On Mirror Symmetry Conjecture for Schoen's Calabi-Yau 3 folds

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Abstracts

In this paper, we verify a part of the Mirror Symmetry Conjecture for Schoen's Calabi-Yau 3-fold, which is a special complete intersection in a toric variety. We calculate a part of the prepotential of the A-model Yukawa couplings of the Calabi-Yau 3-fold directly by means of a theta function and Dedekind's eta function. This gives infinitely many Gromov-Witten invariants, and equivalently infinitely many sets of rational curves in the Calabi-Yau 3-fold. Using the toric mirror construction [Ba-Bo, HKTY, Sti], we also calculate the prepotential of the B-model Yukawa couplings of the mirror partner. Comparing the expansion of the B-model prepotential with that of the A-model prepotential, we check a part of the Mirror Symmetry Conjecture up to a high order.

1 Introduction

Let W be a generic complete intersection variety in $\mathbf{P}^1 \times \mathbf{P}^2 \times \mathbf{P}^2$ which is defined by two equations of multi-degrees $(1,3,0)$ and $(1,0,3)$ respectively. A generic W is a non-singular Calabi-Yau 3-fold, which we call *Schoen's Calabi-Yau 3-fold* [Sch]. The purpose of this paper is to verify a part of the Mirror Symmetry Conjecture for Schoen's Calabi-Yau 3-folds.

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In [COGP], Candelas et al. calculated the prepotential of the B-model Yukawa couplings of the mirror of generic quintic hypersurfaces X in $\mathbf{P}^4$ and under the mirror hypothesis they gave predictions of numbers of rational curves of degree d in X. Their predictions have been verified mathematically only for $d \leq 3$, that is, for numbers of lines, conics and cubic curves (cf. e.g. [E-S]). On the B-model side one can compute as many coefficients as one wants and thus conjecturally count curves of any degree. However it is very hard to calculate the Gromov–Witten invariants directly on the A-model side. In this paper we calculate, (directly on the A-model side), a part of the prepotential of Schoen's Calabi-Yau 3-fold W which gives infinitely many Gromov–Witten invariants of W.
The main strategy of our verification is as follows:

- We will calculate a part of the prepotential of the A-model Yukawa couplings (for genus zero) of Schoen's Calabi-Yau 3-fold by using a structure of fibration $h : W \longrightarrow \mathbf{P}^1$ by abelian surfaces. The theory of Mordell-Weil lattices [Man1, Sh1, Sa] allows us to calculate that part of the prepotential coming from sections of h. Under very plausible assumptions, we can count the "numbers of *pseudo-sections*", which makes it possible for us to obtain a very explicit description of the 1-sectional part of the A-model prepotential (cf. Theorem 4.2) in 19 variables by using a lattice theta function for E_8 and Dedekind's eta function.
- According to Batyrev-Borisov [Ba-Bo] we can construct a mirror partner W^* of W. The prepotential of the B-model Yukawa couplings of W^* can be defined by means of period integrals of W^*. Following the recipe in [HKTY, Sti] we expand the B-model prepotential in 3 variables by using the toric data. These 3-variables correspond to 3 elements of the Picard group of W coming from line bundles on the ambient space $\mathbf{P}^1 \times \mathbf{P}^2 \times \mathbf{P}^2$.
- By identifying the 3 variables with the corresponding 3 variables on the A-model side we have two expansions which should be compared. By a simple computer calculation we can verify the conjecture up to a high order.

To state the results for the A-model side let $f_i : S_i \longrightarrow \mathbf{P}^1$ $(i = 1, 2)$ be two generic rational elliptic surfaces. Then Schoen's generic Calabi-Yau 3-fold can be obtained as the fiber product $h : W = S_1 \times_{\mathbf{P}^1} S_2 \longrightarrow \mathbf{P}^1$. A general fiber of h is a product of two elliptic curves. Hence after fixing the zero section the set of sections of h becomes an abelian group. In this case the group of sections $MW(W)$ is a finitely generated abelian group and admits a Néron-Tate height pairing. Let B be the symmetric bilinear form associated to the Néron-Tate height pairing. According to Shioda, we call the pair $(MW(W), B)$ of the group and the symmetric bilinear form a *Mordell-Weil lattice*. Under the genericity condition for W and a suitable choice of a Néron-Tate height we can easily see that the Mordell-Weil lattice is isometric to $E_8 \times E_8$. (Cf. [Sa]). There is a very suitable set of 19 generators $[F], [L_i], [M_j]$ $(0 \leq i, j \leq 8)$ for the Picard group of W. We introduce the parameters p, q_i, s_j corresponding to these generators. The divisor class $[F]$, which is the class of the fiber, has a special meaning in our context. A homology class η is called k-sectional if the intersection number $([F], [\eta]) = k$. Let Ψ_A denote the prepotential of the A-model Yukawa couplings of W and $\Psi_{A,k}$ its k-sectional part for $k \geq 0$. Then we have an expansion like

$$\Psi_A = \text{topological part} + \sum_{k=0}^{\infty} \Psi_{A,k}.$$

Our main theorem can be stated as follows. For detailed notation, see Section 4.

Theorem 1.1 (*cf. Theorem 4.1, 4.2*). *Assume that Conjecture 5.1 in Section 5 holds. Then for a generic Schoen's Calabi-Yau 3-fold W the 1-sectional prepotential is given by*

$$\begin{aligned} & \Psi_{A,1}(p, q_0, \cdots q_8, s_0, \cdots, s_8) \\ = & \sum_{n=1}^{\infty} \frac{1}{n^3} \cdot (p \cdot \prod_{i=1}^{8}(q_i \cdot s_i))^n \cdot A(n\tau_1, nz_1, \ldots, nz_8) \cdot A(n\tau_2, ny_1, \ldots, ny_8). \end{aligned}$$

where

$$A(\tau, x_1, \ldots, x_8) = \Theta_{E_8}^{root}(\tau, x_1, \ldots, x_8) \cdot [\sum_{m=0}^{\infty} p(m) \cdot \exp(2\pi m i \tau)]^{12}.$$

Here $p(m)$ denotes the number of partitions of m.

(For the definition of the various notations see Theorem 4.1).
Since we can prove $\Psi_{A,0} \equiv 0$ (cf. [Sa]) and $\Psi_{A,k}, k \geq 2$ consists of terms divisible by p^2, we obtain an asymptotic expansion of Ψ_A with respect to p:

$$\Psi_A = \text{topological term} + p \cdot \prod_{i=1}^{8}(q_i s_i) \cdot A(\tau_1, \mathbf{z}) \cdot A(\tau_2, \mathbf{y}) + O(p^2)$$

where $\mathbf{z} = (z_1, \cdots, z_8), \mathbf{y} = (y_1, \cdots, y_8)$.
In Section 9 we show that $\Theta_{E_8}^{root}$ has a very simple expression in terms of the standard Jacobi theta functions.
On the other hand on the B-model side there are only 3 parameters involved in the calculation, because the Batyrev-Borisov construction can only deal with generators of the Picard group of W coming from the ambient toric variety $\mathbf{P}^1 \times \mathbf{P}^2 \times \mathbf{P}^2$. One can easily find the corresponding parameters $p = U_0, U_1 = \exp(2\pi i t_1), U_2 = \exp(2\pi i t_2)$ on the A-model side and one can obtain the following expansion:

$$\Psi_A^{res}(p, t_1, t_2) = \text{topological term} + p \cdot A(3t_1, t_1\gamma) \cdot A(3t_2, t_2\gamma) + O(p^2),$$

where $\gamma = (1, 1, 1, 1, 1, 1, 1, -1)$. We also obtain an expansion of the B-model prepotential

$$\Psi_B(p, t_1, t_2) = \text{topological term} + p \cdot B(t_1) \cdot B(t_2) + O(p^2).$$

Therefore in this context the mirror symmetry conjecture can be stated as

$$\Psi_A^{res}(p, t_1, t_2) \equiv \Psi_B(p, t_1, t_2).$$

From the above asymptotic expansion, we come to a concrete mathematical conjecture:

$$A(3t, t\gamma) \equiv B(t).$$

At this moment we can calculate both sides up to high order of $U = \exp(2\pi i t)$ by using computer programs. One can find the expansion of $A(t)$ up to the order of 50 at the end of Section 6 and also the expansion of $B(t)$ at the end of Section 7.
The rough plan of this paper is as follows. In Section 2 we recall a basic property of Schoen's Calabi-Yau 3-fold and its toric description. In Section 3 we recall the Mordell-Weil lattice which will be essential in the later sections. In Section 4 we first recall the definition of Gromov-Witten invariants and the A-model prepotential. We calculate the Mordell-Weil part of the prepotential in terms of the lattice theta function $\Theta_{E_8}^{root}$ and state the main theorem (Theorem 4.2). Section 5

is devoted to counting the pseudo-sections in W. We also prove Theorem 4.2 here. In Section 6 we restrict the parameters of the A-model prepotential in order to compare the expansion with that of the B-model prepotential of the mirror. A table for the coefficients $\{a_n\}$ of $A(3t, t\gamma)$ is given up to order 50(cf. Table 2 in Section 6). In Section 7 after recalling the formulation of the mirror symmetry conjecture we calculate the B-model prepotential of the mirror of Schoen's example following the recipe of [HKTY, Sti]. We expand the function $B(t)$ whose coefficients $\{b_n\}$ should coincide with $\{a_n\}$ if the mirror symmetry conjecture is true. We check the coincidence up to order 50.
In Appendix I (Section 8) we derive the equation of the mirror according to the Batyrev-Borisov construction [Ba-Bo]. In Appendix II (Section 9) we give a formula for the theta function of the lattice E_8.

Let us mention some papers which are related to our work. In the paper [D-G-W], Donagi, Grassi and Witten calculate the non-perturbative superpotential in F-theory compactification to four dimensions on $\mathbf{P}^1 \times S$, where S is a rational elliptic surface. It is interesting enough to notice that the supperpotential in their case is also described by the lattice theta function for E_8. It is interesting that they also mention the contribution of Dedekind's eta function $\eta(\tau)$ to the superpotential, though we do not know any direct relation between F-theory and the Type II theory. In [G-P], Göttsche and Pandharipande calculated the quantum cohomology of blowing-ups of $\mathbf{P}^2$. Their calculation for the blowing-up of 9-points in *general position* on $\mathbf{P}^2$ is certainly related to our calculation for the rational elliptic surfaces. Moreover, in [Y-Z] S.-T. Yau and Zaslow describe the counting of BPS states of Type II on K3 surfaces. In the paper, they treated rational curves with nodes, which may have some relation to our treatment of pseudo-sections.

2 Schoen's Calabi-Yau 3-folds

Let $f_i : S_i \longrightarrow \mathbf{P}^1$ $(i = 1, 2)$ be two smooth rational surfaces defined over $\mathbf{C}$. In this paper we always assume that an elliptic surface has a section.
In [Sch] C. Schoen showed that the fiber product of two rational elliptic surfaces

$$\begin{array}{ccccc} & W = & S_1 \times_{\mathbf{P}^1} S_2 & & \\ & \swarrow p_1 & & p_2 \searrow & \\ S_1 & & \downarrow h & & S_2 \\ & f_1 \searrow & & \swarrow f_2 & \\ & & \mathbf{P}^1 & & \end{array}$$

becomes a Calabi-Yau 3-fold after small resolutions of possible singularities of the fiber product. In what follows we consider such Calabi-Yau 3-folds which satisfy the following genericity assumption.

Assumption 2.1 *1. The rational elliptic surfaces $f_i : S_i \longrightarrow \mathbf{P}^1$ $(i = 1, 2)$ are generic in the sense that the surfaces S_i are smooth and every singular fiber of f_i is of Kodaira type I_1, that is, a rational curve with one node. Then one can see that each fibration f_i has exactly 12 singular fibers of type I_1. (cf. [Kod]).*

2. Let $\Sigma_i \subset \mathbf{P}^1$ be the set of critical values of f_i. Then we assume that $\Sigma_1 \cap \Sigma_2 = \emptyset$.

Under Assumption 2.1 the fiber product $W = S_1 \times_{\mathbf{P}^1} S_2$ becomes a *nonsingular* Calabi-Yau 3-fold. The following facts are well-known. (See [Kod] or [Man1]).

Lemma 2.1 *Let S_1, S_2, W be as above.*

1. *All fibers of $h : W \longrightarrow \mathbf{P}^1$ have vanishing topological Euler numbers. Hence we have $e(W) = 2(h^{1,1}(W) - h^{2,1}(W)) = 0$.*

2. *A generic rational elliptic surface with section is obtained by blowing-up the 9 base points of a cubic pencil on $\mathbf{P}^2$. Let $\pi_1 : S_1 \longrightarrow \mathbf{P}^2$ and $\pi_2 : S_2 \longrightarrow \mathbf{P}^2$ be the blowing-ups and $E_i, i = 1, \cdots, 9$ and $E'_j, j = 1, \cdots, 9$ the divisor classes of the exceptional curves of π_1 and π_2 respectively. Set $H_i = \pi_i^*(\mathcal{O}_{\mathbf{P}^2}(1))$. Then we have*

$$\begin{aligned} \mathrm{Pic}(S_1) &= \mathbf{Z}H_1 \oplus \mathbf{Z}E_1 \oplus \cdots \oplus \mathbf{Z}E_9, & (1) \\ \mathrm{Pic}(S_2) &= \mathbf{Z}H_2 \oplus \mathbf{Z}E'_1 \oplus \cdots \oplus \mathbf{Z}E'_9. & (2) \end{aligned}$$

3. *Let F_1 and F_2 be the divisor classes of the fibers of f_1 and f_2 respectively. Then we have*

$$F_1 = 3H_1 - \sum_{i=1}^{9} E_i, \quad F_2 = 3H_2 - \sum_{i=1}^{9} E'_i \tag{3}$$

4. *We have the following isomorphism of groups.*

$$\mathrm{Pic}(W) \simeq p_1^*(\mathrm{Pic}(S_1)) \oplus p_2^*(\mathrm{Pic}(S_2))/\mathbf{Z}[p_1^*(F_1) - p_2^*(F_2)] \tag{4}$$

   Hence the Picard number of W is $h^{11}(W) = 19$. Also $h^{21}(W) = 19$ because $e(W) = 0$.

□

We now show that Schoen's Calabi-Yau 3-fold W can also be realized as a complete intersection in the toric variety $\mathbf{P}^1 \times \mathbf{P}^2 \times \mathbf{P}^2$. Let $z_0, z_1, x_0, x_1, x_2, y_0, y_1, y_2$ be the homogeneous coordinates of $\mathbf{P}^1 \times \mathbf{P}^2 \times \mathbf{P}^2$ and let

$$a_0(x_0, x_1, x_2), a_1(x_0, x_1, x_2), b_0(y_0, y_1, y_2), b_1(y_0, y_1, y_2)$$

be generic homogeneous cubic polynomials. Then we can assume that the generic rational elliptic surfaces S_1 and S_2 in Lemma 2.1 are obtained as hypersurfaces in $\mathbf{P}^1 \times \mathbf{P}^2$ as in the following way.

$$\begin{aligned} S_1 &= \{P_1 = z_1 \cdot a_0(x_0, x_1, x_2) - z_0 \cdot a_1(x_0, x_1, x_2) = 0\} \subset \mathbf{P}^1 \times \mathbf{P}^2 \\ S_2 &= \{P_2 = z_1 \cdot b_0(y_0, y_1, y_2) - z_0 \cdot b_1(y_0, y_1, y_2) = 0\} \subset \mathbf{P}^1 \times \mathbf{P}^2 \end{aligned}$$

We have the natural morphisms

$$\begin{array}{ccccc} & & S_i & & \subset \mathbf{P}^1 \times \mathbf{P}^2 \\ & f_i \swarrow & & \searrow \pi_i & \\ \mathbf{P}^1 & & & & \mathbf{P}^2 , \end{array}$$

where $f_1 = (p_1)_{|S_i}, \pi_i = (p_2)_{|S_i}$. Moreover, one can easily see that $W = S_1 \times_{\mathbf{P}^1} S_2$ can be obtained as a complete intersection in the toric variety $\mathbf{P}^1 \times \mathbf{P}^2 \times \mathbf{P}^2$ of types $(1, 3, 0)$, $(1, 0, 3)$:

$$W = \left\{ \begin{array}{l} [z_0 : z_1] \times [x_0 : x_1 : x_2] \times [y_0 : y_1 : y_2] \\ \in \mathbf{P}^1 \times \mathbf{P}^2 \times \mathbf{P}^2 \end{array} \middle| \begin{array}{l} P_1 = 0 \\ P_2 = 0 \end{array} \right\}$$

3 Mordell-Weil lattices

The purpose of this section is a review of results on Mordell-Weil lattices which is needed to calculate a part of the prepotential of the A-model Yukawa couplings of Schoen's Calabi-Yau 3-folds. For more complete treatments the reader may refer to [Man1], [Sh1], [Sa].
We keep the notation and assumptions of the previous section, that is, let $f_i : S_i \longrightarrow \mathbf{P}^1$ be rational elliptic surfaces and let $h : W = S_1 \times_{\mathbf{P}^1} S_2 \longrightarrow \mathbf{P}^1$ be the fiber product.
Let $MW(S_i), i = 1, 2$ and $MW(W)$ denote the set of sections of f_i and h respectively. Since the exceptional curves of the blowing-ups $\pi_i : S_i \longrightarrow \mathbf{P}^2$ are the images of sections of f_i, we denote by e_1 and e'_1 the sections of f_1 and f_2 respectively such that $e_1(\mathbf{P}^1) = E_1$ and $e'_1(\mathbf{P}^2) = E'_1$. We take e_1 and e'_1 as zero sections of f_1 and f_2 respectively. Then $MW(S_1)$ and $MW(S_2)$ become finitely generated abelian groups with the identity elements e_1 and e'_1 respectively. The group $MW(S_i)$ is called the Mordell-Weil group of the rational elliptic surface $f_i : S_i \longrightarrow \mathbf{P}^1$.
Take the line bundles

$$L_0 = E_1 + F_1 \in \mathrm{Pic}(S_1), \quad M_0 = E'_1 + F_2 \in \mathrm{Pic}(S_2).$$

Note that these line bundles are symmetric [3] and numerically effective and $(L_0)^2 = (M_0)^2 = 1$. Hence L_0 and M_0 are nearly ample line bundles and E_1 (resp. E'_1) is the only irreducible effective curve on S_1 (resp. S_2) with $(L_0, E_1)_{S_1} = 0$ (resp. $(M_0, E'_1)_{S_2} = 0$). (Here $(C, D)_{S_i}$ denotes the intersection pairing of curves on the surface S_i. Later we sometimes identify this pairing with the natural pairing $H^2(S_i) \times H_2(S_i) \longrightarrow \mathbf{Z}$ via Poincaré duality.) Thanks to Manin [Man1] we can define Néron-Tate heights with respect to $2L_0$ and $2M_0$, that is, quadratic forms on $MW(S_i)$ by

$$Q_1(\sigma_1) = (2L_0, \sigma_1(\mathbf{P}^1))_{S_1}, \quad Q_2(\sigma_2) = (2M_0, \sigma_2(\mathbf{P}^1))_{S_2} \tag{5}$$

for $\sigma_1 \in MW(S_1)$ and $\sigma_2 \in MW(S_2)$.
Let B_i denote the positive definite symmetric bilinear form associated to the quadratic form Q_i, i.e. $B_i(\sigma, \sigma') = \frac{1}{2}\{Q_i(\sigma + \sigma') - Q_i(\sigma) - Q_i(\sigma')\}$.
According to Shioda [Sh1] we call $(MW(S_i), B_i)$ *the Mordell-Weil lattice* of $f_i : S_i \longrightarrow \mathbf{P}^1$. Noting that our Néron-Tate height coincides with Shioda's [Sh1] we can show the following proposition.

Proposition 3.1 *Under Assumption 2.1 in § 2, we have the following isometry of lattices.*

$$(MW(S_i), B_i) \simeq E_8, \quad (i = 1, 2)$$

where E_8 is the unique positive-definite even unimodular lattice of rank 8.

□

Next we consider the Mordell-Weil group $MW(W)$ of $h : W \longrightarrow \mathbf{P}^1$, whose zero section corresponds to (e_1, e'_1) (cf. (6)). From a property of the fiber product we have the following isomorphism:

$$\begin{array}{ccc} MW(W) & \xrightarrow{\sim} & MW(S_1) \oplus MW(S_2) \\ \sigma & \mapsto & (\sigma_1, \sigma_2) = (p_1 \circ \sigma,\ p_2 \circ \sigma) \end{array} \tag{6}$$

Since the Picard group $\mathrm{Pic}(W)$ can be described as in (4), we will use the following notation for the line bundles on W pulled back by p_1 and p_2:

$$\begin{array}{lll} [F] = p_1^*(F_1) = p_2^*(F_2), & & \\ [H_1] = p_1^*(H_1), & [L_0] = p_1^*(L_0), & [E_i] = p_1^*(E_i), \quad (i = 1, \cdots, 9), \\ [H_2] = p_2^*(H_2), & [M_0] = p_2^*(M_0), & [E'_j] = p_2^*(E'_j), \quad (j = 1, \cdots, 9). \end{array}$$

[3] A line bundle on a fibration of abelian varieties is called symmetric if it is invariant under the pull-back by the inverse automorphism $\mathbf{z} \rightarrow -\mathbf{z}$.

We can easily see that $[J_0] := [L_0] + [M_0]$ is a symmetric numerically effective line bundle on W. It defines a Néron-Tate height on $MW(W)$ as follows. For each $\sigma \in MW(W)$ we set

$$Q_W(\sigma) := ([2J_0], [\sigma(\mathbf{P}^1)])_W. \tag{7}$$

Here $(\quad , \quad)_W$ denotes the natural pairing $H^2(W) \times H_2(W) \to \mathbf{Z}$. Note that the zero section of $MW(W)$ is $0_W = (e_1, e_1')$ and $Q_W(0_W) = 0$. From this we obtain the Mordell-Weil lattice $(MW(W), B_W)$ where B_W denotes the symmetric bilinear form associated to Q_W. Moreover we obtain the following relation for each section $\sigma \in MW(W)$:

$$\begin{aligned} Q_W(\sigma) &= ([2J_0], [\sigma(\mathbf{P}^1)])_W \\ &= (2L_0, [\sigma_1(\mathbf{P}^1)])_{S_1} + (2M_0, [\sigma_2(\mathbf{P}^1)])_{S_2} = Q_1(\sigma_1) + Q_2(\sigma_2). \end{aligned} \tag{8}$$

Therefore we find the following

Proposition 3.2 *The Néron-Tate height with respect to* $[2J_0]$ *on* $MW(W)$ *gives a lattice structure on* $MW(W)$ *which induces the isometry:*

$$(MW(W), B_W) \simeq (MW(S_1), B_1) \oplus (MW(S_2), B_2) \simeq E_8 \oplus E_8.$$

□

There are natural maps

$$\begin{array}{rccl} j: & MW(S_i) & \longrightarrow & H_2(S_i) \\ & \sigma & \mapsto & j(\sigma) = [\sigma(\mathbf{P}^1)] = \text{the homology class of the curve } \sigma(\mathbf{P}^1) \end{array}$$

$$\begin{array}{rccl} j: & MW(W) & \longrightarrow & H_2(W) \\ & \sigma & \mapsto & j(\sigma) = [\sigma(\mathbf{P}^1)]. \end{array}$$

For each section $\sigma_i \in MW(S_i)$, we can always find a birational morphism $\varphi_i : S_i \longrightarrow \overline{S_i}$ which contracts only the image of the section $\sigma_i(\mathbf{P}^1)$. This implies the following lemma.

Lemma 3.1 *The maps* j *are injective.*

□

Note that the maps j are *not* homomorphisms of groups.[4]

Next we will choose other generators of $\mathrm{Pic}(S_i)$. These generators will be used for defining the parameters in which we will expand the prepotential of the A-model Yukawa coupling of Schoen's Calabi-Yau 3-folds. Let $(MW(S_i), B_i)$ be the Mordell-Weil lattices of S_i, which are isometric to the lattice E_8. We choose a set of simple roots $\{\alpha_1, \alpha_2, \cdots \alpha_8\}$ of E_8 whose intersection pairing will be given by the following Dynkin diagram.

[4]However, Shioda [Sh1] obtained a way to modify the map j to obtain a natural homomorphism. See [Sh1] or [Sa].

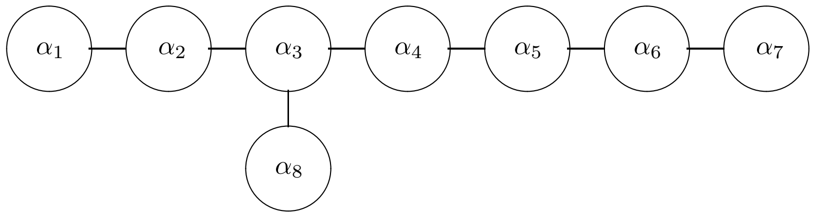

Figure 1.

We also choose $a_1, \cdots, a_8 \in MW(S_1)$ (resp. $b_1, \cdots, b_8 \in MW(S_2)$) corresponding with the roots of $MW(S_1)$ (resp. $MW(S_2)$) so that the numbering of the roots is the same as in Figure 1.
For each section $\sigma \in MW(S_i)$, one can define a translation automorphism $T_\sigma : S_i \longrightarrow S_i$:

$$\begin{array}{ccccc} S_i & & \xrightarrow{T_\sigma} & & S_i \\ & f_i \searrow & & \swarrow f_i & \\ & & \mathbf{P}^1. & & \end{array}$$

Pulling back the line bundles L_0 and M_0 by the translation automorphisms T_{a_i} and T_{b_j} respectively, we define the line bundles

$$L_i = T_{a_i}^*(L_0) \in \mathrm{Pic}(S_1), \quad M_j = T_{b_j}^*(M_0) \in \mathrm{Pic}(S_2), \tag{9}$$

for $1 \leq i, j \leq 8$.
Then for each section $\sigma_i \in MW(S_i)$ we have

$$\begin{aligned} (L_i, j(\sigma_1))_{S_1} &= (T_{a_i}^*(L_0), j(\sigma_1))_{S_1} = (L_0, j(\sigma_1 + a_i))_{S_1} = \frac{1}{2} Q_1(\sigma + a_i) \\ (M_i, j(\sigma_2))_{S_2} &= (T_{b_i}^*(M_0), j(\sigma_2))_{S_2} = (M_0, j(\sigma_2 + b_i))_{S_2} = \frac{1}{2} Q_2(\sigma_2 + b_i) \end{aligned}$$

Now it is easy to see the following:

Lemma 3.2 1. *The classes $F_1, L_0, L_1, \cdots, L_8$ (resp. $F_2, M_0, M_1, \cdots, M_8$) are generators of* $\mathrm{Pic}(S_1)$ *(resp.* $\mathrm{Pic}(S_2)$*).*

2. $\mathrm{Pic}(W)$ *is generated by* $[F], [L_0], \cdots, [L_8], [M_0], [M_1], \cdots, [M_8]$.

3. *For $\sigma \in MW(W)$ set $\sigma_i = \sigma \circ p_i$. Then we have*

$$\begin{aligned} ([F], j(\sigma))_W &= 1 \\ ([L_i], j(\sigma))_W &= \tfrac{1}{2} Q_1(\sigma_1 + a_i), \\ ([M_i], j(\sigma))_W &= \tfrac{1}{2} Q_2(\sigma_2 + b_i). \end{aligned}$$

□

Moreover, in order to see the relation between the A-model and the B-model later, we have to express H_1 and H_2 by $F_1, \{L_i\}$ and $F_2, \{M_j\}$. Obviously, we only have to see the case of H_1.

Recall that the exceptional curves $\{E_i\}$ in (1) are the images of sections of f_1. We denote by $e_i \in MW(S_1)$ the section corresponding to E_i; hence we have $e_i(\mathbf{P}^1) = E_i$. In particular, e_1 is the zero section of $f_1 : S_1 \longrightarrow \mathbf{P}^1$. As for the system of roots, one can take the following elements:

$$a_1 = e_2,\ a_2 = e_3 - e_2,\ a_3 = e_4 - e_3, \cdots, a_7 = e_8 - e_7,$$

and

$$a_8 = e_2 + e_3 - \frac{1}{3}\sum_{i=2}^{9} e_i.$$

Here all sums are taken in the Mordell-Weil group. We denote by $(\sigma) \in H^2(S_1, \mathbf{Z})$ the divisor class of the curve $\sigma(\mathbf{P}^1) \subset S_1$. Since $L_0 = E_1 + F_1 = (e_1) + F_1$, we see that

$$L_i = T^*_{a_i}(L_0) = T^*_{a_i}((e_1) + F_1) = (-a_i) + F_1.$$

Moreover we can see the following relation. (For divisor classes $(-a_i)$ one may refer to [Sa]).

$$\begin{array}{lllll}
L_0 & = & E_1 + F_1 & & \\
L_1 & = & (-a_1) + F_1 & = & 2E_1 - E_2 + 3F_1 \\
L_2 & = & (-a_2) + F_1 & = & E_1 + E_2 - E_3 + 2F_1 \\
L_3 & = & (-a_3) + F_1 & = & E_1 + E_3 - E_4 + 2F_1 \\
L_4 & = & (-a_4) + F_1 & = & E_1 + E_4 - E_5 + 2F_1 \\
L_5 & = & (-a_5) + F_1 & = & E_1 + E_5 - E_6 + 2F_1 \\
L_6 & = & (-a_6) + F_1 & = & E_1 + E_6 - E_7 + 2F_1 \\
L_7 & = & (-a_7) + F_1 & = & E_1 + E_7 - E_8 + 2F_1 \\
L_8 & = & (-a_8) + F_1 & = & \frac{1}{3}\sum_{i=1}^{9} E_i - (E_2 + E_3) + \frac{4}{3}F_1.
\end{array}$$

Recall also the relation (3):

$$F_1 = 3H_1 - \sum_{i=1}^{9} E_i.$$

From these linear relations one easily derives the following:

Lemma 3.3 *One has the following relation in* $\mathrm{Pic}(S_1)$*:*

$$H_1 = 2F_1 + 5L_0 - 2L_1 - L_2 + L_8. \tag{10}$$

□

4 The prepotential of the A-model Yukawa couplings and its 1-sectional part

In this section we summarize a result in ([Sa]) on the Mordell-Weil part of the prepotential of the A-model Yukawa coupling of Schoen's Calabi-Yau 3-folds. The main theorems are Theorem 4.1 and Theorem 4.2.

Following Section 3.3 in [Mo-1], we define the (full) *A-model Yukawa coupling* for a Calabi-Yau 3-fold X as a triple product on $H^2(X, \mathbf{Z})$:

$$\Phi_A(M_1, M_2, M_3) = (M_1, M_2, M_3) + \sum_{0 \not\equiv \eta \in H_2(X,\mathbf{Z})} \Phi_\eta(M_1, M_2, M_3)\frac{q^\eta}{1 - q^\eta}. \tag{11}$$

Here M_1, M_2, M_3 are elements of $H^2(X, \mathbf{Z}) \cong \mathrm{Pic}(X)$ and $\Phi_\eta(M_1, M_2, M_3)$ is the Gromov-Witten Invariant for $\eta \in H_2(X, \mathbf{Z})$ and M_1, M_2, M_3.
Moreover, we have (cf. Section 3.2, [Mo-1]):

$$\Phi_\eta(M_1, M_2, M_3) \quad = \quad n(\eta)(M_1, \eta)(M_2, \eta)(M_3, \eta). \tag{12}$$

Here (M_i, η) denote the natural pairing of $M_i \in H^2(X)$ and $\eta \in H_2(X)$ and $n(\eta)$ denotes the number of simple rational curves $\varphi : \mathbf{P}^1 \longrightarrow X$ with $\varphi_*([\mathbf{P}^1]) = \eta$. A more precise definition by J-holomorphic curves can be found in [McD-S1] and Lecture 3 of [Mo-1].

The full Yukawa coupling Φ_A has the *prepotential* Ψ_A defined by

$$\Psi_A = (\text{topological term}) + \sum_{0 \not\equiv \eta \in H_2(X, \mathbf{Z})} n(\eta)\, \mathrm{Li}_3(q^\eta), \tag{13}$$

where

$$\mathrm{Li}_3(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^3} \tag{14}$$

is the trilogarithm function.
In general it is very difficult to calculate the prepotential of the A-model Yukawa coupling. Even for Schoen's Calabi-Yau 3-fold, we can not calculate the full prepotential, but by using the structure of its Mordell-Weil lattice, we can calculate a part of the prepotential Ψ_A whose summation is taken just over the homology 2-cycles of sections of $h : W \longrightarrow \mathbf{P}^1$. Later we will extend the summation to all homology classes of *pseudo-sections* (see Section 5). (Cf. [Sa]).

Definition 4.1 For Schoen's generic Calabi-Yau 3-fold W we define the *Mordell-Weil part of the prepotential of the A-model Yukawa coupling* by

$$\Psi_{A,MW(W)} = \sum_{\sigma \in MW(X)} n(j(\sigma))\, \mathrm{Li}_3(q^{j(\sigma)}). \tag{15}$$

Here again $j(\sigma)$ denotes the homology class of the image $\sigma(\mathbf{P}^1)$.

Definition 4.2 We define the k-sectional part of the prepotential of the A-model Yukawa coupling by

$$\Psi_{A,k} = \sum_{0 \not\equiv \eta \in H_2(X, \mathbf{Z}),\ (F, \eta) = k} n(\eta)\, \mathrm{Li}_3(q^\eta). \tag{16}$$

Recall that we denote by $[F]$ the divisor class of the fiber of $h : W \longrightarrow \mathbf{P}^1$.

Obviously, we have the expansion

$$\Psi_A = \text{topological term} + \sum_{k=0}^{\infty} \Psi_{A,k}. \tag{17}$$

We are interested in calculating the functions $\Psi_{A,MW(W)}$ and $\Psi_{A,1}$.

Remark 4.1 We will find a difference in $\Psi_{A,MW(W)}$ and $\Psi_{A,1}$, which will be explained in the next section by introducing the notion *pseudo-section.*

We first recall a result in [Sa] on the calculation of $\Psi_{A,MW(W)}$ by using the theta function of the Mordell-Weil lattice. We need to introduce the special formal parameters in order to get explicit expansions of $\Psi_{A,MW(W)}$.
Let $f_i : S_i \longrightarrow \mathbf{P}^1$ be two generic rational elliptic surfaces and let $h : W \longrightarrow \mathbf{P}^1$ be the Calabi-Yau 3-fold as in Section 2. Then from Lemma 3.2 Pic(W) is generated by $[F], [L_0], \cdots, [L_8]$, $[M_0], [M_1], \cdots, [M_8]$. We introduce formal parameters p, q_i, s_j for $0 \leq i, j \leq 8$ corresponding to these generators:

$$[F] \leftrightarrow p, \quad [L_i] \leftrightarrow q_i, \quad [M_j] \leftrightarrow s_j. \tag{18}$$

By using the formal parameters we can associate to $\sigma \in MW(W)$ the monomials

$$q^\sigma = \prod_{i=0}^{8} q_i^{([L_i], j(\sigma))_W}, \quad s^\sigma = \prod_{i=0}^{8} s_i^{([M_i], j(\sigma))_W}. \tag{19}$$

and

$$T^\sigma = p^{([F], j(\sigma))_W} \cdot q^\sigma \cdot s^\sigma = p \cdot q^\sigma \cdot s^\sigma. \tag{20}$$

Here $(\ ,\ )_W : H^2(W) \times H_2(W) \longrightarrow \mathbf{Z}$ is the natural pairing. Note that all line bundles $[F], [L_i], [M_j]$ are numerically effective. Hence all exponents in T^σ are non-negative. Now we can expand $\Psi_{A,MW(W)}$ in the parameters p, q_i, s_j.

Theorem 4.1

$$\begin{aligned} & \Psi_{A,MW(W)}(p, q_0, \cdots q_8, s_0, \cdots, s_8) \\ = \ & \textstyle\sum_{n=1}^{\infty} \frac{1}{n^3} \cdot (p \cdot \prod_{i=1}^{8} (q_i \cdot s_i))^n \cdot \Theta_{E_8}^{root}(n\tau_1, n \cdot \mathbf{z}) \cdot \Theta_{E_8}^{root}(n\tau_2, n \cdot \mathbf{y}) \end{aligned} \tag{21}$$

Here, we set

$$\mathbf{z} = (z_1, \ldots, z_8), \quad \mathbf{y} = (y_1, \ldots, y_8),$$

$$\exp(2\pi i \tau_1) = \prod_{i=0}^{8} q_i, \quad \exp(2\pi i z_i) = q_i \quad \textit{for } 1 \leq i \leq 8 \tag{22}$$

$$\exp(2\pi i \tau_2) = \prod_{i=0}^{8} s_i, \quad \exp(2\pi i y_j) = s_j \quad \textit{for } 1 \leq j \leq 8 \tag{23}$$

and

$$\Theta_{E_8}^{root}(\tau, z_1, \cdots, z_8) = \sum_{\gamma \in E_8} \exp(2\pi i((\tau/2)Q(\gamma) + B(\gamma, \sum_{j=1}^{8} z_j \alpha_j)), \tag{24}$$

where $\{\alpha_1, \cdots, \alpha_8\}$ is the root system of E_8 as in Figure 1 and B is the symmetric bilinear form on E_8.

The following lemma is easy but essential to calculate the prepotential.

Lemma 4.1 *For each section $\sigma \in MW(W)$ the contribution of the homology 2-cycle $j(\sigma) = [\sigma(\mathbf{P}^1)]$ to the Gromov-Witten invariant (12) is one, that is, $n(j(\sigma)) = 1$*

Proof. According to Lemma 3.1 $MW(W)$ can be considered as a subset of $H_2(W, \mathbf{Z})$ via the map j. Moreover the rational curve $C = \sigma(\mathbf{P}^1) \subset W$ has the normal bundle $\mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$. Hence we have $n(j(\sigma)) = 1$. □

Proof of Theorem 4.1. Recalling the isomorphism (6), one can write $\sigma \in MW(W)$ as $(\sigma_1, \sigma_2) \in MW(S_1) \oplus MW(S_2) \simeq E_8 \oplus E_8$. Since $Q_1(a_i) = Q_2(b_j) = 2$ for $1 \leq i, j \leq 8$ we obtain from Lemma 3.2

$$([L_i], [\sigma(\mathbf{P}^1)])_W = \frac{1}{2} Q_1(\sigma_1 + a_i) = \frac{1}{2} Q_1(\sigma_1) + B_1(\sigma_1, a_i) + 1, \tag{25}$$

$$([M_j], [\sigma(\mathbf{P}^1)])_W = \frac{1}{2} Q_2(\sigma_2 + b_j) = \frac{1}{2} Q_2(\sigma_2) + B_2(\sigma_2, b_j) + 1. \tag{26}$$

Therefore one has

$$\begin{aligned} q^\sigma &= \prod_{i=0}^{8} (q_i)^{(1/2)Q_1(\sigma_1 + a_i)} \\ &= (\prod_{i=0}^{8} (q_i))^{(1/2)Q_1(\sigma_1)} \cdot (\prod_{i=1}^{8} (q_i)^{B_1(\sigma_1, a_i)}) \cdot (\prod_{i=1}^{8} q_i) \\ &= (\prod_{i=1}^{8} q_i) \cdot \exp(2\pi i((1/2)Q_1(\sigma_1)\tau_1 + \sum_{i=1}^{8} z_i B(\sigma_1, a_i)), \end{aligned}$$

and a similar expression for s^σ. From these formulas one can obtain

$$\begin{aligned} &(T^\sigma)^n = (p \cdot q^\sigma \cdot s^\sigma)^n \\ &= (p \prod_{i=1}^{8} (q_i \cdot s_i))^n \times \\ &\quad \times \exp(2\pi i n((\tau_1/2)Q_1(\sigma_1) + B_1(\sigma_1, \mathbf{z}) + (\tau_2/2)Q_2(\sigma_2) + B_2(\sigma_2, \mathbf{y})) \end{aligned} \tag{27}$$

where we set $\mathbf{z} = \sum_{i=1}^{8} z_i a_i$ and $\mathbf{y} = \sum_{i=1}^{8} y_i b_i$. Therefore, if we take the summation of $(T^\sigma)^n$ over $\sigma = (\sigma_1, \sigma_2) \in E_8 \oplus E_8$, we obtain the following formula:

$$\begin{aligned} &\sum_{\sigma \in MW(W)} (T^\sigma)^n \\ &\quad = \quad (p \prod_{i=1}^{8} (q_i \cdot s_i))^n \cdot \Theta_{E_8}^{root}(n\tau_1, n \cdot z) \cdot \Theta_{E_8}^{root}(n\tau_2, n \cdot \mathbf{y}) \end{aligned} \tag{28}$$

Now thanks to Lemma 4.1, we can calculate the prepotential as follows:

$$\begin{aligned}\Psi_{A,MW(W)} &= \sum_{\sigma\in MW(W)} \mathrm{Li}_3(T^\sigma) \\ &= \sum_{\sigma\in MW(W)}(\sum_{n=1}^\infty \frac{(T^\sigma)^n}{n^3}) \\ &= \sum_{n=1}^\infty \frac{1}{n^3}\times[\sum_{\sigma\in MW(W)}(T^\sigma)^n)] \\ &= \sum_{n=1}^\infty \frac{1}{n^3}\cdot(p\prod_{i=1}^8(q_i\cdot s_i))^n\times \\ &\qquad \times\Theta_{E_8}^{root}(n\tau_1, n\cdot\mathbf{z})\cdot\Theta_{E_8}^{root}(n\tau_2, n\cdot\mathbf{y}).\end{aligned}$$

This completes the proof of Theorem 4.1. □

For the 1-sectional part $\Psi_{A,1}$ of the prepotential, we can show the following theorem, whose proof can be found in Section 5.

Theorem 4.2 *Assume that Conjecture 5.1 in Section 5 holds. Then, under the same notation as in Theorem 4.1, for a generic Schoen's Calabi-Yau 3-fold W the 1-sectional prepotential is given by*

$$\begin{aligned}&\Psi_{A,1}(p,q_0,\cdots q_8,s_0,\cdots,s_8) \\ = \ &\sum_{n=1}^\infty \frac{1}{n^3}\cdot(p\cdot\prod_{i=1}^8(q_i\cdot s_i))^n\cdot A(n\tau_1,n\cdot\mathbf{z})\cdot A(n\tau_2,n\cdot\mathbf{y}),\end{aligned} \tag{29}$$

where

$$A(\tau,\mathbf{x}) = \Theta_{E_8}^{root}(\tau,\mathbf{x})\cdot[\sum_{m=0}^{\infty}p(m)\cdot\exp(2\pi mi\tau)]^{12}. \tag{30}$$

$$= \Theta_{E_8}^{root}(\tau,\mathbf{x})\cdot[\frac{1}{\prod_{n\geq1}(1-\exp(2\pi ni\tau))}]^{12} \tag{31}$$

Here $p(m)$ denotes the number of partitions of m.

Remark 4.2 In order to identify the theta function $\Theta_{E_8}^{root}(\tau,\mathbf{z})$ in (24) with the theta function $\Theta_{E_8}(\tau,\mathbf{w})$ of (108) in Appendix II we should apply the linear transformation $\mathbf{w}\longrightarrow\mathbf{z}$, for $\mathbf{w}=\sum_{i=1}^8 w_i\epsilon_i$ and $\mathbf{z}=\sum_{i=1}^8 z_i\alpha_i$. Fix an embedding $E_8\subset\mathbf{R}^8$, that is, α_i should have coordinates in $\mathbf{R}^8$. For example, we can choose

$$\alpha_1=\frac{1}{2}(\epsilon_1+\epsilon_8)-\frac{1}{2}(\epsilon_2+\epsilon_3+\epsilon_4+\epsilon_5+\epsilon_6+\epsilon_7)$$

$$\begin{array}{llll}\alpha_2=\epsilon_2-\epsilon_1 & \alpha_3=\epsilon_3-\epsilon_2 & \alpha_4=\epsilon_4-\epsilon_3 & \alpha_5=\epsilon_5-\epsilon_4 \\ \alpha_6=\epsilon_6-\epsilon_5 & \alpha_7=\epsilon_7-\epsilon_6 & \alpha_8=\epsilon_1+\epsilon_2 & \end{array}$$

(Note that the numbering of roots is the same as in Figure 1.)

Remark 4.3 In expansion (17), we see that each term of the expansion of $\Psi_{A,k}$ for $k\geq 2$ is divisible by p^2. Moreover we can see that $\Psi_{A,0}\equiv 0$. (Cf. [Sa]). Therefore, Theorem 4.2 shows that if we expand the full A-model prepotential Ψ_A in the variables in p,q_i,s_j, we have the following expansion:

$$\begin{aligned}&\Psi_A(p,q_0,\cdots q_8,s_0,\cdots,s_8) \\ &= \ \text{topological term} + (p\cdot\prod_{i=1}^8(q_i\cdot s_i))\cdot A(\tau_1,\mathbf{z})\cdot A(\tau_2,\mathbf{y})+O(p^2).\end{aligned} \tag{32}$$

5 Counting Pseudo-Sections and Proof of Theorem 4.2

In Section 4, we see differences between the two prepotentials $\Psi_{A,MW(W)}$ and $\Psi_{A,1}$. Looking at the formulas (21) and (30) one can observe that $\Psi_{A,MW(W)}$ and $\Psi_{A,1}$ are essentially produced by the functions

$$\Psi_{A,MW(W)} \leftrightarrow \Theta_{E_8}^{root}(\tau, \mathbf{x}) \tag{33}$$

$$\Psi_{A,1} \leftrightarrow A(\tau, \mathbf{x}) = \Theta_{E_8}^{root}(\tau, \mathbf{x}) \cdot [\sum_{m=0}^{\infty} p(m) \cdot \exp(2\pi m i \tau)]^{12} \tag{34}$$

As we see in Section 4 the geometric meaning of the function $\Theta_{E_8}^{root}$ is clear, that is, it is the generating function of the contributions of pure sections of $h : W \longrightarrow \mathbf{P}^1$. However, the meaning of the factor

$$[\sum_{m=0}^{\infty} p(m) \cdot \exp(2\pi m i \tau)]^{12} = \exp(\pi i \tau) \cdot \eta(\tau)^{-12}$$

was mysterious at least in the geometric sense.[5] In this section, we give a geometric explanation of this factor assuming one very plausible Conjecture 5.1, and we give a proof of Theorem 4.2. Our answer seems to be very simple and natural at least in a mathematical sense.
For this purpose we give the following:

Definition 5.1 We call a 1-dimensional connected subscheme C of W a *pseudo-section* if $C \subset W$ has no embedded point and

$$([F], C)_W = 1, \tag{35}$$

and the normalization $\tilde{C}_{red}$ of the reduced structure C_{red} is a sum of $\mathbf{P}^1$s.

Example 5.1 The image $\sigma(\mathbf{P}^1)$ of a section $\sigma \in MW(W)$ is a pseudo-section.

Example 5.2 Both rational elliptic surfaces $f_i : S_i \longrightarrow \mathbf{P}^1 (i = 1, 2)$ have 12 singular fibers of type I_1 (in Kodaira's notation [Kod]):

$$D_1, D_2, \cdots, D_{12} \subset S_1, \tag{36}$$

$$D'_1, D'_2, \cdots, D'_{12} \subset S_2. \tag{37}$$

We set $d_i = f_1(D_i) \in \mathbf{P}^1$ and $d'_i = f_2(D'_i) \in \mathbf{P}^1$, the supports of the singular fibers. By Assumption 2.1 in Section 2, the points $d_1, \cdots, d_{12}, d'_1, \cdots, d'_{12}$ are distinct on $\mathbf{P}^1$. Take any $\sigma \in MW(W)$ and set $\sigma_1 = p_1 \circ \sigma \in MW(S_1), \sigma_2 = p_2 \circ \sigma \in MW(S_2)$. Hence $\sigma(\mathbf{P}^1) \subset W_{|\sigma_2(\mathbf{P}^1)}$. Now we take a singular fiber $D_1 \subset W_{|\sigma_2(\mathbf{P}^1)} \simeq S_1$, then

$$\sigma(\mathbf{P}^1) + D_1 \subset W_{|\sigma_2(\mathbf{P}^1)} \subset W$$

is a pseudo-section. Since we have

$$([L_i], D_1)_W = (L_i, D_1)_{S_1} = (L_i, F_1)_{S_1} = 1 \tag{38}$$

$$([M_j], D_1)_W = (M_j, \sigma_2(d_1))_{S_2} = 0, \tag{39}$$

[5] The similar factor are also discussed in the papers [D-G-W] and [Y-Z].

we obtain

$$([L_i], \sigma(\mathbf{P}^1) + D_1)_W = ([L_i], \sigma(\mathbf{P}^1))_W + 1, \tag{40}$$
$$([M_j], \sigma(\mathbf{P}^1) + D_1)_W = ([M_j], \sigma(\mathbf{P}^1))_W. \tag{41}$$

Example 5.3 More generally, to a pure section of $h : W \longrightarrow \mathbf{P}^1$ we can add many rational curves coming from singular fibers of type I_1 and also with multiplicity. Fix a section $\sigma \in MW(W)$ and set $\sigma_1 = p_1 \circ \sigma$, $\sigma_2 = p_2 \circ \sigma$ as before. Consider the following (reduced) closed points:

$$\sigma_1(d_i') \in S_1, \quad \sigma_2(d_i) \in S_2. \tag{42}$$

Moreover, we set

$$D'[\sigma_1, d_i'] = p_1^{-1}(\sigma_1(d_i')) \subset W_{|\sigma_1(\mathbf{P}^1)}(\simeq S_2) \subset W \tag{43}$$
$$D[\sigma_2, d_i] = p_2^{-1}(\sigma_2(d_i)) \subset W_{|\sigma_2(\mathbf{P}^1)}(\simeq S_1) \subset W \tag{44}$$

Note that $D'[\sigma_1, d_i']$ and $D[\sigma_2, d_i]$ are reduced rational curves each of which has one node as its singularities. From (38), (39) it is easy to see that

$$([F], D[\sigma_2, d_i])_W = 0, \quad ([L_i], D[\sigma_2, d_i])_W = 1, ([M_j], D[\sigma_2, d_i])_W = 0, \tag{45}$$
$$([F], D'[\sigma_1, d_i'])_W = 0, \quad ([L_i], D'[\sigma_1, d_i'])_W = 0, ([M_j], D'[\sigma_1, d_i'])_W = 1. \tag{46}$$

We denote by $\mathcal{I}(k_i, \sigma_2(d_i))$ an ideal sheaf on S_2 such that the quotient sheaf

$$\mathcal{O}_{S_2}/\mathcal{I}(k_i, \sigma_2(d_i))$$

is supported on the point $\sigma_2(d_i)$ and length $\mathcal{O}_{S_2}/\mathcal{I}(k_i, \sigma_2(d_i)) = k_i$. We call such an ideal $\mathcal{I}(k_i, \sigma_2(d_i))$ a punctual ideal of colength k_i supported on $\sigma_2(d_i)$. And similarly for $\mathcal{I}(k_j', \sigma_1(d_j'))$. For each of $1 \leq i \leq 12$ (resp. $1 \leq j \leq 12$), let $\mathcal{I}(k_i, \sigma_2(d_i))$ (resp. $\mathcal{I}(k_j', \sigma_1(d_j'))$) be a punctual ideal of colength k_i (resp. k_j') supported on $\sigma_2(d_i)$ (resp. $\sigma_1(d_j')$). We denote by

$$D[\mathcal{I}(k_i, \sigma_2(d_i))] \quad (\text{resp.} D'[\mathcal{I}(k_j', \sigma_1(d_j'))])$$

the one-dimensional subscheme of W defined by the pullback of the ideal sheaf $\mathcal{I}(k_i, \sigma_2(d_i))$ (resp. $\mathcal{I}(k_j', \sigma_1(d_j'))$) via p_2 (resp. p_1). Note that

$$D[\mathcal{I}(k_i, \sigma_2(d_i))]_{red} = D[\sigma_2, d_i], \quad D'[\mathcal{I}(k_j', \sigma_1(d_j'))]_{red} = D'[\sigma_1, d_j'].$$

Now we take the following subscheme of W:

$$C = \sigma(\mathbf{P}^1) + \sum_{i=1}^{12} D[\mathcal{I}(k_i, \sigma_2(d_i))] + \sum_{j=1}^{12} D'[\mathcal{I}(k_j', \sigma_1(d_j'))]. \tag{47}$$

This one dimensional subscheme C in (47) is actually a pseudo-section.

Definition 5.2 The pseudo-section C in (47) is called *of type*

$$(\sigma, k_1, \cdots, k_{12}, k_1', \cdots, k_{12}') \in MW(W) \times (\mathbf{Z}_+)^{24}.$$

Proposition 5.1 *Every pseudo-section C of $h : W \longrightarrow \mathbf{P}^1$ can be written as in* (47).

Proof. Since $([F], C)_W = (F_1, (p_1)_*C))_{S_1} = (F_2, (p_2)_*C))_{S_2} = 1$, it is easy to see that there are sections $\sigma_i \in MW(S_i)$,

$$(p_1)_*(C) = \sigma_1(\mathbf{P}^1) + \text{fibers}, \quad (p_2)_*(C) = \sigma_2(\mathbf{P}^1) + \text{fibers}.$$

Then by definition of a pseudo-section, we can easily see that C can be written in the form of (47) where σ corresponds to (σ_1, σ_2). □

Fix a type $\mu = (\sigma, k_1, \cdots, k_{12}, k'_1, \cdots, k'_{12}) \in MW(W) \times (\mathbf{Z}_+)^{24}$ of a pseudo-section of $h : W \longrightarrow \mathbf{P}^1$. We would like to count the "number" $n(\mu)$ of rational curves which gives the correct contribution to the Gromov-Witten invariant in the formula (12). Since a pseudo-section of type μ is a non-reduced subscheme of W if some k_i or k'_j is greater than 1, it is not easy to determine $n(\mu)$. Of course, the Gromov-Witten invariant should be defined as the number of J-holomorphic curves with a fixed homology class after perturbing the complex structure of W to a generic almost complex structure J ([Mo-1], Theorem 3.3). However at this moment we do not know how to perturb the almost complex structure and how a pseudo-section C of type μ arises as a limit of J-holomorphic curves. (Different J-holomorphic curves for generic J may have the same limit in our complex structure of Schoen's Calabi-Yau 3-fold W.)
Here we propose the following:

Conjecture 5.1 *The contribution $n(\mu)$ of all pseudo-sections of type μ is given by*

$$n(\mu) = e(\mathrm{Hilb}^{\mu}_W) = \text{ Topological Euler number of } (\mathrm{Hilb}^{\mu}_W), \tag{48}$$

where Hilb^{μ}_W is the Hilbert scheme parameterizing pseudo-sections $C \subset W$ of type μ.

Let $\mathbf{C}^2$ be the complex affine space of dimension 2 and denote by $\mathrm{Hilb}^k(\mathbf{C}^2, 0)$ the Hilbert scheme parameterizing the punctual ideal sheaves $\mathcal{I} \subset \mathcal{O}_{\mathbf{C}^2}$ of colength k supported on the origin $0 \in \mathbf{C}^2$

Lemma 5.1 *Fix a type $\mu = (\sigma, k_1, \cdots, k_{12}, k'_1, \cdots, k'_{12})$ of pseudo-section. Then we have a natural isomorphism of schemes*

$$\mathrm{Hilb}^{\mu}_W \simeq \prod_{i=1}^{12} (\mathrm{Hilb}^{k_i}(\mathbf{C}^2, 0)) \times \prod_{j=1}^{12} (\mathrm{Hilb}^{k'_j}(\mathbf{C}^2, 0)) \tag{49}$$

Proof. From the definition of pseudo-section C of type μ in (47), we have the natural morphism φ from Hilb^{μ}_W to

$$\mathrm{Hilb}(\sigma(\mathbf{P}^1) \subset W) \times \prod_{i=1}^{12} (\mathrm{Hilb}^{k_i}(S_2, \sigma_2(d_i))) \times \prod_{j=1}^{12} (\mathrm{Hilb}^{k'_j}(S_1, \sigma_1(d'_j))).$$

of W defined by

$$\begin{aligned} \varphi(C) &= \varphi(\sigma(\mathbf{P}^1) + \sum_{i=1}^{12} D[\mathcal{I}(k_i, \sigma_2(d_i))] + \sum_{j=1}^{12} D'[\mathcal{I}(k'_j, \beta[\sigma_1, d'_j])]) \\ &= (\sigma(\mathbf{P}^1),\ \mathcal{I}[k_i, \sigma_2(d_i)],\ \mathcal{I}[k'_j, \alpha[\sigma_1, d'_j]]). \end{aligned}$$

(Here $\mathrm{Hilb}(\sigma(\mathbf{P}^1) \subset W)$ denotes the connected component of the Hilbert scheme which contains the subscheme $\sigma(\mathbf{P}^1)$ of W.)
Noting that C is connected and $\sigma(\mathbf{P}^1) \subset W$ has no deformation (in particular $\mathrm{Hilb}(\sigma(\mathbf{P}^1) \subset W) = 1pt$) , we can easily see that φ is an isomorphism and obtain (49). □

The following important lemma is a kind suggestion of Kota Yoshioka.

Lemma 5.2 *The Hilbert scheme* $\mathrm{Hilb}^k(\mathbf{C}^2, 0)$ *is irreducible scheme of dimension* $k-1$ *and*

$$e(\mathrm{Hilb}^k(\mathbf{C}^2, 0)) = p(k)$$

where $p(k)$ *denotes the number of partitions of* k.

Proof. The irreduciblity of $\mathrm{Hilb}^k(\mathbf{C}^2, 0)$ is due to Briançon [B]. Moreover $\mathrm{Hilb}^k(\mathbf{C}\{x, y\})$ has a natural S^1-action induced by $(x, y) \to (t^a \cdot x, t^b \cdot y)$ for any weight (a, b). Then for a general choice of a weight (a, b) its fixed points set becomes just the set of monomial ideals of length k. Now a standard argument shows that the topological Euler number of $\mathrm{Hilb}^k(\mathbf{C}^2, 0)$ is equal to the number of fixed points, and it is an easy exercise to see that the number of monomial ideals of $\mathbf{C}[x, y]$ with colength k is equal to $p(k)$. □

From Lemma 5.1 and Lemma 5.2, we obtain the following result.

Corollary 5.1 *Let* μ *and* $n(\mu)$ *as in Conjecture (5.1), then we have*

$$n(\mu) = e(\mathrm{Hilb}^{\mu}_W) = (\prod_{i=1}^{12} p(k_i)) \cdot (\prod_{j=1}^{12} p(k'_j)) \tag{50}$$

□

Lemma 5.3 *Let* $\mu = (\sigma, k_1, \cdots, k_{12}, k'_1, \cdots, k'_{12}), \mu' = (\sigma', l_1, \cdots, l_{12}, l'_1, \cdots, l'_{12})$ *be two types of pseudo-sections. Then a pseudo-section* C *of type* μ *and* C' *of type* μ' *have the same homology class in* $H_2(W)$ *if and only if*

$$\sigma = \sigma', \quad \sum_{i=1}^{12} k_i = \sum_{i=1}^{12} l_i, \quad \sum_{j=1}^{12} k'_j = \sum_{j=1}^{12} l'_j. \tag{51}$$

Proof. The "only if" part is obvious. Lemma 3.1 shows that the first equality in (51) is necessary. Moreover noting that $[D[\mathcal{I}(k_i, \sigma_2, d_i)]]$ is homologous to $k_i \cdot (p_2)^{-1}(\sigma_2(d_i))$ and $[D'(\mathcal{I}[k'_j, \sigma_1, d'_j)]]$ is homologous to $k'_j \cdot [(p_1)^{-1}(\sigma_1(d'_j))]$ we have the other implication. □

Let $\eta \in H_2(W)$ be such that $([F], \eta)_W = 1$, then in order to have non-vanishing contribution $n(\eta)$, η must be the class of a pseudo-section, so write η as $(\sigma, m \cdot p_2^{-1}(1pt), np_1^{-1}(1pt))$. We call η *of type* (σ, m, n).

Proposition 5.2 *For* $\eta \in H_2(W)$ *with* $([F], \eta)_W = 1$ *of type* (σ, m, n), *we have*

$$n(\eta) = n(\sigma, m, n) := (\sum_{k_1+\cdots+k_{12}=n} \prod_{i=1}^{12} p(k_i))(\sum_{k'_1+\cdots+k'_{12}=m} \prod_{j=1}^{12} p(k'_j)). \tag{52}$$

Here k_i *and* k'_j *run over the non-negative integers.*

Proof. From Lemma 5.3 and the remark above, we have

$$n(\eta) = \sum_{\mu} n(\mu)$$

where the summation is taken over the types $\mu = (\sigma, k_1, \cdots, k_{12}, k'_1, \cdots, k'_{12})$ of pseudo-sections such that

$$n = \sum_{i=1}^{12} k_i, \quad m = \sum_{j=1}^{12} k'_j.$$

Combining this with (50), we obtain the assertion (52). □

Proof of Theorem 4.2

Let us fix a homology class $\eta \in H_2(W)$ with $([F], \eta) = 1$ of type (σ, n, m). From (45), (46) it is easy to see that

$$\begin{aligned} ([F], \eta)_W &= 1, && (53) \\ ([L_i], \eta)_W &= ([L_i], j(\sigma))_W + n && (54) \\ ([M_j], \eta)_W &= ([M_j], j(\sigma))_W + m. && (55) \end{aligned}$$

We introduce parameters $z_0 = \log q_0, y_0 = \log s_0$. Note that we have set in Theorem 4.1

$$\tau_1 = \sum_{j=0}^{8} z_j, \quad \tau_2 = \sum_{j=0}^{8} y_j.$$

Moreover just for notation in the proof, we set $v_l = \exp(2\pi i \tau_l)$ for $l = 1, 2$. Recalling the definition of T^σ (cf. (20)), we have

$$\begin{aligned} T^\eta &= \exp(2\pi i (t_0[F] + \sum_{i=0}^{8} z_i[L_i] + \sum_{j=1}^{8} y_j[M_j], \eta)_W) \\ &= p \cdot \exp(2\pi i (\sum_{i=0}^{8} z_i[([L_i], j(\sigma))_W + n] + \sum_{i=0}^{8} y_j[([M_j], j(\sigma))_W + m])) \\ &= T^\sigma \cdot (v_1)^n \cdot (v_2)^m && (56) \end{aligned}$$

Then we have

$$\begin{aligned} &\Psi_{A,1}(p, q_0, \cdots q_8, s_0, \cdots, s_8) \\ &= \sum_{\eta \in H_2(W), ([F], \eta) = 1} n(\eta) \operatorname{Li}_3(T^\eta) \\ &= \sum_{\sigma \in MW(W), n \geq 0, m \geq 0} n(\sigma, n, m) \cdot \operatorname{Li}_3(T^\sigma \cdot (v_1)^n \cdot (v_2)^m) \\ &= \sum_{N=1}^{\infty} [\sum_{\sigma \in MW(W), n \geq 0, m \geq 0} n(\sigma, n, m) \cdot \frac{(T^\sigma)^N \cdot (v_1)^{Nn} \cdot (v_2)^{Nm}}{N^3}] \end{aligned}$$

$$= \sum_{N=1}^{\infty} \frac{1}{N^3} \cdot [\sum_{\sigma \in MW(W)} (T^\sigma)^N] \cdot [\sum_{n \geq 0, m \geq 0} n(\sigma, n, m)(v_1)^{Nn} \cdot (v_2)^{Nm}]. \tag{57}$$

Note that the last equality follows from the fact that $n(\sigma, n, m)$ does not depend on σ.
On the other hand, from equality (52) we have

$$[(\sum_{k=0}^{\infty} p(k)(v_1)^k)(\sum_{k'=0}^{\infty} p(k')(v_2)^{k'})]^{12} = \sum_{n \geq 0, m \geq 0} n(\sigma, n, m)(v_1)^n \cdot (v_2)^m.$$

Moreover as in the proof of Theorem 4.1 we can see that

$$[\sum_{\sigma \in MW(W)} (T^\sigma)^N] = (p \prod_{i=1}^{8} (q_i \cdot s_i))^N \Theta_{E_8}^{root}(N\tau_1, N \cdot \mathbf{z}) \cdot \Theta_{E_8}^{root}(N\tau_2, N \cdot \mathbf{y}).$$

Combining these equalities with (57), we obtain the proof of Theorem 4.2. □

6 The restricted A-model Prepotential

In order to compare the prepotential of the A-model Yukawa coupling of W with the B-model Yukawa coupling of the mirror partner W^*, which we obtain in Section 7, we need to take a special restriction of the variables of the prepotential, that is, we have to specify the parameters which correspond to the line bundles which are induced from the ambient space $\mathbf{P}^1 \times \mathbf{P}^2 \times \mathbf{P}^2$. Let $\iota : W \hookrightarrow \mathbf{P}^1 \times \mathbf{P}^2 \times \mathbf{P}^2$ be the natural embedding. Then we set

$$[F] = \pi_1^*(\mathcal{O}_{\mathbf{P}^1}(1)), \quad [H_1] = \pi_2^*(\mathcal{O}_{\mathbf{P}^2}(1)), \quad [H_2] = \pi_3^*(\mathcal{O}_{\mathbf{P}^2}(1)), \tag{58}$$

and introduce corresponding parameters as follows:

$$\begin{aligned} [F] &\leftrightarrow p = U_0 = \exp(2\pi i t_0), \\ [H_1] &\leftrightarrow U_1 = \exp(2\pi i t_1), \\ [H_2] &\leftrightarrow U_2 = \exp(2\pi i t_2). \end{aligned} \tag{59}$$

Now we consider the following restricted prepotential

$$\Psi_A^{res} = \text{topological term} + \sum_{0 \neq \eta \in H_2(W)} n(\eta) \, \mathrm{Li}_3(U^\eta) \tag{60}$$

where

$$U^\eta = \exp(2\pi i (t_0[F] + t_1[H_1] + t_2[H_2], \eta)_W) \tag{61}$$

$$= p^{([F],\eta)_W} \cdot (U_1)^{([H_1],\eta)_W} \cdot (U_2)^{([H_2],\eta)_W}. \tag{62}$$

Moreover, we can define the k-sectional part and the Mordell-Weil part of the restricted prepotential by

$$\Psi_{A,k}^{res} = \sum_{0 \not\equiv \eta \in H_2(X, \mathbf{Z}), \ (F,\eta) = k} n(\eta) \, \mathrm{Li}_3(U^\eta), \tag{63}$$

$$\Psi^{res}_{A,MW(W)} = \sum_{\sigma\in MW(W)} \mathrm{Li}_3(U^{j(\sigma)}), \tag{64}$$

respectively.

Proposition 6.1

$$\Psi^{res}_{A,MW(W)}(p,t_1,t_2) = \sum_{n=1}^{\infty} \frac{p^n}{n^3} \cdot \Theta_{E_8}(3nt_1, nt_1\gamma) \cdot \Theta_{E_8}(3nt_2, nt_2\gamma) \tag{65}$$

$$\Psi^{res}_{A,1}(p,t_1,t_2) = \sum_{n=1}^{\infty} \frac{p^n}{n^3} \cdot A^{res}(nt_1) \cdot A^{res}(nt_2) \tag{66}$$

where

$$\gamma = (1,1,1,1,1,1,1,-1)$$

and

$$\begin{aligned} A^{res}(t) &= \Theta_{E_8}(3t, t\cdot\gamma)\cdot(\sum_{n=0}^{\infty} p(n)\exp(2\pi i n(3t))^{12} \qquad (67)\\ &= \Theta_{E_8}(3t, t\cdot\gamma)\cdot\frac{\exp(3\pi i t)}{[\eta(3t)]^{12}}\\ &= \Theta_{E_8}(3t, t\cdot\gamma)\cdot\frac{1}{[\prod_{m\geq 1}(1-\exp(2\pi i m(3t)))]^{12}}. \end{aligned}$$

Proof. From Relation (10), we obtain for every $\sigma \in MW(W)$

$$\begin{aligned} ([H_1], j(\sigma))_W &= (H_1, j(\sigma_1))_{S_1}\\ &= (2F_1 + 5L_0 - 2L_1 - L_2 + L_8, j(\sigma_1))_{S_1}\\ &= 2 + 1/2(5Q_1(\sigma_1) - 2Q_1(\sigma_1+a_1) - Q_1(\sigma_1+a_2) + Q_1(\sigma_1+a_8))\\ &= \frac{3}{2}Q_1(\sigma_1) + B_1(\sigma_1, -2a_1 - a_2 + a_8), \end{aligned}$$

and a similar equation for $([H_2], j(\sigma))_W$. Then from Remark 4.2, we see that

$$\begin{aligned} \gamma = -2a_1 - a_2 + a_8 &= -[(\epsilon_1+\epsilon_8) - (\epsilon_2+\epsilon_3+\epsilon_4+\epsilon_5+\epsilon_6+\epsilon_7)] + \\ &\quad -(\epsilon_2-\epsilon_1) + (\epsilon_1+\epsilon_2)\\ &= \epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4+\epsilon_5+\epsilon_6+\epsilon_7-\epsilon_8 \end{aligned}$$

Therefore we see that

$$\begin{aligned} U^{j(\sigma)} &= \exp(2\pi i(t_0 + ([H_1], j(\sigma))_W t_1 + ([H_2], j(\sigma))_W t_2))\\ &= p\cdot\exp(2\pi i(\frac{3t_1}{2}Q_1(\sigma_1) + B_1(\sigma_1, t_1\gamma))\exp(2\pi i(\frac{3t_2}{2}Q_2(\sigma_2) + B_2(\sigma_2, t_2\gamma))) \end{aligned}$$

Then as in the proof of Theorem 4.1, we can obtain Assertion (65). The proof of Assertion (66) is similar. □

Now we consider the expansion of (65) and (66) with respect to the variable p.

$$\Psi^{res}_{A,MW(W)} = p \cdot \Theta_{E_8}(3t_1, t_1\gamma) \cdot \Theta_{E_8}(3t_2, t_2\gamma) + O(p^2) \tag{68}$$

$$\Psi_A - \text{topological term} = \Psi^{res}_{A,1} + O(p^2) = p \cdot A^{res}(t_1) \cdot A^{res}(t_2) + O(p^2) \tag{69}$$

Let us define sequences of positive integers $\{c_n\}$ and $\{a_n\}$ by

$$\Theta_{E_8}(3t, t\gamma) = \sum_{n=0}^{\infty} c_m \exp(2\pi mit) = \sum_{n=0}^{\infty} c_m U^m \tag{70}$$

$$A^{res}(t) = \sum_{n=0}^{\infty} a_m \exp(2\pi mit) = \sum_{n=0}^{\infty} a_m U^m. \tag{71}$$

Let us also expand functions as follows:

$$p \cdot \Theta_{E_8}(3t_1, t_1\gamma) \cdot \Theta_{E_8}(3t_2, t_2\gamma) = \sum_{n_1 \geq 0, n_2 \geq 0} M_{1,n_1,n_2} \cdot p \cdot (U_1)^{n_1}(U_2)^{n_2} \tag{72}$$

$$p \cdot A^{res}(t_1) \cdot A^{res}(t_2) = \sum_{n_1 \geq 0, n_2 \geq 0} N_{1,n_1,n_2} \cdot p \cdot (U_1)^{n_1}(U_2)^{n_2}. \tag{73}$$

The proof of the following proposition follows from (72), (73) and Proposition 6.1.

Proposition 6.2 *1.*

$$M_{1,n_1,n_2} = c_{n_1} \cdot c_{n_2}, \quad N_{1,n_1,n_2} = a_{n_1} \cdot a_{n_2} \tag{74}$$

2. *Let $f : S \longrightarrow \mathbf{P}^1$ be a generic rational elliptic surface as in section 2. Then in the expansion* (70) *the coefficient c_m is the number of sections of $f : S \longrightarrow \mathbf{P}^1$ with degree $(H, [\sigma(\mathbf{P}^1)])_S = m$, that is,*

$$c_m = \sharp\{ \ \sigma \in MW(S) \ \ | \ \ (H, [\sigma(\mathbf{P}^1)])_S = m \ \},$$

   where H is the class of the total transform of a line on $\mathbf{P}^2$.

3. *The integer M_{1,n_1,n_2} (resp.N_{1,n_1,n_2}) is the number of sections $\sigma \in MW(W)$ (resp. pseudo-sections η) of $h : W \longrightarrow \mathbf{P}^1$ with bidegree (n_1, n_2) where $n_i = ([H_i], j(\sigma))_W$ (resp. $n_i = ([H_i], \eta)_W$) is the degree with respect to $[H_i]$.*

□

Remark 6.1 The factorization property of M_{1,n_1,n_2} and N_{1,n_1,n_2} in (74) follows from the fact that sections and pseudo-sections of $h : W \longrightarrow \mathbf{P}^1$ can be split as in (6), (47).

Remark 6.2 Note that the sequences $\{c_m\}$ and $\{a_m\}$ are connected to each other by the formula:

$$\sum_{n=0}^{\infty} a_m U^m = [\sum_{n=0}^{\infty} c_m U^m][\sum_{k=0}^{\infty} p(k) U^{3k}]^{12}.$$

The number a_m can be considered as the number of pseudo-sections C of a generic rational elliptic surface $f : S \longrightarrow \mathbf{P}^1$ of degree m with respect to the divisor class $[H]$. The term

$$[\sum_{k=0}^{\infty} p(k) U^{3k}]^{12}$$

is nothing but the contribution of 12 singular fibers of type I_1, when we count the contribution of one singular fiber of type I_1 with multiplicity k as $p(k)$.

Here we will expand $\Theta_{E_8}(3t, t\gamma)$ and $A^{res}(t)$ in the variable $U = \exp(2\pi i t)$ and give the table of coefficients c_n and a_n up to order 50. (See also the last remark of Section 7). We can use Proposition 9.1 to obtain the following expansion.

Table. 1

$$\Theta_{E_8}(3t, t\gamma) = \sum_{m\geq 0} c_m U^m.$$

n	c_n	n	c_n	n	c_n	n	c_n
0	9	13	8892	26	68922	39	197136
1	36	14	12168	27	68796	40	241920
2	126	15	13104	28	86580	41	227556
3	252	16	17766	29	84168	42	276948
4	513	17	18648	30	103824	43	262080
5	756	18	24390	31	101556	44	319410
6	1332	19	25200	32	127647	45	298116
7	1764	20	33345	33	121212	46	357912
8	2808	21	33516	34	148878	47	341460
9	3276	22	43344	35	143964	48	410958
10	4914	23	43092	36	178776	49	382356
11	5616	24	55692	37	170352	50	458208
12	8190	25	54684	38	205380		

Table 2. (Table for $\{a_n\}$).

$$A^{res}(t) = \sum_{m\geq 0} a_m U^m = (\sum_{n\geq 0} c_n U^n) \cdot (\sum_{k\geq 0} p(k) U^{3k})^{12}.$$

n	a_n	n	a_n	n	a_n
0	9	17	5421132	34	11208455370
1	36	18	9131220	35	16538048640
2	126	19	15195600	36	24282822798
3	360	20	25006653	37	35487134928
4	945	21	40722840	38	51626878470
5	2268	22	65670768	39	74779896240
6	5166	23	104930280	40	107861179482
7	11160	24	166214205	41	154945739844
8	23220	25	261141300	42	221711362038
9	46620	26	407118726	43	316042958880
10	90972	27	630048384	44	448856366490
11	172872	28	968272605	45	635216766732
12	321237	29	1478208420	46	895854679650
13	584640	30	2242463580	47	1259213600736
14	1044810	31	3381344280	48	1764210946995
15	1835856	32	5069259342	49	2463949037340
16	3177153	33	7557818940	50	3430694064888

7 The prepotential of the B-model Yukawa coupling

In this section we study the prepotential of the B-model Yukawa coupling for the mirror W^* of Schoen's example in the sense of Batyrev-Borisov [Ba-Bo] and compare it with the prepotential for the A-model Yukawa coupling of W. Formula (80) gives this B-model prepotential Ψ_B explicitly. In order to determine the B-model prepotential we will basically follow the recipe of [HKTY, Sti] which uses only the toric data of the A-model side. However in order to give an intuitive picture of the mirror W^* we will put here the orbifold construction of the mirror W^* of Schoen's example and the Picard-Fuchs equations of the periods of a holomorphic 3-form of W^*.
Based on the Batyrev-Borisov mirror construction (cf. [Ba-Bo], [HKTY]) for complete intersection Calabi-Yau manifolds in toric varieties we can derive the following

Proposition 7.1 *The family of mirror Calabi-Yau 3-folds of W is obtained by the orbifold construction with group $\mathbf{Z}_3 \times \mathbf{Z}_3$ from the subfamily $W_{\alpha_0,\alpha_1,\beta_0,\beta_1}$ of W:*

$$W_{\alpha_0,\alpha_1,\beta_0,\beta_1} = \left\{ [z_0 : z_1] \times [x_0 : x_1 : x_2] \times [y_0 : y_1 : y_2] \in \mathbf{P}^1 \times \mathbf{P}^2 \times \mathbf{P}^2 \mid P_1 = P_2 = 0 \right\}$$

where

$$\begin{aligned} P_1 &= (x_0^3 + x_1^3 + x_2^3 + \alpha_0 x_0 x_1 x_2) z_1 + \alpha_1 x_0 x_1 x_2 z_0 \,, \\ P_2 &= (y_0^3 + y_1^3 + y_2^3 + \beta_0 y_0 y_1 y_2) z_0 + \beta_1 y_0 y_1 y_2 z_1 \end{aligned}$$

and the group $\mathbf{Z}_3 \times \mathbf{Z}_3$ is generated by

$$\begin{aligned} g_1 &: ([z_0 : z_1], [x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) \\ &\quad \mapsto ([z_0 : z_1], [x_0 : \omega x_1 : \omega^2 x_2], [y_0 : y_1 : y_2]), \\ g_2 &: ([z_0 : z_1], [x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) \\ &\quad \mapsto ([z_0 : z_1], [x_0 : x_1 : x_2], [y_0 : \omega y_1 : \omega^2 y_2]), \end{aligned} \tag{75}$$

with $\omega = e^{2\pi i/3}$. That is, the mirror W^* is

$$W^* = W_{\alpha_0,\alpha_1,\beta_0,\beta_1} / (\mathbf{Z}_3 \times \mathbf{Z}_3).$$

Proof. See Section 8, Appendix I. □

In the equations P_1 and P_2 above we have kept four parameters $\alpha_0, \alpha_1, \beta_0, \beta_1$ for symmetry reasons. However only three of them are essential because of the scaling of the variables z_0, z_1. After the orbifoldization this three parameter deformation describes a three dimensional subspace in the complex structure (B-model) moduli space of W^*. The full complex structure moduli space has dimension 19. Under the mirror symmetry the three dimensional subspace will be mapped to the subspace in the complexified Kähler moduli space parameterized by (t_0, t_1, t_2) in (59). The B-model calculations are local calculations based on the variation of the Hodge structure for the family W^* about the *Large complex structure limit* (LCSL). A mathematical characterization of LCSL is given in [Mo-2]. Here we simply follow a general recipe applicable to CICYs in toric varieties to find a LCSL and write the Picard-Fuchs differential equations [HKTY]. We find that the origin of the local coordinate system $u = (u_0, u_1, u_2)$ with $u_0 = \frac{\alpha_1 \beta_1}{\alpha_0 \beta_0}$, $u_1 = -\frac{1}{\alpha_0^3}$ and $u_2 = -\frac{1}{\beta_0^3}$

is a LCSL, and that the Picard-Fuchs (PF) differential operators about this point are

$$\begin{aligned} D_1 &= (3\theta_{u_1}-\theta_{u_0})\theta_{u_1} - 9u_1(3\theta_{u_1}+\theta_{u_0}+2)(3\theta_{u_1}+\theta_{u_0}+1) \\ &\quad +u_0\theta_{u_1}(3\theta_{u_2}+\theta_{u_0}+1) \ , \\ D_2 &= (3\theta_{u_2}-\theta_{u_0})\theta_{u_2} - 9u_2(3\theta_{u_2}+\theta_{u_0}+2)(3\theta_{u_2}+\theta_{u_0}+1) \\ &\quad +u_0\theta_{u_2}(3\theta_{u_1}+\theta_{u_0}+1) \ , \\ D_3 &= \theta_{u_0}^2 - u_0(3\theta_{u_1}+\theta_{u_0}+1)(3\theta_{u_2}+\theta_{u_0}+1) \ , \end{aligned} \tag{76}$$

with $\theta_{u_i} = u_i \frac{\partial}{\partial u_i}$. We note that if we set $u_0 = 0$ in (76), the operators D_1 and D_2 reduce to the PF equations for the Hesse pencil of elliptic curves. Local solutions about $u = 0$ have several interesting properties. To state these, we denote the three elements $[F], [H_1]$ and $[H_2]$ in the Picard group $\mathrm{Pic}(W)$ by J_0, J_1 and J_2, respectively. By the notation K_{ijk} $(i, j, k = 0, 1, 2)$ we denote the classical intersection numbers among the corresponding divisors. Then the nonzero components are calculated, up to obvious permutations of the indices, by

$$K_{012} = 9 \ , \ K_{112} = K_{122} = 3 \ . \tag{77}$$

Proposition 7.2 *1. The Picard-Fuchs equation (76) has only one regular solution, namely*

$$\Omega^{(0)}(u) := \sum_{m_0,m_1,m_2\geq 0} \frac{(m_0+3m_1)!\,(m_0+3m_2)!}{(m_0!)^2\,(m_1!)^3\,(m_2!)^3} u_0^{m_0} u_1^{m_1} u_2^{m_2} \tag{78}$$

2. All other solutions of (76) have logarithmic regular singularities and have the following form in terms of the classical Frobenius method

$$\begin{aligned} \Omega_i^{(1)}(u) &:= \frac{\partial}{\partial\rho_i}\Omega(u,\rho)|_{\rho=0} \ , \\ \Omega_i^{(2)}(u) &:= \frac{1}{2}\sum_{j,k=0,1,2} K_{ijk}\frac{\partial}{\partial\rho_j}\frac{\partial}{\partial\rho_k}\Omega(u,\rho)|_{\rho=0} \ , \\ \Omega^{(3)}(u) &:= -\frac{1}{3!}\sum_{i,j,k=0,1,2} K_{ijk}\frac{\partial}{\partial\rho_i}\frac{\partial}{\partial\rho_j}\frac{\partial}{\partial\rho_k}\Omega(u,\rho)|_{\rho=0} \ , \end{aligned} \tag{79}$$

with

$$\Omega(u,\rho) := \sum_{m_0,m_1,m_2\geq 0} \frac{(1+\rho_0+3\rho_1)_{m_0+3m_1}(1+\rho_0+3\rho_2)_{m_0+3m_2}}{(1+\rho_0)_{m_0}^2(1+\rho_1)_{m_1}^3(1+\rho_2)_{m_2}^3} u_0^{m_0+\rho_0} u_1^{m_1+\rho_1} u_2^{m_2+\rho_2}$$

and K_{ijk} being the coupling in (77). The notation $(x)_m$ represents the Pochhammer symbol: $(x)_m := x(x+1)\cdots(x+m-1)$. □

Now we are ready to define the B-model prepotential and the mirror map:

Definition 7.1 We define the *B-model prepotential* by

$$\Psi_B(u) = \frac{1}{2}\left(\frac{1}{\Omega^{(0)}(u)}\right)^2 \left\{\Omega^{(0)}(u)\Omega^{(3)}(u) + \sum_i \Omega_i^{(1)}(u)\Omega_i^{(2)}(u)\right\} \ . \tag{80}$$

Definition 7.2 We define the *special coordinates on the B-model moduli space* by

$$t_j = \frac{1}{2\pi i}\frac{\Omega_j^{(1)}(u)}{\Omega^{(0)}(u)}\,, \quad U_j := e^{2\pi i t_j} \quad (j=0,1,2)\,. \tag{81}$$

Then U_0, U_1, U_2 are functions of u_0, u_1, u_2 and $U_j = u_j +$ higher order terms. The inverse map $(u_0(U),\, u_1(U),\, u_2(U))$ is called *the mirror map.*

Conjecture 7.1 (Mirror Conjecture) *The B-model prepotential $\Psi_B(u)$ combined with the mirror map has the expansion*

$$\Psi_B(u(U)) = \frac{(2\pi i)^3}{3!}\sum_{i,j,k=0,1,2} K_{ijk}t_i t_j t_k + \sum_{n_0,n_1,n_2\geq 0} N_{n_0,n_1,n_2}\,\mathrm{Li}_3(U_0^{n_0}U_1^{n_1}U_2^{n_2}) \tag{82}$$

where N_{n_0,n_1,n_2} is the number of rational curves $\varphi : \mathbf{P}^1 \mapsto W$ with $(J_i, \varphi_*([\mathbf{P}^1])) = n_i$, $(i=0,1,2)$. In our context, we can state the conjecture in more precise form as follows:

$$\Psi_A^{res}(U_0,U_1,U_2) = \Psi_B(u(U_0,U_1,U_2)) \tag{83}$$

where $\Psi_A^{res}(U_0,U_1,U_2)$ is the restricted A-model prepotential defined in (60).

Next we briefly sketch the approach of [Sti] for calculating the B-model prepotential by using only toric data of the A-model side. This starts from the observation that Schoen's example W can be embedded in $\mathbf{P}^1\times\mathbf{P}^2\times\mathbf{P}^2$ as the intersection of a hypersurface of degree $(1,3,0)$ and a hypersurface of degree $(1,0,3)$. (cf. Section 2). So W is the zero locus of a (general) section of the rank 2 vector bundle $\mathcal{O}(1,3,0)\oplus\mathcal{O}(1,0,3)$ on $\mathbf{P}^1\times\mathbf{P}^2\times\mathbf{P}^2$. This vector bundle can be constructed as a quotient of an open part of $\mathbf{C}^{10}$ by a 3-dimensional subtorus of $(\mathbf{C}^*)^{10}$ acting by coordinatewise multiplication. The subtorus is the image of the homomorphism $(\mathbf{C}^*)^3 = \mathbf{Z}^3\otimes\mathbf{C}^* \to \mathbf{Z}^{10}\otimes\mathbf{C}^* = (\mathbf{C}^*)^{10}$ given by the 3×10-matrix

$$\mathsf{B} := \begin{pmatrix} -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \tag{84}$$

The open part of $\mathbf{C}^{10}$ is

$$\bigcup_{(i,j,k)\in\{3,4\}\times\{5,6,7\}\times\{8,9,10\}} \mathbf{C}^{10}_{(i,j,k)} \tag{85}$$

with

$$\mathbf{C}^{10}_{(i,j,k)} := \{(x_1,\dots,x_{10})\in\mathbf{C}^{10} \mid x_i\neq 0\,,\ x_j\neq 0\,,\ x_k\neq 0\ \}$$

We view $\{3,4\}\times\{5,6,7\}\times\{8,9,10\}$ as a collection of 18 subsets of $\{1,\dots,10\}$ and note that the complement of the union of these 18 subsets is $\{1,2\}$. As explained in [Sti] this bit of combinatorial input suffices to explicitly write down the hypergeometric function from which one can subsequently compute the B-model prepotential.
This hypergeometric function is a priori a function in 10 variables $v_1,\dots,v_{10}$, which correspond to the a priori 10 coefficients in the equations P_1 and P_2 in proposition 7.1:

$$\begin{aligned} \Phi \;:=\;& (\bar{J}_0+3\bar{J}_1)(\bar{J}_0+3\bar{J}_2)\times v_1^{-1}v_2^{-1}\times\, u_1^{\bar{J}_0}u_2^{\bar{J}_1}u_3^{\bar{J}_2}\times \\ \times & \sum_{m_0,m_1,m_2\geq 0}\frac{(1+\bar{J}_0+3\bar{J}_1)_{m_0+3m_1}\cdot(1+\bar{J}_0+3\bar{J}_2)_{m_0+3m_2}}{(1+\bar{J}_0)_{m_0}{}^2\cdot(1+\bar{J}_1)_{m_1}{}^3\cdot(1+\bar{J}_2)_{m_2}{}^3}\,u_0^{m_0}u_1^{m_1}u_2^{m_2} \end{aligned}$$

with

$$u_0 := v_1^{-1} v_2^{-1} v_3 v_4 \,, \quad u_1 := -\, v_1^{-3} v_5 v_6 v_7 \,, \quad u_2 := -\, v_2^{-3} v_8 v_9 v_{10}$$

and where $\bar{J}_0,\, \bar{J}_1,\, \bar{J}_2$ are elements in the ring

$$\mathcal{R}_{\mathbf{P}^1\times\mathbf{P}^2\times\mathbf{P}^2} := \mathbf{Z}[\bar{J}_0, \bar{J}_1, \bar{J}_2]/(\bar{J}_0^2, \bar{J}_1^3, \bar{J}_2^3).$$

So, $v_1 v_2 \Phi$ is an element of

$$\big((\bar{J}_0 + 3\bar{J}_1)(\bar{J}_0 + 3\bar{J}_2)\mathcal{R}_{\mathbf{P}^1\times\mathbf{P}^2\times\mathbf{P}^2}\big) \otimes \mathbf{Q}[[u_1, u_2, u_3]][\log u_1,\, \log u_2,\, \log u_3]\,.$$

The map multiplication by $(\bar{J}_0 + 3\bar{J}_1)(\bar{J}_0 + 3\bar{J}_2)$ on $\mathcal{R}_{\mathbf{P}^1\times\mathbf{P}^2\times\mathbf{P}^2}$ induces an isomorphism of linear spaces from the ring

$$\mathcal{R}_{toric} := \mathcal{R}_{\mathbf{P}^1\times\mathbf{P}^2\times\mathbf{P}^2}/Ann((\bar{J}_0 + 3\bar{J}_1)(\bar{J}_0 + 3\bar{J}_2))$$

onto the ideal $(\bar{J}_0 + 3\bar{J}_1)(\bar{J}_0 + 3\bar{J}_2)\mathcal{R}_{\mathbf{P}^1\times\mathbf{P}^2\times\mathbf{P}^2}$.
$\mathcal{R}_{\mathbf{P}^1\times\mathbf{P}^2\times\mathbf{P}^2}$ is in fact the cohomology ring of the ambient toric variety $\mathbf{P}^1 \times \mathbf{P}^2 \times \mathbf{P}^2$ and $\mathcal{R}_{toric}$ is a subring of the Chow ring of W. The classes of $\bar{J}_0, \bar{J}_1, \bar{J}_2$ in $\mathcal{R}_{toric}$ correspond to the elements J_0, J_1, J_2 of $\mathrm{Pic}(W)$ defined earlier. One easily checks that $\mathcal{R}_{toric}$ is a free $\mathbf{Z}$-module of rank 8 with basis $\{1\,,\ J_0\,,\ J_1\,,\ J_2\,,\ J_1^2\,,\ J_1 J_2\,,\ J_2^2\,,\ J_1^2 J_2\}$ and that the following relations hold

$$\begin{aligned}
&J_0^2 = J_1^3 = J_2^3 = J_0 J_1^2 = J_0 J_2^2 = 0\,,\\
&J_0 J_1 = 3J_1^2\,, \quad J_0 J_2 = 3J_2^2\,, \quad J_1 J_2^2 = J_1^2 J_2\,, \quad J_0 J_1 J_2 = 3J_1^2 J_2
\end{aligned}$$

Instead of $\bar{\Phi}$ we may as well work with

$$\Omega(u, J) := \sum_{m_0, m_1, m_2 \geq 0} \frac{(1 + J_0 + 3J_1)_{m_0+3m_1} \cdot (1 + J_0 + 3J_2)_{m_0+3m_2}}{(1 + J_0)_{m_0}{}^2 \cdot (1 + J_1)_{m_1}{}^3 \cdot (1 + J_2)_{m_2}{}^3}\, u_0^{m_0+J_0} u_1^{m_1+J_1} u_2^{m_2+J_2}$$

Using the notations (77), (78), (79) and $J_0^\vee := \frac{1}{9}J_1 J_2 - \frac{1}{27}J_0 J_1 - \frac{1}{27}J_0 J_2$, $J_1^\vee := \frac{1}{9}J_0 J_2$, $J_2^\vee := \frac{1}{9}J_0 J_1$ and $vol := \frac{1}{9}J_0 J_1 J_2$ the relation between the two approaches may now be formulated as

Proposition 7.3

$$\Omega(u, J) = \Omega^{(0)}(u) + \sum_{i=0}^{2} \Omega_i^{(1)}(u) J_i + \sum_{i=0}^{2} \Omega_i^{(2)}(u) J_i^\vee - \Omega^{(3)}(u) vol$$

□

$\Omega(u, J)$ is an element of the ring $\mathcal{R}_{toric} \otimes \mathbf{Q}[[u_1, u_2, u_3]][\log u_1,\, \log u_2,\, \log u_3]$. It is 1 modulo $J_0, J_1, J_2, u_0, u_1, u_2$ and hence its logarithm also exists in the ring $\mathcal{R}_{toric} \otimes \mathbf{Q}[[u_1, u_2, u_3]][\log u_1,\, \log u_2,\, \log u_3]$. Expanding $\log\Omega(u, J)$ with respect to the basis $\{1,\, J_0,\, J_1,\, J_2,\, J_0^\vee,\, J_1^\vee,\, J_2^\vee,\, vol\}$ of $\mathcal{R}_{toric}$ one finds

$$\log\Omega(u, J) = \log\Omega^{(0)}(u) \;+\; \sum_{j=0}^{2} \log U_j\, J_j \;+\; \sum_{j=0}^{2} P_j\, J_j^\vee \;+\; P\, vol$$

with U_j as in (81) and hence $\log U_j = 2\pi i t_j$. A straightforward computation shows (see also (80) and (82))

$$
\begin{aligned}
P &= -\left(\frac{\Omega^{(3)}(u)}{\Omega^{(0)}(u)} + \sum_{j=0}^{2} \frac{\Omega_j^{(1)}(u)}{\Omega^{(0)}(u)} \frac{\Omega_j^{(2)}(u)}{\Omega^{(0)}(u)} - \frac{(2\pi i)^3}{3} \sum_{m,j,k=0,1,2} K_{mjk} t_m t_j t_k\right) \\
&= -2\left(\Psi_B(u) - \frac{(2\pi i)^3}{3!} \sum_{m,j,k=0,1,2} K_{mjk} t_m t_j t_k\right) \\
&= -2 \sum_{n_0,n_1,n_2 \geq 0} N_{n_0,n_1,n_2} \operatorname{Li}_3(U_0^{n_0} U_1^{n_1} U_2^{n_2}) \qquad (86)
\end{aligned}
$$

Proposition 7.4 *Let the numbers N_{n_0,n_1,n_2} be defined by (86). Then*

$$N_{0,n_1,n_2} = 0 \quad \text{for all } \ n_1, n_2 \geq 0 \qquad (87)$$

Proof. Note that modulo u_0

$$
\begin{aligned}
& u_0^{-J_0} u_1^{-J_1} u_2^{-J_2} \Omega(u, J) \equiv \\
& \equiv \left(\sum_{m_1 \geq 0} \frac{(1 + J_0 + 3J_1)_{3m_1}}{{(1 + J_1)_{m_1}}^3} u_1^{m_1}\right) \left(\sum_{m_2 \geq 0} \frac{(1 + J_0 + 3J_2)_{3m_2}}{{(1 + J_2)_{m_2}}^3} u_2^{m_2}\right)
\end{aligned}
$$

and take logarithms. The logarithms involve no mixed terms $J_1 J_2$. This shows $P \equiv 0 \bmod u_0$. □

As explained in [F, Sti] a theorem of Bryant and Griffiths shows

$$P_j = -\frac{1}{2} U_j \frac{\partial P}{\partial U_j}$$

for $j = 0, 1, 2$. Hence

$$P_j = \sum_{n_0,n_1,n_2 \geq 0} n_j N_{n_0,n_1,n_2} \operatorname{Li}_2(U_0^{n_0} U_1^{n_1} U_2^{n_2}) \qquad (88)$$

where $\operatorname{Li}_2(x) := \sum_{n \geq 1} \frac{x^n}{n^2}$ is the dilogarithm function.
It follows from (88) and (87) that we can get all numbers N_{n_0,n_1,n_2} from P_0. The computations are now greatly simplified by observing:

Lemma 7.1 *In $\mathcal{R}_{toric}$ the intersection of the* **Z***-module with basis* $\{1, J_1, J_2, J_1 J_2\}$ *and the ideal generated by* J_0, J_1^2, J_2^2 *is* 0. □

So for studying $\log U_1$, $\log U_2$ and P_0 we may reduce modulo the ideal (J_0, J_1^2, J_2^2); i.e. replace $\mathcal{R}_{toric}$ by $\mathbf{Z}[\bar{J}_1, \bar{J}_2]/(\bar{J}_1^2, \bar{J}_2^2)$. From now on we use J_1 resp. J_2 to denote the classes of $\bar{J}_1$ resp. $\bar{J}_2$ in the latter ring; so we have in particular from now on

$$J_1^2 = J_2^2 = 0$$

Let

$$\tilde{\Omega}(u,J_1,J_2) := \sum_{m_0,m_1,m_2\geq 0} \frac{(1+3J_1)_{m_0+3m_1}\cdot(1+3J_2)_{m_0+3m_2}}{m_0!^{\,2}\cdot(1+J_1)_{m_1}{}^3\cdot(1+J_2)_{m_2}{}^3}\, u_0^{m_0}u_1^{m_1}u_2^{m_2}$$

Then clearly

$$\begin{aligned}\log\tilde{\Omega}(u,J_1,J_2) \quad &= \quad \log\Omega^{(0)}(u) \;+\; (\log U_1 - \log u_1)\,J_1 + \\ &\quad + (\log U_2 - \log u_2)\,J_2 \;+\; \tfrac{1}{9}P_0\,J_1J_2\end{aligned} \tag{89}$$

We have the following expansion of $\tilde{\Omega}(u,J_1,J_2)$ w.r.t. u_0

$$\tilde{\Omega}(u,J_1,J_2) = \phi_0(u_1,J_1)\phi_0(u_2,J_2) + \phi_1(u_1,J_1)\phi_1(u_2,J_2)u_0 + \mathcal{O}(u_0^2),$$

where we define

$$\begin{aligned}\phi_0(w,\rho) \quad &:= \quad \sum_{n\geq 0}\frac{(1+3\rho)_{3n}}{(1+\rho)_n{}^3}w^n\;, \\ \phi_1(w,\rho) \quad &:= \quad \sum_{n\geq 0}\frac{(1+3\rho)_{1+3n}}{(1+\rho)_n{}^3}w^n.\end{aligned} \tag{90}$$

Note

$$\phi_1(w,\rho) = (1+3\rho)\phi_0(w,\rho) + 3w\frac{\partial}{\partial w}\phi_0(w,\rho)\,.$$

This shows that modulo u_0^2

$$\begin{aligned}&\log\tilde{\Omega}(u,J_1,J_2) \equiv \log\phi_0(u_1,J_1) \;+\; \log\phi_0(u_2,J_2) + \\ &+\left(1+3J_1+3u_1\frac{\partial}{\partial u_1}\log\phi_0(u_1,J_1)\right)\left(1+3J_2+3u_2\frac{\partial}{\partial u_2}\log\phi_0(u_2,J_2)\right)u_0\end{aligned} \tag{91}$$

Comparing (89) and (91) we see that we have proved:

Proposition 7.5 *Define for $j=1,2$ the function $\bar{U}_j$ by*

$$\log\bar{U}_j := \log u_j \;+\; \frac{\partial}{\partial\rho}\log\phi_0(u_j,\rho)|_{\rho=0}\,.$$

Then

$$\begin{aligned}\log U_j \quad &= \quad \log\bar{U}_j \;+\; \mathcal{O}(u_0) \\ \frac{1}{9}P_0 \quad &= \quad 9\left(u_1\frac{\partial}{\partial u_1}\log\bar{U}_1\right)\left(u_2\frac{\partial}{\partial u_2}\log\bar{U}_2\right)u_0 \;+\; \mathcal{O}(u_0^2)\end{aligned} \tag{92}$$

□

Before we can draw conclusions for the numbers N_{1,n_1,n_2} we must first analyse U_0 modulo u_0^2. Let

$$\tilde{\tilde{\Omega}}(u,J_0) := \sum_{m_0,m_1,m_2\geq 0}\frac{(1+J_0)_{m_0+3m_1}\cdot(1+J_0)_{m_0+3m_2}}{(1+J_0)_{m_0}{}^2\cdot m_1!^{\,3}\cdot m_2!^{\,3}}\,u_0^{m_0}u_1^{m_1}u_2^{m_2}$$

with as before $J_0^2 = 0$. Then

$$\log \tilde{\tilde{\Omega}}(u, J_0) \;=\; \log \Omega^{(0)}(u) + (\log U_0 - \log u_0)\, J_0$$

Let

$$\xi(w, \rho) := \sum_{n \geq 0} \frac{(1+\rho)_{3n}}{n!^{\,3}} w^n \tag{93}$$

Then

$$\tilde{\tilde{\Omega}}(u, J_0) = \xi(u_1, J_0) \cdot \xi(u_2, J_0) \;+\; \mathcal{O}(u_0)$$

and hence

$$U_0 = u_0 \cdot \psi(u_1) \cdot \psi(u_2) \;+\; \mathcal{O}(u_0^2) \tag{94}$$

where

$$\psi(w) := \exp\left(\frac{\partial}{\partial \rho} \log \xi(w, \rho)|_{\rho=0}\right)$$

By combining (88), (92) and (94) we find

Corollary 7.1

$$\sum_{n_1, n_2 \geq 0} N_{1,n_1,n_2} \bar{U}_1^{n_1} \bar{U}_2^{n_2} \;=\; 81 \left(\frac{1}{\psi(u_1)} u_1 \frac{\partial}{\partial u_1} \log \bar{U}_1 \right) \left(\frac{1}{\psi(u_2)} u_2 \frac{\partial}{\partial u_2} \log \bar{U}_2 \right)$$

The number N_{1,n_1,n_2} factorizes as

$$N_{1,n_1,n_2} = b_{n_1} b_{n_2} \quad ,$$

where the numbers b_n are defined by

$$\sum_{n \geq 0} b_n \bar{U}_1^n \;:=\; 9 \left(\frac{1}{\psi(u_1)} u_1 \frac{\partial}{\partial u_1} \log \bar{U}_1 \right). \tag{95}$$

□

Corollary 7.2 *Let $\{b_n\}$ be the sequence of integers defined by the expansion* (95)*. We obtain the asymptotic expansion of the B-model prepotential as follows:*

$$\Psi_B(U_0, U_1, U_2) = \textit{topological term} + U_0 B(t_1) B(t_2) + \mathcal{O}(U_0^2) \tag{96}$$

where $B(t)$ is defined by the series

$$B(t) = \sum_{n \geq 0} b_n \exp(2\pi i n t) = \sum_{n \geq 0} b_n U^n.$$

From the asymptotic expansions of (69) and (96) we obtain the following precise identity between two functions, which actually follows from the Mirror Conjecture 7.1.

Conjecture 7.2 *We will obtain the following identity*

$$\boxed{A^{res}(t) \equiv B(t)}$$

or equivalently

$$\boxed{\textstyle\sum_{n \geq 0} b_n U^n = \Theta_{E_8}(3t, t\gamma) \prod_{n \geq 1} (1 - U^{3n})^{-12}}$$

where $U = \exp(2\pi i t)$ and $\gamma = (1, 1, 1, 1, 1, 1, 1, -1)$.

Unfortunately we are unable to prove Conjecture 7.2. However since we can explicitly expand the right hand side of (95), we can obtain the expansion of $B(t)$ by using a computer and compare the result with the expansion of $A^{res}(t)$.

Proposition 7.6 *Conjecture 7.2 is true up to order* U^{50}.

To get started on the computer one may notice:

$$\phi_0(u,0)=\xi(u,0) = \sum_{n\geq 0}\frac{(3n)!}{n!^{\,3}}u^n \tag{97}$$

$$\frac{\partial}{\partial u}\phi_0(u,\rho)|_{\rho=0} = \sum_{n\geq 0}\frac{(3n)!}{n!^{\,3}}3(g(3n)-g(n))u^n \tag{98}$$

$$\frac{\partial}{\partial u}\xi(u,\rho)|_{\rho=0} = \sum_{n\geq 0}\frac{(3n)!}{n!^{\,3}}g(3n)u^n \tag{99}$$

where

$$g(n)=\sum_{k=1}^{n}\frac{1}{k}\,, \qquad g(3n)=\sum_{k=1}^{3n}\frac{1}{k}$$

A simple PARI program then yields:

$$\begin{aligned}
&B(t)=9\frac{1}{\psi(u)}u\frac{\partial}{\partial u}\log U = \\
&9+36U+126U^2+360U^3+945U^4+2268U^5+5166U^6+11160U^7 \\
&+23220U^8+46620U^9+90972U^{10}+172872U^{11}+321237U^{12} \\
&+584640U^{13}+1044810U^{14}+1835856U^{15}+3177153U^{16}+5421132U^{17} \\
&+9131220U^{18}+15195600U^{19}+25006653U^{20}+40722840U^{21} \\
&+65670768U^{22}+104930280U^{23}+166214205U^{24}+261141300U^{25} \\
&+407118726U^{26}+630048384U^{27}+968272605U^{28}+1478208420U^{29} \\
&+2242463580U^{30}+3381344280U^{31}+5069259342U^{32}+7557818940U^{33} \\
&+11208455370U^{34}+16538048640U^{35}+24282822798U^{36} \\
&+35487134928U^{37}+51626878470U^{38}+74779896240U^{39} \\
&+107861179482U^{40}+154945739844U^{41}+221711362038U^{42} \\
&+316042958880U^{43}+448856366490U^{44}+635216766732U^{45} \\
&+895854679650U^{46}+1259213600736U^{47}+1764210946995U^{48} \\
&+2463949037340U^{49}+3430694064888U^{50}+O(U^{51})
\end{aligned} \tag{100}$$

Comparing this expansion (100) with Table 2 in Section 6, we see that $a_n=b_n$ for $n\leq 50$. □

8 Appendix I: B-model equation

In this appendix we derive the equations stated in proposition 7.1 for the mirror W^* of Schoen's example W. We use the mirror construction of Batyrev-Borisov [Ba-Bo] by means of reflexive

Gorenstein cones of index 2. As explained in [Sti] the story in [Ba-Bo] about split Gorenstein cones and NEF partitions can for examples like W be reformulated in terms of triangulations of the polytope Δ on the mirror side; more specifically, W can be embedded in $\mathbf{P}^1 \times \mathbf{P}^2 \times \mathbf{P}^2$ as the intersection of a hypersurface of degree $(1,3,0)$ and a hypersurface of degree $(1,0,3)$. This leads to the matrix B in (84) and to the set $\{3,4\} \times \{5,6,7\} \times \{8,9,10\}$ in (85). To get the reflexive Gorenstein cone Λ from which the mirror of Schoen's example can be constructed one should take a 7×10 -matrix $\mathsf{A} = (a_{ij})$ with rank 7 and with integer entries such that $\mathsf{A} \cdot \mathsf{B}^t = 0$. We take

$$\mathsf{A} := \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}$$

Let $\mathsf{a}_1, \ldots, \mathsf{a}_{10} \in \mathbf{Z}^7$ be the columns of A. Then

$$\Lambda := \mathbf{R}_{\geq 0}\mathsf{a}_1 + \ldots + \mathbf{R}_{\geq 0}\mathsf{a}_{10} \ \subset \mathbf{R}^7$$

The polytope Δ is the convex hull of the points $\mathsf{a}_1, \ldots, \mathsf{a}_{10}$ in $\mathbf{R}^7$. With $\{3,4\} \times \{5,6,7\} \times \{8,9,10\}$ viewed as a collection of subsets of $\{1, \ldots, 10\}$ the complements of these 18 subsets are the index sets for the maximal simplices in a triangulation of Δ.
Let $\mathbf{S}_\Lambda$ denote the subalgebra of the algebra of Laurent polynomials

$$\mathbf{C}[u_1^{\pm 1}, \ldots, u_7^{\pm 1}]$$

generated by the monomial $u_1^{m_1} \cdot \ldots \cdot u_7^{m_7}$ with $(m_1, \ldots, m_7) \in \Lambda \cap \mathbf{Z}^7$. Giving such a monomial degree $m_1 + m_2$ makes $\mathbf{S}_\Lambda$ a graded ring. The scheme $\mathbf{P}_\Lambda := \mathrm{Proj}\mathbf{S}_\Lambda$ is a projective toric variety of dimension 6. A global section of $\mathcal{O}_{\mathbf{P}_\Lambda}(1)$ is given by a Laurent polynomial (with coefficients $v_1, \ldots, v_{10}$)

$$\begin{aligned} \mathsf{s} \;\; = \;\; & u_1(v_1 + v_5 u_4 u_5 + v_6 u_4^{-1} + v_7 u_5^{-1} + v_3 u_3) + \\ & u_2(v_2 + v_8 u_6 u_7 + v_9 u_6^{-1} + v_{10} u_7^{-1} + v_4 u_3^{-1}) \end{aligned}$$

For generic coefficients $v_1, \ldots, v_{10}$ the zero locus of s in $\mathbf{P}_\Lambda$ is a generalized Calabi-Yau manifold of dimension 5 in the sense of [Ba-Bo]. This is one mirror of W suggested by [Ba-Bo].
As in [Ba-Bo] Section 4, one can also realize a mirror as a complete intersection Calabi-Yau threefold in a 5-dimensional toric variety, as follows. $\mathbf{P}_\Lambda$ is a compactification of the torus $(\mathbf{C}^*)^7/\mathbf{C}^*$ where $\mathbf{C}^* := \{(u,u,1,1,1,1,1) \in (\mathbf{C}^*)^7\}$. The morphism

$$(\mathbf{C}^*)^7 \quad \rightarrow \quad \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^3 \times \mathbf{P}^3$$

given by

$$\begin{aligned} (u_1, \ldots, u_7) \mapsto & ([u_1 : u_1 u_3], [u_2 : u_2 u_3^{-1}], \\ & [u_1 : u_1 u_4 u_5 : u_1 u_4^{-1} : u_1 u_5^{-1}], [u_2 : u_2 u_6 u_7 : u_2 u_6^{-1} : u_2 u_7^{-1}]) \end{aligned}$$

extends to a morphism $\mathbf{P}_\Lambda \to \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^3 \times \mathbf{P}^3$. The image is

$$V := \left\{ \begin{array}{l} [p_0 : p_1] \times [q_0 : q_1] \times [s_0 : s_1 : s_2 : s_3] \times [t_0 : t_1 : t_2 : t_3] \\ \in \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^3 \times \mathbf{P}^3 \\ p_0 q_0 = p_1 q_1, \quad s_0^3 = s_1 s_2 s_3, \quad t_0^3 = t_1 t_2 t_3 \end{array} \right\}$$

As noted in [Ba-Bo] Cor.3.4 the complement of the generalized Calabi-Yau 5-fold $\mathsf{s} = 0$ in $\mathbf{P}_\Lambda$ is a $\mathbf{C}$-bundle over the complement in V of the complete intersection Calabi-Yau 3-fold with equations

$$\begin{aligned}(v_1 s_0 + v_5 s_1 + v_6 s_2 + v_7 s_3) p_0 + v_3 s_0 p_1 &= 0\\ (v_2 t_0 + v_8 t_1 + v_9 t_2 + v_{10} t_3) q_0 + v_4 t_0 q_1 &= 0\end{aligned}$$

This complete intersection Calabi-Yau 3-fold itself is another realization for a mirror of W.
Now note that the morphism

$$\mathbf{P}^1 \times \mathbf{P}^2 \times \mathbf{P}^2 \quad \to \quad \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^3 \times \mathbf{P}^3 \ ,$$

$$\begin{aligned}&([z_0 : z_1], [x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) \mapsto\\ &([z_0 : z_1], [z_1 : z_0], [x_0 x_1 x_2 : x_0^3 : x_1^3 : x_2^3], [y_0 y_1 y_2 : y_0^3 : y_1^3 : y_2^3])\end{aligned}$$

realizes V also as the quotient of $\mathbf{P}^1 \times \mathbf{P}^2 \times \mathbf{P}^2$ by the group $\mathbf{Z}_3 \times \mathbf{Z}_3$ acting as in Proposition 7.1. This completes the proof of Proposition 7.1.

9 Appendix II: The Theta function of the E_8 lattice

Let Λ be a lattice of rank d with positive definite quadratic form $Q : \Lambda \longrightarrow \mathbf{Z}$. We can fix an embedding $\Lambda \hookrightarrow \mathbf{R}^d$ such that the quadratic form Q is induced by the usual Euclidean inner product $(\ ,\ )$. Let $\mathcal{H} = \{\tau \in \mathbf{C} | \mathrm{Im}(\tau) > 0\}$ be the upper half plane. We denote by $\mathbf{w} = (w_1, \cdots, w_d)$ the standard complex coordinates of $\mathbf{C}^d = \mathbf{R}^d \otimes \mathbf{C}$. We define the theta function associated to the lattice Λ by

$$\Theta_\Lambda(\tau, \mathbf{w}) = \sum_{\sigma \in \Lambda} \exp(2\pi i((\tau/2) Q(\sigma) + (\sigma, \mathbf{w}))). \tag{101}$$

For certain Calabi-Yau 3-folds with a fibration by abelian surfaces one can calculate a part of the prepotential of the Yukawa coupling arising from the sections of the fibration by using the theta function associated to the Mordell-Weil lattice [Sa]. Since the Mordell-Weil lattice of a generic Schoen's example is isometric to $E_8 \times E_8$, we would like to calculate the theta function of E_8 and write it in an explicit form.
For that purpose we fix a standard embedding of D_8 and E_8 into $\mathbf{R}^8$. ([C-S] p. 117 $\sim$ p. 121). Let $e_1, e_2, \cdots, e_8$ be the standard orthonormal basis of $\mathbf{R}^8$. An element of $\mathbf{R}^8$ is written as $\sum_{i=1}^8 x_i e_i$. We define lattices in $\mathbf{R}^8$

$$\mathbf{Z}^8 := \left\{ \sum_{i=1}^8 x_i e_i, x_i \in \mathbf{Z} \right\} \supset D_8 := \left\{ \sum_{i=1}^8 x_i e_i \in \mathbf{Z}^8, \sum_{i=1}^8 x_i \equiv 0\,(2) \right\},$$

$$E_8 = D_8 \cup (D_8 + s_0), \quad s_0 = \frac{1}{2} \sum_{i=1}^8 e_i,$$

The inner product $(\ ,\ )$ induces positive definite bilinear forms on these lattices and E_8 and D_8 have integral bases whose intersection matrices are the Cartan matrices of E_8 and D_8 respectively. The theta function for the one dimensional lattice $\Lambda = \mathbf{Z}$ with $Q(n) = n^2$ is the Jacobi theta function:

$$\vartheta(\tau, w) := \Theta_{\mathbf{Z}}(\tau, w) = \sum_{n \in \mathbf{Z}} \exp(\pi i n^2 \tau + 2\pi i n w). \tag{102}$$

We also have the following 4 theta functions (cf. [Mum1]):

$$\begin{aligned} \vartheta_{0,0}(\tau,w) &= \vartheta(\tau,w) && (103)\\ \vartheta_{0,1}(\tau,w) &= \vartheta(\tau,w+\frac{1}{2}) && (104)\\ \vartheta_{1,0}(\tau,w) &= \exp(\frac{\pi i\tau}{4}+\pi iw)\cdot\vartheta(\tau,w+\frac{\tau}{2}) && (105)\\ \vartheta_{1,1}(\tau,w) &= \exp(\frac{\pi i\tau}{4}+\pi i(w+\frac{1}{2}))\cdot\vartheta(\tau,w+\frac{\tau+1}{2}) && (106) \end{aligned}$$

Proposition 9.1 *Let* $\mathbf{w}=(w_1,w_2,\cdots,w_8)\in\mathbf{C}^8$.

$$\begin{aligned} \Theta_{\mathbf{Z}^8}(\tau,\mathbf{w}) &= \prod_{i=1}^{8}\vartheta_{0,0}(\tau,w_i) && (107)\\ \Theta_{E_8}(\tau,\mathbf{w}) &= \frac{1}{2}\sum_{(a,b)\in(\mathbf{Z}/2\mathbf{Z})^2}\prod_{i=1}^{8}\vartheta_{a,b}(\tau,w_i) && (108) \end{aligned}$$

Proof. Straightforward exercise. See also [D-G-W]. □

Recall $\gamma=(1,1,1,1,1,1,1,-1)$. The above formulas show (cf.[Mum1]):

$$\begin{aligned} \vartheta_{0,0}(\tau,-w)=\vartheta_{0,0}(\tau,w)\,, &\qquad \vartheta_{0,1}(\tau,-w)=\vartheta_{0,1}(\tau,w)\,,\\ \vartheta_{1,0}(\tau,-w)=\vartheta_{1,0}(\tau,w)\,, &\qquad \vartheta_{1,1}(\tau,-w)=-\vartheta_{1,1}(\tau,w) \end{aligned}$$

and hence

$$\Theta_{E_8}(3t,t\gamma)=\frac{1}{2}\{\vartheta_{0,0}(3t,t)^8+\vartheta_{0,1}(3t,t)^8+\vartheta_{1,0}(3t,t)^8-\vartheta_{1,1}(3t,t)^8\} \qquad (109)$$

Next note:

$$\begin{aligned} \vartheta_{0,0}(3t,t) &= \exp(-\pi it/3)\sum_{m\equiv\pm1\,(3)}\exp(\pi itm^2/3)\\ \vartheta_{0,1}(3t,t) &= -\exp(-\pi it/3)\sum_{m\equiv\pm1\,(3)}(-1)^m\exp(\pi itm^2/3)\\ \vartheta_{1,0}(3t,t) &= \exp(-\pi it/3)\sum_{m\equiv\pm1\,(6)}\exp(\pi itm^2/12)\\ \vartheta_{1,1}(3t,t) &= -i\exp(-\pi it/3)\sum_{m\equiv\pm1\,(6)}\chi(m)\,\exp(\pi itm^2/12) \end{aligned}$$

where the summations run over $m\in\mathbf{N}$ with the indicated restrictions and $\chi(m)=1$ (resp. $=-1$) if $m\equiv\pm1 \bmod 12$ (resp. $\equiv\pm5 \bmod 12$). Another useful observation is that the Jacobi product formula for $\vartheta_{1,1}(\tau,w)$ (see [Mum1]) implies

$$\vartheta_{1,1}(3t,t)=-i\exp(-\pi it/4)\prod_{m\geq1}(1-\exp(2\pi imt))$$

Now the computer can do its work and compute the expansion of $\Theta_{E_8}(3t,t\gamma)$.

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