STABLE COHOMOLOGY OF THE MAPPING CLASS GROUP WITH SYMPLECTIC COEFFICIENTS AND OF THE UNIVERSAL ABEL–JACOBI MAP

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Abstract. The irreducible representations of the complex symplectic group of genus g are indexed by nonincreasing sequences of integers $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ with $\lambda_k = 0$ for $k > g$. A recent result of N.V. Ivanov implies that for a given partition λ, the cohomology group of a given degree of the mapping class group of genus g with values in the representation associated to λ is independent of g if g is sufficiently large. We prove that this stable cohomology is the tensor product of the stable cohomology of the mapping class group and a finitely generated graded module over $\mathbb{Q}[c_1, \dots, c_{|\lambda|}]$, where $\deg(c_i) = 2i$ and $|\lambda| = \sum_i \lambda_i$. We describe this module explicitly. In the same sense we determine the stable rational cohomology of the moduli space of compact Riemann surfaces with s given ordered distinct (resp. not necessarily distinct) points as well as the stable cohomology of the universal Abel–Jacobi map. These results take into account mixed Hodge structures.

1. Introduction

The mapping class group $\Gamma^s_{g,r}$ can be defined in terms of a compact connected oriented surface S_g of genus g on which are given $s + r$ (numbered) distinct points $(x_i)_{i=1}^{r+s}$: it is then the connected component group of the group of orientation preserving diffeomorphisms of S_g which fix each x_i and are the identity on the tangent space of S_g at x_i for $i = s+1, \dots, s+r$. It is customary to omit the suffix r resp. s when it is zero.

Harer's stability theorem says essentially that $H^k(\Gamma^s_{g,r}; \mathbb{Z})$ only depends on s if g is large compared to k. For a more precise statement it is convenient to make a definition first. There is a natural outer homomorphism $\Gamma^s_{g,r+1} \to \Gamma^s_{g+1,r}$ (that is, an orbit of homomorphisms under the inner automorphism group of the target group, so that there is well-defined map on homology) and there is a forgetful homomorphism $\Gamma^s_{g,r+1} \to \Gamma^s_{g,r}$. For a coefficient ring R and an integer $g_0 \geq 0$, we define $N(g_0; R)$ as the maximal integer N such that both induce isomorphisms on homology with coefficients in R in degree $\leq N$ for all $g \geq g_0$ and $s, r \geq 0$. Harer showed in [3] that $N(g; \mathbb{Z}) \geq \frac{1}{3}g$ and Ivanov [5], [6] improved this to $N(g; \mathbb{Z}) \geq \frac{1}{2}g - 1$. Recently, Harer proved that $N(g; \mathbb{Q}) \leq \frac{2}{3}g$ and that $N(g; \mathbb{Q}) \geq \frac{2}{3}g$ is almost true: it holds, provided that we restrict to the mapping class groups with $r \geq 1$ [4]. It is likely that in fact $N(g; \mathbb{Q}) \geq \frac{2}{3}g - 1$. We will be mostly concerned with $N(g; \mathbb{Q})$ and so we shall write for this number $N(g)$ instead. A consequence of

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the stability property is that for every integer $s \geq 0$ we have a stable cohomology algebra $H^\bullet(\Gamma_\infty^s; R)$ (where as before, we omit the superscript s if it is equal to 0). As the notation suggests, this is indeed the cohomology of a group Γ_∞^s, namely the group of compactly supported mapping classes of an oriented connected surface of infinite genus relative s given numbered points. (The number of ends of this surface may be arbitrary.)

Consider the symplectic vector space $V_g := H^1(S_g; \mathbb{Q})$. Its symplectic form is preserved by the natural action of the mapping class group Γ_g on V_g and so any finite dimensional representation U of the algebraic group $Sp(V_g)$ can be regarded as a representation of Γ_g; in particular we have defined the cohomology groups $H^k(\Gamma_g; U)$. A basic fact of representation theory is that the isomorphism classes of the irreducible complex representations of $Sp(V_g)$ are in a natural bijective correspondence with g-tuples of nonnegative integers $(a_1, \dots, a_g)$. For instance, the kth basis vector $(0, \dots, 0, 1, 0 \dots, 0)$ corresponds to the kth exterior power of V_g. This result goes back to Weyl, who gave in addition a functorial construction of such a representation inside $\mathrm{Sym}^{a_1}(\wedge^1 V_g) \otimes \cdots \otimes \mathrm{Sym}^{a_g}(\wedge^g V_g)$. He labeled this representation by the sequence $(a_1 + \cdots + a_g, a_2 + a_3 + \cdots + a_g, \dots, a_{g-1} + a_g, a_g)$. We will follow his convention, so for us a numerical partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ with at most g parts (that is, $\lambda_k = 0$ for $k > g$) determines an irreducible representation $S_{\langle\lambda\rangle}(V_g)$. It follows from a recent result of Ivanov [6] that for fixed k and λ, the cohomology groups $H^k(\Gamma_{g,1}; S_{\langle\lambda\rangle}(V_g))$ are independent of g if $k \leq N(g) - |\lambda|$, where $|\lambda| := \sum_i \lambda_i$ is the size of the partition. We shall give an independent proof of this in the undecorated case $r = s = 0$ which generalizes in a straightforward manner to the case of arbitrary r and s and we determine these stable cohomology groups as $H^\bullet(\Gamma_\infty; \mathbb{Q})$-modules at the same time.

(1.1) Theorem. *For every numerical partition λ of s, there is a graded finitely generated $\mathbb{Q}[c_1, \dots, c_s]$-module $B_\lambda^\bullet$ (where c_i has degree $2i$) and a natural homomorphism*

$$H^\bullet(\Gamma_\infty; \mathbb{Q}) \otimes B_\lambda^\bullet \to H^\bullet(\Gamma_g; S_{\langle\lambda\rangle}(V_g)),$$

which is an isomorphism in degree $\leq N(g) - |\lambda|$.

To describe $B_\lambda^\bullet$, denote the coordinates of s-space $\mathbb{Q}^s$ by $u_1, \dots, u_s$ so that $\mathbb{Q}[u_1, \dots, u_s]$ is its algebra of regular functions. We grade this algebra by giving each coordinate u_i weight 2. A *diagonal* of $\mathbb{Q}^s$ is by definition an intersection of the hyperplanes $u_i = u_j$; this includes the intersection with empty index set, that is, $\mathbb{Q}^s$ itself. It is clear that a partition P of the set $\{1, \dots, s\}$ determines (and is determined by) a diagonal Δ_P. Notice that the algebra of regular functions $\mathbb{Q}[\Delta_P]$ is a quotient of $\mathbb{Q}[u_1, \dots, u_s]$ by a graded ideal, so that it inherits a grading.

Denote by $l_i(P)$ the number of parts of P of cardinality i, and consider

$$\bigoplus_P t^{-s} u^{\mathrm{codim}(P) + 2l_1(P) + l_2(P)} \mathbb{Q}[\Delta_P], \tag{1}$$

where $\mathrm{codim}(P)$ is short for $\mathrm{codim}(\Delta_P)$ and t resp. u has formal degree 1 resp. 2. (The difference between t^2 and u becomes manifest in the Hodge theory: t has Hodge type $(0,0)$, whereas u has Hodge type $(1,1)$.) We regard this as a graded module over $\mathbb{Q}[u_1, \dots, u_s]$ that has a natural action of the symmetric group $\mathfrak{S}_s$. We tensorize this module with the signum representation of $\mathfrak{S}_s$ and denote the resulting graded $\mathbb{Q}[u_1, \dots, u_s]$-module with $\mathfrak{S}_s$-action by $B_s^\bullet$. Now recall that every numerical partition λ of s determines an equivalence class (λ) of irreducible representations of $\mathfrak{S}_s$ and that this gives a bijection between these two sets. For instance, the coarsest partition (s) labels the trivial representation, whereas the finest partition (1^s) corresponds to the signum representation. Passing from a partition λ to the conjugate partition λ' corresponds to taking the tensor product with the signum representation. In the case at hand we have a decomposition of $B_s^\bullet$ into isotypical subspaces:

$$B_s^\bullet = \oplus_\lambda B_\lambda^\bullet \otimes (\lambda), \quad \text{with } B_\lambda^\bullet = \mathrm{Hom}_{\mathfrak{S}_s}((\lambda), B_s^\bullet).$$

Clearly, this is also a decomposition into graded $\mathbb{Q}[u_1, \dots, u_s]^{\mathfrak{S}_s}$-submodules. We identify the latter ring with $\mathbb{Q}[c_1, \dots, c_s]$, where c_k is the kth elementary symmetric function in the u_i's. This completes the description of $B_\lambda^\bullet$.

It follows from the work of M. Saito that $H^\bullet(\Gamma_g; S_{\langle\lambda\rangle}(V_g))$ carries a natural mixed Hodge structure. It is known that $H^\bullet(\Gamma_\infty; \mathbb{Q})$ has a natural mixed Hodge structure as well (see (2.5)). We will find that if we put a Hodge structure on $B_\lambda^\bullet$ by giving its degree $2d - |\lambda|$-part Hodge type (d,d), then the homomorphism of (1.1) is a morphism of mixed Hodge structures. M. Pikaart has recently shown that $H^n(\Gamma_\infty; \mathbb{Q})$ is pure of weight n; this implies that $H^n(\Gamma_g; S_{\langle\lambda\rangle}(V_g))$ is pure of weight $n + |\lambda|$ in the stable range $n \le N(g) - |\lambda|$.

Example 1. The s-th symmetric power of V_g corresponds to $S_{\langle s\rangle}(V_g)$ and so the stable cohomology of Γ_g with values in $\mathrm{Sym}^s(V_g)$ is the tensor product of $H^\bullet(\Gamma_\infty)$ with $B^\bullet(1^s)$, that is, the isotypical subspace of expression (1) corresponding to the signum representation of $\mathfrak{S}_s$. Any partition different from the partition into singletons is invariant under a transposition and so will not contribute. This leaves us therefore with the (1^s)-isotypical subspace of $t^{-s}u^{2s}\mathbb{Q}[u_1, \dots, u_s]$. This is the free $\mathbb{Q}[c_1, \dots, c_s]$-module generated by the element $t^{-s}u^{2s}\prod_{i>j}(u_i - u_j)$, hence is of the form $t^{-s}u^{\frac{1}{2}s(s+3)}\mathbb{Q}[c_1, \dots, c_s]$. We find that

$$H^\bullet(\Gamma_\infty; \mathrm{Sym}^s(V_\infty)) = t^{-s}u^{\frac{1}{2}s(s+3)}H^\bullet(\Gamma_\infty; \mathbb{Q})[c_1, \dots, c_s].$$

Example 2. For $g \ge s$, the primitive subspace $\mathrm{Pr}^s(V_g)$ of the s-th exterior power of V_g corresponds to $S_{\langle 1^s\rangle V_g}$ and so the stable cohomology of Γ_g with values in $\mathrm{Pr}^s(V_g)$ is the tensor product of $H^\bullet(\Gamma_\infty; \mathbb{Q})$ with $\mathfrak{S}_s$-invariant part of (1). This is naturally written as a sum over the numerical partitions of s. Here a numerical partition is best described by means of the exponential notation: $(1^{l_1}2^{l_2}3^{l_3}\cdots)$, where l_k is the number of parts of cardinality k (so that $\sum_k kl_k = s$). Its contribution is then

$$\otimes_{k\ge1} t^{-l_k k} u^{l_k \max(2,k-1)}\mathbb{Q}[c_1, c_2, \dots, c_{l_k}].$$

If we sum over all sequences $(l_1, l_2, \dots)$ of nonnegative integers which become eventually zero, we get

$$\otimes_{k\ge1}\big(\oplus_{l=0}^\infty t^{-lk}u^{l\max(2,k-1)}\mathbb{Q}[c_1, c_2, \dots, c_l]\big).$$

and $B^\bullet_{(1^s)}$ is the t^{-s}-part of this expression. For instance,

$$B^\bullet_{(1^2)} = t^{-2}(u^4\mathbb{Q}[c_1,c_2] \oplus u^2\mathbb{Q}[c_1]),$$
$$B^\bullet_{(1^3)} = t^{-3}(u^6\mathbb{Q}[c_1,c_2,c_3] \oplus u^4\mathbb{Q}[c_1]\otimes\mathbb{Q}[c_1] \oplus u^2\mathbb{Q}[c_1]).$$

Since $H^1(\Gamma_\infty;\mathbb{Q}) = 0$, it follows in particular that $H^1(\Gamma_\infty, S_{\langle 1^3\rangle})$ has dimension one. (It is easy to see that $H^1(\Gamma_\infty; , S_{\langle\lambda\rangle}) = 0$ for all other numerical partitions λ.)

(1.2) The cohomology groups of Γ_g with values in a symplectic representation have a geometric interpretation as the cohomology of a local system (in fact, of a variation of polarized Hodge structure). The Teichmüller space $\mathcal{T}_g$ of conformal structures on S_g modulo isotopy is a *contractible* complex manifold (of complex dimension $3g-3$, when $g \geq 2$). The action of Γ_g on it is properly discontinuous and a subgroup of finite index acts freely. The orbit space $\mathcal{M}_g := \Gamma_g\backslash\mathcal{T}_g$ is naturally interpreted as the coarse moduli space of smooth complex projective curves. Via this interpretation, it gets the structure of a normal quasi-projective variety. It follows from the preceding that $\mathcal{M}_g$ has the rational cohomology of Γ_g. More generally, if U is a rational representation of Γ^s_g, then we have a natural isomorphism

$$H^\bullet(\Gamma_g;U) \cong H^\bullet(\mathcal{M}_g;\mathbb{U}),$$

where $\mathbb{U}$ is the sheaf on $\mathcal{M}_g$ which is the quotient of the trivial local system $\mathcal{T}_g\times U \to \mathcal{T}_g$ by the (diagonal) action of Γ_g. This applies in particular to the representations $S_\lambda(V_g)$. This representation appears in the cohomology of degree $s := |\lambda|$ of the configuration space of s numbered (not necessarily distinct) points on S_g. So if $\mathcal{C}^s_g \to \mathcal{M}_g$ denotes the s-fold fiber product of the universal curve, then its sth direct image contains the local system (in the orbifold sense) associated to $S_\lambda(V_g)$ as a direct summand. By a theorem of Deligne, the Leray spectral sequence of the forgetful map $\mathcal{C}^s_g \to M_g$ degenerates at the E_2-term and thus the stable cohomology of this representation is realized inside $H^\bullet(\mathcal{C}^s_g;\mathbb{Q})$. This fact will be used in an essential way.

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2. Stable cohomology of $\mathcal{M}^s_g$ and $\mathcal{C}^s_g$

We first state and prove an immediate consequence of the stability theorems. For $s \geq 1$, the class of a (Dehn) twist about x_s generates an infinite cyclic central subgroup of $\Gamma^{s-1}_{g,r+1}$. The quotient group can be identified with $\Gamma^s_{g,r}$ and so we have a Gysin sequence

$$\cdots \to H^{k-2}(\Gamma^s_{g,r};\mathbb{Z}) \xrightarrow{u_s\cup} H^k(\Gamma^s_{g,r};\mathbb{Z}) \to H^k(\Gamma^{s-1}_{g,r+1};\mathbb{Z}) \to \cdots,$$

where $u_s \in H^2(\Gamma^s_{g,r};\mathbb{Z})$ is the first Chern class. Similarly, x_i determines a first Chern class $u_i \in H^2(\Gamma^s_{g,r};\mathbb{Z})$ for $i = 1,\dots,s$. These classes are clearly stable and we shall not make any notational distinction between the u_i's and their stable representatives.

(2.1) Proposition. *The stable cohomology algebra over of the mapping class groups of surfaces with s distinct numbered points is a graded polynomial algebra on the stable cohomology ring of the absolute mapping class groups. More precisely, there is a natural $\mathfrak{S}_s$-equivariant graded ring homomorphism*

$$H^\bullet(\Gamma_\infty;\mathbb{Z})[u_1,\dots,u_s]\to H^\bullet(\Gamma^s_{g,r};\mathbb{Z}),$$

which is an isomorphism in degree $\le N(g;\mathbb{Z})$. In particular, the rational stable cohomology algebra of the mapping class groups of surfaces with s unlabeled points is a graded polynomial $H^\bullet(\Gamma_\infty;\mathbb{Q})$-algebra on the elementary symmetric functions $c_1,\dots,c_s$ of the u_i's.

Proof. The composite of the forgetful maps $\Gamma^{s-1}_{g,r}\to\Gamma^s_{g,r-1}$ and $\Gamma^s_{g,r-1}\to\Gamma^{s-1}_{g,r-1}$ induces an isomorphism on $H^k(-;\mathbb{Z})$ for large g. So in this range the forgetful map induces a surjection $H^k(\Gamma^s_{g,r-1};\mathbb{Z})\to H^k(\Gamma^{s-1}_{g,r};\mathbb{Z})$. If we feed this in the above Gysin sequence, we get short exact sequence

$$0\to H^{\bullet-2}(\Gamma^s_\infty;\mathbb{Z})\xrightarrow{u\cup}H^\bullet(\Gamma^s_\infty;\mathbb{Z})\to H^\bullet(\Gamma^{s-1}_\infty;\mathbb{Z})\to 0$$

so that $H^\bullet(\Gamma^s_\infty;\mathbb{Z})\cong H^\bullet(\Gamma^{s-1}_\infty;\mathbb{Z})[u]$ as algebra's. The assertion now follows with induction on s.

Let us fix a finite set X. We denote by $\mathcal{C}^X_g$ the moduli space of pairs (C,x) where C is a compact Riemann surface of genus g and $x:X\to C$ is a map. Let $j:\mathcal{M}^X_g\subset\mathcal{C}^X_g$ be the open subset defined by the condition that x be injective. Just as $\mathcal{M}_g$ is a virtual classifying space for Γ_g, $\mathcal{M}^X_g$ is one for $\Gamma^{|X|}_g$; in particular, $\Gamma^{|X|}_g$ and $\mathcal{M}^X_g$ have the same rational cohomology. This enables us to restate (2.1) in more geometric terms. Let $\mathcal{C}_g\to\mathcal{M}_g$ be the universal curve and denote by θ its relative tangent sheaf. For every $i\in X$, the map $(C,x)\mapsto x(i)$ defines a projection of $\mathcal{C}^X_g$ onto the $\mathcal{C}_g$; denote by θ_i the pull-back of θ under this map. One easily recognizes the first Chern class of $\tau_i|\mathcal{M}^X_g$ as the first Chern class defined above. So (2.1) implies:

(2.2) Proposition. *The ring homomorphism*

$$H^\bullet(\mathcal{M}_g;\mathbb{Q})[u_i:i\in X]\to H^\bullet(\mathcal{M}^X_g;\mathbb{Q}),\quad u_i\mapsto c_1(\theta_i)|\mathcal{M}^X_g.$$

is an isomorphism in degree $\le N(g)$.

We will use this proposition to prove that the rational cohomology of $\mathcal{C}^X_g$ also stabilizes.

We begin with attaching to X a graded commutative $\mathbb{Q}[u_i:i\in X]$-algebra. It is convenient to introduce an auxiliary graded commutative $\mathbb{Q}$-algebra $\tilde{A}^\bullet_X$ first. The latter is defined by the following presentation: for each nonempty subset I of X, $\tilde{A}^\bullet_X$ has a generator u_I of degree two (we also write u_i for $u_{\{i\}}$), and these are subject to the relations $u_Iu_J=u_iu_{I\cup J}$ if $i\in I\cap J$. So if $i\in I$, then $u_Iu_I=u_iu_I$. It is then easy to see that the $\mathbb{Q}[u_i:i\in X]$-submodule generated by u_I is defined by the relations $(u_i-u_j)u_I=0$ whenever $i,j\in I$. The monomials $\prod_I u_I^{r_I}$ for which I runs over the members of a partition of X form an additive basis of $\tilde{A}^\bullet_X$. (To make the indexing effective, let us agree that we only allow r_I to be zero if I is a singleton.)

We then let $A^\bullet_X$ be the $\mathbb{Q}[u_i : i \in X]$-subalgebra of $\tilde{A}^\bullet_X$ generated by the elements $a_I := u_I^{|I|-1}$, where I runs over the subsets of X with at least two elements. These generators obey the relations

$$u_i a_I = u_j a_I \text{ if } i, j \in I,$$
$$a_I a_J = u_i^{|I\cap J|-1} a_{I\cup J} \text{ if } i \in I \cap J.$$

and it is easy to see we thus obtain a presentation of $A^\bullet_X$ as a graded commutative $\mathbb{Q}[u_i : i \in X]$-algebra. Notice that as a $\mathbb{Q}[u_i : i \in X]$-algebra, $A^\bullet_X$ is already generated by the a_I's with $|I| = 2$. For every partition P of X we put $a_P := \prod_{I\in P;|I|\geq 2} a_I$ (with the convention that $a_P = 1$ if P is the partition into singletons). These elements generate $A^\bullet_X$ as a $\mathbb{Q}[u_i : i \in X]$-module. In fact,

$$A^\bullet_X = \bigoplus_{P|X} \mathbb{Q}[u_I : I \in P] a_P.$$

We give $\tilde{A}^\bullet_X$ a (rather trivial) Hodge structure: its degree $2p$-part has Hodge type (p, p). Clearly, $A^\bullet_X$ is then a Hodge substructure.

Given a partition P of X, then the pairs $(C, x : X \to C)$ for which every member of P is contained in a fiber of x define a closed subvariety $i_P : \mathcal{C}_g(P) \subset \mathcal{C}_g^X$. Those for which P is the partition defined by x make up a Zariski-open subvariety $\mathcal{M}_g(P) \subset \mathcal{C}_g(P)$. Notice that $\mathcal{C}_g(P)$ resp. $\mathcal{M}_g(P)$ can be identified with $\mathcal{C}_g^{X/P}$ resp. $\mathcal{M}_g^{X/P}$ (where X/P stands for quotient of X by the equivalence relation defined by P, or rather the set of parts of P), and that this is a submanifold in the orbifold sense.

For every nonempty $I \subset X$, let P_I be the partition of X whose parts are I and the singletons in $X - I$. So the corresponding orbifold $\mathcal{C}_g(P_I)$ has codimension $|I| - 1$ in $\mathcal{C}_g^X$.

(2.3) Theorem. *There is an algebra homomorphism*

$$\phi_g^X : H^\bullet(\mathcal{M}_g; \mathbb{Q}) \otimes A^\bullet_X \to H^\bullet(\mathcal{C}_g^X; \mathbb{Q})$$

that extends the natural homomorphism $H^\bullet(\mathcal{M}_g; \mathbb{Q}) \to H^\bullet(\mathcal{C}_g^X; \mathbb{Q})$*, sends* $1 \otimes u_i$ *to* $c_1(\theta_i)$ *and sends* $1 \otimes a_I$ *to the Poincaré dual of the class of* $\mathcal{C}_g(P_I)$*. This is an* $\mathfrak{S}_X$*-equivariant algebra homomorphism and is also a morphism of mixed Hodge structures. Moreover,* ϕ_g^X *is an isomorphism in degree* $\leq N(g)$*.*

Proof. For the first statement we must show that if $I, J \subset X$ have at least two elements, $i \in I \cap J$, $j \in I$ and P is a partition of X, then

$$i_{P!}(1) = \prod_{I'\in P;|I'|\geq 2} i_{P_{I'}!}(1),$$
$$c_1(\theta_i) i_{P_I!}(1) = c_1(\theta_j) i_{P_I!}(1),$$
$$i_{P_I!}(1) i_{P_J!}(1) = c_1(\theta_i)^{|I\cap J|-1} i_{P_{I\cup J}!}(1).$$

The first identity is geometrically clear and the second follows from the fact that θ_i and θ_j have isomorphic restrictions to $\mathcal{C}_g(P_I)$. To derive the last identity, we use the following lemma (the proof of which is left to the reader):

(2.4) Lemma. *Let U and V be closed complex submanifolds of a complex manifold M whose intersection $W := U \cap V$ is also a complex manifold. Suppose that any tangent vector of M which is tangent to both U and V is tangent to W. Then $i_{U!}(1)i_{V!}(1) = i_{W!}(e)$, where e is the euler class of the cokernel of the natural monomorphism $\nu_W \to \nu_U|W \oplus \nu_V|W$.*

Completion of the proof of (2.3). We apply the orbifold version of this lemma to $M = \mathcal{C}_g^X$, $U = \mathcal{C}_g(P_I)$, $V = \mathcal{C}_g(P_J)$ so that $W = \mathcal{C}_g(P_{I\cup J})$. The desired assertion then follows if we use the fact that the bundles appearing in the monomorphism $\nu_W \to \nu_U|W \oplus \nu_V|W$ are all direct sums of copies of $\theta_i|W$.

The second statement of the theorem is clear. To prove the last, let U_k resp. S_k denote the union of the strata $\mathcal{M}_g(P)$ of codim $\leq k$ resp. $= k$. We prove with induction on k that the homomorphism

$$\bigoplus_{\text{codim } P\leq k} H^\bullet(\mathcal{M}_g;\mathbb{Q}) \otimes \mathbb{C}[u_I : I \in P]a_P \to H^\bullet(U_k;\mathbb{Q})$$

is an isomorphism in degree $\leq N(g)$. For $k = 0$ this is (2.2). If $k \geq 1$, then consider the Gysin sequence of the pair (U_k, S_k):

$$\cdots \to H^{n-2k}(S_k;\mathbb{Q}) \to H^n(U_k;\mathbb{Q}) \to H^n(U_{k-1};\mathbb{Q}) \to \cdots .$$

In degrees $\leq N(g)$ the isomorphism of $\bigoplus_{\text{codim } P\leq k} H^\bullet(\mathcal{M}_g;\mathbb{Q}) \otimes \mathbb{C}[u_I : I \in P]a_P$ onto $H^n(U_{k-1};\mathbb{Q})$ factorizes over $H^n(U_k;\mathbb{Q})$. So the Gysin sequence splits in this range. Since $\bigoplus_{\text{codim } P=k} H^\bullet(\mathcal{M}_g;\mathbb{Q}) \otimes \mathbb{C}[u_I : I \in P]$ maps isomorphically onto $H^\bullet(S_k;\mathbb{Q})$ in degree $\leq N(g)$, the theorem follows.

Remark. For a curve C, the image of $A_X^\bullet$ in $H^\bullet(C^X;\mathbb{Q})$ is contained in the Hodge ring of C^X. If C is general, then it is in fact equal to it.

(2.5) A virtual classifying space of Γ_g^{r+s} is the moduli space of $r+s$-pointed curves $\mathcal{M}_g^{r+s}$. It carries $r+s$ relative tangent bundles $\theta_1,\dots,\theta_{r+s}$ and the total space of the $(\mathbb{C}^*)^r$-bundle $\mathcal{M}_{g,r}^s \to \mathcal{M}_g^{r+s}$ defined by $\theta_{1+s},\dots,\theta_{r+s}$ is a virtual classifying space for $\Gamma_{g,r}^s$. The comparison maps that enter in the statement of the stability theorem admit simple descriptions in these algebro-geometric terms and so preserve Hodge structures. This is probably well-known, but since we do not know a reference, we explain this. Clearly, the forgetful homomorphism $\Gamma_{g,r+1}^s \to \Gamma_{g,r}^s$ corresponds to the obvious projection $\mathcal{M}_{g,r+1}^s \to \mathcal{M}_{g,r}^s$. To exhibit the outer homomorphism $\Gamma_{g,r+1}^s \to \Gamma_{g+1,r}^s$, we first note that $\mathcal{M}_g^{r+s+1} \times \mathcal{M}_1^1$ parametrizes a codimension one stratum of the Knudsen–Deligne–Mumford compactification of $\mathcal{M}_{g+1}^{r+s}$: a pair $((C;x_1,\dots,x_{r+s+1}),(E;O))$ determines a stable $r+s$-pointed genus $(g+1)$-curve by identifying x_{r+s+1} with O. The normal bundle of this stratum is just the exterior tensor product of the relative tangent bundles of the factors. Denote the complement of its zero section by U and let U_e be a general fibre of the projection of $U \to \mathcal{M}_1^1$. We can identify U_e with $\mathcal{M}_{g,1}^{r+s}$ and there is a natural restriction homomorphism

$$H^\bullet(\mathcal{M}_{g+1}^{r+s};\mathbb{Q}) \to H^\bullet(U;\mathbb{Q}) \to H^\bullet(U_e;\mathbb{Q}) \cong H^\bullet(\mathcal{M}_{g,1}^{r+s};\mathbb{Q}).$$

The $(\mathbb{C}^*)^r$-bundles that lie over these spaces determine likewise a homomorphism $H^\bullet(\mathcal{M}^s_{g+1,r};\mathbb{Q})\to H^\bullet(\mathcal{M}^s_{g,1+r};\mathbb{Q})$. This is the one we were after.

Thus $H^\bullet(\Gamma_\infty;\mathbb{Q})$ acquires a canonical mixed Hodge structure. The composite map

$$H^\bullet(\Gamma_\infty;\mathbb{Q})\otimes A_X\to H^\bullet(\Gamma_g;\mathbb{Q})\otimes A_X\cong H^\bullet(\mathcal{M}_g;\mathbb{Q})\otimes A_X\to H^\bullet(\mathcal{C}^X_g;\mathbb{Q})$$

is evidently a morphism of mixed Hodge structures.

For every $i\in X$ we have a projection $f_i:\mathcal{C}^X_g\to\mathcal{C}^{X-\{i\}}_g$ and an obvious inclusion $A^\bullet_{X-\{i\}}\hookrightarrow A^\bullet_X$ (with image the linear combinations of monomials in which the u_I with $i\in I$ occur with exponent 0). The two are related:

(2.6) Lemma. *The map $f_i^*:H^\bullet(\mathcal{C}^{X-\{i\}}_g;\mathbb{Q})\to H^\bullet(\mathcal{C}^X_g;\mathbb{Q})$ is covered by the inclusion $A^\bullet_{X-\{i\}}\hookrightarrow A^\bullet_X$ tensorized with the identity of $H^\bullet(\Gamma_\infty;\mathbb{Q})$.*

The proof is straightforward.

Denote by $\overline{H}^\bullet(\mathcal{C}^X_g;\mathbb{Q})$ the quotient of $H^\bullet(\mathcal{C}^X_g;\mathbb{Q})$ by the subspace spanned by the images of $f_i^*, u_if_i^*$, $i\in X$.

(2.7) Corollary. *Put*

$$A'^\bullet_X:=\bigoplus_{P|X}(\prod_{\{i\}\in P}u_i^2)\mathbb{Q}[u_I:I\in P]a_P.$$

Then there is a natural homomorphism of mixed Hodge structures

$$H^\bullet(\Gamma_\infty;\mathbb{Q})\otimes A'^\bullet_X\to\overline{H}^\bullet(\mathcal{C}^X_g;\mathbb{Q})$$

which is an isomorphism in degree $\leq N(g)$.

3. The Schur–Weyl functor

(3.1) We continue to denote by X a finite nonempty set. The product S_g^X comes with an obvious $\mathfrak{S}_X$-action and so there is a resulting action of $\mathfrak{S}_X$ on $H^\bullet(S_g^X;\mathbb{Q})$. Any diffeomorphism f of S_g induces a diffeomorphism of S_g^X commuting with this action. Thus is obtained an action of the product $\Gamma_g\times\mathfrak{S}_X$ on $H^\bullet(S_g^X;\mathbb{Q})$. As before, $V_g:=H^1(S_g;\mathbb{Q})$ and $\mathrm{Sp}(V_g)$ denotes its symplectic group. It is easily seen that the $\Gamma_g\times\mathfrak{S}_X$-action factorizes through one of $\mathrm{Sp}(V_g)\times\mathfrak{S}_X$.

Let us write the total cohomology $H^\bullet(S_g;\mathbb{Q})$ as $\mathbb{Q}\oplus V_gt\oplus\mathbb{Q}u$ where u is the canonical class (we assume $g\geq 2$ here) and t shifts the degree by 1. The Künneth rule gives an isomorphism

$$H^\bullet(S_g^X;\mathbb{Q})=\bigoplus_{I,J\subset X;I\cap J=\emptyset}V_g^{\otimes I}t^Iu^J,$$

where t^I should be thought of as a generator of the signum representation of $\mathfrak{S}_I$ placed in degree $|I|$ and $u^J=\prod_{j\in J}u_j$. The multiplication in the right-hand side obeys the Koszul sign rule and is given by contractions stemming from the symplectic form on V_g. If we define $\overline{H}^\bullet(S_g^X;\mathbb{Q})$ as in the relative case, that is, as the quotient of $H^\bullet(S_g^X;\mathbb{Q})$ by the span of the images of $H^\bullet(S_g^{X-\{i\}};\mathbb{Q}), u_iH^\bullet(S_g^{X-\{i\}};\mathbb{Q})$, $i\in X$, then we see that in terms of the Künneth decomposition this reduces to the single summand $V_g^{\otimes X}t^X$.

(3.2) Since $\pi:\mathcal{C}_g^X\to\mathcal{M}_g$ is a projective morphism whose total space is an orbifold, it follows from a theorem of Deligne [1] that its Leray spectral sequence degenerates at the E_2-term over $\mathbb{Q}$. In other words, $H^\bullet(\mathcal{C}_g^X;\mathbb{Q})$ has a canonical decreasing filtration $L^\bullet$, the *Leray filtration*, such that there is a natural isomorphism

$$\mathrm{Gr}_L^k\, H^n(\mathcal{C}_g^X;\mathbb{Q})\cong H^k(\mathcal{M}_g;R^{n-k}\pi_*\mathbb{Q}).$$

The maps f_i^* in (2.6) are strict with respect to the Leray filtrations. So if we combine this with (2.7) we find:

(3.3) Corollary. *There is a natural graded $\mathfrak{S}_s$-equivariant map of $H^\bullet(\Gamma_\infty;\mathbb{Q})$-modules*

$$H^\bullet(\Gamma_\infty;\mathbb{Q})\otimes A'^\bullet_X\to H^\bullet(\Gamma_g;V_g^{\otimes X})t^X$$

which is an isomorphism in degree $\leq N(g)$.

We want to decompose $V_g^{\otimes X}$ as a $Sp(V_g)\times\mathfrak{S}_X$-representation. Following Weyl this is done in two steps. The first step involves Weyl's representation $V_g^{\langle X\rangle}$ whose definition we presently recall. Let $\omega\in V_g\otimes V_g$ correspond to the symplectic form on V_g. For every ordered pair $(i,j)\in X$ with $i\neq j$, we have a natural homomorphism $V_g^{\otimes(X-\{i,j\})}\to V_g^{\otimes X}$ defined by placing ω in the (i,j)-slot. Reversing the order gives minus this map. So the natural assertion is that we have a map

$$\bigoplus_{I\subset X;|I|=2} V_g^{\otimes(X-I)}t^I\to V_g^{\otimes X}.$$

It is easy to see that this map is injective; its cokernel is by definition $V_g^{\langle X\rangle}$. Notice that $V_g^{\langle X\rangle}$ is in a natural way a representation of $\mathrm{Sp}(V_g)\times\mathfrak{S}_X$. The second step is the decomposition of $V_g^{\langle X\rangle}$: Weyl proved the remarkable fact that

$$V_g^{\langle X\rangle}\cong\bigoplus_\lambda S_{\langle\lambda\rangle}(V_g)\boxtimes(\lambda),$$

where λ runs over the numerical partitions of $|X|$ in at most g parts and (λ) denotes the corresponding equivalence class of irreducible representations of $\mathfrak{S}_X$. In particular, all these irreducible representations appear with multiplicity one. A modern account of the proof and of related results can be found in the book by Fulton and Harris [2].

(3.4) Theorem. *If $A''^\bullet_X\subset A'^\bullet_X$ is the $\mathbb{Q}[u_i:i\in X]$-submodule defined by*

$$A''_X=\bigoplus_{P|X}(\prod_{\{i\}\in I}u_i^2)(\prod_{I\in P;|I|=2}u_I)\mathbb{Q}[u_I:I\in P]a_P,$$

then there is a natural graded $\mathfrak{S}_X$-equivariant map of $H^\bullet(\Gamma_g, \mathbb{Q})[u_i : i \in X]$-modules

$$H^\bullet(\Gamma_\infty; \mathbb{Q}) \otimes A''^\bullet_X \to H^\bullet(\Gamma_g; V_g^{\langle X\rangle})t^X,$$

and this map is an isomorphism in degree $\leq N(g)$.

Proof. If $I \subset X$ is a two-element subset, then the Poincaré dual of the hyperdiagonal $\mathcal{C}_g(P_I) \subset \mathcal{C}_g^X$ is u_I. The restriction of this element to the fiber S_g^X is $(\sum_{i\in I} u_i) + \omega t^I$. So the map

$$H^{\bullet-2}(\Gamma_g; V_g^{\otimes(X-I)})t^{X-I} \to H^\bullet(\Gamma_g; V_g^{\otimes X})t^X$$

defined by multiplication with ωt^I is covered by the map $H^\bullet(\Gamma_\infty; \mathbb{Q}) \otimes A'^\bullet_{X-I} \to H^\bullet(\Gamma_\infty; \mathbb{Q}) \otimes A'^\bullet_X$ which is multiplication by $u_I - \sum_{i\in I} u_i$. The theorem follows.

Proof of (1.1). We decompose both members of the stable isomorphism (3.4) according to the action of $\mathfrak{S}_X$. Let λ be a numerical partition of $|X|$ and let λ' be the conjugate partition. According to Weyl's decomposition theorem, the (λ)-isotypical component of $H^\bullet(\Gamma_g, V_g^{\langle X\rangle})$ is $H^\bullet(\Gamma_g, S_{\langle\lambda\rangle}(V_g))$. Note that this is also the isotypical component of type λ' of $H^\bullet(\Gamma_g; V_g^{\langle X\rangle})t^X$. On the other hand, it is easily seen that the (λ')-isotypical component of $A''^\bullet_X$ can be identified with $t^s B^\bullet(\lambda)$.

Since the Leray spectral sequence (3.2) is a spectral sequence of mixed Hodge structures, the homomorphism of (1.1) is actually a morphism of mixed Hodge structures. The theorem follows.

4. Stable cohomology of the universal Abel–Jacobi map

If we give S_g a complex structure, then S_g becomes a compact Riemann surface C of genus g, so that we have defined an Abel–Jacobi map $\mathrm{Sym}^s(C) \to \mathrm{Pic}^s(C)$. The induced map on cohomology has been determined by Macdonald:

(4.1) Proposition. (Macdonald [7]) *Identify $H^\bullet(\mathrm{Pic}^s(C); \mathbb{Q})$ with the exterior algebra on V_g so that the Abel–Jacobi map determines an algebra homomorphism $\wedge^\bullet V_g \to H^\bullet(\mathrm{Sym}^s(C); \mathbb{Q})$. Let $\wedge^\bullet V_g[y] \to H^\bullet(\mathrm{Sym}^s(C); \mathbb{Q})$ be the extension that sends the indeterminate y of degree two to the sum of the fundamental classes of the factors. Then this map is surjective and its its kernel is the degree $> s$-part of the ideal $\mathcal{I}$ in $\wedge^\bullet V_g[y]$ generated by $\{v \wedge v' - (v.v')y : v, v' \in V_g\}$.*

If $s > 2g-2$, then the Abel–Jacobi map is the projectivization of a vector bundle of rank $s+1-g$ over $\mathrm{Pic}(C)$ and we can interpret the image of y as the first Chern class of the associated line bundle over $\mathrm{Sym}^s(C)$. Macdonald also expresses the Poincaré duals of the diagonals of C^s in terms of this presentation.

It is our aim to make a corresponding discussion for the universal situation in the stable range.

The morphism $\mathcal{C}_g \to \mathcal{M}_g$ defines a relative Picard bundle $\mathrm{Pic}(\mathcal{C}_g/\mathcal{M}_g) \to \mathcal{M}_g$ (in the orbifold sense) whose connected components are still indexed by the degree: $\mathrm{Pic}^k(\mathcal{C}_g/\mathcal{M}_g)$, $k \in \mathbb{Z}$. The degree 0-component is called the *universal Jacobian* and is also denoted $\mathcal{J}_g$.

(4.2) Lemma. *For every $k \in \mathbb{Z}$, there is a natural isomorphism*

$$H^\bullet(\mathrm{Pic}^k(\mathcal{C}_g/\mathcal{M}_g);\mathbb{Q}) \cong H^\bullet(\mathcal{J}_g;\mathbb{Q}).$$

Proof. The relative canonical sheaf defines a section of $\mathrm{Pic}^{2g-2}(\mathcal{C}_g/\mathcal{M}_g)$. On a suitable Galois cover $\tilde{\mathcal{M}}_g \to \mathcal{M}_g$ (with Galois group G, say) this section becomes divisible by $2g-2$ and thus produces a section of $\mathrm{Pic}^1(\tilde{\mathcal{C}}_g/\tilde{\mathcal{M}}_g)$. This determines an isomorphism $\mathrm{Pic}^k(\tilde{\mathcal{C}}_g/\tilde{\mathcal{M}}_g) \cong \mathrm{Pic}^0(\tilde{\mathcal{C}}_g/\tilde{\mathcal{M}}_g)$. Although this isomorphism will not in general be G-equivariant, it will differ from any G-translate by a section of $\mathrm{Pic}^0(\tilde{\mathcal{C}}_g/\tilde{M}_g)$ of finite order, and so the induced map on rational cohomology is G-equivariant. By passing to the G-invariants we obtain an isomorphism $H^\bullet(\mathrm{Pic}^k(\mathcal{C}_g/\mathcal{M}_g);\mathbb{Q}) \cong H^\bullet(\mathcal{J}_g;\mathbb{Q})$. One checks that this map does not depend on choices.

Let X be a finite nonempty set as before. We wish to determine the subalgebra of $\mathfrak{S}_X$-invariants of $A_X^\bullet$, at least stably. Recall that an additive basis of $A_X^\bullet$ consists of the set of elements of the form $\prod_{I\in P} u_I^{r_I}$, where P runs over the partitions of X and $r_I \geq |I|-1$. Let us define a partial ordering on the collection of partitions of X by: $P \leq Q$ if $P=Q$ or if for the smallest number k such that the k-element members of P and Q do not coincide every k-element member of P is a k-element member of Q. This determines a partial ordering on the set of monomials: $\prod_{I\in P} u_I^{r_I} \leq \prod_{J\in Q} u_J^{s_J}$ if $P<Q$ or if $P=Q$ and $r_I \leq s_I$ for all I. The defining relations for $A_X^\bullet$ show that a product of two monomials associated to partitions P and Q is a monomial associated to a partition that dominates both P and Q.

Denote by $S := \sum_{\sigma\in\mathfrak{S}_X}\sigma$ the symmetrizer operator (acting on A_X). Given a partition P of X, then the smallest terms in $\prod_{I\in P} S(u_I^{r_I})$ are monomials associated to a partition that is a $\mathfrak{S}_X$-translate of P. In this expression the part associated to P is the subsum corresponding to the partial symmetrizer $S_P := \sum_{\sigma\in\mathfrak{S}_X;\sigma(P)=P}\sigma$. If l_k is the number of members of P with k elements, then the image of S_P can be identified with

$$\mathbb{Q}[c_1,c_2,\dots,c_{l_1}] \otimes \bigotimes_{k\geq 2; l_k>0} (c_{l_k}^{k-1}\mathbb{Q}[c_1,c_2,\dots,c_{l_k}]).$$

Here c_l in the tensor factor with index k is to be thought of as the lth elementary symmetric function in the u_I's with $I\in P$ and $|I|=k$ (and so has degree $2l$); the appearance of $c_{l_k}^{k-1}$ comes from the condition $r_I \geq |I|-1$. So if we put

$$C_\infty^\bullet := \mathbb{Q}[c_1,c_2,\cdots] \otimes \bigotimes_{k\geq 2}(\mathbb{Q}\oplus\bigoplus_{l\geq 1} c_l^{k-1}\mathbb{Q}[c_1,c_2,c_3,\dots,c_l]),$$

then we find:

(4.3) Corollary. *There is natural surjective homomorphism of graded algebra's $C_\infty^\bullet \to (A_X^\bullet)^{\mathfrak{S}_X}$. Its restriction to the $\mathbb{Q}$-span of all the monomials involving the variables $c_{l_1}^{(k_1)},\dots,c_{l_r}^{(k_r)}$ (where $c_l^{(k)}$ denotes the variable c_l that occurs in the kth tensor power) with $\sum_i k_i l_i \leq |X|$ is a linear isomorphism.*

If Y is another finite set with $|Y| \geq |X|$, then any injection $f: X \hookrightarrow Y$ induces an algebra homomorphism $A_X^\bullet \to A_Y^\bullet$. Since all such injections are in the same

$\mathfrak{S}_Y$-orbit they give rise to the same algebra homomorphism $(A_X^\bullet)^{\mathfrak{S}_X} \to (A_Y^\bullet)^{\mathfrak{S}_Y}$. In particular, $(\tilde{A}_X^\bullet)^{\mathfrak{S}_X}$ only depends on $|X|$. So if we write $\tilde{C}_{|X|}^\bullet$ for this algebra, then we have a direct system $\cdots \to C_s^\bullet \to C_{s+1}^\bullet \to \cdots$. The corollary shows that the limit of this direct system can be identified with $C_\infty^\bullet$. It follows from (2.3) that we have an algebra homomorphism

$$H^\bullet(\Gamma_\infty;\mathbb{Q}) \otimes C_s^\bullet \to H^\bullet((\mathcal{C}_g^s);\mathbb{Q})^{\mathfrak{S}_s} \cong H^\bullet((\mathcal{C}_g^s)^{\mathfrak{S}_s};\mathbb{Q})$$

that is an isomorphism in degree $\leq N(g)$. In the limit this yields a homomorphism

$$H^\bullet(\Gamma_\infty;\mathbb{Q}) \otimes C_\infty^\bullet \to H^\bullet((\mathcal{C}_g^s)^{\mathfrak{S}_s};\mathbb{Q})$$

that is an isomorphism in degree $\leq \min(2s, N(g))$. The image of $c_1 \otimes 1 \otimes 1 \otimes \cdots$ is easily seen to be proportional to the element y appearing in Macdonald's theorem. We put

$$C'^\bullet_\infty := \mathbb{Q}[c_2, c_3, \cdots] \otimes \bigotimes_{k\geq 2}\big(\mathbb{Q} + \oplus_{l\geq 1} c_l^{k-1}\mathbb{Q}[c_1, c_2, c_3, \dots, c_l]\big).$$

(4.4) Theorem. *The algebra homomorphism above fits in a commutative square of algebra homomorphisms*

$$\begin{array}{ccc} H^\bullet(\Gamma_\infty;\mathbb{Q}) \otimes C_\infty^\bullet & \longrightarrow & H^\bullet((\mathcal{C}_g^s)^{\mathfrak{S}_s};\mathbb{Q}) \\ \cup & & \uparrow \\ H^\bullet(\Gamma_\infty;\mathbb{Q}) \otimes C'^\bullet_\infty & \longrightarrow & H^\bullet(\mathrm{Pic}^s(\mathcal{C}_g/\mathcal{M}_g);\mathbb{Q}), \end{array}$$

in which the right vertical map is induced by the Abel–Jacobi map. The lower horizontal map is an isomorphism in degree $\leq \min(s, N(g))$ so that in the limit we have an isomorphism

$$H^\bullet(\Gamma_\infty;\mathbb{Q}) \otimes C'^\bullet_\infty \cong \bigoplus_{s=0}^{\infty} H^\bullet(\Gamma_\infty; \wedge^s)t^s.$$

Proof. If we combine Macdonald's theorem with the Leray spectral of sequence of the map $(\mathcal{C}_g^s)^{\mathfrak{S}_s} \to \mathcal{M}_g$, then we see that the map

$$H^\bullet(\mathrm{Pic}^s(\mathcal{C}_g/\mathcal{M}_g);\mathbb{Q})[y] \to H^\bullet((\mathcal{C}_g^s)^{\mathfrak{S}_s};\mathbb{Q})$$

that sends y to $c_1 \otimes 1 \otimes 1 \otimes \cdots$ is an isomorphism in degree $\leq s$. The theorem follows from this.

References

1. P. Deligne, *Théorème de Lefschetz et critères de dégénérescence de suites spectrales*, Inst. Hautes Études Sci. Publ. Math. **35** (1968), 259–278.
2. W. Fulton and J. Harris, *Representation theory*, Graduate Texts in Math., vol. 129, Springer Verlag, New York, 1991.
3. J. Harer, *Stability of the homology of the mapping class groups of orientable surfaces*, Ann. of Math. **121** (1985), 215–249.

4. J. Harer, *Improved stability for the homology of the mapping class groups of surfaces*, preprint Duke University (1993).
5. N.V. Ivanov, *Complexes of curves and the Teichmüller modular group*, Uspekhi Mat. Nauk **42** (1987), 110-126 (Russian); English transl. in Russian Math. Surveys **42** (1987), 55–107.
6. N.V. Ivanov, *On the homology stability for Teichmüller modular groups: closed surfaces and twisted coefficients*, Mapping class groups and moduli spaces of Riemann surfaces (C.F. Bödigheimer and R.M. Hain, eds.), Contemp. Math., vol. 150, AMS, 1993, pp. 149–194.
8. I.G. Macdonald, *Symmetric products of an algebraic curve*, Topology **1** (1962), 319–343.

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