CORRESPONDENCES BETWEEN MODULI SPACES OF CURVES

EDUARD LOOIJENGA

Abstract. The moduli space of (possibly ramified) covers of nonsingular complex projective curves of a fixed topological type defines a correspondence between moduli spaces of pointed curves. We study the action of such a correspondence on the cohomology of these moduli spaces, where we pay special attention to what happens in the stable range.

1. Introduction

In this paper we begin the study of correspondences between moduli spaces of curves. The idea is simple enough: there is a moduli stack parameterizing unramified covers $\tilde{C} \to C$ of smooth (say, complex) projective curves of a fixed topological type. If the genera are $\tilde{g}$ and g respectively, then this moduli space defines a one-to-finite correspondence from $\mathcal{M}_g$ to $\mathcal{M}_{\tilde{g}}$. The case $g = 1$ is at the same time special and classical: then $\tilde{g} = 1$ also and we are dealing with Hecke correspondences. They generate an algebra. But if $g \geq 2$, then $\tilde{g} > g$, and the correspondences no longer form an algebra, but only an additive category (with an object for every genus $g \geq 2$). Yet, by virtue of Harer's stability theorem, which states that $H^k(\mathcal{M}_g)$ is independent of g when g is large compared to k, such correspondences will act on the cohomology of $\mathcal{M}_g$ in the stable range. It is not difficult to compute this action on a monomial in the so-called Miller-Morita-Mumford classes; we find that they are in fact common eigenvectors for these correspondences (Proposition 3.1). The precise result is best stated in terms of the Hopf algebra structure on the stable cohomology: the Miller-Morita-Mumford classes generate a Hopf subalgebra, the *tautological subalgebra*, and a correspondence defined by a degree d cover acts in the stable range of this tautological subalgebra as the Hopf algebra automorphism ψ_d which on the primitive part is multiplication by d.

Now Mumford has conjectured that the stable cohomology algebra is no bigger than the tautological subalgebra. As his conjecture is still open, one may try to show that correspondences act in the same way on the stable cohomology as they do on the tautological part. (And if one does not believe in the Mumford conjecture, one may think of Proposition 3.1 as a possible means to distinguish Miller-Morita-Mumford classes from other primitive classes.) At any rate, we prove that such a result is true for the simplest covers, namely the Galois covers of prime degree p, and then only in a suitable p-adic sense. But our motivation has another, though related source as well: The images of these correspondences define interesting algebraic cycles of high codimension on moduli spaces of stable curves that apparently have

1991 *Mathematics Subject Classification.* Primary 14H10, 14H30, 14E10.

Key words and phrases. Moduli space of curves, Correspondence.

Research supported by the Institut des Hautes Études Scientifiques in Bures-sur-Yvette (France).

not been looked at before. Our ultimate goal is to understand these cycles and to provide a motivic context (both in statement and proof) for the stability theorems. At present, this is still a dream.

In the present paper we limit ourselves to characteristic zero. But as the case of genus one suggests, it may be expected that in positive characteristic we have new features, especially regarding the interplay between Hecke correspondences and the Frobenius.

Another observation we left unpursued is that the correspondences considered here have analogues for the Kontsevich moduli spaces that support the Gromov-Witten invariants.

Let us now briefly review the contents of the sections. In Section 2 we set up the Hecke category framework associated with unramified covers of smooth complex projective curves. The following section describes the action of the correspondences on the tautological subalgebra. In Section 4 we associate to a finite abelian group A a correspondence in every genus and we show that if we do this for A cyclic of prime order p, then the action of these correspondences on the stable cohomology converges in a p-adic sense to ψ_p if the genus of the base curve tends to ∞. We also consider for a positive integer d the abelian cover of degree d^{2g} of C given by $H_1(C;\mathbb{Z}/d)$. This defines a correspondence which is in fact a morphism. Its composite with $\psi_{d^{2g}}^{-1}$ stabilizes and stably this gives a Hopf algebra endomorphism. The proof is postponed to Section 5, where we state and prove more general results involving correspondences associated to possibly nonabelian ramified covers. (For optimal use of the stability theorem it is best to allow the covers to ramify.) We here also point out that these correspondences naturally act on the local systems defined by conformal blocks.

The final Section 6 raises what we believe is an interesting question: according to Kontsevich the cohomology of the disjoint union of moduli spaces $\mathcal{S}_n\backslash\mathcal{M}_g^n$ can be identified with the stable primitive cohomology of a graded Lie algebra pair; certain correspondences therefore act on the latter in a way that shift the grade and the dimension by the same amount. The problem is to understand this action in terms of the Lie algebra pair.

It is appropriate to mention here work of Biswas-Nag-Sullivan. Although their paper [2] differs in spirit and goals from ours, there is a relationship all the same. They investigate among other things identities (e.g., a form of the Mumford isomorphism) on curves that are invariant under finite unramified covers.

Acknowledgements. This paper has its origin in some correspondence with Carel Faber around 1993, during which Proposition 3.1 was established. My interest in the matter was revived by some email exchanges with Dick Hain in Spring 1997.

Most of this work was done while I was visiting and supported by the Institut des Hautes Études Scientifiques in Bures-sur-Yvette (France). I am very grateful to this Institute and its staff for providing such marvelous working conditions and I thank my home University for granting me a sabbatical leave during that period.

Finally, I thank Carel Faber for his comments on the penultimate e-script version of this paper.

Notation: If X is a space or a group, then $H_\bullet(X)$ denotes its rational homology and $H_\bullet^{\mathbb{Z}}(X)$ resp. $H^\bullet_{\mathbb{Z}}(X)$ its integral homology resp. cohomology modulo torsion. Similarly, if X is variety, then $\mathrm{CH}_k(X)$ stands for the kth Chow group for rational equivalence, tensorized with $\mathbb{Q}$. If X is smooth of pure dimension d, then

$\mathrm{CH}^k(X) := \mathrm{CH}_{d-k}(X)$, a definition that extends in an evident manner to the case where X is the quotient of a smooth variety by a finite group action.

It is always understood that the surfaces under consideration are oriented, and that the maps between them respect the orientations.

2. Correspondences for mapping class groups: closed surfaces

2.1. **The mapping class category for closed surfaces.** For any integer $g \geq 2$, we fix a connected oriented closed differentiable surface S_g of genus $g \geq 2$ with base point $* \in S_g$. We write π_g for its fundamental group. The orientation of S_g defines a natural generator of $H_2(\pi_g;\mathbb{Z})$, referred to as the *orientation of* π_g. The mapping class group $\Gamma(S_g,*)$ of $(S_g,*)$ can be identified with the group $\mathrm{Aut}^+(\pi_g)$ of orientation preserving automorphisms of π_g and the mapping class group $\Gamma(S_g)$ of S_g with the quotient $\mathrm{Out}^+(\pi_g)$ of $\mathrm{Aut}^+(\pi_g)$ by the subgroup of inner automorphisms $\mathrm{Int}(\pi_g)$. We define the *mapping class category* Γ as follows: its objects are the positive integers and for two such integers k,l, the morphism set $\Gamma(k,l)$ is the set of orientation preserving monomorphisms $\pi_{l+1} \to \pi_{k+1}$ modulo inner automorphisms of π_{k+1}. Composition is defined in the obvious way. The theory of covering maps shows that to give an orientation preserving monomorphism $\pi_{l+1} \to \pi_{k+1}$ up to inner automorphism amounts to giving an orientation preserving covering map $S_{l+1} \to S_{k+1}$ up to isotopy. The Hurwitz formula tells us that its degree is equal to l/k. So if $\mathbb{N}^\times$ denotes the semigroup of positive integers under multiplication, then we have a natural forgetful functor $\Gamma \to \mathbb{N}^\times$. Notice that by taking $k = l$, we see that $\Gamma(k,k) = \mathrm{Aut}_\Gamma(k)$ can be identified with $\Gamma(S_{k+1})$.

2.2. **The category of mapping class correspondences.** Assigning to k the cohomology of the mapping class group $\mathrm{Aut}_\Gamma(k)$ does not define a functor from Γ to the category of abelian groups. In order that this be the case, we must replace the composition law of Γ by a convolution, and this leads to the notion of a Hecke category which we presently define.

For $f \in \Gamma(k,l)$, let $\tilde{f} : \pi_{l+1} \to \pi_{k+1}$ be a representative and $F : (S_{l+1},*) \to (S_{k+1},*)$ a corresponding covering. If $F' : (S_{l+1},*) \to (S_{k+1},*)$ is another covering, then we say that F' is *equivalent* to F if there exist sense preserving diffeomorphisms of source and target which carry one onto the other. So the equivalence class of F is described by the double coset $\Gamma(S_{k+1},*)\tilde{f}\Gamma(S_{l+1},*)$, or what amounts to the same, by $D_f := \Gamma(l,l)f\Gamma(k,k)$.

Let us write $\mathrm{Aut}_\Gamma(f)$ for the group of $(u,\tilde{u}) \in \Gamma(k,k)\times\Gamma(l,l)$ satisfying $fu = \tilde{u}f$. It can be identified with the connected component group of the group of pairs $(U,\tilde{U})$ with U a sense preserving diffeomorphism of S_{k+1} and $\tilde{U}$ a sense preserving diffeomorphism of S_{l+1} which is a lift over F of U. If U is the identity, then $\tilde{U}$ must be a covering transformation and vice versa. In particular, the kernel of the projection $p : \mathrm{Aut}_\Gamma(f) \to \Gamma(k,k)$ is finite. Since nontrivial covering transformations define nontrivial mapping classes, the other projection $q : \mathrm{Aut}_\Gamma(f) \to \Gamma(l,l)$ is injective.

Lemma 2.1. *The image of* $\mathrm{Aut}_\Gamma(f)$ *in* $\Gamma(k,k)$ *is a subgroup of finite index. This index is equal to the number of* $\Gamma(l,l)$*-cosets in the double coset* $D_f = \Gamma(l,l)f\Gamma(k,k)$.

Proof. Let $d = l/k$ be the degree of f. Then the image of $\tilde{f}$ is a subgroup of π_{k+1} of index d. The collection $\tilde{\mathcal{C}}$ of subgroups of π_{k+1} of index d is finite. It is acted on by the mapping class group $\mathrm{Aut}^+(\pi_{k+1})$, and the orbit set $\mathcal{C} := \mathrm{Int}(\pi_{k+1})\backslash\tilde{\mathcal{C}}$

is acted on by $\mathrm{Out}^+(\pi_{k+1}) = \Gamma(k,k)$. The image of $\tilde f$ defines an element $[f] \in \mathcal{C}$ and it is clear that $u \in \Gamma(k,k)$ extends to an automorphism of f if and only if it fixes $[f]$. Moreover, the assignment $\psi \in D_f \mapsto [\psi] \in \mathcal{C}$ induces a bijection from $\Gamma(l,l)f\Gamma(k,k)/\Gamma(l,l)f$ onto the $\Gamma(k,k)$-orbit of $[f]$. Both assertions follow. □

We call the integer appearing in this lemma the *mass* of D_f and denote it by $\mu(D_f)$. Notice that this mass is one if and only if the image of $\tilde f$ is $\mathrm{Aut}^+(\pi_{k+1})$-invariant. This is why we then say that f (or D_f) is *invariant.*

Example 2.2. Consider the case when f is represented by a monomorphism $\pi_{l+1} \to \pi_{k+1}$ whose image is a normal subgroup with cyclic quotient of order $d = l/k$. So f defines an order d subgroup $C(f) \subset H^1(S_{k+1};\mathbb{Z}/d)$. This subgroup only depends on the coset $\Gamma(l,l)f$ and conversely, the subgroup determines the coset. If we let f run over the double coset $D_f := \Gamma(l,l)f\Gamma(k,k)$, then $C(f)$ runs over all such subgroups. So the mass of D_f is the number of order d subgroups of $(\mathbb{Z}/d)^{2k+2}$, i.e., $\phi_{2k+2}(d)/\phi_1(d)$, where $\phi_r(m)$ is a generalized Euler indicator: it is the number of elements in $(\mathbb{Z}/m)^r$ of exact order m:

$$\phi_r(m) = m^r \prod_{p|m, p \text{ prime}} (1-p^{-r}). \tag{1}$$

The lemma implies that the composite of two left cosets is a finite union of left cosets: if $f \in \Gamma(k,l)$ and $\psi \in \Gamma(l,m)$ and $D_g = \cup_i \Gamma(m,m)g_i$, then

$$\Gamma(m,m)g\Gamma(l,l)f = \cup_i \Gamma(m,m)g_i f.$$

If the cosets $\Gamma(m,m)g_i$ are mutually disjoint, then so are the $\Gamma(m,m)g_i f$.

We introduce a "Hecke quotient" of Γ. Let $\mathcal{L}(k,l)$ denote the abelian group of $\mathbb{Z}$-valued functions on $\Gamma(k,l)$ spanned by the characteristic functions $E_{\Gamma(l,l)f}$ of the left cosets. We have a natural bilinear map $\mathcal{L}(l,m) \times \mathcal{L}(k,l) \to \mathcal{L}(k,m)$ defined by convolution: given $f \in \Gamma(k,l)$ and $\psi \in \Gamma(l,m)$, then define the image of $E_{\Gamma(m,m)\psi} \otimes E_{\Gamma(l,l)f}$ as the characteristic function of $\Gamma(m,m)\psi\Gamma(l,l)f$. This is in $\mathcal{L}(k,m)$ by the remark above. The composition is obviously associative and so we have an additive category $\mathcal{L}$. The homomorphism $\mathcal{L}(k,l) \to \mathbb{Z}$ which takes the value one on every generator $E_{\Gamma(l,l)f}$ will be called the *mass functional* and denoted by μ also.

The functor $\Gamma \to \mathbb{N}^\times$ extends to an additive functor from $\mathcal{L}$ to the free additive category $\mathbb{Z}[\mathbb{N}^\times]$ generated by $\mathbb{N}^\times$. The ring of additive endofunctors of $\mathbb{Z}[\mathbb{N}^\times]$ has as an additive basis the functors "multiplication by d", ψ_d (here d runs over the positive integers). This is clearly a polynomial algebra with generators the ψ_p with p prime. We may (and often will) think of this ring as a quotient category of $\mathbb{Z}[\mathbb{N}^\times]$ which identifies the unique morphism $k \to kd$ with ψ_d. We denote the composite functor $\mathcal{L} \to \mathbb{Z}[\psi_p : p \text{ prime}]$ by χ and call it the *Adams character.* So if $f \in \Gamma(k,l)$, then $\chi(E_{\Gamma(l,l)f}) = \psi_{l/k}$.

We introduced the category $\mathcal{L}$ only for an auxiliary purpose, for we will rather be concerned with a subcategory of it. This subcategory $\mathcal{H}$ shall have the same object set (i.e., the positive integers), but the morphism set $\mathcal{H}(k,l)$ is to be the submodule of $\mathcal{L}(k,l)$ spanned by the characteristic functions E_{D_f} of the double cosets D_f. Convolution preserves these submodules so that a subcategory is defined indeed. Notice that $\chi(E_{D_f}) = \mu(D_f)\psi_{l/k}$.

Another useful category is the subcategory of $\mathcal{L}\otimes\mathbb{Q}$ generated by the characteristic functions of double cosets divided by their mass:

$$T_{D_f} := \mu(D_f)^{-1}E_{D_f} \in \mathcal{L}(k,l)\otimes\mathbb{Q}. \tag{2}$$

It is clear that this category, which we denote by $\tilde{\mathcal{H}}$, contains $\mathcal{H}$. Notice that $\mu(D_f) = \mu(E_{D_f})$, so that the linear extension of the mass functional takes unit value on T_{D_f}. The Adams character naturally extends to $\tilde{\mathcal{H}}$ and maps T_{D_f} to $\psi_{l/k}$.

The Hecke category $\mathcal{H}$ has a universal property: suppose that we are given a covariant functor Y from Γ to the category of spaces. So $Y(k)$ is a space with $\Gamma(k,k)$-action and we may form its orbit space $\bar{Y}(k)$. Then every double coset $D = \Gamma(l,l)f\Gamma(k,k)$ defines a one-to-finite correspondence $E_D : \bar{Y}(k) \rightrightarrows \bar{Y}(l)$: it assigns to the orbit $\Gamma(k,k)x$ the finite union of $\Gamma(l,l)$-orbits $Y(\Gamma(l,l)f\Gamma(k,k))x = (\Gamma(l,l)Y(f_i)x)_i$, where $\Gamma(l,l)f_i$ runs over the distinct left cosets in D. Often such a one-to-finite correspondence defines a map on homology and so we get a covariant additive functor from $\mathcal{H}$ to the category of $\mathbb{Q}$-vector spaces.

The basic example is the Teichmüller functor $\mathcal{X}$ which assigns to k the Teichmüller space $\mathcal{X}_{k+1}$ of conformal structures on S_{k+1} modulo isotopies. An isotopy class of orientation preserving coverings $F : S_{l+1} \to S_{k+1}$ defines an analytic morphism $\mathcal{X}_{k+1} \to \mathcal{X}_{l+1}$ (defined by lifting the conformal structure). This in fact defines a functor from Γ to an analytic category. The mapping class group $\Gamma(k,k)$ acts properly discontinuously on $\mathcal{X}_{k+1}$ and the orbit space $\mathcal{M}_{k+1}$ can be regarded as the moduli space of nonsingular complex projective curves of genus $k+1$. It is naturally a quasi-projective orbifold. A double coset $D = \Gamma(l,l)f\Gamma(k,k)$ defines a one-to-finite correspondence E_D from $\mathcal{M}_{k+1} \to \mathcal{M}_{l+1}$. This correspondence is finite algebraic; a more concrete description is

$$\mathcal{M}_{k+1} \xleftarrow{p} \mathcal{M}(D) \xrightarrow{q} \mathcal{M}_{l+1}. \tag{3}$$

Here $\mathcal{M}(D)$ is the moduli space of pointed unramified covers $\tilde{C} \to C$, with C a nonsingular complex projective connected curve, which on fundamental groups induce a map equivalent to f, and the projections are the obvious forgetful maps. Note that since the curve $\tilde{C}$ has in general a nontrivial group of automorphisms, q may map to the singular locus of $\mathcal{M}_{l+1}$.

Lemma 2.3. *The map $p : \mathcal{M}(D) \to \mathcal{M}_{k+1}$ is an unramified covering in the orbifold sense of degree $\mu(D)$ and $q : \mathcal{M}(D) \to \mathcal{M}_{l+1}$ is finite and generically injective. Assigning to D the correspondence qp^{-1} defines an additive functor from $\mathcal{H}$ to the category of correspondences.*

Proof. Let C be a nonsingular complex projective curve of genus $k+1$. Then the fiber of p over $[C] \in \mathcal{M}_{k+1}$ parameterizes the unramified covers of C equivalent to $F : S_{l+1} \to S_{k+1}$. These are $\mu(D)$ in number by Lemma 2.1. A nonsingular complex-projective curve $\tilde{C}$ of genus $l+1$ is in at most a finite number of ways realized as an unramified cover of a curve of genus $k+1$ and if that number is positive, then generically it is one. □

So E_D acts as $q_*p^!$. Since $p_*p^!$ is multiplication by $\mu(D)$, the map $\mu(D)^{-1}p^!$ realizes the homology of $\mathcal{M}_{k+1}$ as a direct summand of the homology of $\mathcal{M}(D)$. So from a motivic point of view it is perhaps better to replace E_D by T_D, which acts as $\mu(D)^{-1}q_*p^!$. We refer to T_D as the *normalized Hecke correspondence* defined by

D. Notice that it respects the augmentations. The map T_D also acts on the Chow groups (after tensorizing them with $\mathbb{Q}$).

There is a natural extension of the diagram (3) to the Deligne-Mumford compactifications:

$$\overline{\mathcal{M}}_{k+1} \xleftarrow{\bar p} \overline{\mathcal{M}}(D) \xrightarrow{\bar q} \overline{\mathcal{M}}_{l+1}.$$

Here $\overline{\mathcal{M}}(D)$ is simply the normalization of $\mathcal{M}(D)$ over $\overline{\mathcal{M}}_{k+1}$. This variety parameterizes (perhaps noneffectively) *admissible covers* of stable curves (in the sense of [6], in particular, ramification is only allowed in singular points and locally given by $w^k = xy = 0$). This implies that q extends to a morphism $\overline{\mathcal{M}}(D) \to \overline{\mathcal{M}}_{l+1}$. The corresponding extension of E_D resp. T_D is denoted $\overline{E}_D$ resp. $\overline{T}_D$.

Question 2.4. Is the class in $\mathrm{CH}^{3(l-k)}(\overline{\mathcal{M}}_{l+1})$ defined by the image of $\overline{T}_D$ in the tautological algebra of $\overline{\mathcal{M}}_{l+1}$ (in the sense of Section 5 of [3]).

3. Action of the correspondences on the tautological classes

Let $f : \mathcal{C} \to S$ be a family of stable complex curves. Denote by ω_f its relative dualizing sheaf and let $K(f) \in \mathrm{CH}^1(\mathcal{C})$ be its first Chern class. Then the nth Mumford class of this family is $\kappa(f)_n := f_*(K(f)^{n+1}) \in \mathrm{CH}^n(S)$. This definition extends to the orbifold setting and so we have also defined for $g \geq 2$, $\bar K \in \mathrm{CH}^1(\overline{\mathcal{M}}_g^1)$, $K \in \mathrm{CH}^1(\mathcal{M}_g^1)$ and a universal Mumford class $\bar\kappa_n \in \mathrm{CH}^n(\overline{\mathcal{M}}_g)$, $\kappa_n \in \mathrm{CH}^n(\mathcal{M}_g)$. (The genus is deliberately left out from the notation.)

Proposition 3.1. *Let $f \in \Gamma(k,l)$. Then $\overline{T}_{D_f}$ takes the monomial $\bar\kappa_{n_1}\bar\kappa_{n_2}\cdots\bar\kappa_{n_r}$ to $(\frac{l}{k})^r\bar\kappa_{n_1}\bar\kappa_{n_2}\cdots\bar\kappa_{n_r}$.*

It is easy to see that this proposition follows from the following assertion:

Proposition 3.2. *Suppose we are given a family of stable complex curves $\pi : \mathcal{C} \to S$ as above with smooth base, a finite surjective map $q : \tilde S \to S$ of degree μ and an unramified covering $p : \tilde{\mathcal{C}} \to q^*\mathcal{C}$ of degree d:*

$$\begin{array}{ccccc}
\tilde{\mathcal{C}} & \xrightarrow{p} & q^*\mathcal{C} & \xrightarrow{\pi^* q} & \mathcal{C} \\
\tilde\pi\downarrow & & q^*\pi\downarrow & & \pi\downarrow \\
\tilde S & = & \tilde S & \xrightarrow{q} & S,
\end{array}$$

where $\tilde\pi = (q^*\pi)p$. Then $(\pi^*q)_*p_*$ maps $K(\tilde\pi)^n$ to $\mu d K(\pi)^n$ and q_* maps the monomial $\kappa_{n_1}(\tilde\pi)\kappa_{n_2}(\tilde\pi)\cdots\kappa_{n_r}(\tilde\pi))$ to $\mu d^r\kappa_{n_1}(\pi)\kappa_{n_2}(\pi)\cdots\kappa_{n_r}(\pi)$.

Proof. First observe that $\omega_{\tilde\pi}$ is the coherent pull-back of ω_π. So $K(\tilde\pi)^n$ is the pull-back of $K(\pi)^n$. Since p_*p^* resp. q_*q^* is multiplication by d resp. μ, it follows that $(\pi^*q)_*p_*(K(\tilde\pi)^n) = \mu d K(\pi)^n$. Applying π_* to this identity yields $q_*\bar\kappa_{n-1}(\tilde\pi) = \mu d\bar\kappa_{n-1}(\pi)$. In order to generalize this to any monomial in the kappa's, we consider the diagram of r-fold fibered products:

$$\begin{array}{ccccc}
\tilde{\mathcal{C}}^{(r)} & \xrightarrow{p^{(r)}} & q^*\mathcal{C}^{(r)} & \longrightarrow & \mathcal{C}^{(r)} \\
\downarrow & & \downarrow & & \downarrow \\
\tilde S & = & \tilde S & \xrightarrow{q} & S.
\end{array}$$

If $\pi_i : \mathcal{C}^{(r)} \to \mathcal{C}^{(r-1)}$ omits the ith factor, then we have defined $K(\pi_i) \in \mathrm{CH}^1(\mathcal{C}(r))$. Since $K(\pi_1)^{n_1+1}K(\pi_2)^{n_2+1}\cdots K(\pi_r)^{n_r+1} \in \mathrm{CH}(\mathcal{C}(r))$ pulls back to the corresponding class on $\tilde{\mathcal{C}}^{(r)}$, and since $p^{(r)}$ has degree d^r, an argument as above shows that q_* has the asserted effect on $\kappa_{n_1}(\tilde{\pi})\kappa_{n_2}(\tilde{\pi})\cdots\kappa_{n_r}(\tilde{\pi})$. □

According to Harer ([4], [5]), the homology groups $H_s(\Gamma(S_g);\mathbb{Z})$ can for fixed s be canonically identified with each other when $s \leq cg$ for some positive constant c (c about $\frac{2}{3}$ will do). The more precise result can be stated as follows. If S is a connected orientable compact surface possibly with boundary, then let $\Gamma(S)$ denote the mapping class group of S relative its boundary. An embedding $S \to S'$ induces a homomorphism of mapping class groups $\Gamma(S) \to \Gamma(S')$, and Harer shows in fact that this map induces an isomorphism on integral homology in degree $\leq cg(S)$, where $g(S)$ denotes the genus of S. We denote the corresponding stable (co)homology groups mod torsion by $H_\bullet^{\mathbb{Z}}(\Gamma_\infty)$ resp. $H_{\mathbb{Z}}^\bullet(\Gamma_\infty)$. E. Miller observed that $H_\bullet^{\mathbb{Z}}(\Gamma_\infty)$ comes naturally with the structure of a Hopf algebra; the coproduct is the standard one (it comes from the diagonal embedding) and the product from embedding two surfaces S_1 and S_2 as above with nonempty boundary in a third, S, say. This yields a homomorphism $\Gamma(S_1)\times\Gamma(S_2) \to \Gamma(S)$ and thus produces in the stable range a map $H_r^{\mathbb{Z}}(\Gamma_\infty)\otimes H_s^{\mathbb{Z}}(\Gamma_\infty) \to H_{r+s}^{\mathbb{Z}}(\Gamma_\infty)$. Using Harer's theorem, it is easily seen to be independent of choices. It defines the (Hopf) product. As a Hopf algebra, $H_\bullet^{\mathbb{Z}}(\Gamma_\infty)$ is graded-bicommutative. Since $H_{\mathbb{Z}}^\bullet(\Gamma_\infty)$ is the graded dual of $H_\bullet^{\mathbb{Z}}(\Gamma_\infty)$, it is also a Hopf algebra. Miller and Morita have shown that the cohomological Mumford class κ_n stabilizes and defines a nonzero primitive integral element of $H_{\mathbb{Z}}^{2n}(\Gamma_\infty)$ (which is also denoted by κ_n): The coproduct sends κ_n to $\kappa_n\otimes 1 + 1\otimes\kappa_n$. So these elements generate a Hopf subalgebra $H_{\mathrm{taut}}^\bullet(\Gamma_\infty;\mathbb{Z})$ of $H_{\mathbb{Z}}^\bullet(\Gamma_\infty)$, called the *tautological algebra*. Mumford has conjectured that the two coincide after we tensorize with $\mathbb{Q}$.

As any commutative Hopf algebra, $H_\bullet^{\mathbb{Z}}(\Gamma_\infty)$ comes with "Adams operations": for a positive integer n, the nth Adams operation ψ_n is the composite

$$\psi_n : H_\bullet^{\mathbb{Z}}(\Gamma_\infty) \to H_\bullet^{\mathbb{Z}}(\Gamma_\infty)^{\otimes n} \to H_\bullet^{\mathbb{Z}}(\Gamma_\infty),$$

where the first map is $(n-1)$-fold iteration of the coproduct and the second is multiplication. This map is simply the Hopf algebra endomorphism which on the primitive part is multiplication by n. So ψ_n is invertible on $H_\bullet(\Gamma_\infty)$ and $\psi_n\psi_m = \psi_{nm}$. Thus we have defined a ring homomorphism, the *Adams action*,

$$\mathbb{Z}[\psi_p : p \text{ prime}] \to \mathrm{End}(H_\bullet^{\mathbb{Z}}(\Gamma_\infty)). \tag{4}$$

For any double coset $D = \Gamma(l,l)f\Gamma(k,k)$, E_D induces an endomorphism of H_s if s is in the stable range with respect to $k+1$. Then Proposition 3.1 amounts to the statement that in the stable range T_D acts as $\psi^*_{\deg(D)}$ on the tautological classes. So we can reformulate 3.1 as follows:

Proposition 3.3. *In the stable range, the Hecke category $\tilde{\mathcal{H}}$ preserves the tautological algebra and the action factors through the Adams character: it is given by composing χ with the Adams action.*

4. Hecke operators attached to finite abelian covers

As long as the Mumford conjecture is open, it is worthwhile to attempt to prove that the above proposition holds on all of $H^\bullet(\Gamma_\infty)$. We have not succeeded in this, but we will show that in certain cases a weak form of this property holds.

Fix a finite abelian group A. Given a positive integer k, then every element u of $H^1(S_{k+1};A)$ can be thought of as a homomorphism $\pi_{k+1}\to A$ and thus defines an abelian covering of S_{k+1}. The degree d_u of this covering is of course the order of the image of the map $\pi_{k+1}\to A$. We thus get a well-defined double coset $D(u)$ and hence an element $E_{D(u)}\in\mathcal{H}(k,d_uk)$ and its normalization $T_{D(u)}\in\tilde{\mathcal{H}}(k,d_uk)$. We now let s be in the stable range with respect to k and consider the expression

$$\mathcal{T}_{A,k}:=|A|^{-2k-2}\sum_{u\in H^1(S_{k+1},A)}\psi_{|A|/d_u}T_{D(u)}, \tag{5}$$

viewed as an endomorphism of $H_s(\Gamma_\infty)$.

Proposition 4.1. *The element $\mathcal{T}_{A,k}$ is independent of k and the resulting endomorphism $\mathcal{T}_A$ of $H_\bullet(\Gamma_\infty)$ is in fact an algebra endomorphism.*

Notice that we do not claim that $\mathcal{T}_A$ preserves the coproduct. We postpone the proof of 4.1 to 5.4, where we will in fact prove a nonabelian generalization of this result. The case that interests us most is when A is cyclic of order d. We then write $T_{d,k}$ for the associated normalized operator. (It follows from 2.2 that the unnormalized operator $E_{d,k}$ is $\phi_{2k+2}(d)/\phi(d)T_{d,k}$, where ϕ_r is the Euler indicator defined in (1) that generalizes the usual one $\phi=\phi_1$.) So if we denote the linear combination (5) by $\mathcal{T}_d$, then

$$\mathcal{T}_d=d^{-2k-2}\sum_{m|d}\phi_{2k+2}(m)\psi_{d/m}T_{m,k}. \tag{6}$$

For instance, when $d=p$ is prime, then $\mathcal{T}_{p,k}=p^{-2k-2}((p^{2k+2}-1)T_{p,k}+\psi_p)$. By means of Möbius inversion we can express $T_{d,k}$ as a weighted average of operators $\psi_*\mathcal{T}_*$: if $\mathcal{P}(d)$ is the set of primes dividing d, then

$$T_{d,k}\prod_{p\in\mathcal{P}(d)}(p^{2k+2}-1)=\sum_{I\subset\mathcal{P}(d)}(-1)^{|I|}\operatorname{pr}(\mathcal{P}(d)-I)^{2k+2}\psi_{\operatorname{pr}(I)}\mathcal{T}_{d/\operatorname{pr}(I)}. \tag{7}$$

Here $\operatorname{pr}(I)$ stands for the product of the members of I (the empty product is 1 by convention). This formula gives $T_{d,k}$ a sense on all of $H_\bullet(\Gamma_\infty)$.

Corollary 4.2. *Let p be a prime and let d be a positive integer not divisible by p.*

- (i) *For every positive integer n the action of $T_{p^n,k}$ on $H_\bullet(\Gamma_\infty)$ converges to $\psi_p\mathcal{T}_{p^{n-1}}$ in the p-adic topology as $k\to\infty$.*
- (ii) *The action of $T_{pd,k}-\psi_pT_{d,k}$ on $H_\bullet(\Gamma_\infty)$ converges to zero in the p-adic topology as $k\to\infty$.*

Proof. Formula (7) gives:

$$(p^{2k+2}-1)T_{p^n,k}=p^{2k+2}\mathcal{T}_{p^n}-\psi_p\mathcal{T}_{p^{n-1}}.$$

Taking the limit for $k\to\infty$ yields the first assertion. For the second statement we note that

$$T_{pd,k}(p^{2k+2}-1)\prod_{l\in\mathcal{P}(d)}(l^{2k+2}-1)=$$
$$\sum_{I\subset\mathcal{P}(dp)}(-1)^{|I|}\operatorname{pr}(\mathcal{P}(dp)-I)^{2k+2}\psi_{\operatorname{pr}(I)}\mathcal{T}_{pd/\operatorname{pr}(I)}.$$

Reducing this identity modulo p^{2k+2} reduces the sum on the righthand side to a sum over the subsets I of $\mathcal{P}(dp)$ containing p. Writing this as a sum over the subsets of J of $\mathcal{P}(d)$, we get

$$\sum_{J\subset\mathcal{P}(d)} -(-1)^{|J|}\operatorname{pr}(\mathcal{P}(d)-J)^{2k+2}\psi_p\psi_{\operatorname{pr}(J)}\mathcal{T}_{d/\operatorname{pr}(J)}.$$

We recognize this expression as $-\psi_p T_{d,k}\prod_{l\in\mathcal{P}(d)}(l^{2k+2}-1)$ and so we find that

$$T_{dp,k}\equiv\psi_p T_{d,k}\mod p^{2k+2}.$$

The statement follows. □

By Corollary 4.2, $\mathcal{T}_{p,k}: H_r(\mathcal{M}_{k+1})\to H_r(\mathcal{M}_{pk+1})$ is an isomorphism when k is large enough. Fix a positive integer d not divisible by the prime p. Then the direct limit of the system

$$\Sigma_p(d): H_r(\mathcal{M}_{d+1})\xrightarrow{T_{p,d}} H_r(\mathcal{M}_{pd+1})\xrightarrow{T_{p,pd}}\cdots\longrightarrow H_r(\mathcal{M}_{p^sd+1})\xrightarrow{T_{p,p^sd}}\ldots$$

is isomorphic to $H_r(\Gamma_\infty)$, but the isomorphism is not canonical (it depends on the choice of a sufficiently large integer of the form dp^s). However, there is one if we tensorize with the p-adic numbers:

Theorem 4.3. *The direct limit of the system $\Sigma_p(d)\otimes\mathbb{Q}_p$:*

$$H_r(\mathcal{M}_{d+1};\mathbb{Q}_p)\xrightarrow{T_{p,d}} H_r(\mathcal{M}_{pd+1};\mathbb{Q}_p)\xrightarrow{T_{p,pd}}\cdots\longrightarrow H_r(\mathcal{M}_{p^sd+1};\mathbb{Q}_p)\xrightarrow{T_{p,p^sd}}\ldots$$

is canonically isomorphic to the stable cohomology $H_r(\Gamma_\infty;\mathbb{Q}_p)$.

Proof. For k in the stable range with respect to r, we define $u_k: H_r(\Gamma_\infty)\to H_r(\mathcal{M}_{k+1})$ as the composite of ψ_k with the natural isomorphism. Every such u_k is an isomorphism. If we identify the terms of large index of $\Sigma_p(d)$ with some $H_r(\mathcal{M}_{p^sd+1})$, then it is clear that $(u_{p^{t+1}d}-T_{p,p^td}u_{p^td})_t$ converges to zero in the p-adic topology. Hence $(u_{p^td})_t$ induces an isomorphism of $H_r(\Gamma_\infty;\mathbb{Q}_p)$ onto the direct limit of $\Sigma_p(d)\otimes\mathbb{Q}_p$. □

A weakness of Theorem 4.3 is that it does not relate the isomorphisms for different choices of d. For instance, one would like to say that for a prime $\ell\neq p$, the Hecke correspondences $T_{\ell,*}$ map the system $\Sigma_p(d)$ to $\Sigma_{d\ell}(p)$, or at least in the p-adic limit. In other words, we would like the correspondences $T_{p,*}$ and $T_{\ell,*}$ to commute in a weak sense. Related to this is the question of whether the endomorphisms $\mathcal{T}_n$ commute for various n.

There is another series of Hecke correspondences of interest. For a positive integer d, the natural map $\pi_{k+1}\to H_1(S_{k+1};\mathbb{Z}/d)$ defines an invariant covering of S_{k+1} of degree $d^{2(k+1)}$. The corresponding double coset is a left coset and so the associated Hecke correspondence $T_k(d)$ is in fact a morphism from $\mathcal{M}_{k+1}$ to $\mathcal{M}_{k\,d^{2(k+1)}+1}$.

Proposition 4.4. *The action of $\psi^{-1}_{d^{2k+2}}T_k(d)$ on $H_\bullet(\mathcal{M}_{k+1})$ in the stable range is independent of k. The resulting endomorphism $\mathcal{S}(d)$ of $H_\bullet(\Gamma_\infty)$ is in fact a Hopf algebra endomorphism.*

The proof will be given in section 5 as a special case of a more general statement. The endomorphism $\mathcal{S}(d)$ acts as the identity on the tautological subalgebra, but I do not know whether it is in fact the identity. I do not even know whether the $\mathcal{S}(d)$'s mutually commute for varying d.

5. Correspondences via ramified covers

There is a much bigger category of correspondences acting on the stable cohomology of the mapping class groups if we allow the coverings to ramify. This makes fuller use of the stability theorem, since we will now also deal with the algebro-geometric analogues of surfaces with boundary. For this, we choose in $S_g - \{*\}$ a sequence $x_1, x_2, x_3, \dots$ of distinct points and for every x_i a sense preserving linear isomorphism $v_i : T_{x_i} S_g \to \mathbb{C}$. The real oriented blow-up of S_g in $x_1, \dots, x_n$ gives a surface $S_{g,n}$ with boundary and the mapping class group $\Gamma(S_{g,n})$ can be identified with the group $\Gamma(S_g; v_1, \dots, v_n)$ of connected components of the group of sense preserving diffeomorphisms of S_g which fix v_i for $i = 1, \dots, n$. The relevance of this remark is that to $(S_g; v_1, \dots, v_n)$ there is naturally associated a moduli space: let $\mathcal{X}_{g,n}$ be the Teichmüller space of conformal structures on S_g extending $v_1, \dots, v_n$, up to isotopy relative $(v_1, \dots, v_n)$. This is a contractible complex manifold of complex dimension $3g - 3 + 2n$ and it is clearly acted on by the mapping class group $\Gamma(S_g; x_1, \dots, x_n)$ resp. $\Gamma(S_g; v_1, \dots, v_n)$. The natural map $\mathcal{X}_{g,n} \to \mathcal{X}_g$ is an equivariant analytic submersion. In between we have the Teichmüller space $\mathcal{X}_g^n$ of conformal structures on S up to isotopy relative $(x_1, \dots, x_n)$; it is a contractible complex manifold of complex dimension $3g - 3 + n$. The action of the relevant mapping class group is properly discontinuous so that the orbit space $\mathcal{M}_g^n$ of $\mathcal{X}_g^n$ resp. $\mathcal{M}_{g,n}$ of $\mathcal{X}_{g,n}$ is an orbifold. This orbit space is the moduli space of tuples $(C; x_1, \dots, x_n)$ resp. $(C; t_1, \dots, t_n)$, where C is a nonsingular complex projective curve of genus g, and $x_1, \dots, x_n$ are distinct points of C resp. $t_1, \dots, t_n$ are nonzero tangent vectors at distinct points of C. (This notation may lead to confusion, since our $\mathcal{M}_g^n$ is what many algebraic geometers denote by $\mathcal{M}_{g,n}$, but it is in agreement with Harer's convention, which presently suits me better.). This moduli space interpretation brings us in the category of quasi-projective orbifolds: the maps

$$\mathcal{M}_{g,n} \to \mathcal{M}_g^n \to \mathcal{M}_g \tag{8}$$

are quasi-projective, and the first map is naturally a principal $(\mathbb{C}^\times)^n$-bundle in the orbifold sense.

Notice that $\mathcal{M}_g^n$ resp. $\mathcal{M}_{g,n}$ is a virtual classifying space for $\Gamma(S_g; x_1, \dots, x_n)$ resp. $\Gamma(S_g; v_1, \dots, v_n)$ and that (8) gives

$$\Gamma(S_{g,n}) \cong \Gamma(S_g; v_1, \dots, v_n) \to \Gamma(S_g; x_1, \dots, x_n) \to \Gamma(S_g)$$

on orbifold fundamental groups. Following Harer, the composite map induces an isomorphism on integral homology in the stable range. We conclude:

Proposition 5.1. *The projection $\mathcal{M}_{g,n} \to \mathcal{M}_g$ induces an isomorphism on rational cohomology in the stable range.*

An n-tuple of positive integers $d = (d_1, \dots, d_n)$ determines a subgroup $\mu_d := \mu_{d_1} \times \cdots \times \mu_{d_n}$ of $(\mathbb{C}^\times)^n$. The orbit space $\mu_d \backslash \mathcal{M}_{g,n}$ can be interpreted as the moduli space of tuples $(C; u_1, \dots, u_n)$ with C a nonsingular complex projective curve of genus g and u_i is a nonzero tangent vector in $T^{\otimes d_i} C$ such that the base points of $u_1, \dots, u_n$ are distinct. It is a virtual classifying space whose orbifold fundamental group $\Gamma(S_g; v^{d_1}, \dots, v^{d_n})$ can be decribed as the connected component group of the group of sense preserving diffeomorphisms of S_g which fix $x_1, \dots, x_n$ and the maps $v_1^{d_1}, \dots, v_n^{d_n}$.

The permutation group $\mathcal{S}_n$ acts on $\mathcal{M}_{g,n}$ by re-indexing the tangent vectors. This action is trivial on $\mathcal{M}_g$ and so the induced map $\mathcal{S}_n\backslash\mathcal{M}_{g,n} \to \mathcal{M}_g$ also induces an isomorphism on rational cohomology in the stable range.

5.1. **Correspondences involving ramification.** Let $f : S_{h,m} \to S_{g,n}$ be a finite, unramified, sense preserving covering. This covering determines a ramified covering $S_h \to S_g$, which we also denote by f. We require compatibility with respect to our decorations v_i: if f ramifies in x_j, $1 \leq j \leq m$, of order d_j, and sends it to x_i, $1 \leq i \leq n$, then we want the pull-back of v_i under the d_j-jet of f at x_j to be equal to $v_j^{d_j}$. We remark that for $n > 0$ such a covering can be described as a functor between certain subgroupoids of the fundamental groupoids of $S_{h,m}$ and $S_{g,n}$ and that this can be used to define a mapping class category as we did in the undecorated case. Since we can do without that, we will not elaborate.

Let us define a *decorated* curve as a triple (C, X, v), where C is a complex projective curve, X a finite subset of its smooth part, and v a trivialization of the tangent bundle of C restricted to X. If we take C to be connected smooth of genus g and fix the cardinality n of X, then the moduli space of such curves can be identified with $\mathcal{S}_n\backslash\mathcal{M}_{g,n}$. Allowing (C, X) to be stable as a pointed curve yields $\mathcal{S}_n\backslash\overline{\mathcal{M}}_{g,n}$.

A *cover* $(\tilde{C}, \tilde{X}, \tilde{v}) \to (C, X, v)$ of two nonsingular decorated curves is a morphism $\pi : \tilde{C} \to C$ such that

(i) $\pi^{-1}(X) = \tilde{X}$,
(ii) X contains the discriminant of π,
(iii) if π ramifies of order d at $\tilde{p} \in \tilde{X}$, then v and $\tilde{v}^{\otimes d}$ define the same trivialization of $T_{\tilde{p}}\tilde{C}^{\otimes d}$.

It should be clear what we mean when we say that such a cover is equivalent to f. Let $\mathcal{M}(f)$ denote the moduli space of covers between nonsingular decorated curves which are equivalent to f. We then have forgetful morphisms

$$\mathcal{S}_n\backslash\mathcal{M}_{g,n} \xleftarrow{p} \mathcal{M}(f) \xrightarrow{q} \mathcal{S}_m\backslash\mathcal{M}_{h,m}$$

As in the undecorated case, p is a finite covering. In particular, qp^{-1} is a one-to-finite correspondence. We call the degree of p the *mass* of f and denote it by $\mu(f)$. Similarly we define the $\mathbb{Q}$-correspondence

$$T(f) := \mu(f)^{-1}qp^{-1} : \mathcal{S}_n\backslash\mathcal{M}_{g,n} \rightrightarrows \mathcal{S}_m\backslash\mathcal{M}_{h,m}$$

It acts on homology as $T(f)_* = \mu(f)^{-1}q_*p^!$. In view of the discussion at the beginning of this section, this defines an action on $H_\bullet(\Gamma_\infty)$ in the stable range.

The normalization $\bar{p} : \overline{\mathcal{M}}(f) \to \mathcal{S}_n\backslash\overline{\mathcal{M}}_{g,n}$ of $\mathcal{M}(f)$ over $\mathcal{S}_n\backslash\overline{\mathcal{M}}_{g,n}$ can still be interpreted as a moduli space (namely as one of admissible covers between stable decorated curves), and this shows that q extends to a morphism $\bar{q} : \overline{\mathcal{M}}(f) \to \mathcal{S}_m\backslash\overline{\mathcal{M}}_{h,m}$. The latter is finite and so $T(f)$ extends as a $\mathbb{Q}$-correspondence $\bar{T}(f) : \mathcal{S}_n\backslash\overline{\mathcal{M}}_{g,n} \rightrightarrows \mathcal{S}_m\backslash\overline{\mathcal{M}}_{h,m}$.

Remark 5.2. Our correspondences lift to the local systems defined by conformal blocks. To explain, let us begin with going quickly through the relevant definitions (for details we refer to [1] and [8] and the references cited therein). First of all, we need a connected nonsingular complex projective curve C, a finite subset $X \subset C$ so that $U := C - X$ is affine and a semisimple complex Lie algebra $\mathfrak{g}$. Put $\mathfrak{g}(U) := \mathfrak{g} \otimes \mathcal{O}(U)$ and let $\mathfrak{g}_X$ be the completion of $\mathfrak{g}(U)$ along X, in other words, $\mathfrak{g}_X = \prod_{x\in X} \mathfrak{g}_x$, where $\mathfrak{g}_x$ is $\mathfrak{g}$ tensorized with the quotient field of the completed local ring of (C,x). This Lie algebra has a natural central extension $\hat{\mathfrak{g}}_X$ by $\mathbb{C}$, defined by the cocycle

$$(\sum_{x\in X} Y_x\otimes f_x, \sum_{x\in X} Z_x\otimes g_x)\mapsto \sum_{x\in X}\langle Y_x, Z_x\rangle \operatorname{Res}_x(f_x dg_x)$$

where $\langle\, ,\,\rangle$ denotes the normalized Killing form of $\mathfrak{g}$. This is called an *affine Lie algebra*. Since the sum of the residues of a rational function on C is zero, the inclusion $\mathfrak{g}(U)\subset\mathfrak{g}_X$ composed with the obvious linear map $\mathfrak{g}_X\subset\hat{\mathfrak{g}}_X$ is a homomorphism of Lie algebras. Now fix a positive integer l (the *level*) and choose an irreducible highest weight representation H of $\mathfrak{g}_X$ on which the generator of the center (defined by the above cocycle) acts as scalar multiplication by l. (Perhaps we should remark that we have a natural embedding of $\hat{\mathfrak{g}}_X$ in a product of affine Lie algebras $\prod_{x\in X}\hat{\mathfrak{g}}_x$ and that each of these factors appears as a subalgebra; so a representation H as above is tantamount to giving for each $x\in X$ an irreducible standard highest (integral) weight representation H_x of $\hat{\mathfrak{g}}_x$ of level l; H is then the tensor product of these.) The associated *conformal block* is by definition the dual of the space of $\mathfrak{g}(U)$-coinvariants of H: $V:=(H_{\mathfrak{g}(U)})^*$. This is a finite dimensional space whose associated projective space $\mathbb{P}(V)$ only depends on the isomorphism class of H. If a decoration is given, then this is even true for V itself and so we have a vector bundle $\mathcal{V}$ over $\mathcal{M}_{g,n}$. A remarkable feature of this bundle is that it comes with a natural flat connection, in other words, it naturally underlies a local system.

Now suppose that we have a cover $\pi:(\tilde C,\tilde X,\tilde v)\to(C,X,v)$ of decorated curves and put $\tilde U:=\pi^{-1}U$. In order to pull back the conformal block V on U to one on $\tilde U$, we must manufacture a standard representation $\tilde H$ of $\mathfrak{g}_{\tilde X}$ out of H. There is an obvious choice for this: induce H to a representation of $\mathfrak{g}_{\tilde X}$ (via the inclusion $\mathfrak{g}_X\subset\mathfrak{g}_{\tilde X}$); it is a highest weight representation which has a maximal standard quotient—this is our $\tilde H$. There is a natural map $H_{\mathfrak{g}(U)}\to\tilde H_{\mathfrak{g}(\tilde U)}$. This gives rise to a vector bundle homomorphism $\tilde{\mathcal{V}}\to\mathcal{V}$ over $\mathcal{M}(f)$. This bundle map can be shown to be a homomorphism of local systems.

There is also a direct image construction, which is simply gotten by restricting $\tilde H$ to $\mathfrak{g}(U)$. This leads to a vector bundle homomorphism $\mathcal{V}\to\tilde{\mathcal{V}}$ over $\mathcal{M}(f)$ which is also a homomorphism of local systems.

5.2. **The Hopf product and its algebro-geometric incarnation.** If S_i, $i=1,2$, are two oriented connected, compact surfaces with nonempty boundary, then an embedding of their disjoint union in an oriented connected compact surface S defines a homomorphism $\Gamma(S_1)\times\Gamma(S_2)\to\Gamma(S)$ and in the stable range this defines the Hopf product $\bullet$ on $H_\bullet^{\mathbb{Z}}(\Gamma_\infty)$. In particular, an embedding of n disjoint copies of S_1 in S, defines a homomorphism $\Gamma(S_1)^n\to\Gamma(S)$, which composed with the diagonal embedding $\Gamma(S_1)\to\Gamma(S_1)^n$, induces on $H_\bullet^{\mathbb{Z}}(\Gamma_\infty)$ the Adams operator ψ_n in the stable range.

Now let $f:\tilde S\to S$ be a connected covering of degree d. Denote by $\Gamma(f)$ the group of pairs $(\tilde h,h)\in\Gamma(\tilde S)\times\Gamma(S)$ with $f\tilde h=hf$. The projection $\Gamma(f)\to\Gamma(S)$ has finite kernel and maps onto a subgroup of finite index. So the rational homology of $\Gamma(S)$ appears as a direct summand of the rational homology of $\Gamma(f)$. The restriction of the second projection to this summand defines a correspondence $T[f]:H_\bullet(\Gamma(S))\to H_\bullet(\Gamma(\tilde S))$ that takes counit to counit. Choose a connected component $\tilde S_i$ of $f^{-1}S_i$. Then the degree d_i of the covering $f_i:\tilde S_i\to S_i$ will divide d and $f^{-1}S_i$ consists of

d/d_i copies of $\tilde{S}_i$. We have defined similarly $T[f_i] : H_\bullet(\Gamma(S_i)) \to H_\bullet(\Gamma(\tilde{S}_i))$. It is now clear that for $a_i \in H_{r_i}(\Gamma(S_i)) \cong H_{r_i}(\Gamma_\infty)$ in the stable range with respect to $g(S_i)$, we have the product formula

$$T[f](a_1 \bullet a_2) = \psi_{d/d_1} T[f_1](a_1) \bullet \psi_{d/d_2} T[f_2](a_2). \tag{9}$$

Now recall that $T(f)$ is the average of all $T[hf]$, where h runs over a system of representatives of cosets of the image of $\Gamma(f)$ in $\Gamma(S)$. Since h will in general not respect the subsurfaces S_i, we may not, in the above expression, replace brackets by parentheses.

The Hopf product is realized inside the Deligne-Mumford compactification in the following way. Out of two smooth once-pointed complex projective curves (C_1, x_1) and (C_2, x_2) of genus g_1 resp. g_2, we can fabricate a stable once-pointed curve (C, x) of genus $g = g_1 + g_2$ by attaching them both to $\mathbb{P}^1$: identify x_1 resp. x_2 with 0 resp. ∞ and taking for x the image of $1 \in \mathbb{P}^1$. This clearly defines a map $k : \mathcal{M}^1_{g'} \times \mathcal{M}^1_{g_2} \to \overline{\mathcal{M}}_{g,1}$, simply use the standard affine differential of $\mathbb{P}^1$ to decorate x. The image of k is of complex codimension two and lies in the locus where the boundary divisor has exactly two branches. It has a normal bundle ν_k in the orbifold sense, the normal space to the point defined by (C, x) being naturally identified with the direct sum $T_{x_1}C_1 \oplus T_{x_2}C_2$. Each summand defines normal vectors pointing along a branch of the boundary divisor and so a vector $(t_1, t_2) \in T_{x_1}C_1 \oplus T_{x_2}C_2$ points to the interior if and only if both t_1 and t_2 are nonzero. In other words, if we start with stable once decorated curves (C_i, v_i), then we not only get a boundary point of $\mathcal{M}^1_g$, but a normal vector pointing to the interior as well. Let $E(k)$ denote the normal bundle of the stratum of $\overline{\mathcal{M}}_{g,1}$ containing the image of k, and let $E'(k)$ be the $(\mathbb{C}^\times)^2$-subbundle of normal vectors pointing towards the interior. So we just defined a lift of k,

$$\tilde{k} : \mathcal{M}_{g_1,1} \times \mathcal{M}_{g_2,1} \to E'(k).$$

We should think of $E'(k)$ as the intersection with $\mathcal{M}_{g,1}$ of a regular neighborhood of the stratum containing the image of k. In particular, there is a natural map on homology $H_\bullet(E'(k)) \to H_\bullet(\mathcal{M}_{g,1})$. Its composite with $\tilde{k}$,

$$H_\bullet(\mathcal{M}_{g_1,1}) \otimes H_\bullet(\mathcal{M}_{g_2,1}) \to H_\bullet(\mathcal{M}_{g,1}),$$

realizes the Hopf product. A geometric picture of its behaviour with respect to correspondences is obtained by composing this map with a correspondence $\overline{T}(f)$: $\overline{\mathcal{M}}_{g,1} \rightrightarrows \mathcal{S}_m \backslash \overline{\mathcal{M}}_{h,m}$.

5.3. **The stable Hecke operator attached to a finite group.** Next consider the simplest case when S_1, S_2 and S have a single boundary circle and S minus the union of the interiors of S_1 and S_2 is a three holed sphere P. Choose a base point $* \in \partial S$ and choose a path in P connecting $*$ with a point $*_i \in \partial S_i$ so that $\pi_1(S_i, *_i)$ can be identified with a subgroup of $\pi(S, *)$. Notice that $\pi(S, *)$ is the free product of these two subgroups. We now fix a finite group G of order d. Every homomorphism $u : \pi_1(S, *) \to G$ defines a G-covering $f_u : \tilde{S}_u \to S$ whose degree d_u is the order of image(u). Put

$$\mathcal{T}_{S,G} := |\operatorname{Hom}(\pi_1(S, *), G)|^{-1} \sum_{u \in \operatorname{Hom}(\pi_1(S,*),G)} \psi_{d/d_u} T[f_u], \tag{10}$$

viewed as homomorphism from the stable range homology of $\Gamma(S)$ to the stable range homology of $\Gamma(\tilde{S})$. So this is a weighted average of the operators $\psi_{d/d_u} T[f_u]$.

In particular, $\mathcal{T}_{S,G}$ sends counit to counit. Our reason for introducing these maps is their nice behavior with respect to the Hopf product:

Proposition 5.3. *We have $\mathcal{T}_{S,G}(1) = 1$ and if $a_i \in H_{s_i}(\Gamma(S_i))$ is in the stable range, then $\mathcal{T}_{S,G}(a_1 \bullet a_2) = \mathcal{T}_{S_1,G}(a_1) \bullet \mathcal{T}_{S_2,G}(a_2)$.*

Proof. The first assertion is clear and included for the purpose of reference only. Since $\pi(S, *)$ is the free product of $\pi_1(S_1, *_1)$ and $\pi_1(S_2, *_2)$, we have a natural bijection between $\mathrm{Hom}(\pi_1(S, *), G)$ and $\mathrm{Hom}(\pi_1(S_1, *_1), G) \times \mathrm{Hom}(\pi_1(S_2, *_2), G)$. The second assertion now follows from the product formula (9). □

Corollary 5.4. *The operator $\mathcal{T}_{S,G}$ in $H_s(\Gamma_\infty)$ for $s \leq cg(S)$ is independent of S and the resulting endomorphism $\mathcal{T}_G$ of $H_\bullet(\Gamma_\infty)$ is an algebra homomorphism.*

Proof. By taking $a_2 = 1$ in the above proposition, we see that $\mathcal{T}_{S,G}$ and $\mathcal{T}_{S_1,G}$ act identically in the stable range. The first statement follows and the second is clear. □

I do not know whether $\mathcal{T}_G$ preserves the coproduct. By taking G abelian we get Proposition 4.1.

5.4. **Stable covers and the stable Hecke operators they define.** Remember that $S_{g,1}$ is a connected oriented compact surface of genus g with a single boundary component. Choose a base point in this boundary component and let $\pi_{g,1}$ denote the fundamental group of $S_{g,1}$ relative this base point. Let be given for every g a cofinite subgroup of $\pi_{g,1}$. We say that the collection $(I(g) \subset \pi_{g,1})_g$ is *stable* if for each g,

(i) $I(g)$ is a $\Gamma(S_{g,1})$-invariant subgroup of $\pi_{g,1}$ of finite index and
(ii) there exists a sense preserving embedding of $S_{g,1}$ in $S_{g+1,1}$ mapping base point to base point such that the preimage of $I(g+1)$ is equal to $I(g)$.

The homomorphisms $\pi_{g,1} \to \pi_{h,1}$, $h \geq g$, that arise from embeddings of $S_{g,1}$ in $S_{h,1}$ lie in a single $\Gamma(S_{h,1})$-orbit and so for a stable sequence $(I(g))_g$, the preimage of $I(h)$ under any such homomorphisms is equal to $I(g)$. (A natural subgroup of $\pi_{g,n}$ of finite index is now defined for every n: embed $S_{g,n}$ in some $S_{h,1}$ such that base point goes to base point and the closure of every connected component of the complement $S_{h,1} - S_{g,n}$ meets $S_{g,n}$ in a single boundary circle. Such embeddings belong to a single homotopy class and so the preimage of $I(g)$ in $\pi_{g,n}$ is independent of choices.) We may think of the sequence $(I(g))_g$ as being given by a subgroup I of the fundamental group of a connected pointed surface $(S, *)$ of infinite genus with the property that

(i) I is invariant under all compactly supported mapping classes of $(S, *)$ and
(ii) I has cofinite intersection with the fundamental group of any compact subsurface $S' \subset S$ containing the base point.

Geometrically, a stable sequence amounts to giving a finite $\Gamma(S_{g,1})$-invariant covering $pr_g : \tilde{S}_{g,1} \to S_{g,1}$ for every g such that the pull-back of pr_h under an embedding $S_{g,1} \to S_{h,1}$ has each connected component $S_{g,1}$-isomorphic to $pr_g : \tilde{S}_{g,1} \to S_{g,1}$. This can also be described by a covering $pr : \tilde{S} \to S$ of our infinite genus surface S satisfying the two properties corresponding to the ones above. We call $(pr_g)_g$ a *stable sequence of coverings*. An intersection of two stable sequences is again stable.

Example 5.5. Let d be a positive integer and take for $I(g)$ the kernel of the homomorphism $\pi_{g,1} \to H_1(\pi_{g,1}; \mathbb{Z}/d)$. This defines a stable sequence.

Example 5.6. Fix a finite group G and define $I_G(g)$ as the intersection of the kernels of all group homomorphisms $\pi_{g,1} \to G$. This is certainly an invariant group. Since $\pi_{g,1}$ is finitely generated there are only finitely many homomorphisms from $\pi_{g,1}$ to the finite group G and so $I_G(g)$ is of finite index in $\pi_{g,1}$. The sequence $(I_G(g))_g$ is stable: since every homomorphism $\pi_{g,1} \to G$ can be extended to $\pi_{g+1,1}$, the pull-back of $I_G(g+1)$ is equal to $I_G(g)$. Notice that for $G = \mathbb{Z}/d$ we recover the previous example.

This example also shows that the stable sequences define a system of subgroups of $\pi_{g,1}$ of finite index that is cofinal among all such subgroups. For if π is any subgroup of $\pi_{g,1}$ of finite index, then the intersection of all its conjugates is a normal subgroup $N \subset \pi_{g,1}$ of finite index contained in π. If we put $G := \pi_{g,1}/N$, then clearly, $I_G(g) \subset N$.

Let $(I(g))_g$ and $(pr_g : \tilde{S}_{g,1} \to S_{g,1})_g$ be as above. Then $\tilde{S}_{g,1}$ is isomorphic to some $S_{g',n'}$ and so pr_g defines a correspondence that is almost a morphism $T_{I(g)} : \mathcal{M}_{g,1} \rightrightarrows \mathcal{M}_{g',n'}$: the ambiguity lies only in the decoration, so that its composite with $\mathcal{M}_{g',n'} \to \mathcal{M}_{g'}$ is a morphism indeed. In particular, it acts in the stable range as a cohomomorphism. Now let $S_{g_1,1}$ and $S_{g_2,1}$ be disjointly embedded in $S_{g,1}$ in a sense preserving way. If $d(g)$ denotes the index of $I(g)$ in $\pi_{g,1}$, then the restriction of pr_g to $S_{g_i,1}$ consists of $d(g)/d(g_i)$ copies of pr_{g_i}. So if $a_i \in H_{s_i}(\Gamma_\infty)$ with s_i in the stable range with respect to g_i, then we have a product formula just as in (9)

$$T_{I(g)}(a_1 \bullet a_2) = \psi_{d(g)/d(g_1)} T_{I(g_1)}(a_1) \bullet \psi_{d(g)/d(g_2)} T_{I(g_2)}(a_2). \tag{11}$$

From this we derive:

Proposition 5.7. *The action of* $\psi^{-1}_{[\pi_{g,1}:I(g)]} T_{I(g)}$ *on* $H_s(\Gamma_\infty)$ *is (for* g *in the stable range with respect to* s*) independent of* g*. The resulting endomorphism* $\mathcal{S}_I$ *of* $H_\bullet(\Gamma_\infty)$ *is a Hopf algebra endomorphism.*

Proof. Apply $\psi^{-1}_{d(g)}$ to formula (11). Then substitution of $a_2 = 1$ shows that $\psi^{-1}_{d(g_1)} T_{I(g_1)} = \psi^{-1}_{d(g)} T_{I(g)}$. This proves the first assertion. The formula also shows that $\mathcal{S}_I$ is an algebra homomorphism. We already noticed that it is a cohomomorphism. □

Proposition 4.4 follows if this proposition is applied to the stable sequence of Example 5.5.

Problem 5.8. I do not know whether the Hopf algebra endomorphism $\mathcal{S}_I$ is in fact an automorphism, let alone whether it is the identity (by 3.3 it is so on the tautological subalgebra).

6. Correspondences acting on a Lie algebra

We may think of $\mathcal{S}_n \backslash \mathcal{M}_g^n$ as the moduli space of connected smooth affine curves of genus g with $n > 0$ punctures. Let us collect those with prescribed (negative) Euler characteristic $-e < 0$:

$$\mathcal{A}_e := \bigsqcup_{2g-2+n=e, n>0} \mathcal{S}_n \backslash \mathcal{M}_g^n.$$

If we drop the condition that the curves be connected, we get

$$\mathcal{B}_e := \bigsqcup_{k_1+2k_2+\cdots ek_e=e} \mathcal{A}_1^{k_1} \times \mathcal{A}_2^{k_2} \times \cdots \times \mathcal{A}_e^{k_e}.$$

According to Kontsevich, the cohomology of $\mathcal{A}_e$ has a remarkable interpretation. Consider $\mathbb{Q}^{2r}$ with the standard symplectic element $\omega_r := e_1 \wedge e_2 + \cdots + e_{2r-1} \wedge e_{2r} \in \wedge^2(\mathbb{Q}^{2r})$. Regard $\mathbb{Q}^{2r}$ as a graded vector space which is homogeneous of degree -1 and consider the derivations of the tensor algebra of $\mathbb{Q}^{2r}$ which kill ω_r and have degree ≤ 0. This is a graded Lie algebra which we denote by $\mathfrak{g}_r$. Its degree zero summand $\mathfrak{g}_{r,0}$ can be identified with the symplectic Lie $\mathbb{Q}$-algebra $\mathfrak{sp}_{2r}(\mathbb{Q})$. Then is defined the relative cohomology $H^\bullet(\mathfrak{g}_r, \mathfrak{g}_{r,0})$. Then the relative cohomology $H^\bullet(\mathfrak{g}_r, \mathfrak{g}_{r,0})$ is defined. The grading of $\mathfrak{g}_r$ defines one on each $H^s(\mathfrak{g}_r, \mathfrak{g}_{r,0})$ and the latter has degrees ≥ 0 only. The natural embeddings $(\mathfrak{g}_r, \mathfrak{g}_{r,0}) \subset (\mathfrak{g}_{r+1}, \mathfrak{g}_{r+1,0})$ of graded pairs of Lie algebras define graded maps $H^s(\mathfrak{g}_{r+1}, \mathfrak{g}_{r+1,0}) \to H^s(\mathfrak{g}_r, \mathfrak{g}_{r,0})$. These can be proved to stabilize and the limit $H^\bullet(\mathfrak{g}_\infty, \mathfrak{g}_{\infty,0})$ is naturally a bigraded Hopf algebra (the coproduct is induced by the obvious maps $(\mathfrak{g}_{r_1}, \mathfrak{g}_{r_1,0}) \times (\mathfrak{g}_{r_2}, \mathfrak{g}_{r_2,0}) \to (\mathfrak{g}_{r_1+r_2}, \mathfrak{g}_{r_1+r_2,0})$). Let $H^\bullet_{\rm pr}(\mathfrak{g}_\infty, \mathfrak{g}_{\infty,0})$ denote its primitive part. Kontsevich [7] (see also [3]) proves that we have a canonical isomorphism

$$H_k(\mathcal{A}_e) \cong H^{2e-k}_{\rm pr}(\mathfrak{g}_\infty, \mathfrak{g}_{\infty,0})_{2e}.$$

This is equivalent to saying that we have canonical isomorphisms

$$H_k(\mathcal{B}_e) \cong H^{2e-k}(\mathfrak{g}_\infty, \mathfrak{g}_{\infty,0})_{2e}$$

whose direct sum is an isomorphism of Hopf algebras. (The Hopf algebra structure on the homology of the disjoint union of the $\mathcal{B}_e$'s comes from 'taking disjoint union of curves'.)

Via this isomorphism correspondences give rise to operations in $H^\bullet_{\rm pr}(\mathfrak{g}_\infty, \mathfrak{g}_{\infty,0})_\bullet$. For instance, if we are given for every connected oriented surface of Euler characteristic $-e$ a connected covering of degree d up to homeomorphism, then this determines a correspondence $H_\bullet(\mathcal{A}_e) \to H_\bullet(\mathcal{A}_{de})$, and so a linear map

$$t : H^\bullet_{\rm pr}(\mathfrak{g}_\infty, \mathfrak{g}_{\infty,0})_{2e} \to H^{\bullet+2(d-1)e}_{\rm pr}(\mathfrak{g}_\infty, \mathfrak{g}_{\infty,0})_{2de}.$$

This map increases both grade and dimension by $2(d-1)e$ and so has the formal appearance of a dth power operation.

Problem 6.1. Give an interpretation of the maps t (for some natural choices of covers, say) in terms of the Lie pair $(\mathfrak{g}_\infty, \mathfrak{g}_{\infty,0})$.

We believe that such an interpretation could be very useful. Perhaps this is also the place to remark that to the best of the our knowledge none of the finer structure that exists on the homology of $\mathcal{A}_e$ (the coproduct, the Hodge decomposition, ...) has been transcribed to the Lie algebra side. And now that we are at it, a similar interpretation for the homology of the Deligne-Mumford compactification $\bar{\mathcal{A}}_e$ is conspicuously missing.

References

[1] A. Beauville: *Conformal blocks, fusion rules and the Verlinde formula*, in Proc. Hirzebruch 65 Conference on Algebraic Geometry, Israel Math. Conf. Proc. 9 (1996), 75–96.

[2] I. Biswas, S. Nag, D. Sullivan: *Determinant bundles, Quillen metrics and Mumford isomorphisms over the universal commensurability Teichmüller space,* Acta Math. 176 (1996), 145–169.

[3] R. Hain and E. Looijenga: *Mapping Class Groups and Moduli Spaces of Curves*, in Algebraic Geometry—Santa Cruz 1995, Proc. Symp. Pure Math. 62, Part 2 (1997), 97–142.

[4] J. Harer: *Stability of the homology of the mapping class groups of orientable surfaces*, Ann. of Math. 121 (1985), 215–249.

[5] J. Harer: *Improved stability for the homology of the mapping class groups of surfaces*, Duke University preprint, 1993. (Available from `http://www.math.duke.edu/preprints/`).

[6] J. Harris and D. Mumford: *On the Kodaira dimension of the moduli space of curves*, Invent. Math. 67 (1982), 23–86.

[7] M. Kontsevich: *Feynman diagrams and low-dimensional topology*, in: Proc. First Eur. Math. Congr. at Paris (1992), Vol. II, 97–121, Birkhäuser Verlag, Basel (1994).

[8] K. Ueno: *Introduction to Conformal Field Theory with Gauge Symmetries*, in: Geometry and Physics, Proc. of a Conf. at Aarhus Univ. (1995), 603–745. Lecture Notes in Pure and Appl. Math. Vol. 184, Marcel Dekker, Inc., New York (1997).

Faculteit Wiskunde en Informatica, University of Utrecht, Postal Box 80.010, NL-3508 TA Utrecht, The Netherlands

E-mail address: `looijeng@math.uu.nl`