z})$ module obtained by inducing the trivial representation up to $Sp_2(\mathbb{Z})$ from the stabilizer $(\mathbb{Z}/2) \times (SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z}))$ of a decomposition of $H_1(S_2; \mathbb{Z})$ into two symplectic modules each of rank 2. (We shall sketch a proof in the next subsection.) It is still unknown whether, for any $g \geq 3$, T_g is finitely presented.

Problem 7.5. Determine whether T_g is finitely presented when g is sufficiently large.

Next, we describe Johnson's computation of the torsion in $H_1(T_g; \mathbb{Z})$. Denote the field of two elements by \mathbb{F}_2 . Recall that an \mathbb{F}_2 quadratic form on $H_1(S_g; \mathbb{F}_2)$ associated to the mod two symplectic form $\langle \cdot, \cdot \rangle$ on $H_1(S_g; \mathbb{F}_2)$ is a function $\omega : H_1(S_g; \mathbb{F}_2) \rightarrow \mathbb{F}_2$ satisfying

$$\omega(a + b) = \omega(a) + \omega(b) + \langle a, b \rangle.$$

The difference between any two such is an element of $H^1(S_g; \mathbb{F}_2)$. This makes the set Ω_g of such quadratic forms an affine space over the \mathbb{F}_2 vector space $H^1(S_g; \mathbb{F}_2)$. Denote the algebra of \mathbb{F}_2 valued functions on Ω_g by $S\Omega_g$. All such functions are given by polynomials and so we have a filtration

$$\mathbb{F}_2 = S_0\Omega_g \subset S_1\Omega_g \subset S_2\Omega_g \subset \cdots \subset S\Omega_g,$$

where $S_d\Omega_g$ denotes the space of polynomial functions of degree $\leq d$. Since $f = f^2$ for each $f \in S\Omega_g$, the associated graded algebra is naturally isomorphic to the exterior algebra $\wedge^\bullet H_1(S_g; \mathbb{F}_2)$. The algebra $S\Omega_g$ has as a distinguished element which is called the *Arf invariant*, denoted here by arf . If $a_1, \dots, a_g, b_1, \dots, b_g$ is a symplectic basis of $H^1(S; \mathbb{F}_2)$, then arf is defined by

$$\text{arf} : \omega \mapsto \sum_i \omega(a_i)\omega(b_i).$$

It is an element of $S_2\Omega_g$, and its zero set Ψ_g is an affine quadric in Ω_g . Let $S_d\Psi_g$ denote the image of $S_d\Omega_g$ in the set of \mathbb{F}_2 valued functions on Ψ_g .

Theorem 7.6 (Johnson [44]). *There are natural isomorphisms*

$$\begin{aligned} \sigma_{g,1} : H_1(T_{g,1}; \mathbb{F}_2) &\xrightarrow{\sim} S_3\Omega_g, & \sigma_g^1 : H_1(T_g^1; \mathbb{F}_2) &\xrightarrow{\sim} S_3\Omega_g/\mathbb{F}_2 \text{arf}, \\ \sigma_g : H_1(T_g; \mathbb{F}_2) &\xrightarrow{\sim} S_3\Psi_g \end{aligned}$$

which are equivariant with respect to the $Sp_g(\mathbb{F}_2)$ -action. These induce natural isomorphisms

$$H_1(T_{g,1}; \mathbb{Z})_{\text{tor}} \cong S_2\Omega_g, \quad H_1(T_g^1; \mathbb{Z})_{\text{tor}} \cong H_1(T_g; \mathbb{Z})_{\text{tor}} \cong S_2\Psi_g.$$

Moreover, the natural isomorphisms

$$\phi_{g,r}^n : H_1(T_{g,r}^n; \mathbb{F}_2)/H_1(T_{g,r}^n; \mathbb{Z})_{\text{tor}} \xrightarrow{\sim} [H_1(T_{g,r}^n; \mathbb{Z})/\text{torsion}] \otimes \mathbb{F}_2$$

correspond, under the isomorphisms $\sigma_{g,r}^n$ and $\tau_{g,r}^n$, to the obvious isomorphisms

$$\begin{aligned} \phi_{g,1} : S_3\Omega_g/S_2\Omega_g &\xrightarrow{\sim} \wedge^3 H_1(S_g; \mathbb{F}_2), & \phi_g^1 : S_3\Omega_g/(\mathbb{F}_2 \text{arf} + S_2\Omega_g) &\xrightarrow{\sim} \wedge^3 H_1(S_g; \mathbb{F}_2), \\ \phi_g : S_3\Psi_g/S_2\Psi_g &\xrightarrow{\sim} \wedge^3 H_1(S_g; \mathbb{F}_2)/([S_g] \wedge H_1(S_g; \mathbb{F}_2)). \end{aligned}$$

The homomorphisms τ_g^1 and τ_g admit direct conceptual definitions that we will give later. Here we give a formula for the image of the standard generators of $T_{g,1}$ in $\wedge^3 H_1(S_g; \mathbb{Z})$ and in $S_3\Omega_g$ under $\tau_{g,1}$ and $\sigma_{g,1}$, respectively.

Let (α_1, α_2) be a separating pair of simple loops. Let t be the corresponding element of $T_{g,1}$ — recall that this is the product of the Dehn twist about α_1 and the *inverse* of the Dehn twist about α_2 . The two loops decompose S_g into two pieces S' and S'' , say, where we suppose that S' contains the point x_1 . We orient α_1 and α_2 as boundary components of S'' . The resulting cycles are opposite in $H_1(S''; \mathbb{Z})$: $[\alpha_2] = -[\alpha_1]$, and each spans the radical of the intersection pairing on this group. So there is a well-defined element in $\wedge^2 H_1(S''; \mathbb{Z})/[\alpha_1] \wedge H_1(S''; \mathbb{Z})$ representing the intersection pairing on $H_1(S''; \mathbb{Z})$. Its wedge with $[\alpha_1]$ can be regarded as an element of $\wedge^3 H_1(S''; \mathbb{Z})$. Since the inclusion $S'' \subset S_g$ induces an injection on first

homology, we can also view the latter as an element of $\wedge^3 H_1(S_g; \mathbb{Z})$. This is the element $\tau_{g,1}(t)$; it is clear that it only depends on the image of t in T_g^1 .

Next we associate to t a function $\sigma_t : \Omega_g \rightarrow \mathbb{F}_2$ as follows. If $\omega \in \Omega_g$ takes the value 1 on $[\alpha]$, then we put $\sigma_t(\omega) = 0$; if it takes the value 0 on $[\alpha]$, then the restriction of ω to $H_1(S''; \mathbb{F}_2)$ factors through a nondegenerate quadratic function on $H_1(S''; \mathbb{F}_2)/\mathbb{F}_2[\alpha]$. Then $\sigma_{g,1}(t)(\omega)$ is its Arf invariant. It can be shown that $\sigma_{g,1}(t)$ lies in $S_3\Omega$.

Now suppose that t is the element of $T_{g,1}$ associated to a separating simple loop α . Denote the pieces S' and S'' as before. In this case, $\tau_{g,1}(t)$ is trivial and $\sigma_{g,1}(t)$ is the element of $S\Omega_g$ that assigns to ω the Arf invariant of its restriction to $H_1(S''; \mathbb{F}_2)$. Notice that if α is a simple loop around x_1 , then $\sigma_{g,1}(t)$ is just the function arf. (This explains why we mod out by this function when passing from $T_{g,1}$ to T_g^1 .)

Without a base point there is no way of telling S' and S'' apart. It is because of this ambiguity that we have to restrict functions to Ψ_g in order to obtain a well defined function.

A diffeomorphism of S_g onto a smooth projective curve C determines a natural isomorphism between Ω_g and the space of theta characteristics of C (i.e., square roots of the canonical bundle K_C ; see for instance Appendix B of [2]). This suggests that Johnson's computation should have an algebro-geometric interpretation, if not interesting applications to the geometry of curves.

Problem 7.7. Give an algebro-geometric construction of the epimorphism $T_g \rightarrow S_3\Omega$.

Van Geemen has suggested such a construction (unpublished).

7.1. Torelli space and period space. The group T_g acts freely on \mathcal{X}_g . The quotient \mathcal{T}_g is therefore a complex manifold. It is called *Torelli space*. According to the discussion at the beginning of Section 3, \mathcal{T}_g is then a classifying space for T_g so that there is a canonical isomorphism $H_\bullet(T_g; \mathbb{Z}) \cong H_\bullet(\mathcal{T}_g; \mathbb{Z})$. Torelli space has a moduli interpretation; it is the moduli space of smooth projective curves C of genus g together with a symplectic isomorphism

$$\gamma : H_1(S_g; \mathbb{Z}) \rightarrow H_1(C; \mathbb{Z}).$$

There are also decorated versions $\mathcal{T}_{g,r}^n$ of Torelli space. Their points are points of $\mathcal{M}_{g,r}^n$ together with a symplectic isomorphism γ of $H_1(S; \mathbb{Z})$ with the first homology of the curve corresponding to the point of \mathcal{M}_g . It is clear that the map $\mathcal{T}_{g,r}^n \rightarrow \mathcal{M}_{g,r}^n$ is Galois with Galois group $Sp_g(\mathbb{Z})$.

Denote the Siegel space associated to V_g by \mathfrak{h}_g . To be precise, \mathfrak{h}_g is the set of pure Hodge structures on V_g with Hodge numbers $(-1, 0)$ and $(0, -1)$, polarized by the intersection form. This is a contractible complex manifold of dimension $g(g+1)/2$ on which the group $Sp_g(\mathbb{R})$ acts properly and transitively. We can also regard \mathfrak{h}_g as the moduli space of pairs consisting of a g dimensional principally polarized abelian variety A plus a symplectic isomorphism

$$\gamma : H_1(S; \mathbb{Z}) \rightarrow H_1(A; \mathbb{Z}).$$

This interprets the $Sp_g(\mathbb{Z})$ orbit space of \mathfrak{h}_g as the moduli space of principally polarized abelian varieties of dimension g , \mathcal{A}_g . We regard \mathcal{A}_g as an orbifold with orbifold fundamental group $Sp_g(\mathbb{Z})$, although $Sp_g(\mathbb{Z})$ does not act faithfully on \mathfrak{h}_g . The kernel of this action is $\{\pm 1\}$.

Assigning to a smooth projective curve the Hodge structure on its first homology group defines a map $\mathcal{T}_g \rightarrow \mathfrak{h}_g$, the *period map* for T_g . It is an isomorphism in genus 1, an open imbedding when $g = 2$, and 2:1 with ramification along the hyperelliptic locus when $g \geq 3$.¹⁰ The reason for this is that for all abelian varieties we have the equality

$$[A; \gamma] = [A; -\gamma]$$

of points of \mathfrak{h}_g as $-\text{id}$ is an automorphism of each abelian variety. On the other hand, we have the equality

$$[C; \gamma] = [C; -\gamma]$$

of points of \mathcal{T}_g if and only if C is hyperelliptic.

Mess's result (mentioned at the beginning of the section) can now be deduced from this: \mathcal{T}_2 is the complement in \mathfrak{h}_2 of the locus of principally polarized abelian varieties that are products of two elliptic curves. The locus of such reducible abelian varieties is a countable disjoint union of copies of $\mathfrak{h}_1 \times \mathfrak{h}_1$. The group $Sp_2(\mathbb{Z})$ permutes them transitively, and each is stabilized by a product of two copies of $SL_2(\mathbb{Z})$ and an involution that switches the two copies of the upper half plane. Mess's result follows easily using the stratified Morse theory of Goresky and MacPherson — use distance from a generic point of \mathfrak{h}_2 as the Morse function. Since each component of $\mathfrak{h}_2 - \mathcal{T}_2$ is a totally geodesic divisor, the distance function has a unique critical point (necessarily a minimum) on each stratum. It follows that \mathcal{T}_2 has the homotopy type of a wedge of circles, one for each component of $\mathfrak{h}_2 - \mathcal{T}_2$.

The period map gives, after passage to $Sp_g(\mathbb{Z})$ orbit spaces, a morphism $\mathcal{M}_g \rightarrow \mathcal{A}_g$, the period mapping for \mathcal{M}_g . This period mapping extends to the partial completion $\widetilde{\mathcal{M}}_g$ of \mathcal{M}_g and the resulting map $\widetilde{\mathcal{M}}_g \rightarrow \mathcal{A}_g$ is proper.

Now assume $g \geq 3$ and denote the image of the period map $\mathcal{T}_g \rightarrow \mathfrak{h}_g$ by \mathcal{S}_g . This space is the quotient of \mathcal{T}_g by the subgroup $\{\pm 1\}$ of $Sp_g(\mathbb{Z})$. Consequently

$$H^\bullet(\mathcal{S}_g) \cong H^\bullet(T_g)^{\{\pm 1\}}.$$

Observe that \mathcal{S}_g is a locally closed analytic subvariety of \mathfrak{h}_g , but not closed. The $\{\pm 1\}$ cover $\mathcal{T}_g \rightarrow \mathcal{S}_g$ extends as a $\{\pm 1\}$ cover $\overline{\mathcal{T}}_g \rightarrow \overline{\mathcal{S}}_g$ over the closure of \mathcal{S}_g in \mathfrak{h}_g , and the $\{\pm 1\}$ action on the total space is the restriction of an $Sp_g(\mathbb{Z})$ action. Both $\overline{\mathcal{T}}_g$ and $\overline{\mathcal{S}}_g$ are rather singular along the added locus (which is of codimension 3). If we pass to $Sp_g(\mathbb{Z})$ orbit spaces, then the natural map

$$\widetilde{\mathcal{M}}_g \rightarrow Sp_g(\mathbb{Z}) \backslash \overline{\mathcal{T}}_g \cong Sp_g(\mathbb{Z}) \backslash \overline{\mathcal{S}}_g$$

resolves these singularities in an orbifold sense. A resolution of a normal analytic variety always induces a surjection on fundamental groups and so it follows from (7.1) that the fundamental group of $\overline{\mathcal{T}}_g$ is abelian and is $Sp_g(\mathbb{Z})$ equivariantly a quotient of $\Lambda^3 H_1(S_g; \mathbb{Z}) / ([S_g] \wedge H_1(S_g; \mathbb{Z}))$.

Problem 7.8. Understand the topology of \mathcal{S}_g and its closure $\overline{\mathcal{S}}_g$ in \mathfrak{h}_g . In particular, how close is \mathcal{S}_g to being a finite complex? (Observe that if it has a finite 2-skeleton, then T_g is finitely presented.)

Related, but formally independent of this problem, is the question of whether the cohomology of T_g stabilizes in a suitable sense:

¹⁰It is stated incorrectly in [24] that $\mathcal{T}_2 \rightarrow \mathfrak{h}_2$ is an unramified 2:1 map onto its image.

Question 7.9. Is $H^k(T_g)$ expressible as an $Sp_g(\mathbb{Z})$ module in a manner that is independent of g if g is large enough? For example, from Johnson's Theorem, we know that $H^1(T_g)$ is the third fundamental representation of Sp_g for all $g \geq 3$.

7.2. The Johnson homomorphism. The proof of Johnson's Theorem is non-trivial and uses geometric topology, but the homomorphism τ_g^1 is easily described.

Since T_g is torsion free, the projection $\mathcal{T}_g^1 \rightarrow \mathcal{T}_g$ defines the universal curve over \mathcal{T}_g . Denote the corresponding bundle of jacobians by $\mathcal{J}_g \rightarrow \mathcal{T}_g$. Since the local system of first homology groups associated to the universal curve is canonically framed, this jacobian bundle $\mathcal{J}_g \rightarrow \mathcal{T}_g$ is analytically trivial as a bundle of Lie groups: we have a natural trivializing projection $p : \mathcal{J}_g \rightarrow \text{Jac } S_g$, where $\text{Jac } S_g := H_1(S_g; \mathbb{R}/\mathbb{Z})$ is the "jacobian" of the reference surface.

The usual Abel-Jacobi map, which assigns to an ordered pair of points (x, y) on a smooth curve C the divisor class of $(x) - (y)$, induces a morphism

$$\mathcal{T}_g^1 \times_{\mathcal{T}_g} \mathcal{T}_g^1 \rightarrow \mathcal{J}_g.$$

over \mathcal{T}_g . This provides a correspondence

$$\begin{array}{ccc} \mathcal{T}_g^1 \times_{\mathcal{T}_g} \mathcal{T}_g^1 & \longrightarrow & \mathcal{J}_g \xrightarrow{p} \text{Jac } S_g \\ \downarrow pr_2 & & \\ \mathcal{T}_g^1 & & \end{array}$$

from \mathcal{T}_g^1 to $\text{Jac } S_g$. It induces homomorphisms

$$H_k(T_g^1) \cong H_k(\mathcal{T}_g^1) \rightarrow H_{k+2}(\text{Jac } S_g).$$

The first of these is the Johnson homomorphism

$$\tau_g^1 : H_1(T_g^1) \rightarrow H_3(\text{Jac } S_g)$$

for T_g^1 . Since $\mathcal{T}_g^1 \rightarrow \mathcal{T}_g$ is a fibration of Eilenberg-MacLane spaces, we have an exact sequence of fundamental groups:

$$1 \rightarrow \pi_g \rightarrow T_g^1 \rightarrow T_g \rightarrow 1.$$

This induces an exact sequence

$$H_1(S_g; \mathbb{Z}) \rightarrow H_1(T_g^1; \mathbb{Z}) \rightarrow H_1(T_g; \mathbb{Z}) \rightarrow 0$$

on homology. Since $\text{Jac } S_g$ is a topological group with torsion free homology, its integral homology has a product — the *Pontrjagin product*. It is not difficult to check that the composite

$$H_1(S_g; \mathbb{Z}) \rightarrow H_1(T_g^1; \mathbb{Z}) \rightarrow H_3(\text{Jac } S_g; \mathbb{Z})$$

is the map given by Pontrjagin product with the class $[S_g]$. It follows that there is a natural homomorphism

$$H_1(T_g; \mathbb{Z}) \rightarrow H_3(\text{Jac } S_g; \mathbb{Z}) / ([S_g] \times H_1(S_g; \mathbb{Z})).$$

This is the Johnson homomorphism τ_g for T_g .

Johnson's Theorem, alone and in concert with Saito's theory of Hodge modules, has several interesting applications to the geometry of moduli spaces of curves as we shall see in subsequent sections.

7.3. Monodromy of roots of the canonical bundle. In this subsection we assume that $g \geq 2$. Suppose that C is a smooth projective curve of genus g . Since its canonical bundle K_C is of degree $2g - 2$ and since $\text{Pic}^0 C$ is a divisible group, K_C has n th roots whenever n divides $2g - 2$. Any two such n th roots will differ by an n torsion point of $\text{Pic}^0 C$. Because of this, n th roots of K_C are rigid under deformation. It follows that they form a locally constant sheaf (in the orbifold sense) Root^n over \mathcal{M}_g . The fiber over C , denoted $\text{Root}^n C$, is a principal homogenous space over $H_1(C; \mathbb{Z}/n)$, the group of n torsion points of $\text{Pic}^0 C$.

Choose a conformal structure on S_g . Denote the corresponding algebraic curve by C . Sipe [77] determined the monodromy representation

$$\rho^n : \Gamma_g \rightarrow \text{Aut Root}^n(C).$$

of this sheaf. Before giving it, we make some remarks. Since the Torelli group acts trivially on the n torsion of $\text{Pic}^0 C$, it follows that the restriction of ρ^n to T_g factors through a representation $T_g \rightarrow H_1(S; \mathbb{Z}/(2g - 2)) \rightarrow H_1(S; \mathbb{Z}/n)$. However, the action of Γ_g on the set $\text{Root}^2 C$ of square roots of K_C (the set of theta characteristics of C) factorizes through $Sp_g(\mathbb{Z})$ also (even through $Sp_g(\mathbb{F}_2)$) — this is because there is a canonical correspondence between square roots of K_C and \mathbb{F}_2 quadratic forms on $H_1(S; \mathbb{F}_2)$ associated to the intersection form. It follows that the image of the monodromy representation ρ^n will be contained in an extension of $Sp_g(\mathbb{Z}/n)$ by a subgroup of $2 \cdot H^1(S_g; \mathbb{Z}/n)$. In fact, it is all of this group.

Theorem 7.10 (Sipe [77]). *The monodromy group of Root^n is an extension of $Sp_g(\mathbb{Z}/n)$ by the subgroup $2 \cdot H^1(S_g; \mathbb{Z}/n)$ of $H_1(S_g; \mathbb{Z}/n)$.*

The subgroup $2 \cdot H^1(S_g; \mathbb{Z}/n)$ appears as a quotient of the Torelli group T_g . In [24] it is shown that the restriction of the monodromy representation to T_g is the composite of the Johnson homomorphism with a natural surjection

$$\wedge^3 H_1(S_g; \mathbb{Z}) / ([S_g] \wedge H_1(S_g; \mathbb{Z}) \rightarrow H_1(S_g; \mathbb{Z}/(g - 1)) \rightarrow 2 \cdot H_1(S_g; \mathbb{Z}/n).$$

7.4. Picard groups of level covers. Denote the moduli space of smooth projective genus g curves with a level l structure by $\mathcal{M}_g[l]$. This is convenient shorthand for the notation $\mathcal{M}_g[Sp_g(\mathbb{Z}/l)]$ introduced in Section 3. Denote the kernel of the reduction mod l map

$$Sp_g(\mathbb{Z}) \rightarrow Sp_g(\mathbb{Z}/l)$$

by $Sp_g(\mathbb{Z})[l]$, and its full inverse image in Γ_g by $\Gamma_g[l]$. Then $\mathcal{M}_g[l]$ is the quotient of Teichmüller space \mathcal{X}_g by $\Gamma_g[l]$. As in the case of \mathcal{M}_g , there is a canonical isomorphism

$$H^\bullet(\mathcal{M}_g[l]) \cong H^\bullet(\Gamma_g[l]).$$

This holds with rational coefficients for all l , and arbitrary coefficients whenever $\Gamma_g[l]$ is torsion free, which holds whenever $Sp_g(\mathbb{Z})[l]$ is torsion free — $l \geq 3$.

We know from (7.4) that

$$H^1(\Gamma_g[l]) \cong H^1(\mathcal{M}_g[l]) = 0$$

when $g \geq 3$. By standard arguments (cf. [24, §5]), this implies that

$$c_1 : \text{Pic } \mathcal{M}_g[l] \otimes \mathbb{Q} \rightarrow H^2(\mathcal{M}_g[l])$$

is injective, and therefore that $\text{Pic } \mathcal{M}_g[l]$ is finitely generated when $g \geq 3$.

The stable cohomology of an arithmetic group depends only on the ambient real algebraic group [5]. Based on this, one might expect that the natural map

$$H^k(\Gamma_g) \rightarrow H^k(\Gamma_g[l])$$

is an isomorphism for all $l \geq 0$, once the genus g is sufficiently large compared to the degree k . It follows from Johnson's work that this is true when $k = 1$ (cf. [24]), but the only evidence for it when $k > 1$ is Harer's computation of the second homology of the spin mapping class groups [33], and Foisy's theorem from which Harer's computation now follows:

Theorem 7.11 (Foisy [15]). *For all $g \geq 3$, the natural map $H^2(\Gamma_g) \rightarrow H^2(\Gamma_g[2])$ is an isomorphism. Consequently, $\text{Pic } \mathcal{M}_g[2]$ is finitely generated of rank 1.*

Question 7.12. Is $\text{Pic } \mathcal{M}_g[l]$ rank 1 for all $g \geq 3$ and all $l \geq 1$?

This would be the case if we knew that the $Sp_g(\mathbb{Z})$ action on $H^2(T_g)$ extended to an algebraic action of Sp_g , for we could then invoke Borel's computation of the stable cohomology of arithmetic groups [5].

7.5. Normal Functions. Each rational representation V of Sp_g gives rise to an orbifold local system \mathbb{V} over $\mathcal{M}_g[l]$. Such a local system underlies an admissible variation of Hodge structure. First, if V is irreducible, then V underlies a variation of Hodge structure unique up to Tate twist ([24, (9.1)]). Every polarized \mathbb{Q} variation of Hodge structure whose monodromy representation comes from a rational representation of Sp_g has the property that each of its isotypical components is an admissible variation of Hodge structure of the form $A_\lambda \otimes \mathbb{V}(\lambda)$, where A_λ is a Hodge structure and $\mathbb{V}(\lambda)$ is a variation of Hodge structure corresponding to the Sp_g module with highest weight λ — cf. [24, (9.2)].

For a Hodge structure V of weight -1 one defines the corresponding *intermediate jacobian* JV by

$$JV = V_{\mathbb{C}} / (F^0V + V_{\mathbb{Z}}).$$

Its interest comes from the fact that it parametrizes the extensions of \mathbb{Z} by V in the MHS category: if E is an extension of the \mathbb{Z} (with its trivial Hodge structure of weight zero) by V , then choose an integral lift $e \in E$ of 1 and consider the image of e in

$$E_{\mathbb{C}} / (F^0E + V_{\mathbb{Z}}) \cong V_{\mathbb{C}} / (F^0V + V_{\mathbb{Z}}).$$

This is independent of the lift and yields a complete invariant of the extension. There is an inverse construction that makes JV support a variation of mixed Hodge structure \mathbb{E} that is universal as an extension of the trivial Hodge structure \mathbb{Z} by the constant Hodge structure V :

$$0 \rightarrow \mathbb{V}_{JV} \rightarrow \mathbb{E} \rightarrow \mathbb{Z}_{JV} \rightarrow 0$$

(see [7]). This immediately generalizes to a relative setting: if \mathbb{V} is an admissible variation of \mathbb{Z} Hodge structure of weight -1 over a smooth variety X , then we have a corresponding bundle $\pi : \mathcal{J}\mathbb{V} \rightarrow X$ of intermediate jacobians over X supporting a universal extension

$$0 \rightarrow \pi^*\mathbb{V} \rightarrow \mathbb{E} \rightarrow \pi^*\mathbb{Z}_X \rightarrow 0.$$

A section σ of $\mathcal{J}\mathbb{V}$ over X determines an extension of Hodge structures:

$$0 \rightarrow \mathbb{V} \rightarrow \sigma^*\mathbb{E} \rightarrow \mathbb{Z}_X \rightarrow 0.$$

A *normal function* is a section of $\mathcal{J}\mathbb{V}$ such that the corresponding extension \mathbb{E} is an admissible variation of mixed Hodge structure. The normal functions arising from algebraic cycles are normal functions in this sense — cf. [24, §6].

We briefly recall Griffiths' construction of a normal function associated to a family of homologically trivial algebraic cycles. First we consider the case where the base is a point. Suppose that X is a smooth projective variety. A homologically trivial algebraic d -cycle Z in X canonically determines an extension of \mathbb{Z} by $H_{2d+1}(X; \mathbb{Z}(-d))$ by pulling back the exact sequence

$$0 \rightarrow H_{2d+1}(X; \mathbb{Z}(-d)) \rightarrow H_{2d+1}(X, |Z|; \mathbb{Z}(-d)) \rightarrow H_{2d}(|Z|; \mathbb{Z}(-d)) \rightarrow \dots$$

of MHSs along the inclusion

$$\mathbb{Z} \rightarrow H_{2d}(|Z|; \mathbb{Z}(-d))$$

that takes 1 to the class of Z . So an integral lift of 1 is given by an integral singular $2d+1$ chain W in X whose boundary is Z . Integration identifies $JH_{2d+1}(X; \mathbb{Z}(-d))$ with the *Griffiths intermediate Jacobian*

$$J_d(X) := \mathrm{Hom}_{\mathbb{C}}(F^d H^{2d+1}(X); \mathbb{C}(-d)) / H^{2d+1}(X; \mathbb{Z}(-d)),$$

and under this isomorphism the extension class in question is just given by integration over W .

Families of homologically trivial cycles give rise to normal functions: Suppose that $\mathcal{X} \rightarrow T$ is a family of smooth projective varieties over a smooth base T and that \mathcal{Z} is an algebraic cycle in \mathcal{X} which is proper over T of relative dimension d . Then the local system whose fiber over $t \in T$ is $H_{2d+1}(X_t; \mathbb{Z}(-d))$ naturally underlies a variation of Hodge structure \mathbb{V} over T of weight -1 so that we can form the d th *relative intermediate jacobian* $\mathcal{J}_d(\mathcal{X}/T) \rightarrow T$, whose fiber over $t \in T$ is $J_d(X_t)$. The family of cycles \mathcal{Z} defines a section of this bundle which is a normal function.

Theorem 7.13 (Hain [24]). *Suppose that \mathbb{V} is an admissible variation of Hodge structure of weight -1 over $\mathcal{M}_g[l]$ whose monodromy representation factors through a rational representation of Sp_g . If $g \geq 3$, then the space of normal functions associated to \mathbb{V} is finitely generated of rank equal to the number of copies of the variation $\mathbb{V}(\lambda_3)$ of weight -1 that occur in \mathbb{V} .*

The theorem implies that, up to torsion and multiples, there is only one normal function over \mathcal{M}_g associated to a variation of Hodge structure whose monodromy factors through a rational representation of Sp_g . So what is the generator of these normal functions?

To answer this question, recall that if C is a smooth projective curve of genus g and $x \in C$, we have the Abel-Jacobi morphism

$$C \rightarrow \mathrm{Jac} C, \quad y \mapsto (y) - (x).$$

Denote the image 1-cycle in $\mathrm{Jac} C$ by C_x and the cycle $i_* C_x$ by C_x^- , where $i : \mathrm{Jac} C \rightarrow \mathrm{Jac} C$ takes u to $-u$. The cycle $C_x - C_x^-$ is homologous to zero, and therefore defines a point $\nu^1(C, x)$ in $J_1(\mathrm{Jac} C)$. Pontrjagin product with the class of C induces a homomorphism

$$A : \mathrm{Jac} C \rightarrow J_1(\mathrm{Jac} C).$$

We call the cokernel of A the *primitive first intermediate Jacobian* $J_1^{\mathrm{PF}}(\mathrm{Jac} C)$ of $\mathrm{Jac} C$. The family of such primitive intermediate jacobians over \mathcal{M}_g is the unique

one (up to isogeny) associated to the variation of Hodge structure of weight -1 over \mathcal{M}_g whose associated Γ_g module is $V(\lambda_3)$. It is not difficult to show that

$$\nu^1(C, x) - \nu^1(C, y) = 2A(x - y).$$

It follows that the image of $\nu^1(C, x)$ in $J_1^{\text{pr}}(\text{Jac } C)$ is independent of x . This is the value of the normal function associated with $C - C^-$ over $[C]$.

We can do better and realize half of this generator by a generalized normal function as follows. Let A be a principally polarized abelian variety of dimension $g \geq 3$. The polarization determines a distinguished element ω of $H_2(A; \mathbb{Z})$. If Z and Z' are two piecewise smooth cycles representing ω , then their difference is the boundary of a piecewise smooth 3-chain W on A . Represent the dual of $H_3(A; \mathbb{R})$ by translation invariant 3-forms on A . Then integrating these forms over W determines an element of $H_3(A; \mathbb{R})$. Another choice of W gives a class that differs from this one by an element of $H_3(A; \mathbb{Z})$, and so we have a well-defined element $[Z - Z']$ of $H_3(A; \mathbb{R}/\mathbb{Z})$. Notice that the latter torus is naturally identified with the first intermediate jacobian $J_1(A)$ of A . We declare Z and Z' to be equivalent if $[Z - Z'] = 0$ and denote the space of piecewise smooth cycles representing ω modulo this equivalence relation by $D(A)$. This is clearly a torsor of $J_1(A)$ and so it has a natural complex structure. In view of its connection with Deligne cohomology, we call it the *Deligne torsor* of A . This torsor contains naturally a subtorsor $D(A)[2]$ of the 2-torsion in $J_1(A)$, $J_1(A)[2] \cong H_3(A; \frac{1}{2}\mathbb{Z}/\mathbb{Z})$: Let $a = (a_1, a_{-1}, \dots, a_g, a_{-g})$ be a symplectic basis of $H_1(A; \mathbb{Z})$. Each basis element a_i is uniquely represented by a homomorphism $\alpha_i : S^1 \rightarrow A$ and so ω is represented by the 2-cycle $\sum_{i=1}^g \alpha_i \times \alpha_{-i}$. This cycle defines an element $z(a) \in D(A)$. It is easily verified that $z(a)$ only depends on the mod two reduction of a and that if a runs over all symplectic bases, $z(a)$ runs over an entire orbit $D(A)[2]$ of $J_1(A)[2]$. (So $J_1(A)[2] \setminus D(A)$ has a canonical point which identifies it with $J_1(A)$.) The group $Sp(H_1(A; \mathbb{Z}))$ acts on $D(A)$ as an affine transformation group in a way that is easily made explicit. The lifts of these transformations to a universal covering of $D(A)$ form a group of affine symplectic transformations. It is an extension of $Sp(H_1(A; \mathbb{Z}))$ by $H_3(A; \mathbb{Z})$ which splits if we enlarge the extension to $H_3(A; \frac{1}{2}\mathbb{Z})$.

The Pontrjagin product with ω defines a homomorphism $A \rightarrow J_1(A)$ which gives rise to corresponding primitive notions: the *primitive Deligne torsor* $D^{\text{pr}}(A) := A \setminus D(A)$ is a torsor of the primitive intermediate Jacobian $J_1^{\text{pr}}(A) := A \setminus J_1(A)$. We have corresponding universal Deligne torsors over \mathcal{A}_g which we denote $\mathcal{D}_g \rightarrow \mathcal{A}_g$ and $\mathcal{D}_g^{\text{pr}} \rightarrow \mathcal{A}_g$. By the above argument, these torsors become trivial on the Galois cover of \mathcal{A}_g representing principally polarized abelian varieties with a level 2 structure. The torsors themselves are nontrivial, for it can be shown that the orbifold fundamental groups of these torsors are nonsplit extensions of the integral symplectic group of genus g .

For C a nonsingular projective curve of genus $g \geq 3$ and $x \in C$, the Abel-Jacobi morphism $C \rightarrow \text{Jac } C$ defined by $y \mapsto (y) - (x)$ defines a cycle in the homology class of the natural polarization of $\text{Jac } C$ and so we get an element $[(C, x)]$ of $D(\text{Jac } C)$. Its image in $D^{\text{pr}}(\text{Jac } C)$ is independent of x and so can be denoted by $[C]$. Universally this produces holomorphic lifts of the period map:

$$\nu_g^1 : \mathcal{M}_g^1 \rightarrow \mathcal{D}_g \text{ and } \nu_g : \mathcal{M}_g \rightarrow \mathcal{D}_g^{\text{pr}}.$$

We call ν_g the *fundamental normal function* on \mathcal{M}_g .

7.6. Picard group of the generic curve with a level l structure. The classification of normal functions (7.13) implies that there are no sections of Pic^0 of infinite order defined over $\mathcal{M}_g[l]$ when $g \geq 3$. This, combined with Sipe's computation (7.10) of the monodromy of roots of the canonical bundle allows one to determine the Picard group of the generic point of $\mathcal{M}_g[l]$. The case $l = 1$ was the subject of the Franchetta Conjecture which was deduced from Harer's computation of Γ_g by Beauville (unpublished) and by Arbarello and Cornalba [3].

Theorem 7.14 (Hain [25]). *The Picard group of the generic curve of genus $g \geq 3$ with a level l structure is of rank 1, has torsion subgroup isomorphic to the l torsion points $H_1(\text{Jac } S_g; \mathbb{Z}/l)$, and, modulo torsion, is generated by the canonical bundle if l is odd, and a theta characteristic if l is even.*

8. RELATIVE MALCEV COMPLETION

Fundamental groups of smooth algebraic varieties are quite special as we know from the work of Morgan [62] and others. The least trivial restrictions on these groups come from Hodge theory and Galois theory. Since Γ_g is the (orbifold) fundamental group of \mathcal{M}_g , a smooth orbifold, Hodge theory and Galois theory should have something interesting to say about its structure. To put a MHS on a group one needs to linearize it. One way to do this is to replace the group by some kind of algebraic envelope and put a MHS on the coordinate ring of this (pro)algebraic group.

In this section we introduce these linearizations and use them to establish a relation between the fundamental normal function and a remarkable central extension that is hidden in a quotient of the mapping class group. Here the impact of mixed Hodge theory is not yet felt, but we are setting the stage for Section 9 where it is omnipresent.

8.1. Classical Malcev completion. Suppose that π is a finitely generated group. The classical Malcev (or unipotent) completion of π consists of a pronilpotent group $\mathcal{U}(\pi)$ (over \mathbb{Q}) and a homomorphism $\pi \rightarrow \mathcal{U}(\pi)$. It is characterized by the following universal mapping property: if U is a unipotent group, and $\phi : \pi \rightarrow U$ is a homomorphism, there is a unique homomorphism of pronilpotent groups $\mathcal{U}(\pi) \rightarrow U$ through which ϕ factors. There are several well known constructions of the unipotent completion, which can be found in [23], for example. Each (pro)unipotent group U is isomorphic to its Lie algebra \mathfrak{u} , a (pro)nilpotent Lie algebra via the exponential map. Thus, to give the Malcev group $\mathcal{U}(\pi)$ associated to π it suffices to give its associated pronilpotent Lie algebra $\mathfrak{u}(\pi)$. This Lie algebra is called the *Malcev Lie algebra associated to π* . It comes with a natural descending filtration whose k th term $\mathfrak{u}^{(k)}(\pi)$ is the closed ideal of $\mathfrak{u}(\pi)$ generated by its k -fold commutators ($k = 1, 2, \dots$) and it is complete with respect to this filtration. We will refer to this filtration as the *Malcev filtration*.

When π is the fundamental group $\pi_1(X, x)$ of a smooth complex algebraic variety, $\mathfrak{u}(\pi)$ has a canonical MHS which was first constructed by Morgan [62]. If X is also complete, or more generally, when $H_1(X)$ has a pure Hodge structure of weight -1 , then the weight filtration is the Malcev filtration:

$$W_{-k}\mathfrak{u}(\pi_1(X, x)) = \mathfrak{u}^{(k)}(\pi_1(X, x)).$$

Alternatively, this MHS determines and is determined by a MHS on the coordinate ring $\mathcal{O}(\mathcal{U}(\pi))$ of the associated Malcev group.

We shall denote the Malcev completion of $\pi_g^n = \pi_1(S_g^n, x_0)$ by \mathfrak{p}_g^n .

8.2. Relative Malcev completion. The Malcev completion of a group π is trivial when $H_1(\pi)$ vanishes, for then π has no non-trivial unipotent quotients. Since the first homology of Γ_g vanishes for all g , its Malcev completion will be trivial. Deligne has defined the notion of Malcev completion of a group π relative to a Zariski dense homomorphism $\rho : \pi \rightarrow S$, where S is a reductive algebraic group defined over a base field F (that we assume to be of characteristic zero).

The *Malcev completion of π relative to $\rho : \pi \rightarrow S$* is a proalgebraic F -group $\mathcal{G}(\pi, \rho)$, which is an extension

$$1 \rightarrow \mathcal{U} \rightarrow \mathcal{G}(\pi, \rho) \rightarrow S \rightarrow 1$$

of S by a prounipotent group, together with a lift $\tilde{\rho} : \pi \rightarrow \mathcal{G}(\pi, \rho)$ of ρ .¹¹ It is characterized by the following universal mapping property: if G is an F -group which is an extension of S by a unipotent group U , and if $\phi : \pi \rightarrow G$ is a homomorphism, then there is a unique homomorphism $\mathcal{G}(\pi, \rho) \rightarrow G$ through which ϕ factors:

$$\phi : \pi \xrightarrow{\tilde{\rho}} \mathcal{G}(\pi, \rho) \rightarrow G.$$

Since S is reductive, we should think of \mathcal{U} as the prounipotent radical of $\mathcal{G}(\pi, \rho)$. One can show, for instance, that \mathcal{U} has a Levi supplement so that $\mathcal{G}(\pi, \rho)$ is a semidirect product of S and \mathcal{U} . The Lie algebra $\mathfrak{g}(\pi, \rho)$ of $\mathcal{G}(\pi, \rho)$ also comes with a Malcev filtration with respect to which it is complete: $\mathfrak{g}(\pi, \rho)^{(0)} = \mathfrak{g}(\pi, \rho)$, and for $k \geq 1$, $\mathfrak{g}(\pi, \rho)^{(k)}$ is the closed ideal generated by k -fold commutators in the Lie algebra of \mathcal{U} .

We will often write $\mathcal{G}(\pi)$ instead of $\mathcal{G}(\pi, \rho)$ when the representation ρ is clear from the context. We shall denote the completion of the (orbifold) fundamental group of a pointed orbifold (X, x) with respect to a Zariski dense reductive representation $\rho : \pi_1(X, x) \rightarrow S$ by $\mathcal{G}(X, x; \rho)$, or simply $\mathcal{G}(X, x)$ when ρ is clear from the context.

When S is trivial, we recover the classical Malcev completion. The universal property of the Malcev completion of $\ker \rho$ yields a natural homomorphism of proalgebraic F -groups $\mathcal{U}(\ker \rho)(F) \rightarrow \mathcal{G}(\pi)$. In general, it is neither surjective nor injective as the following two examples show.

Example 8.1. The fundamental group of the symplectic Lie group $Sp_g(\mathbb{R})$ is infinite cyclic and hence so is its universal cover $\widehat{Sp}_g(\mathbb{R}) \rightarrow Sp_g(\mathbb{R})$. This universal cover is not an algebraic group (which follows for instance from the fact that the complexification of $Sp_g(\mathbb{R})$, $Sp_g(\mathbb{C})$, is simply connected). The preimage $\widehat{Sp}_g(\mathbb{Z})$ of $Sp_g(\mathbb{Z})$ in this covering contains the universal central extension of $Sp_g(\mathbb{Z})$ by \mathbb{Z} . Now take for π this central extension and for ρ its natural homomorphism to $Sp_g(\mathbb{C})$. The corresponding relative Malcev completion is then reduced to $Sp_g(\mathbb{C})$ itself, so that the homomorphism from $\mathcal{U}(\mathbb{Z})(\mathbb{C})$ (which is just the abelian group \mathbb{C}) to $\mathcal{G}(\widehat{Sp}_g(\mathbb{Z}))$ is trivial. We will see that this example is realized inside a quotient of the mapping class group.

¹¹In many cases the completion of π over an algebraic closure \bar{F} of F is the set of \bar{F} points of the completion of π over F . This is the case for the mapping class groups when $g \geq 3$, but we do not know whether this is true in general, except when S is trivial.

Example 8.2. In this example, $\ker \rho$ is trivial, but $\mathcal{U}(\pi)$ is not. The basic fact we need (see [25, (10.3)]) is that there is always a natural S equivariant isomorphism

$$H_1(\mathbf{u}(\pi)) \cong \prod_{\alpha \in \bar{S}} H_1(\pi; V_\alpha) \otimes V_\alpha^*,$$

where V_α denotes a representation with highest weight α . For π we take Γ , a finite index subgroup of $SL_2(\mathbb{Z})$, for S we take $SL_2(\mathbb{Q})$, and for ρ we take the natural inclusion. Denote the n th power of the fundamental representation of SL_2 by $S^n V$. For all such Γ , there is an infinite number of integers $n \geq 0$ such that $H^1(\Gamma; S^n V)$ is non-trivial.¹² It follows that $\mathcal{U}(\Gamma)$ has an infinite dimensional H_1 , even though $\ker \rho$ is trivial.

This example suggests the following problem:

Problem 8.3. Investigate the relationship between the theory of modular forms associated to a finite index subgroup Γ of $SL_2(\mathbb{Z})$ and the completion of Γ relative to the inclusion $\Gamma \hookrightarrow SL_2(\mathbb{Q})$.

8.3. The relative Malcev completion of Γ_g . The natural homomorphism $\rho : \Gamma_{g,r}^n \rightarrow Sp_g$ has Zariski dense image. Denote the completion of $\Gamma_{g,r}^n$ relative to ρ by $\mathcal{G}_{g,r}^n$, its pronilpotent radical by $\mathcal{U}_{g,r}^n$ and their Lie algebras by $\mathfrak{g}_{g,r}^n$ and $\mathfrak{u}_{g,r}^n$.

The following theorem indicates the presence of essentially one copy of the universal central extension of $Sp_g(\mathbb{Z})$ in quotients of each mapping class group of genus g when $g \geq 3$.

Theorem 8.4 (Hain [23]). *When $g \geq 2$, the homomorphism*

$$(2) \quad \mathcal{U}(T_{g,r}^n) \rightarrow \mathcal{U}_{g,r}^n$$

is surjective. When $g \geq 3$, its kernel is a central subgroup isomorphic to the additive group.

This phenomenon is intimately related to the cycle $C - C^-$ and its normal function as we shall now explain.

8.4. The central extension. The existence of the central extension has both a group theoretic and a geometric explanation. It is also related to the Casson invariant through the work of Morita [65, 66]. We begin with the group theoretic one.

The group analogue of the Malcev filtration for the Torelli group T_g is the most rapidly descending central series of T_g with torsion free quotients:

$$T_g = T_g^{(1)} \supset T_g^{(2)} \supset T_g^{(3)} \supset \dots$$

Note that $T_g^{(1)}/T_g^{(2)}$ is the maximal torsion free abelian quotient of T_g , which is

$$V(\lambda_3)_g \mathbb{Z} := \wedge^3 H_1(S_g; \mathbb{Z}) / ([S_g] \times H_1(S_g; \mathbb{Z}))$$

by Johnson's Theorem (7.1). The group $\Gamma_g/T_g^{(3)}$ can be written as an extension

$$(3) \quad 1 \rightarrow T_g^{(2)}/T_g^{(3)} \rightarrow \Gamma_g/T_g^{(3)} \rightarrow \Gamma_g/T_g^{(2)} \rightarrow 1.$$

It turns out that this sequence contains a multiple of the universal central extension of $Sp_g(\mathbb{Z})$ by \mathbb{Z} .

¹²This is easily seen when Γ is free, for example. In general it is related to the theory of modular forms.

Since $V_g(\lambda_3)$ is a rational representation of Sp_g , and since the surjection

$$\wedge^2 V_g(\lambda_3)_{\mathbb{Z}} \rightarrow T_g^{(2)}/T_g^{(3)}$$

induced by the commutator is $Sp_g(\mathbb{Z})$ equivariant, it follows that $T_g^{(2)}/T_g^{(3)} \otimes \mathbb{Q}$ is also a rational representation of Sp_g . Because $V_g(\lambda_3)$ is an irreducible symplectic representation, there is exactly one copy of the trivial representation in $\wedge^2 V_g(\lambda_3)$. This copy of the trivial representation survives in $T_g^{(2)}/T_g^{(3)} \otimes \mathbb{Q}$ [23] so that there is an $Sp_g(\mathbb{Z})$ equivariant projection $T_g^{(2)}/T_g^{(3)} \rightarrow \mathbb{Z}$. Pushing the extension (3) out along this map gives an extension

$$(4) \quad 0 \rightarrow \mathbb{Z} \rightarrow E \rightarrow \Gamma_g/T_g^{(2)} \rightarrow 1$$

Note that E is a quotient of Γ_g . We will manufacture a multiple of the universal central extension of $Sp_g(\mathbb{Z})$ from this group that turns out to be the obstruction to the map $\mathcal{U}(T_g) \rightarrow \mathcal{U}_g$ being injective. (Full details can be found in [23].)

The group $\Gamma_g/T_g^{(2)}$ can be written as an extension

$$(5) \quad 0 \rightarrow V_g(\lambda_3)_{\mathbb{Z}} \rightarrow \Gamma_g/T_g^{(2)} \rightarrow Sp_g(\mathbb{Z}) \rightarrow 1.$$

Morita [67] showed that this extension is *semisplit*, that is, if we replace $V_g(\lambda_3)_{\mathbb{Z}}$ by $\frac{1}{2}V_g(\lambda_3)_{\mathbb{Z}}$, it splits. (This can also be seen using the normal function of $C - C^-$.)

Theorem 8.5 (Morita [66], Hain [23]). *The extension of $Sp_g(\mathbb{Z})$ by \mathbb{Z} obtained by pulling back the extension (4) along a semisplitting of (5) contains the universal central extension of $Sp_g(\mathbb{Z})$.*

The geometric picture uses the fundamental normal function ν_g . The lifted period maps ν_g and ν_g^1 to the Deligne torsors are easily seen to extend over the partial completions $\widetilde{\mathcal{M}}_g$ resp. $\widetilde{\mathcal{M}}_g^1$:

$$\tilde{\nu}_g : \widetilde{\mathcal{M}}_g \rightarrow \mathcal{D}_g^{\text{pr}}, \quad \tilde{\nu}_g^1 : \widetilde{\mathcal{M}}_g^1 \rightarrow \mathcal{D}_g.$$

The orbifold fundamental group of $\mathcal{D}_g^{\text{pr}}$, resp. $\mathcal{D}_{g,1}$, is an extension of $Sp_g(\mathbb{Z})$ by $V_g(\lambda_3)_{\mathbb{Z}}$, resp. $\wedge^3 V_{g,\mathbb{Z}}$, as both the base and fiber are Eilenberg MacLane spaces with these groups as orbifold fundamental groups. But by 7.2 the orbifold fundamental group of $\widetilde{\mathcal{M}}_g$, resp. $\widetilde{\mathcal{M}}_g^1$, also has such a structure. Indeed:

Theorem 8.6. *For $g \geq 3$, the normal functions $\tilde{\nu}_g : \widetilde{\mathcal{M}}_g \rightarrow \mathcal{D}_g^{\text{pr}}$ and $\tilde{\nu}_g^1 : \widetilde{\mathcal{M}}_g^1 \rightarrow \mathcal{D}_g$ induce an isomorphism on orbifold fundamental groups. (The former can be identified with $\Gamma_g/T_g^{(2)}$ and the latter with $\Gamma_g^1/(T_g^1)^{(2)}$.)*

From this theorem we recover the fact that (5) is semisplit, not split. But we get more, since it should also lead to a description of that extension.

The extension (4) can also be realized geometrically.

Proposition 8.7 (Hain [23]). *There is a canonical (locally homogeneous) line bundle \mathcal{B}_g over the bundle $\mathcal{D}_{g,1}^{\text{pr}} \rightarrow \mathcal{A}_g$ that realizes the central extension (4) via the isomorphism of the previous proposition as an extension of orbifold fundamental groups. In particular, both $\tilde{\nu}^* \mathcal{B}_g$ and $\nu^* \mathcal{B}_g$ have nonzero rational first Chern class. The bundle $\nu^* \mathcal{B}_g$ is canonically metrized and its square is isomorphic (as a metrized line bundle) to the metrized line bundle associated to the archimedean height of the cycle $C - C^-$.*

9. HODGE THEORY OF THE MAPPING CLASS GROUP

One reason that mixed Hodge theory is so powerful is that the MHS category is abelian. In many situations this turns out to have topological implications for algebraic varieties that are difficult, if not impossible, to obtain directly. A somewhat related (but less exploited) property is that a MHS is canonically split over \mathbb{C} . This implies that the weight filtration (which often has a topological interpretation) splits in a way that is compatible with all the algebraic structure naturally present. So, for many purposes, there is no loss of information regarding this algebraic structure if we pass to the corresponding weight graded object. For example, the Malcev filtration on the Malcev Lie algebra of a smooth projective variety is minus the weight filtration, and it therefore splits over \mathbb{C} in a natural way. This splitting is natural in the sense that it respects the Lie algebra structure and is preserved under all base point preserving morphisms. But if we vary the complex structure on X or the base point x , then the splitting will, in general, vary with it.

A basic example is the Malcev Lie algebra \mathfrak{p}_g^1 of $\pi_g^1 = \pi_1(S_g^1, x_0)$. The group π_g^1 is free on $2g$ generators and it is a classical fact that the graded of \mathfrak{p}_g^1 with respect to the Malcev filtration is just the free Lie algebra generated by V_g . If S_g is given a conformal structure, then V_g has a pure Hodge structure of weight -1 and the weight filtration of \mathfrak{p}_g^1 is minus the Malcev filtration. The splitting allows us to identify $\mathfrak{p}_g^1 \otimes \mathbb{C}$ with the completion of $\text{Lie}(V_g) \otimes \mathbb{C}$.

We shall come back to this example in Section 10.3. But for now we will focus on the relative Malcev completions introduced in the previous section.

9.1. Hodge theory of $\mathcal{G}_{g,r}^n$. A choice of a conformal structure on S_g and nonzero tangent vectors at x_{n+1}, \dots, x_{n+r} determines a point x_o of the moduli space $\mathcal{M}_{g,r}^n$. We can thus identify $\Gamma_{g,r}^n$ with the orbifold fundamental group of $(\mathcal{M}_{g,r}^n, x_o)$. This induces an isomorphism of $\mathcal{G}_{g,r}^n$ with $\mathcal{G}(\mathcal{M}_{g,r}^n, x_o)$, the completion of $\pi_1(\mathcal{M}_{g,r}^n, x_o)$ with respect to the standard symplectic representation. We shall write $\mathcal{G}_{g,r}^n(x_o)$ for $\mathcal{G}(\mathcal{M}_{g,r}^n, x_o)$ and denote its pronipotent radical by $\mathcal{U}_{g,r}^n(x_o)$. There is a general Hodge de Rham theory of relative Malcev completion [25]. Applying it to $(\mathcal{M}_{g,r}^n, x_o)$, one obtains the following result:

Theorem 9.1 (Hain [26]). *For each choice of a base point x_o of $\mathcal{M}_{g,r}^n$, there is a canonical MHS on the coordinate ring $\mathcal{O}(\mathcal{G}_{g,r}^n(x_o))$ which is compatible with its Hopf algebra structure. Consequently, the Lie algebra $\mathfrak{g}_{g,r}^n(x_o)$ of $\mathcal{G}_{g,r}^n(x_o)$ and the Lie algebra $\mathfrak{u}_{g,r}^n(x_o)$ of its pronipotent radical both have a natural MHS.*

Denote the Malcev Lie algebra of the subgroup of $\pi_1(\mathcal{M}_{g,r}^n, x_o)$ corresponding to the Torelli group $T_{g,r}^n$ by $\mathfrak{t}_{g,r}^n(x_o)$. The normal function of $C - C^-$ can be used to lift the MHS from $\mathfrak{u}_{g,r}^n(x_o)$ to $\mathfrak{t}_{g,r}^n(x_o)$.

Theorem 9.2 (Hain [26]). *For each $g \geq 3$ and for each choice of a base point x_o of $\mathcal{M}_{g,r}^n$, there is a canonical MHS on $\mathfrak{t}_{g,r}^n(x_o)$ which is compatible with its bracket. Moreover, the canonical central extension*

$$0 \rightarrow \mathbb{Q}(1) \rightarrow \mathfrak{t}_{g,r}^n(x_o) \rightarrow \mathfrak{u}_{g,r}^n(x_o) \rightarrow 0$$

is an extension of MHSs, and the weight filtration equals the Malcev filtration.

9.2. A presentation of \mathfrak{t}_g . We denote the Malcev Lie algebra of $T_{g,r}^n$ by $\mathfrak{t}_{g,r}^n$. The existence of a MHS on $\mathfrak{t}_{g,r}^n(x_o)$ implies that, after tensoring with \mathbb{C} , there is a

canonical isomorphism

$$\mathfrak{t}_{g,r}^n(x_o) \otimes \mathbb{C} \cong \prod_m \mathrm{Gr}_{-m}^W \mathfrak{t}_{g,r}^n(x_o) \otimes \mathbb{C}.$$

Since the left hand side is (noncanonically) isomorphic to $\mathfrak{t}_{g,r}^n \otimes \mathbb{C}$, to give a presentation of $\mathfrak{t}_{g,r}^n \otimes \mathbb{C}$, it suffices to give a presentation of its associated graded. It follows from Johnson's Theorem (7.1) that each graded quotient of the lower central series of \mathfrak{t}_g is a representation of the algebraic group Sp_g . We will give a presentation of $\mathrm{Gr}_{\bullet}^W \mathfrak{t}_g$ in the category of representations of Sp_g . Recall that $\lambda_1, \dots, \lambda_g$ is a set of fundamental weights of Sp_g . For a nonnegative integral linear combination of the fundamental weights $\lambda = \sum_{i=1}^g n_i \lambda_i$ we denote by $V_g(\lambda)$ the representation of Sp_g with highest weight λ .

For all $g \geq 3$, the representation $\wedge^2 V_g(\lambda_3)$ contains a unique copy of $V_g(2\lambda_2) + V_g(0)$. Denote the Sp_g invariant complement of this by R_g . Since the quadratic part of the free Lie algebra $\mathrm{Lie}(V_g)$ is $\wedge^2 V_g$, we can view R_g as being a subspace of the quadratic elements of $\mathrm{Lie}(V_g(\lambda_3))$.

As mentioned earlier, it is unknown whether any T_g is finitely presented when $g \geq 3$. But the following theorem says that its de Rham incarnation is:

Theorem 9.3 (Hain [26]). *For all $g \geq 3$, \mathfrak{t}_g is isomorphic to the completion of its associated graded $\mathrm{Gr}_{\bullet}^W \mathfrak{t}_g$. When $g \geq 6$, this has presentation*

$$\mathrm{Gr}_{\bullet}^W \mathfrak{t}_g = \mathrm{Lie}(V_g(\lambda_3)) / (R_g),$$

where R_g is the set of quadratic relations defined above. When $3 \leq g < 6$, the relations in $\mathrm{Gr}_{\bullet}^W \mathfrak{t}_g$ are generated by the quadratic relations R_g , and possibly some cubic relations. In particular, $\mathfrak{t}_{g,r}^n$ is finitely presented whenever $g \geq 3$.

Note that this, combined with (9.2) gives a presentation of $\mathrm{Gr}_{\bullet}^W \mathfrak{u}_g$ when $g \geq 6$:

Corollary 9.4. *For all $g \geq 3$, \mathfrak{u}_g is isomorphic to the completion of its associated graded $\mathrm{Gr}_{\bullet}^W \mathfrak{u}_g$. When $g \geq 6$, this has quadratic presentation*

$$\mathrm{Gr}_{\bullet}^W \mathfrak{u}_g = \mathrm{Lie}(V_g(\lambda_3)) / (R_g + V_g(0)),$$

where R_g is the set of quadratic relations defined above and where $V_g(0)$ is the unique copy of the trivial representation in $\wedge^2 V_g(\lambda_3)$. When $3 \leq g < 6$, the relations in $\mathrm{Gr}_{\bullet}^W \mathfrak{u}_g$ are generated by the quadratic relations $R_g + V_g(0)$, and possibly some cubic relations. In particular, $\mathfrak{u}_{g,r}^n$ is finitely presented whenever $g \geq 3$.

The proof that the relations in the presentation of \mathfrak{t}_g are generated by quadratic relations when $g \geq 6$ and quadratic and cubic ones when $g \geq 3$ is not topological, but uses deep Hodge theory and, surprisingly, intersection homology. The key ingredients are a result of Kabanov, which we state below, and M. Saito's theory of Hodge modules.

We define the *Satake compactification* $\overline{\mathcal{M}}_g^{\mathrm{sat}}$ of \mathcal{M}_g as the closure of \mathcal{M}_g inside the (Baily-Borel-)Satake compactification of \mathcal{A}_g .

Theorem 9.5 (Kabanov [46]). *For each irreducible representation V of Sp_g , the natural map*

$$IH^2(\overline{\mathcal{M}}_g^{\mathrm{sat}}; \mathbb{V}) \rightarrow H^2(\mathcal{M}_g; \mathbb{V})$$

is an isomorphism when $g \geq 6$. Here \mathbb{V} denotes the generically defined local system corresponding to V .

Such a local system \mathbb{V} is, up to a Tate twist, canonically a variation of Hodge structure. Saito's purity theorem then implies that $H^2(\mathcal{M}_g; \mathbb{V})$ is pure of weight $2+$ the weight of \mathbb{V} when $g \geq 6$. It is this purity result that forces $H^2(\mathfrak{t}_g)$ to be of weight 2, and implies that no higher order relations are needed.

9.3. Understanding \mathfrak{t}_g . Even though we have a presentation of \mathfrak{t}_g , we still do not have a good understanding of its graded quotients, either as vector spaces or as Sp_g modules. There is an exact sequence

$$0 \rightarrow \mathfrak{p}_g \rightarrow \mathfrak{t}_g^1 \rightarrow \mathfrak{t}_g \rightarrow 0$$

of Lie algebras (recall that \mathfrak{p}_g stands for the Malcev Lie algebra of $\pi_g = \pi_1(S, x_0)$).

It is the de Rham incarnation of the exact sequence of fundamental groups associated to the universal curve. Fix a conformal structure on (S, x_0) . Then this sequence is an exact sequence of MHSs. Since Gr^W is an exact functor, and since $\mathrm{Gr}_\bullet^W \mathfrak{p}_g$ is well understood, it suffices to understand $\mathrm{Gr}_\bullet^W \mathfrak{t}_g^1$.

There is a natural representation

$$(6) \quad \mathfrak{t}_g^1 \rightarrow \mathrm{Der} \mathfrak{p}_g$$

It is a morphism of MHS, and therefore determined by the graded Lie algebra homomorphism

$$\mathrm{Gr}_\bullet^W \mathfrak{t}_g^1 \rightarrow \mathrm{Der} \mathrm{Gr}_\bullet^W \mathfrak{p}_g.$$

One can ask how close it is to being an isomorphism. Since this map is induced by the natural homomorphism

$$(7) \quad \Gamma_g^1 \rightarrow \varprojlim \mathrm{Aut} \mathbb{C}\pi_g / I^m,$$

the homomorphism (6) factors through the projection $\mathfrak{t}_g^1 \rightarrow \mathfrak{u}_g^1$, and therefore cannot be injective. On the other hand, we have the following (reformulated) result of Morita:

Theorem 9.6 (Morita [68]). *There is a natural Lie algebra surjection*

$$\mathrm{Tr}_M : W_{-1} \mathrm{Der} \mathrm{Gr}_\bullet^W \mathfrak{p}_g \rightarrow \bigoplus_{k \geq 1} S^{2k+1} H_1(S)$$

onto an abelian Lie algebra whose composition with (6) is trivial. Here S^m denotes the m^{th} symmetric power.

One may then hope that the sequence

$$0 \rightarrow \mathbb{C} \rightarrow \mathrm{Gr}_\bullet^W \mathfrak{t}_g^1 \rightarrow W_{-1} \mathrm{Der} \mathrm{Gr}_\bullet^W \mathfrak{p}_g \rightarrow \bigoplus_{k \geq 1} S^{2k+1} H_1(S) \rightarrow 0$$

is exact. However, there are further obstructions to exactness at $W_{-1} \mathrm{Der} \mathrm{Gr}_\bullet^W \mathfrak{p}_g$ which were discovered by Nakamura [70]. They come from Galois theory and use the fact that \mathcal{M}_g^1 is defined over \mathbb{Q} .¹³ On the other hand, one can ask:

Question 9.7. Is the map $\mathfrak{u}_g^1 \rightarrow \mathrm{Der} \mathfrak{p}_g$ injective? Equivalently, is \mathcal{G}_g^1 the Zariski closure of the image of the representation (7)?

A good understanding of \mathfrak{t}_g may help in understanding the stable cohomology of Γ_g as we shall explain in the next subsection.

¹³Actually, Nakamura proves his result for a corresponding sequence for $\mathfrak{t}_{g,1}$, but his obstructions most likely appear in this case too.

9.4. **Torelli Lie algebras and the cohomology of Γ_g .** Each Malcev Lie algebra \mathfrak{g} can be viewed as a complete topological Lie algebra. A basis for the neighbourhoods of 0 being the terms $\mathfrak{g}^{(k)}$ of the Malcev filtration. One can define the continuous cohomology of such a \mathfrak{g} to be

$$H^\bullet(\mathfrak{g}) := \varinjlim H^\bullet(\mathfrak{g}/\mathfrak{g}^{(k)}).$$

If \mathfrak{g} has a MHS, then so will $H^\bullet(\mathfrak{g})$. The continuous cohomology of $\mathfrak{t}_{g,r}^n$, $\mathfrak{u}_{g,r}^n$, etc. each has an action of Sp_g . The general theory of relative Malcev completion [25] gives a canonical homomorphism

$$(8) \quad H^\bullet(\mathfrak{u}_{g,r}^n)^{Sp_g} \rightarrow H^\bullet(\mathcal{M}_{g,r}^n)$$

One can ask how much of the cohomology of $\mathcal{M}_{g,r}^n$ is captured by this map.

Fix a base point x_o of $\mathcal{M}_{g,r}^n$. Then $\mathfrak{t}_{g,r}^n$, etc. all have compatible MHSs, and these induce MHSs on their continuous cohomology groups. These groups have the property that the weights on H^k are $\geq k$.

Theorem 9.8 (Hain [26]). *The map (8) is a morphism of MHS.*

Since $H_1(T_g)$ is a quotient of \mathfrak{u}_g , there is an induced map

$$(9) \quad H^\bullet(H_1(T_g)) \rightarrow H^\bullet(\mathfrak{u}_g).$$

This is also a morphism of MHS. The following result follows directly from [22, (9.2)], the presentation (9.3) of \mathfrak{t}_g , and the existence of the MHS on \mathfrak{u}_g .

Proposition 9.9. *If $g \geq 3$, the map (9) surjects onto the lowest weight subring*

$$\bigoplus_{k \geq 0} W_k H^k(\mathfrak{u}_g)$$

of $H^\bullet(\mathfrak{u}_g)$, and the kernel is generated by the ideal generated by the unique copy of $V_g(2\lambda_2)$ in $H^2(H_1(T_g))$.

Similar results hold when $r + n > 0$ — cf. [26, §14.6].

The following result of Kawazumi and Morita tells us that the image of the lowest weight subring of $H^\bullet(\mathfrak{u}_g)^{Sp_g}$ contains no new cohomology classes.

Theorem 9.10 (Kawazumi-Morita [48]). *The image of the natural map*

$$H^\bullet(H_1(T_g))^{Sp_g} \rightarrow H^\bullet(\mathcal{M}_g)$$

is precisely the subring generated by the κ_i 's.

If we combine this with the previous two results and Pikaart's Purity Theorem (4.2), we obtain the following strengthening of the theorem of Kawazumi and Morita (and obtained independently by Morita, building on our work):

Theorem 9.11. *When $k \leq g/2$, the image of $H^k(\mathfrak{u}_g)^{Sp_g} \rightarrow H^k(\mathcal{M}_g)$ is the degree k part of the subring generated by the κ_i 's.*

To continue the discussion further, it seems useful to consider cohomology with symplectic coefficients.

9.5. Cohomology with symplectic coefficients. The irreducible representations of Sp_g are parametrized by Young diagrams with $\leq g$ rows (and no indexing of the boxes), in other words, by nonincreasing sequences of nonnegative integers whose terms with index $> g$ are zero. So any such sequence $\alpha = (\alpha_1, \alpha_2, \dots)$ defines an irreducible representation of Sp_h for all $h \geq g$. We will denote the representation of Sp_g corresponding to α by $V_{g,\alpha}$, and the corresponding (orbifold) local system over \mathcal{M}_g by $\mathbb{V}_{g,\alpha}$.

A theorem of Ivanov [36] (that in fact pertains to more general local systems) implies that, when $r \geq 1$, the group $H^k(\Gamma_{g,r}^n; V_{g,\alpha})$ is independent of g once g is large enough. In the case at hand we have a more explicit result that we state here for the undecorated case (a case that Ivanov actually excludes).

Theorem 9.12 (Looijenga [58]). *Let $\alpha = (\alpha_1, \alpha_2, \dots)$ be a nonincreasing sequence of nonnegative integers that is eventually zero, and let c_1, c_2, \dots be weighted variables with $\deg(c_i) = 2i$. Put $|\alpha| := \sum_{i \geq 1} \alpha_i$. Then there exists a finitely generated, evenly graded $\mathbb{Q}[c_1, \dots, c_{|\alpha|}]$ -module A_α^\bullet (that can be described explicitly) and a graded homomorphism of $H^\bullet(\Gamma_\infty)$ modules*

$$A_\alpha^\bullet[-|\alpha|] \otimes H^\bullet(\Gamma_\infty) \rightarrow H^\bullet(\Gamma_g; V_{g,\alpha})$$

that is an isomorphism in degree $\leq cg - |\alpha|$. It is also a MHS morphism if we take A_α^{2k} to be pure of type (k, k) . In particular, we have

$$A_\alpha^\bullet[-|\alpha|] \otimes H^\bullet(\Gamma_\infty) \cong H^\bullet(\Gamma_\infty; V_\alpha)$$

both as MHSs and as graded $H^\bullet(\Gamma_\infty)$ modules. So, by (4.2), $H^k(\Gamma_\infty; V_\alpha)$ is pure of weight $k + |\alpha|$.

It is useful to try to understand all cohomology groups with symplectic coefficients at the same time. To do this we take a leaf out of the physicist's book and consider the "generating function"

$$(10) \quad \bigoplus_\alpha H^\bullet(\Gamma_g; V_\alpha^*) \otimes V_\alpha$$

where α ranges over all partitions with $\leq g$ rows, and $*$ denotes dual. This is actually a graded commutative ring as the Peter-Weyl Theorem implies that the coordinate ring \mathcal{O}_g of Sp_g is

$$\mathcal{O}_g = \bigoplus_\alpha (\text{End } V_\alpha)^* \cong \bigoplus_\alpha V_\alpha^* \otimes V_\alpha.$$

The mapping class group acts on \mathcal{O}_g by composing the right translation action of Sp_g on \mathcal{O}_g with the canonical representation $\Gamma_g \rightarrow Sp_g$. The corresponding cohomology group $H^\bullet(\Gamma_g; \mathcal{O})$ is then the "generating function" (10). Note that \mathcal{O}_g is a variation of Hodge structure of weight 0, so the group $H^k(\Gamma_g; \mathcal{O}_g)$ is stably of weight k by the above theorem.

There is a canonical algebra homomorphism

$$H^\bullet(\mathfrak{u}_g) \rightarrow H^\bullet(\Gamma_g; \mathcal{O}_g)$$

whose existence follows from the de Rham theory of relative completion suggested by Deligne — cf. [25]. The map (8) of the previous subsection is just its invariant part. This map is a MHS morphism for each choice of complex structure on S . The α isotypical part of both sides stabilizes as g increases. It is natural to ask:

Question 9.13. Is this map stably an isomorphism?

This has been verified by Hain and Kabanov (unpublished) in degrees ≤ 2 for all weights, and in degree 3 and weight 3. If the answer is yes, or even if one has surjectivity, then it will follow from the theorem of Kawazumi and Morita (9.10) that the stable cohomology of \mathcal{M}_g is generated by the κ_i 's. A consequence of injectivity and Pikaart's Purity Theorem would be that for each k , $H^k(\mathbf{u}_g)$ is pure of weight k once the genus is sufficiently large. This is equivalent to the answer to the following question being affirmative.

Question 9.14. Are $H^\bullet(\mathbf{u}_g)$ and $U \mathrm{Gr}_\bullet^W \mathbf{u}_g$ stably Koszul dual?

Note that $U \mathrm{Gr}_\bullet^W \mathbf{u}_g$ and the lowest weight subalgebra of $H^\bullet(\mathbf{u}_g)$ have dual quadratic presentations.

10. ALGEBRAS RELATED TO THE COHOMOLOGY OF MODULI SPACES OF CURVES

The ribbon graph description is the root of a number of ways of constructing (co)homology classes on moduli spaces of curves from certain algebraic structures. These constructions have in common that they actually produce cellular (co)chains on \mathbb{M}_g^n , and so they are recipes that assign numbers to 'oriented' ribbon graphs. The typical construction, due to Kontsevich [52], goes like this: assume that we are given a complex vector space V , a symmetric tensor $p \in V \otimes V$, and linear forms $T_k : V^{\otimes k} \rightarrow \mathbb{C}$ that are cyclically invariant. If Γ is a ribbon graph, then the decomposition of $X(G)$ into σ_1 and σ_0 -orbits gives isomorphisms

$$\otimes_{s \in X_1(\Gamma)} V^{\otimes \mathrm{or}(s)} \cong V^{\otimes X(G)} \cong \otimes_{v \in X_0(\Gamma)} V^{\otimes \mathrm{out}(v)},$$

where $\mathrm{or}(s)$ stands for the two-element set of orientations of the edge s and $\mathrm{out}(v)$ for the set of oriented edges that have v as initial vertex. Now $p^{\otimes X_1(\Gamma)}$ defines a vector of the lefthand side and a tensor product of certain T_k 's defines a linear form on the righthand side. Evaluation of the linear form on the vector gives a number, which is clearly an invariant of the ribbon graph. Since this invariant does not depend on an orientation on the set of edges of G , it cannot be used directly to define a cochain on the combinatorial moduli spaces. To this end we need some sign rules so that, for instance, the displayed isomorphisms acquire a sign. The tensors p and T_k are sometimes referred to as the *propagator* and the *interactions*, respectively, to remind us of their physical origin.

If p is nondegenerate, then we may use it to identify V with its dual. In this case T_k defines a linear map $V^{\otimes(k-1)} \rightarrow V$. The properties one needs to impose on propagator and interactions in order that the above recipe produce a cocycle on \mathbb{M}_g^n is that they define a $\mathbb{Z}/2$ graded A_∞ algebra with inner product. A similar recipe assigns cycles on \mathbb{M}_g^n to certain $\mathbb{Z}/2$ graded differential algebras. The cocycles can be evaluated on the cycles and this, in principle, gives a method of showing that some of the classes thus obtained are nonzero. We shall not be more precise, but instead refer to [52] or [1] for an overview. A simple example is to take $V = \mathbb{C}$, $p := 1 \otimes 1$ and $T_k(z^k)$ arbitrary for $k \geq 3$ odd, and zero otherwise. Kontsevich asserts that the classes thus obtained are all tautological.

10.1. Outer space. In Section 6 we encountered a beautiful combinatorial model for a virtual classifying space of the mapping class group Γ_g^n . There is a similar, but simpler, combinatorial model that does the same job for the outer automorphism group of a free group.

We fix an integer $r \geq 2$ and consider connected graphs G with first Betti number equal to r and where each vertex has degree ≥ 3 . Let us call these graphs *r-circular graphs*. The maximal number of edges (resp. vertices) such a graph can have is $3r - 3$ (resp. $2r - 2$). These bounds are realized by all trivalent graphs of this type. Notice that an r -circular graph G has fundamental group isomorphic to the free group on r generators, F_r . We say that G is *marked* if we are given an isomorphism $\phi : F_r \rightarrow \pi_1(G, \text{base point})$ up to inner automorphism. The group $\text{Out}(F_r)$ permutes these markings simply transitively. There is an obvious notion of isomorphism for marked r -circular graphs. We shall denote the collection of isomorphism classes by \mathcal{G}_r .

Let $(G, [\phi])$ represent an element of \mathcal{G}_r . The metrics on G that give G total length 1 are parameterized by the interior of a simplex $\Delta(G)$. We fit these simplices together in a way analogous to the ribbon graph case: if s is an edge of G that is not a loop, then collapsing it defines another element $(G/s, [\phi]/s)$ of \mathcal{G}_r . We may then identify $\Delta(G/s)$ with a face of $\Delta(G)$. After we have made these identifications we end up with a simplicial complex $\widehat{\mathcal{O}}_r$. The union of the interiors of the simplices $\Delta(G)$ (indexed by \mathcal{G}_r) will be denoted by \mathcal{O}_r ; it is the complement of a closed subcomplex of $\widehat{\mathcal{O}}_r$. This construction is due to Culler-Vogtmann [9]. We call \mathcal{O}_r the *outer space of order r* for reasons that will become apparent in a moment. Observe that $\widehat{\mathcal{O}}_r$ comes with a simplicial action of $\text{Out}(F_r)$. We denote the quotient of $\widehat{\mathcal{O}}_r$ (resp. \mathcal{O}_r) by $\text{Out}(F_r)$ by $\widehat{\mathbb{G}}_r$ (resp. \mathbb{G}_r). It is easy to see that $\widehat{\mathbb{G}}_r$ is a finite orbicomplex. The open subset \mathbb{G}_r is the moduli space of metrized r -circular graphs. It has a spine of dimension $2r - 3$.

Theorem 10.1 (Culler-Vogtmann [9], Gersten). *The outer space of order r is contractible and a subgroup of finite index of $\text{Out}(F_r)$ acts freely on it. Hence \mathbb{G}_r is a virtual classifying space for $\text{Out}(F_r)$ and $\text{Out}(F_r)$ has virtual homological dimension $\leq 2r - 3$.*

In contrast to the ribbon graph case, \mathcal{O}_r is not piecewise smooth. If we choose $2g - 1 + n$ free generators for the fundamental group of our reference surface S_g^n , then each ribbon graph without vertices of degree ≤ 2 determines an element of \mathbb{G}_{2g-1+n} : simply forget the ribbon structure. The ribbon data is finite and it is therefore not surprising that forgetting the ribbons defines a finite map

$$\widehat{f} : \mathcal{S}_n \backslash \widehat{\mathcal{M}}_g^n \rightarrow \widehat{\mathbb{G}}_{2g-1+n}$$

of orbicomplexes. Here \mathcal{S}_n stands for the symmetric group, which acts in the obvious way on $\widehat{\mathcal{M}}_g^n$. Following Strebel's theorem, the preimage of \mathbb{G}_{2g-1+n} can be identified with $\mathcal{S}_n \backslash (\mathcal{M}_g^n \times \text{int } \Delta^{n-1})$. We denote the resulting map by

$$f : \mathcal{S}_n \backslash (\mathcal{M}_g^n \times \text{int } \Delta^{n-1}) \rightarrow \mathbb{G}_{2g-1+n}.$$

It induces the evident map

$$f_* : H_k(\Gamma_g^n)_{\mathcal{S}_n} \rightarrow H_k(\text{Out } F_{2g-1+n})$$

on rational homology. It is unclear whether there is such an interpretation for the induced map on cohomology with compact supports. We remark that $\mathcal{M}_g^n \times \text{int } \Delta^{n-1}$ is canonically oriented, but that its \mathcal{S}_n -orbit space is not (since transpositions reverse this orientation). Poincaré duality therefore takes the form

$$H_k(\mathcal{M}_g^n)_{\mathcal{S}_n} \cong H_k(\mathcal{S}_n \backslash (\mathcal{M}_g^n \times \text{int } \Delta^{n-1}); \epsilon) \cong H_c^{6g-7+3n-k}(\mathcal{S}_n \backslash (\mathcal{M}_g^n \times \text{int } \Delta^{n-1}); \epsilon),$$

where ϵ is the signum representation of \mathcal{S}_n . If δ denotes the (signum) character of $\text{Out}(F_r)$ on $\wedge^r H_1(F_r)$, then the adjoint of f_* is a map

$$H_c^{6g-7+3n-k}(\mathbb{G}_{2g-1+n}; \delta) \rightarrow H_c^{6g-7+3n-k}(\mathcal{S}_n \backslash (\mathcal{M}_g^n \times \text{int } \Delta^{n-1}); \epsilon) \cong H_k(\mathcal{M}_g^n)_{\mathcal{S}_n}.$$

So, when $m \geq 1$, we have maps

$$H_c^{i+m-1}(\mathbb{G}_{m+1}; \delta) \rightarrow \bigoplus_{2g-2+n=m} H_{2m-i}(\mathcal{M}_g^n)_{\mathcal{S}_n} \rightarrow H_{2m-i}(\mathbb{G}_{m+1}), \quad i = 0, 1, \dots$$

There is a remarkable interpretation of this sequence that we will discuss next.

10.2. Three Lie algebras. We describe Kontsevich's three functors from the category of symplectic vector spaces to the category of Lie algebras and their relation with the cohomology of the moduli spaces \mathcal{M}_g^n . The basic references are [51] and [52].

We start out with a finite dimensional \mathbb{Q} vector space V endowed with a non-degenerate antisymmetric tensor $\omega_V \in V \otimes V$. Let $\text{Ass}(V)$ be the tensor algebra (i.e., the free associative algebra) generated by V . We grade it by giving V degree -1 . The Lie subalgebra generated by V is free and so we denote it by $\text{Lie}(V)$. It is well-known that $\text{Ass}(V)$ may be identified with the universal enveloping algebra of $\text{Lie}(V)$. If we mod out $\text{Ass}(V)$ by the two-sided ideal generated by the degree ≤ -2 part of $\text{Lie}(V)$, we obtain the symmetric algebra $\text{Com}(V)$ of V . Define $\mathfrak{g}_{\text{ass}}(V)$ (resp. $\mathfrak{g}_{\text{lie}}(V)$) to be the Lie algebra of derivations of $\text{Ass}(V)$ (resp. $\text{Lie}(V)$) of degree ≤ 0 that kill ω_V . Since each derivation of $\text{Lie}(V)$ extends canonically to its universal enveloping algebra, we have an inclusion $\mathfrak{g}_{\text{lie}}(V) \subset \mathfrak{g}_{\text{ass}}(V)$. There is also a corresponding Lie algebra $\mathfrak{g}_{\text{com}}(V)$ of derivations of degree ≤ 0 of $\text{Com}(V)$ that kill ω_V . Here we regard the latter as a two-form on the affine space $\text{Spec } \text{Com}(V)$. This Lie algebra is a quotient of $\mathfrak{g}_{\text{ass}}(V)$. All three Lie algebras are graded and have as degree zero summand the Lie algebra $\mathfrak{sp}(V)$ of the group $\text{Sp}(V)$ of symplectic transformations of V . A simple verification shows that the degree -1 summands have as $\mathfrak{sp}(V)$ representations the following natural descriptions:

$$\mathfrak{g}_{\text{com}}(V)_{-1} \cong S^3(V), \quad \mathfrak{g}_{\text{ass}}(V)_{-1} \cong S^3(V) \oplus \wedge^3 V, \quad \mathfrak{g}_{\text{lie}}(V)_{-1} \cong \wedge^3 V.$$

These Lie algebras are functorial with respect to symplectic injections $(V, \omega_V) \hookrightarrow (W, \omega_W)$. Note that $\text{Sp}(V)$ acts trivially on this cohomology of the Lie algebra in question because $\mathfrak{sp}(V) \subset \mathfrak{g}_*(V)$. This implies that $H^k(\mathfrak{g}_*(V))$, $*$ \in $\{\text{lie, ass, com}\}$, depends only on $\dim V$. We form the inverse limit:

$$H^k(\mathfrak{g}_*) := \varprojlim_V H^k(\mathfrak{g}_*(V)).$$

The sum over k , $H^\bullet(\mathfrak{g}_*)$, has the structure of a connected graded bicommutative Hopf algebra; the coproduct comes from the direct sum operation on symplectic vector spaces. It is actually bigraded: apart from the cohomological grading there is another coming from the grading of the Lie algebras. Notice that the latter grading has all its degrees ≥ 0 . The primitive part $H_{\text{pr}}^\bullet(\mathfrak{g}_*)$ inherits this bigrading. Furthermore, the natural maps

$$H^\bullet(\mathfrak{g}_{\text{com}}) \rightarrow H^\bullet(\mathfrak{g}_{\text{ass}}) \rightarrow H^\bullet(\mathfrak{g}_{\text{lie}})$$

are homomorphisms of bigraded Hopf algebras. Consequently, we have induced maps between the bigraded pieces of their primitive parts.

Theorem 10.2 (Kontsevich [51], [52]). *For $\ast \in \{\text{lie}, \text{ass}, \text{com}\}$ we have*

$$H_{\text{pr}}^k(\mathfrak{g}_\ast)_0 = H_{\text{pr}}^k(\mathfrak{sp}_\infty) \cong \begin{cases} \mathbb{Q} & \text{for } k = 3, 7, 11, \dots; \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, $H_{\text{pr}}^k(\mathfrak{g}_\ast)_l = 0$ when l is odd and, when $m > 0$, we have a natural diagram

$$\begin{array}{ccccc} H_{\text{pr}}^k(\mathfrak{g}_{\text{com}})_{2m} & \longrightarrow & H_{\text{pr}}^k(\mathfrak{g}_{\text{ass}})_{2m} & \longrightarrow & H_{\text{pr}}^k(\mathfrak{g}_{\text{lie}})_{2m} \\ \cong \downarrow & & \cong \downarrow & & \cong \downarrow \\ H_c^{k+m-1}(\mathbb{G}_{m+1}, \delta) & \xrightarrow{f_c^\ast} & \oplus_{2g-2+n=m} H_{2m-k}(\mathcal{M}_g^n) \mathcal{S}_n & \xrightarrow{f_\ast} & H_{2m-k}(\mathbb{G}_{m+1}) \end{array}$$

which commutes up to sign and whose rows are complexes. The maps in the top row are the natural maps and the bottom row is the sequence defined in Section 10.1.

The proof is an intelligent application of classical invariant theory. For each of the three Lie algebras one writes down the standard complex. The subcomplex of invariants with respect to the symplectic group is quasi-isomorphic to the full complex. Weyl's invariant theory furnishes a natural basis for this subcomplex. Kontsevich then observes that this makes the subcomplex naturally isomorphic to a cellular chain (or cochain) complex of one of the cell complexes \mathbb{G}_\ast and \mathbb{M}_\ast whose (co)homology appears in the bottom row.

The diagram in this theorem suggests that the sequence of natural transformations $\text{Lie} \rightarrow \text{Ass} \rightarrow \text{Com}$ is self dual in some sense. This can actually be pinned down by looking at the corresponding operads: Ginzburg and Kapranov [18] observed that these operads have ‘quadratic relations’ and they proved the self duality of the operad sequence in a Koszul sense.

However, our main reason for displaying this diagram is that it pertains to the cohomology of the moduli spaces of curves in two apparently unrelated ways. The first one is evident. The \mathcal{S}_n coinvariants of the homology of \mathcal{M}_g^n features in the middle column, but the righthand column has something to do with the cohomology of a ‘linearization’ of Γ_∞ : we will see that $\mathfrak{g}_{\text{lie}}$ is intimately related to the Lie algebra of the relative Malcev completions discussed in Section 8. We explain this in the next subsection after a giving a restatement of Kontsevich's Theorem.

In this restatement the Lie algebra cohomology of $\mathfrak{g}_\ast(V)$ is replaced by the relative Lie algebra cohomology of the pair $(\mathfrak{g}_\ast(V), \mathfrak{k}(V))$, where $\mathfrak{k}(V)$ is a maximal compact Lie subalgebra of $\mathfrak{g}_\ast(V)_0$, and therefore of $\mathfrak{g}_\ast(V)$ ($\mathfrak{k}(V)$ is a unitary Lie algebra of rank $\dim V/2$). As above, the Lie algebra cohomology $H^k(\mathfrak{g}_\ast(V), \mathfrak{k}(V))$ depends only on the dimension of V and stabilizes once $\dim V$ is sufficiently large. We denote the inverse limit of these groups by $H^\bullet(\mathfrak{g}_\ast, \mathfrak{k}_\infty)$. Likewise, we denote the stable cohomology of the pair $(\mathfrak{sp}_g, \mathfrak{k}_g)$ by $H^\bullet(\mathfrak{sp}_\infty, \mathfrak{k}_\infty)$. By a theorem of Borel [5], this is naturally isomorphic to the stable cohomology of \mathcal{A}_g and is a polynomial algebra generated by classes c_1, c_3, c_5, \dots , where c_k has degree $2k$.

Combining Kontsevich's Theorem 10.2 with Borel's computation and an elementary spectral sequence argument, we obtain the following result. (Use the fact that $(\mathfrak{sp}_g, \mathfrak{k}_g)$ is both a sub and a quotient of $(\mathfrak{g}_\ast(V), \mathfrak{k}(V))$.)

Corollary 10.3. *We have*

$$H_{\text{pr}}^k(\mathfrak{g}_\ast, \mathfrak{k}_\infty)_0 = H_{\text{pr}}^k(\mathfrak{sp}_\infty, \mathfrak{k}_\infty) \cong \begin{cases} \mathbb{Q} & \text{for } k = 2, 6, 10, \dots; \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, when $m > 0$, the natural maps $H^\bullet(\mathfrak{g}_*, \mathfrak{k}_\infty)_m \rightarrow H^\bullet(\mathfrak{g}_*)_m$ are isomorphisms.

10.3. Relation with the relative Malcev completion. We begin with an observation. For a symplectic vector space V and $*$ \in {lie, ass, com}, denote by $\mathfrak{g}_*^b(V)$ the subalgebra of $\mathfrak{g}_*(V)$ generated by its summands of weight 0 and -1 . Kontsevich's computation shows:

Proposition 10.4. *The graded cohomology groups $H^k(\mathfrak{g}_*^b(V), \mathfrak{k}(V))_l$ stabilize and the sum of the stable terms is a bigraded bicommutative Hopf algebra $H^\bullet(\mathfrak{g}_*^b, \mathfrak{k}_\infty)_\bullet$. In addition, the restriction map $H_{\text{pr}}^k(\mathfrak{g}_*, \mathfrak{k}_\infty)_l \rightarrow H_{\text{pr}}^k(\mathfrak{g}_*^b(V), \mathfrak{k}_{\text{lie}}(V))_l$ is an isomorphism when $l \leq k$.*

The case of interest here is that of lie where $\mathfrak{g}_{\text{lie}}^b(V)_0 = \mathfrak{sp}(V)$ and $\mathfrak{g}_{\text{lie}}^b(V)_{-1} \cong \wedge^3 V$. Denote by z_m the element of $H_{\text{pr}}^{2m}(\mathfrak{g}_{\text{lie}})_{2m}$ that corresponds, via Theorem 10.2, to $1 \in H_0(\mathbb{C}_{m+1})$. The preceding proposition yields:

Corollary 10.5. *We have a natural isomorphism of bigraded Hopf algebras*

$$\sum_{l \leq k} H^k(\mathfrak{g}_{\text{lie}}^b, \mathfrak{k}_\infty)_l \cong H^\bullet(\mathfrak{sp}_\infty, \mathfrak{k}_\infty)[z_1, z_2, \dots] \cong \mathbb{C}[c_1, c_3, c_5, \dots, z_1, z_2, z_3, \dots]$$

where each z_i and c_j is primitive.

The graded Lie algebra $\mathfrak{g}_{\text{lie}}^b(V)$ is the semi-direct product of $\mathfrak{sp}(V)$ and its elements of positive weight, which we shall denote by $\mathfrak{u}_{\text{lie}}^b$. Consequently, there are natural inclusions

$$H^\bullet(\mathfrak{u}_{\text{lie}}^b(V))^{Sp} \hookrightarrow H^\bullet(\mathfrak{g}_{\text{lie}}^b(V), \mathfrak{k}(V)) \text{ and } H^\bullet(\mathfrak{sp}(V), \mathfrak{k}(V)) \hookrightarrow H^\bullet(\mathfrak{g}_{\text{lie}}^b(V), \mathfrak{k}(V)).$$

Together these induce an algebra homomorphism

$$H^\bullet(\mathfrak{sp}(V), \mathfrak{k}(V)) \otimes H^\bullet(\mathfrak{u}_{\text{lie}}^b(V))^{Sp} \rightarrow H^\bullet(\mathfrak{g}_{\text{lie}}^b(V), \mathfrak{k}(V))$$

which is compatible with stabilization.

Proposition 10.6. *Upon stabilization, these maps induce an isomorphism*

$$H^\bullet(\mathfrak{sp}_\infty, \mathfrak{k}_\infty) \otimes H^\bullet(\mathfrak{u}_{\text{lie}}^b)^{Sp} \rightarrow H^\bullet(\mathfrak{g}_{\text{lie}}^b, \mathfrak{k}_\infty).$$

Next we relate the graded Lie algebra $\mathfrak{g}_{\text{lie}}^b$ to the filtered Lie algebra $\mathfrak{g}_{g,1}$ of the relative Malcev completion $\mathcal{G}_{g,1}$ of $\Gamma_{g,1} \rightarrow Sp_g$. Recall from Section 2 that π_g^1 is freely generated by $2g$ generators named $\alpha_\pm, \dots, \alpha_{\pm g}$ so that the commutator $\beta := (\alpha_1, \alpha_{-1}) \cdots (\alpha_g, \alpha_{-g})$ represents a simple loop around x_1 . Using Latin letters for the logarithms of the images of elements of π_g^1 in its Malcev completion, we find that

$$b \equiv [a_1, a_{-1}] + \cdots + [a_g, a_{-g}] \pmod{(\mathfrak{p}_g^1)^{(3)}}.$$

So the image of b in $\text{Gr}^2 \mathfrak{p}_g^1 \cong \wedge^2 V_g$ is the symplectic form ω_S .

The obvious homomorphism $\Gamma_{g,1} \rightarrow \text{Aut}(\pi_g^1)$ induces a Lie algebra homomorphism

$$(11) \quad \mathfrak{g}_{g,1} \rightarrow \text{Der } \mathfrak{p}_g^1.$$

whose image we denote by $\overline{\mathfrak{g}}_{g,1}$.¹⁴

¹⁴It is possible that this map is injective so that $\mathfrak{g}_{g,1} \cong \overline{\mathfrak{g}}_{g,1}$. Note that it is not surjective — see [68] and [70], and Section 9.3.

Notice that $\bar{\mathfrak{g}}_{g,1}$ is contained in the subalgebra $\text{Der}(\mathfrak{p}_{g,1}, b)$ consisting of those derivations that kill b . Since (11) is (Malcev) filtration preserving, it induces Lie algebra homomorphisms

$$\text{Gr}^\bullet \mathfrak{g}_{g,1} \rightarrow \text{Gr}^\bullet \bar{\mathfrak{g}}_{g,1} \rightarrow \text{Der}^\bullet(\text{Gr} \mathfrak{p}_g^1, \omega_S).$$

Notice that the last term is just $\mathfrak{g}_{\text{lie}}(V_g)$.

In view of (10.6), to construct an algebra homomorphism

$$H^\bullet(\mathfrak{g}_{\text{lie}}^b, \mathfrak{k}_\infty) \rightarrow H^\bullet(\Gamma_\infty)$$

it suffices to construct an algebra homomorphism $H^\bullet(\mathfrak{u}_{\text{lie}}^b)^{Sp_g} \rightarrow H^\bullet(\Gamma_\infty)$.

At this stage we need Hodge theory. Choose a conformal structure on S_g . Then, by (9.1), there are natural MHSs on $\mathfrak{g}_{g,1}$ and $\mathfrak{p}_{g,1}$ whose weight filtrations are the Malcev filtrations and such that (11) is a MHS morphism. Hence the image $\bar{\mathfrak{g}}_{g,1}$ has a natural MHS. Since $\text{Gr}^\bullet \mathfrak{g}_{g,1}$ is generated by its summands in degree 0 and 1, the same is true for $\text{Gr}^\bullet \bar{\mathfrak{g}}_{g,1}$. On the other hand, the summands in degree 0 and 1 of $\text{Gr}^\bullet \bar{\mathfrak{g}}_{g,1}$ are equal to the summands of weight 0 and -1 of $\mathfrak{g}_{\text{lie}}(V_g)$, and so the graded Lie algebra $\text{Gr}^\bullet \bar{\mathfrak{g}}_{g,1}$ may be identified with $\mathfrak{g}_{\text{lie}}^b(V_g)^\bullet$ (except that the indexing of the summands differs by sign).

Denote the pronilpotent radical $W_{-1}\bar{\mathfrak{g}}_{g,1}$ of $\bar{\mathfrak{g}}_{g,1}$ by $\bar{\mathfrak{u}}_{g,1}$. We know from Section 9 that the homomorphisms

$$(12) \quad H^\bullet(\bar{\mathfrak{u}}_{g,1})^{Sp_g} \rightarrow H^\bullet(\mathfrak{u}_{g,1})^{Sp_g} \rightarrow H^\bullet(\Gamma_{g,1})$$

are morphisms of MHS. After weight grading these become bigraded algebra homomorphisms

$$H^\bullet(\mathfrak{u}_{\text{lie}}^b(V_g)^\bullet)^{Sp_g} \rightarrow H^\bullet(\text{Gr}_\bullet^W \mathfrak{u}_{g,1})^{Sp_g} \rightarrow \text{Gr}_\bullet^W H^\bullet(\Gamma_{g,1}).$$

The sequence (12) stabilizes with g to a sequence of Hopf algebras in the MHS category. The corresponding weight graded sequence is

$$H^\bullet((\mathfrak{u}_{\text{lie}}^b)^\bullet)^{Sp_g} \rightarrow H^\bullet(\text{Gr}_\bullet^W \mathfrak{u}_{\infty,1})^{Sp_g} \rightarrow \text{Gr}_\bullet^W H^\bullet(\Gamma_\infty).$$

Each term in this sequence is a Hopf algebra and each map a Hopf algebra homomorphism. But by Pikaart's Purity Theorem we know that the last term is pure of weight k in degree k , so that we can replace it by $H^\bullet(\Gamma_\infty)$ and obtain a map $H^\bullet(\mathfrak{u}_{\text{lie}}^b)^{Sp_g} \rightarrow H^\bullet(\Gamma_\infty)$. We therefore have a Hopf algebra homomorphism

$$H^\bullet(\mathfrak{g}_{\text{lie}}^b, \mathfrak{k}_\infty) \cong H^\bullet(\mathfrak{sp}_\infty, \mathfrak{k}_\infty) \otimes H^\bullet(\mathfrak{u}_{\text{lie}}^b)^{Sp_g} \rightarrow H^\bullet(\Gamma_\infty).$$

If we compose the natural restriction map $H^\bullet(\mathfrak{g}_{\text{lie}}^b, \mathfrak{k}_\infty) \rightarrow H^\bullet(\mathfrak{g}_{\text{lie}}^b, \mathfrak{k}_\infty)$ with the above maps we get a homomorphism

$$H^\bullet(\mathfrak{g}_{\text{lie}}^b, \mathfrak{k}_\infty) \rightarrow H^\bullet(\Gamma_\infty).$$

Kontsevich asked (at the end of [52]) about the meaning of this map.¹⁵ This can now be answered by invoking the theorem of Kawazumi and Morita (9.10), or rather a weaker form, which says that z_i is mapped to a nonzero multiple of κ_i . This result was obtained with Kawazumi and Morita.

¹⁵Actually, Kontsevich asks this for with $H^\bullet(\mathfrak{g}_{\text{lie}})$ in place of the relative Lie algebra cohomology. However, there does not seem to be a natural homomorphism to $H^\bullet(\Gamma_\infty)$ in this case.

Theorem 10.7. *There is a natural Hopf algebra homomorphism*

$$H^\bullet(\mathfrak{g}_{\text{lie}}, \mathfrak{k}_\infty) \rightarrow H^\bullet(\Gamma_\infty).$$

The left hand side is a polynomial algebra generated by primitive elements z_1, z_2, \dots and c_1, c_3, \dots where z_i has degree $2i$ and c_j has degree $2j$. The image of this homomorphism is precisely the subalgebra generated by the κ_i s. The kernel is generated by elements of the form

$$c_{2k+1} - a_k z_{2k+1} - P_{2k+1}, \quad k \in \{0, 1, 2, \dots\}$$

where P_{2k+1} is a polynomial in the z_i and c_j with no linear terms, and a_k is a non zero rational number.

Here we have used the fact, due to Mumford [69], that the image of c_{2k+1} in $H^\bullet(\mathcal{M}_{g,1})$ is a polynomial in the odd κ_i 's. The theorem indicates that no new stable classes are to be expected from Hodge theory — that is, a de Rham version of the Mumford conjecture holds. Recent work of Kawazumi and Morita attempts to explain the kernel of the homomorphism $H^\bullet(\overline{\mathfrak{g}}_\infty) \rightarrow H^\bullet(\Gamma_\infty)$ in terms of secondary characteristic classes of surface bundles. The first element of the kernel is the difference $c_1 - 12z_1$. Its restriction to the Torelli group can be interpreted as the Casson Invariant (cf. [65].)

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