Cyclic Operads, Dendroidal Structures, Higher Categories

Cyclische Operaden, Boomachtige Structuren,
Hogere Categorieën

(met een samenvatting in het Nederlands)

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Andor Lukács

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Promotor: Prof. dr. I. Moerdijk

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Beoordelingscommissie: Prof. dr. C. Casacuberta
Prof. dr. G. Cornelissen
Prof. dr. K. Hess Bellwald
Prof. dr. R. Vogt

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Introduction

In this thesis we address certain questions in the general theory of operads. In the first part we study the homotopy theory of cyclic operads: we extend the work of C. Berger and I. Moerdijk [3–5] to the cyclic case. The central objects of the second part of the thesis are dendroidal sets, a generalisation of simplicial sets introduced by I. Moerdijk and I. Weiss [32] in order to study higher categories as well as weakened notions of operads. We contribute to the theory of dendroidal sets by extending the classical Dold-Kan correspondence between simplicial abelian groups and chain complexes to the dendroidal case, and by comparing the dendroidal notion of weak n-categories for $n = 2, 3$ with the respective classical notions: bicategories and tricategories.

0.1 Background of the thesis

Operads

As of today, operads are best viewed as tools for describing algebraic structures, usually in categories with a good notion of homotopy. Operads first appeared in the literature in a topological framework, in the Ph.D. thesis of R. Vogt. A particular example of an operad was also given implicitly in J. Stasheff's paper [35] on the homotopy associativity of H-spaces, and operads were used in the work of J. M. Boardman and R. Vogt [6] on homotopy invariant algebraic structures. The terminology "operad" was introduced by J. P. May and made its debut in

May's work on the classification of iterated loop spaces [31]. May in his book uses operads to catalog hierarchies of higher homotopies coming from an iterated loop space.

An important and influential aspect of topological operads appeared in [6], where Boardman and Vogt studied homotopy invariant algebraic structures. Intuitively, if a topological space X has a certain algebraic structure, then the associated homotopy invariant structure is an algebraic structure that can be naturally given to any topological space Y that is homotopy equivalent to X, and this structure is respected by the homotopy equivalence between X and Y. For example, if X is a topological monoid then any such Y will naturally be an A_∞-space. As Boardman and Vogt showed, operads are the appropriate ingredients to describe such homotopy invariant structures: usually X is an algebra over an operad P, and to obtain the corresponding homotopy invariant algebraic structure one has to replace P by a *cofibrant model* WP. The algebras over the operad WP will have the required homotopy invariance properties.

As a consequence, to describe the relation between Boardman and Vogt's cofibrant model WP and the original operad P rigourously, one has to consider the homotopy theory of operads themselves. The right context for such a consideration is abstract homotopy theory: ideally, one would want topological operads to form a model category in the sense of Quillen [34] and WP to be a cofibrant replacement of P in this model category. Berger and Moerdijk in [3, 4] showed that under certain circumstances, operads in a symmetric monoidal model category form themselves a model category, and Boardman and Vogt's W-construction can be extended to this general case.

Cyclic operads

In the last two decades, various generalisations of operads came into existence, one of them being cyclic operads. These structures can be viewed also as operads with extra structure. They were introduced in the category of chain complexes by Getzler and Kapranov [14], where the authors gave a unifying view on cyclic homology for various kinds of algebras. It turns out that all these algebras can be viewed as algebras over some cyclic operads, and the extra cyclic structure of the operad contains all the information needed to define cyclic homology for these algebras and to derive a generalised version of Connes' SBI-sequence to compute cyclic homology.

Cyclic operads were anticipated also by a theorem of M. Kontsevich [23] relating the homology of certain infinite dimensional Lie algebras to graph homology. The ideas of Kontsevich have been extensively studied in the paper of J. Conant and K. Vogtmann [10], this time with the explicit use of cyclic operads.

A natural question that arises is whether the model structure on operads, as well as the general W-construction provided by Berger and Moerdijk's work can be extended to cyclic operads.

Higher categories

The idea of working with weakened notions of categories is central in many branches of mathematics. One would often like to form certain "categories" where the composition of arrows is not strictly associative, but only up to some coherent higher cells that one would like to be part of the structure. Important examples of such structures in the literature are homotopy n-types. Roughly speaking, a homotopy n-type in topological spaces is the equivalence class of a space X such that all the homotopy groups $\pi_k(X)$ are trivial when $k > n$. These classes are taken with respect to weak homotopy. Describing algebraic models for homotopy n-types is a classical problem in algebraic topology. For $n = 2, 3$ the first such models were given by Whitehead and Mac Lane [29, 41]. Following the influence of Grothendieck and R. Brown who emphasized that groupoids should provide the natural framework for homotopy types, higher categorical algebraic models of homotopy 3-types were given and studied by Leroy [26], Joyal and Tierney [21], Berger [2].

If we work out the topological intuitions coming from the interplay between spaces, maps and homotopies between maps, we arrive to the abstract notion of bicategories, first defined by Bénabou in [1]. Bicategories are structures that consist of 0-cells (objects), 1-cells (arrows) and 2-cells; 1-cells are composable but not strictly associatively, the failure of this is measured by some natural 2-cells. We can iterate the process to arrive to a definition of tricategories and so on, the n-th step of this process would give us a notion of weak n-categories. The problem we encounter is that each step we take for defining a one-level higher notion increases radically the complexity of the necessary coherence conditions between the higher dimensional cells. These steps also diminish the intuition on the nature of the coherence axioms. As a result, there exist a plethora of different definitions of weak n-categories in the literature. Comparing the different notions of weak n-categories is one of the main problems in higher category theory. One of the issues can be formulated as follows. Roughly speaking, the right place to compare two notions of weak n-categories would be inside a weak $n+1$-category, but how do we decide which notion of weak $n+1$-categories to use for this comparison?

To deal with these problems, one can consider stricter- or non-iterative approaches to define weak n-categories. The idea is that the resulting stricter notions should be enough to deal with the applications on the one hand, and the slogan is that "*weak n-categories are strictifiable up to some extent*" on the other. Examples of this approach include Baez and Dolan's notion of (∞, n)-categories, Tamsamani categories, etc. We would like to mention explicitly one such example, originating in the observation that the category of categories embeds to simplicial sets, via the nerve functor:

$$N \colon \mathcal{C}at \longrightarrow s\mathcal{S}ets.$$

Certain simplicial sets which are not in the image of the nerve functor behave much like categories, except that "composition" of arrows is well defined only up to some higher degree terms in that simplicial set. A. Joyal in [20] called these simplicial sets quasi-categories, although the notion was already introduced by Boardman and Vogt under the name of restricted Kan complex in [6]. A quasi-category is an $(\infty, 1)$-category in Baez and Dolan's sense, i.e. all the degree 2- or higher cells are invertible. An important fact about quasi-categories is that they are exactly the fibrant objects in the Joyal model structure for the category of simplicial sets.

Dendroidal sets

Operads (or rather coloured operads) can be viewed as generalisations of categories, where we consider arrows that can have multiple inputs as opposed to one. This approach opens a new way to study operads in the context of homotopy theory: for example, one can ask if there exists a presheaf category that extends the category of operads in the same way as simplicial sets extend the category of categories via the nerve functor. The question was studied in the papers of Moerdijk and Weiss [32, 33]: the category of dendroidal sets satisfies the requirements and fits in a commutative diagram of categories

$$\begin{tikzcd} \mathcal{C}at \arrow[r, "N"] \arrow[d, tail] & sSets \arrow[d, tail] \\ \mathcal{O}p \arrow[r, "N_d"] & dSets \end{tikzcd}$$

Since dendroidal sets are an extension of simplicial sets, suitable for studying the homotopy theory of operads, the theory of dendroidal sets inherits a lot of questions from the theory of simplicial sets. For example, this extension allows us to consider quasi-operads in the category of dendroidal sets, i.e. analogs of Joyal's quasi-categories. One can then ask whether the Joyal model structure on the category of simplicial sets extends to that of dendroidal sets in such a way, that the fibrant objects of this model category are exactly the quasi-operads. Cisinski and Moerdijk in [9] gave a positive answer to this question. Another important problem influenced by simplicial sets that is still open is to find a dendroidal analog of the geometric realization of simplicial sets.

One nice feature of dendroidal sets, observed in [32], is that they contribute to the theory of higher categories with a new compact definition of weak n-categories.

0.2 Outline of the thesis

The thesis is organised as follows:

Chapter 1 introduces operads and cyclic operads in an unorthodox manner: we consider a coordinate-free description of these notions. This point of view is not necessary for the results we later obtain, but often it simplifies notation and gives a clearer picture of what is going on in the underlying combinatorics. We prove in Theorem 1.2.7 that our approach is equivalent to the classical one. In the second part of the chapter we discuss certain free constructions, coloured operads and their algebras and give those examples of algebras that are going to be used later in the thesis.

In Chapter 2 we prove the existence of a model structure on the category of cyclic operads in a symmetric monoidal model category, under certain assumptions. A model structure is obtained in two separate cases. Theorem 2.1.6 considers the case when the cyclic operads in question are reduced, i.e. they contain no nullary operations:

Theorem 2.1.6. *Suppose that $\mathcal{E}$ is a symmetric monoidal model category with unit I and the following assumptions are satisfied:*

- *$\mathcal{E}$ is cofibrantly generated and I is cofibrant;*
- *$\mathcal{E}$ has a symmetric monoidal fibrant replacement functor;*
- *$\mathcal{E}$ admits a commutative Hopf interval.*

Then there is a cofibrantly generated model structure on the category $\mathcal{C}yc\mathcal{O}p$ of reduced cyclic operads in $\mathcal{E}$. This model structure is induced by the free-forgetful adjunction between the category of reduced cyclic collections in $\mathcal{E}$ and the category $\mathcal{C}yc\mathcal{O}p$.

Our method follows closely that of Berger and Moerdijk in [3]: we use the transfer principle to transport the model structure from the category of cyclic collections. The second case we consider is that of unreduced cyclic operads, the main theorem is stated in Theorem 2.1.8. In this case we need stronger assumptions for the transfer to work. An alternative proof can also be given for Theorem 2.1.8, in view of the constructions of algebras over coloured operads from Chapter 1: we can use a theorem due to Berger and Moerdijk [5] that establishes model structures on certain categories of algebras over an operad.

In the second half of the chapter we construct a cofibrant resolution of cyclic operads that extends the general Boardman-Vogt W-construction of Berger and Moerdijk to the cyclic case. A particular instance of this construction, when the underlying category is the category of differential graded vector spaces is considered as an example. In this case the W-construction can be considered as an extension of the classical bar-cobar resolution of operads to the cyclic case.

Chapter 3 is a short introduction to dendroidal sets. This chapter also contains the necessary definitions and terminology that will be used in the next chapters, when we address some questions about dendroidal sets. In Section 3.2 we analyse in detail the dendroidal identities, which are analogs of the simplicial identities. There are two possible approaches to obtain these dendroidal identities: the coordinate-free approach is the one we use frequently later on in the thesis, especially in Chapter 4. In Subsection 3.2.4 we observe that the planar structure on the objects of the category Ω^{π} gives rise to a non-coordinate-free description of the dendroidal identities. This second approach is more closely related to the classical simplicial identities, and we compare the resulting dendroidal relations with the simplicial ones in the end of Subsection 3.2.4.

In Chapter 4 we provide a dendroidal analog of the classical Dold-Kan correspondence. Recall that the classical Dold-Kan correspondence establishes an equivalence between the category of simplicial abelian groups and the category of positively graded chain complexes. We construct a category of "planar dendroidal complexes” that is equivalent to the category of planar dendroidal abelian groups and fits into a commutative diagram

$$\begin{tikzcd} s\mathcal{A}b \arrow[r, shift left, "i_!"] \arrow[d, shift right, "N_s"'] & pd\mathcal{A}b \arrow[l, shift left, "i^*"] \arrow[d, shift right, "N"'] \\ \mathcal{C}h \arrow[r, shift left, "j_!"] \arrow[u, shift right, "\Gamma_s"'] & pd\mathcal{C}omp \arrow[l, shift left, "j^*"] \arrow[u, shift right, "\Gamma"'] \end{tikzcd}$$

where $pd\mathcal{A}b$ denotes the category of planar dendroidal abelian groups, $pd\mathcal{C}omp$ is the category of planar dendroidal complexes, the functor N is the dendroidal generalisation of the simplicial normalized chain complex functor N_s and the horizontal functors are inclusions and restrictions, respectively. The main result of the chapter is stated in

Theorem 4.2.3. *The functors $N\colon pd\mathcal{A}b \longrightarrow pd\mathcal{C}omp$ and $\Gamma\colon pd\mathcal{C}omp \longrightarrow pd\mathcal{A}b$ form an equivalence of categories. Moreover, the following relations hold:*

$$\begin{array}{rclcrcl} N_s i^* & = & j^* N & \qquad & N i_! & = & j_! N_s \\ \Gamma_s j^* & = & i^*\Gamma & & \Gamma j_! & = & i_! \Gamma_s \\ i^* i_! & = & \mathrm{id} & & j^* j_! & = & \mathrm{id} \\ N\Gamma & = & \mathrm{id} & & N_s\Gamma_s & = & \mathrm{id} \\ \Gamma N & \cong & \mathrm{id} & & \Gamma_s N_s & \cong & \mathrm{id}\,. \end{array}$$

In Chapter 5 we compare the classical notions of bicategories and tricategories with dendroidal weak 2- and 3-categories. In the case of dendroidal weak 2-categories and bicategories the comparison is categorical: there is an obvious notion of the quasi-category of dendroidal weak 2-categories, denoted in the thesis by $i^*(w\mathcal{C}at^2)$, which we prove to have equivalent homotopy category to the category of bicategories.

Theorem 5.3.9. *The category of unbiased bicategories $ubi\mathcal{C}tg$ and $ho(i^*(w\mathcal{C}at^2))$ are isomorphic. Hence the category of classical bicategories is equivalent to the category of dendroidal weak 2-categories.*

To treat the case in degree 3 the situation is a bit different: on the one hand, there is again an obvious notion of the quasi-category of dendroidal weak 3-categories, but the objects of this quasi-category are stricter than general tricategories. By the work of Gordon, Power and Street [16] every tricategory is triequivalent to a Gray-category, hence for practical reasons it is enough to consider Gray-categories instead of general tricategories. We introduce another semistrict notion of tricategories that form a category $\mathcal{T}ricat_1$ with strict trihomomorphisms as maps. We prove that

Theorem 5.4.6. *The category $ho(i^*(w\mathcal{C}at^3))$ is equivalent to $\mathcal{T}ricat_1$.*

1

Operads and cyclic operads

In the first chapter of the thesis we are going to introduce operads and cyclic operads. Our approach is somewhat different from the usual one appearing in the literature, in that we are considering a coordinate-free description. *The first four sections deal with the definitions and also indicate the equivalence of our approach with the standard ones. In Section 1.5 we describe the various free constructions for cyclic operads, which generalise the existing ones for operads. These free constructions provide the backbone when dealing with the model structure and Boardman-Vogt resolution of cyclic operads in Chapter 2. In Section 1.6 we introduce coloured operads and their algebras. The key part of this section is the description of the coloured operad whose algebras are cyclic operads, since this allows us to give a different proof for the existence of a model structure on the category of cyclic operads, in view of [5].*

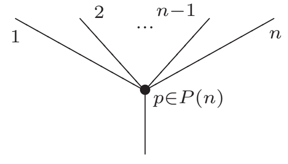
1.1 Preliminaries

1.1.1 An intuitive description of operads and cyclic operads

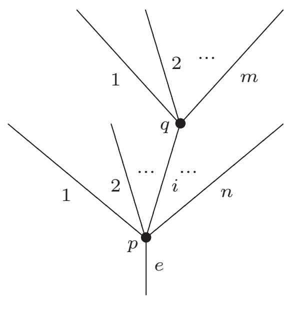
An operad in the category of vector spaces over a field $\Bbbk$ consists of a space $P(n)$ together with a right action of the symmetric group Σ_n on $P(n)$ for every $n \in \mathbb{N}$, an identity element $\mathrm{id} \in P(1)$ and linear composition maps

$$\circ_i \colon P(n) \otimes P(m) \longrightarrow P(n+m-1), \quad i = 1, 2, \ldots, n$$

for all n and m. The nature of the axioms this data has to satisfy comes from the intuition that the space $P(n)$ is thought of as a space of operations with n inputs and one output:



The action of the groups Σ_n permutes the inputs and the composition $p \circ_i q$ of two operations gives a new operation by using the output of q as the i-th input of p. This new operation can be visualised as grafting the tree for q on the i-th leaf of the tree for p:



The unit $\mathrm{id} \in P(1)$ can be thought of as an operation which takes one input and gives it back as output.

The axioms that the operad P has to satisfy are the formal consequences of the above intuition. In fact, the intuition can be made to a rigourous example of an operad: for any vector space V define

$$\mathrm{End}_V(n) := \mathcal{V}ect_{\Bbbk}(\underbrace{V \otimes \cdots \otimes V}_{n \text{ times}}, V)$$

and follow the description above to define the rest of the structure. This operad is called the *endomorphism operad on V*. It has a prominent role in the theory of operads not only because it models the abstract definition of operads, but also because it can "realize" on the space V the algebraic structure encoded in an operad P. To be more precise, note that any map of operads $\alpha\colon P \longrightarrow \mathrm{End}_V$ takes an "abstract" n-ary operation of $P(n)$ to a "concrete" n-ary operation $V\otimes\cdots\otimes V \longrightarrow V$ and the various compatibility conditions for α impose algebraic relations between these concrete operations on the End_V side. For particular operads in $\mathcal{V}ect_{\Bbbk}$ one can describe in this way various kinds of $\Bbbk$-algebras (e.g. associative, commutative, Lie, Poisson, Leibnitz, etc). This provides a justification for the following terminology: in the literature a vector space V together with an operad map $\alpha\colon P \longrightarrow \mathrm{End}_V$ is called a *P-algebra*.

A rigourous definition, along the lines of the intuition given above, can be found in [30], although the standard definition (the original one by J.P. May in [31]) differs from our approach. The usual definition collects the $\circ_i$ composition maps for a given $P(n)$, $i = 1, 2, \ldots, n$ under one big composition map

$$P(n) \otimes \big(P(m_1) \otimes P(m_2) \otimes \cdots \otimes P(m_n)\big) \xrightarrow{\ \gamma\ } P(m_1 + m_2 + \cdots + m_n)\ .$$

An $n+1$-tuple of operations $(p, q_1, q_2, \ldots, q_n)$ is sent by γ to a new operation which we usually write as $p(q_1, q_2, \ldots, q_n)$ and visualise as n trees corresponding to the operations q_i, grafted upon the leaves of the tree corresponding to the operation p. The equivalence of the two definitions follows from the existence of the unit-operation $\mathrm{id} \in P(1)$. For example, the operation

$$\circ_i\colon P(n) \otimes P(m) \longrightarrow P(n + m - 1)$$

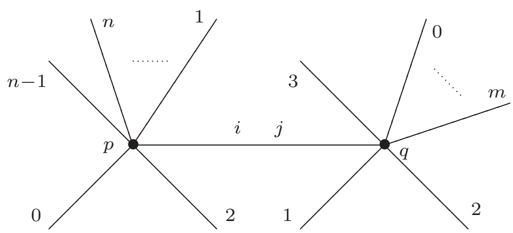
can be obtained from

$$\gamma\colon P(n) \otimes \big(P(1) \otimes \cdots P(1) \otimes \underset{i\text{-th}}{P(m)} \otimes P(1) \otimes \cdots \otimes P(1)\big) \longrightarrow P(m + n - 1).$$

Cyclic operads are on the one hand generalisations of operads, on the other hand they can be viewed as operads with extra structure. If you take the first viewpoint, a cyclic operad in $\mathcal{V}ect_{\Bbbk}$ consists of spaces $P(n)$, the elements of which are thought of as operations with n inputs and one output, but the output cannot be clearly distinguished from the inputs. If one wants to visualise such an operation, it should be done with trees without a distinguished edge, and the labels can vary from 0 to n. The composition maps take the form

$${}_i\circ_j\colon P(n) \otimes P(m) \longrightarrow P(n + m - 1),\quad i = 0, 1, \ldots n \text{ and } j = 0, 1, \ldots m$$

and in the composite $p\,{}_i\circ_j\,q$ the index j indicates that the leaf with label j of the tree for q is considered as the output:



Each space $P(n)$ is equipped with a right action of the symmetric group of the set $\{0, 1, \ldots, n\}$ which is isomorphic to Σ_{n+1} and there is a unit $\mathrm{id} \in P(1)$. This structure is subject to certain axioms originating again from the intuitive approach.

We can describe cyclic operads also as operads with extra structure. To do so, first we observe that if P is a cyclic operad in the sense given above, the operations $\circ_i := {}_i\circ_0$ and the restrictions of the Σ_{n+1} actions to Σ_n ones make P an operad. The axioms that the cyclic operad P has to satisfy induce some extra conditions on P, and a minimal set of conditions which completely describe P as a cyclic operad can be thought of as an alternative definition of cyclic operads (see [30] for these conditions).

In the following subsections we are going to introduce the key ingredients to our approach of studying operads and cyclic operads.

1.1.2 The unordered tensor product

Since we want to define operads and cyclic operads in a coordinate-free manner, it is useful to have a notion of a coordinate-free or unordered tensor product. Such a tensor product can be defined in arbitrary symmetric monoidal categories. To motivate our objective, first consider a (non-symmetric) monoidal category $(\mathcal{E}, \otimes, I, a, l, r)$. It is immediate that there is no need to bracket multiple tensor products like $A_1 \otimes \cdots \otimes A_n$ in $\mathcal{E}$, because the associativity constraint a provides a unique natural isomorphism between any two possible bracketing. If $\mathcal{E}$ is also symmetric with respect to a natural isomorphism $s_{AB} \colon A \otimes B \longrightarrow B \otimes A$, one can make sense of the unordered tensor product in $\mathcal{E}$:
Suppose that X is a finite set and $(A_x)_{x \in X}$ are objects of $\mathcal{E}$. Intuitively, we can think of $\bigotimes\limits_{x \in X} A_x$ as an ordered tensor with a chosen ordering of X, i.e. a bijection $f \colon X \longrightarrow \{0, 1, 2, \ldots, n\} = [n]$. If $g \colon X \longrightarrow [n]$ is another ordering, then there is a unique way to go from f to g, with a natural isomorphism expressed with the various s maps. Driven by this intuition, one can define the unordered tensor product $\bigotimes\limits_{x \in X} A_x$ rigourously as the coequalizer of all the maps

$$\tilde{\sigma} \colon \bigoplus_{f : X \to [n]} \bigotimes_f A_f \longrightarrow \bigoplus_{f : X \to [n]} \bigotimes_f A_f$$

where $\bigotimes\limits_f A_f := A_{f^{-1}(0)} \otimes \cdots \otimes A_{f^{-1}(n)}$ and $\sigma \in \Sigma_{[n]}$ induces by symmetry the maps $\widetilde{\sigma_f} \colon \bigotimes\limits_f A_f \longrightarrow \bigotimes\limits_{\sigma f} A_{\sigma f}$. This colimit is in fact any ordered tensor $\bigotimes\limits_f A_f$ together with the chosen order f.

1.1.3 The categories $\mathcal{F}in_*$ and $\mathcal{F}in$

If one examines the standard definition of operads, one can see that the role of the natural numbers in the definition is to keep track of the arity of the abstract operations as well as to label the inputs of these operations. This approach has certain disadvantages which becomes apparent when we compose two abstract operations. For example, to get the labels of the resulting operation right, one has to adjust the labels of the composed operations accordingly. This adjustment gives rise to not wanted technicalities when proving that something is an operad: one will need to use block permutations to prove equivariance and associativity for example.

A possible remedy to this problem can be given by labeling our operations in P with finite sets, and when a composition occurs just take the disjoint union of the reoccurring labels for the new operation. This approach is not new, it has been used in the past for example by V. Hinich and A. Vaintrob in [18]. They credit P. Deligne and J. S. Milne for the formalism (see [11]). The coordinate-free approach to operads was used also by P. van der Laan in his thesis [38].

Denote by $\mathcal{F}in_*$ the category of finite pointed sets (X, x_0) and basepoint-preserving bijections. To any ordered pair $((X, x_0), (Y, y_0)) \in \mathcal{F}in_* \times \mathcal{F}in_*$ and $x \in X$, $x \neq x_0$ we render $(X \sqcup_x Y, x_0) \in \mathcal{F}in_*$, defined as

$$X \sqcup_x Y = X \sqcup Y \setminus \{x, y_0\}.$$

The following properties of the $\sqcup_x$ operations are going to be important for the definition of operads:

Associativity. If $(X, x_0), (Y, y_0), (Z, z_0) \in \mathcal{F}in_*$ and $x, x' \in X$, $y \in Y$ such that $x_0 \neq x \neq x' \neq x_0$ and $y \neq y_0$ then

$$(X \sqcup_x Y) \sqcup_y Z = X \sqcup_x (Y \sqcup_y Z),$$
$$(X \sqcup_x Y) \sqcup_{x'} Z = (X \sqcup_{x'} Z) \sqcup_x Y.$$

Equivariance. If $\sigma\colon (X, x_0) \longrightarrow (X', x'_0)$ and $\tau\colon (Y, y_0) \longrightarrow (Y', y'_0)$ are maps in $\mathcal{F}in_*$ and $x \in X$, $x \neq x_0$ then σ and τ induce a map

$$\sigma \circ_x \tau\colon (X \sqcup_x Y, x_0) \longrightarrow (X' \sqcup_{\sigma(x)} Y', x'_0)$$

in $\mathcal{F}in_*$, defined as

$$\sigma \circ_x \tau|_{X \setminus \{x\}} = \sigma|_{X \setminus \{x\}},$$
$$\sigma \circ_x \tau|_{Y \setminus \{y_0\}} = \tau|_{Y \setminus \{y_0\}}.$$

Unit. For any pointed set with two elements $(U, u_0) = (\{u, u_0\}, u_0)$ and any other pointed set (X, x_0) together with an element $x \in X \setminus \{x_0\}$ there are maps

$$e_{ux_0}\colon (X, x_0) \longrightarrow (U \sqcup_u X, u_0) \quad \text{and} \quad e_{ux}\colon (X, x_0) \longrightarrow (X \sqcup_x U, x_0),$$

where e_{ux_0} sends x_0 to u_0 and is the identity elsewhere, and e_{ux} sends x to u and is the identity elsewhere.

To define cyclic operads we will use the category $\mathcal{F}in$, which has finite sets as objects and bijections as arrows. For an ordered pair of sets $(X, Y) \in \mathcal{F}in \times \mathcal{F}in$ and elements $x \in X, y \in Y$ we define

$$X\,{}_x{\sqcup}_y\,Y := X \sqcup Y \setminus \{x, y\}.$$

The ${}_x{\sqcup}_y$ operations have similar properties to the pointed-set versions $\sqcup_x$.

1.2 Operads

1.2.1 Definitions and examples

Let $(\mathcal{E}, \otimes, I, a, l, r, s)$ be a symmetric monoidal category.

Definition 1.2.1. A contravariant functor $P\colon \mathcal{F}in_*^{\mathrm{op}} \longrightarrow \mathcal{E}$ is called a *collection* or a $\mathcal{F}in_*$*-module* in $\mathcal{E}$.

If P is a collection in $\mathcal{E}$ then for any map $\sigma\colon (X, x_0) \longrightarrow (X', x_0')$ in $\mathcal{F}in_*$ the induced map $P(\sigma)\colon P(X', x_0') \longrightarrow P(X, x_0)$ can be considered as acting on the right on $P(X', x_0')$. We will write instead of $P(\sigma)$ just σ and if there are elements $u \in P(X', x_0')$, we will write $u\sigma$ instead of $P(\sigma)(u) \in P(X, x_0)$.

Definition 1.2.2. An *operad* in $\mathcal{E}$ is a collection $P\colon \mathcal{F}in_*^{\mathrm{op}} \longrightarrow \mathcal{E}$ with structure maps

$$\circ_x\colon P(X, x_0) \otimes P(Y, y_0) \longrightarrow P(X \sqcup_x Y, x_0)$$

for any (X, x_0), $(Y, y_0) \in \mathcal{F}in_*$ and $x \in X$, $x \neq x_0$, which satisfy the following three conditions:

Associativity. For any (X, x_0), (Y, y_0) and $(Z, z_0) \in \mathcal{F}in_*$, and any $x, x' \in X$, $y \in Y$ such that $x_0 \neq x \neq x' \neq x_0$ and $y \neq y_0$ the following diagrams commute:

$$\begin{tikzcd}
P(X, x_0) \otimes P(Y, y_0) \otimes P(Z, z_0) \arrow[r, "\circ_x \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes \circ_y"'] & P(X \sqcup_x Y, x_0) \otimes P(Z, z_0) \arrow[d, "\circ_y"] \\
P(X, x_0) \otimes P(Y \sqcup_y Z, y_0) \arrow[r, "\circ_x"] & P(X \sqcup_x Y \sqcup_y Z, x_0)
\end{tikzcd}$$

$$\begin{tikzcd}
P(X, x_0) \otimes P(Y, y_0) \otimes P(Z, z_0) \arrow[r, "\circ_x \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes s"'] & P(X \sqcup_x Y, x_0) \otimes P(Z, z_0) \arrow[dd, "\circ_{x'}"] \\
P(X, x_0) \otimes P(Z, z_0) \otimes P(Y, y_0) \arrow[d, "\circ_{x'} \otimes \mathrm{id}"'] & \\
P(X \sqcup_{x'} Z, x_0) \otimes P(Y, y_0) \arrow[r, "\circ_x"] & P(X \sqcup_x Y \sqcup_{x'} Z, x_0)
\end{tikzcd}$$

where $s : P(Y, y_0) \otimes P(Z, z_0) \longrightarrow P(Z, z_0) \otimes P(Y, y_0)$ is the symmetry of $\mathcal{E}$.

Equivariance. For any $\sigma\colon (X, x_0) \longrightarrow (X', x_0')$, $\tau\colon (Y, y_0) \longrightarrow (Y', y_0')$ maps in $\mathcal{F}in_*$ and $x \in X$, $x \neq x_0$ the following diagram commutes:

$$\begin{tikzcd}
P(X', x_0') \otimes P(Y', y_0') \arrow[r, "\circ_{\sigma(x)}"] \arrow[d, "\sigma \otimes \tau"'] & P(X' \sqcup_{\sigma(x)} Y', x_0') \arrow[d, "\sigma \circ_x \tau"] \\
P(X, x_0) \otimes P(Y, y_0) \arrow[r, "\circ_x"] & P(X \sqcup_x Y, x_0)
\end{tikzcd}$$

If there are elements, this property translates to

$$u\sigma \circ_x v\tau = (u \circ_{\sigma(x)} v)(\sigma \circ_x \tau)$$

for any $u \in P(X', x_0')$ and $v \in P(Y', y_0')$.

Unit. For any set with two elements $(U, u_0) = (\{u, u_0\}, u_0) \in \mathcal{F}in_*$ there is a map $\eta_{(U,u_0)} : I \longrightarrow P(U, u_0)$, for which the compositions

$$I \otimes P(X, x_0) \xrightarrow{\eta_U \otimes \mathrm{id}} P(U, u_0) \otimes P(X, x_0) \xrightarrow{\circ_u} P(U \sqcup_u X, u_0) \xrightarrow{e_{ux_0}} P(X, x_0)\,,$$

$$P(X, x_0) \otimes I \xrightarrow{\mathrm{id} \otimes \eta_U} P(X, x_0) \otimes P(U, u_0) \xrightarrow{\circ_x} P(X \sqcup_x U, x_0) \xrightarrow{e_{ux}} P(X, x_0)$$

are the left and right identities in the monoidal category $\mathcal{E}$ for any $(X, x_0) \in \mathcal{F}in_*$.

The following diagram commutes for any two-point sets (X, x_0) and (X', x_0'):

$$\begin{tikzcd}
I \arrow[r, "\eta_X"] \arrow[d, equal] & P(X, x_0) \arrow[d, "\alpha"] \\
I \arrow[r, "\eta_{X'}"] & P(X', x_0')
\end{tikzcd}$$

where $\alpha : (X', x_0') \longrightarrow (X, x_0)$ is the obvious (unique) map.

A collection P is called a *pseudo operad* if it satisfies only the associativity and equivariance conditions.

Definition 1.2.3. Let P and Q be operads in $\mathcal{E}$. A *morphism of operads* $\mu\colon P \longrightarrow Q$ is an equivariant natural transformation from P to Q which is compatible with all the operations $\circ_x$ and unit maps η_U. Explicitly, such a μ is a family of maps $\mu_{(X,x_0)}\colon P(X, x_0) \longrightarrow Q(X, x_0)$, such that the following diagrams commute for any possible choice of x, σ and η:

$$\begin{tikzcd}
P(X', x_0') \arrow[r, "\mu_{X'}"] \arrow[d, "\sigma"'] & Q(X', x_0') \arrow[d, "\sigma"] \\
P(X, x_0) \arrow[r, "\mu_X"] & Q(X, x_0)
\end{tikzcd}
\qquad
\begin{tikzcd}
P(X, x_0) \otimes P(Y, y_0) \arrow[r, "\circ_x"] \arrow[d, "\mu_X \otimes \mu_Y"'] & P(X \sqcup_x Y, x_0) \arrow[d, "\mu_{X \sqcup_x Y}"] \\
Q(X, x_0) \otimes Q(Y, y_0) \arrow[r, "\circ_x"] & Q(X \sqcup_x Y, x_0)
\end{tikzcd}$$

$$\begin{tikzcd}
I \arrow[r, "\eta_U"] \arrow[d, equal] & P(U, u_0) \arrow[d, "\mu_U"] \\
I \arrow[r, "\eta_U"] & Q(U, u_0)
\end{tikzcd}$$

With these maps, operads in $\mathcal{E}$ form the category $\mathcal{O}p$.

Example 1.2.4. One can define an *endomorphism operad* in $\mathcal{V}ect_{\Bbbk}$ for any vector space V: for a pointed set (X, x_0) let

$$\mathrm{End}_V(X, x_0) := \underline{\mathcal{V}ect_{\Bbbk}}(V^{\otimes X\setminus\{x_0\}}, V)$$

where $V^{\otimes X\setminus\{x_0\}}$ denotes the unordered tensor product. The right action of the pointed bijection $\sigma : (X', x_0') \longrightarrow (X, x_0)$ permutes the factors of the tensor by precomposing the orderings $f : X \setminus \{x_0\} \longrightarrow \{1, 2, \dots, |X \setminus \{x_0\}|\}$ with σ. For $f \in \mathrm{End}_V(X, x_0)$, $g \in \mathrm{End}_V(Y, y_0)$ the composition $f \circ_x g \in \mathrm{End}_V(X \sqcup_x Y, x_0)$ is defined by

$$f \circ_x g := f \circ (\mathrm{id}^{\otimes X\setminus\{x,x_0\}} \otimes g).$$

The term in the bracket of the previous formula is an unordered tensor product of maps, over the set $X \setminus \{x_0\}$, where g is the map corresponding to the label x.

One can generalise this construction to define an endomorphism operad associated to an object in any closed symmetric monoidal category $\mathcal{E}$.

Example 1.2.5. Let $\mathcal{E}$ be the category $\mathcal{S}ets$. The *commutative operad* Com is defined by setting $Com(X, x_0) := *$ for every pointed set (X, x_0) with at least two elements and $Com(\{x_0\}, x_0) := \emptyset$. Note that here $*$ stands for a fixed set with one element.

Example 1.2.6. It is shorter to define the *associative operad* in $\mathcal{S}ets$ with the use of the classical definition of operads. Let $As(0) := \emptyset$, $As(n) := \Sigma_n$ and define the right actions of Σ_n on $As(n)$ by composition. For $\sigma \in As(n)$ and $\tau \in As(m)$ let $\sigma \circ_i \tau \in As(m + n - 1)$,

$$\sigma \circ_i \tau(k) := \begin{cases} \sigma(k) & \text{if } \sigma(k) < \sigma(i), \\ \tau(k - i + 1) + \sigma(i) & \text{if } i \le k < i + m - 1, \\ \sigma(k) + m - 1 & \text{if } \sigma(k) > \sigma(i). \end{cases}$$

The intuition behind this definition is that σ acts on n blocks of length 1, except of the i-th block which is of length m. This i-th block thus starts at i and ends at $i + m - 1$, and before applying σ to it, we act by τ.

1.2.2 Equivalence with the classical definition

In this subsection we are going to prove that the category of operads arising from our definition is equivalent to the classical one, given in terms of the $\circ_i$ operations (see [30], pp. 46) which in turn is equivalent to the original definition of May [31].

In the following we are going to denote the pointed set $(\{0, 1, \dots, n\}, 0) \in \mathcal{F}in_*$ by $\langle n\rangle$. Instead of $P(\langle n\rangle)$ let us write $P(n)$. If P is an operad in $\mathcal{E}$ then any composition map $\circ_x : P(X, x_0) \otimes P(Y, y_0) \longrightarrow P(X \sqcup_x Y, x_0)$ gives rise to a new one $\circ_i : P(n) \otimes P(m) \longrightarrow P(\langle n\rangle \sqcup_i \langle m\rangle)$ via the actions of some pointed bijections $\sigma : \langle n\rangle \longrightarrow (X, x_0)$ with $\sigma(i) = x$ and $\tau : \langle m\rangle \longrightarrow (Y, y_0)$, because of the equivariance condition:

$$\circ_x = (\sigma \circ_i \tau)^{-1}(\circ_i)(\sigma \otimes \tau)).$$

This property suggests to study more the structures induced by the operad axioms on the objects $P(n)$. Define the *renumbering map* $\varphi_i : \langle n+m-1\rangle \longrightarrow \langle n\rangle \sqcup_i \langle m\rangle$,

$$\varphi_i(k) := \begin{cases} k \in \langle n\rangle & \text{if } k < i, \\ (k-m+1) \in \langle n\rangle & \text{if } k > i+m-1, \\ (k-i+1) \in \langle m\rangle & \text{if } i \leq k \leq i+m-1. \end{cases} \tag{1.2.1}$$

We can infer that the composition of φ_i with $\circ_i$ defines a new operation, denoted by $\bullet_i$ which is written only in terms of the sets $\langle n\rangle$:

$$\bullet_i := \varphi_i \circ_i : P(n) \otimes P(m) \longrightarrow P(n+m-1).$$

These $\bullet_i$ operations satisfy the associativity and equivariance conditions of [30] as well as the unit conditions of the classical definition. Let us denote temporarily our category of operads with $\mathcal{O}p_{\mathcal{F}in_*}$ and the classical one by $\mathcal{O}p_\Sigma$. In the latter case Σ denotes the skeleton of $\mathcal{F}in_*$ formed by the pointed sets $(\{0,1,2,\ldots,n\},0)$ and an object of $\mathcal{O}p_\Sigma$ is a functor $\Sigma^{\mathrm{op}} \longrightarrow \mathcal{E}$ with extra structure.

Theorem 1.2.7. *The categories $\mathcal{O}p_{\mathcal{F}in_*}$ and $\mathcal{O}p_\Sigma$ are equivalent.*

Proof. The restriction functor $R\colon \mathcal{F}in_* \longrightarrow \Sigma$ together with the extension functor $E\colon \Sigma \longrightarrow \mathcal{F}in_*$ form an equivalence of categories and $RE = \mathrm{id}$. We denote by $E^\# : \mathcal{E}^{\mathcal{F}in_*} \longrightarrow \mathcal{E}^\Sigma$ and $R^\# : \mathcal{E}^\Sigma \longrightarrow \mathcal{E}^{\mathcal{F}in_*}$ the induced functors.

A collection $P : \mathcal{F}in_* \longrightarrow \mathcal{E}$ defines a $\mathcal{F}in_*$-operad if and only if $E^\#(P)$ is a Σ-operad. Moreover, $\mu : P \longrightarrow Q$ is a map of $\mathcal{F}in_*$-operads if and only if $E^\#(\mu)$ is a map of Σ-operads.

For any Σ-operad P' we have that $E^\# R^\#(P') = P'$, thus $R^\#(P')$ is an operad and we can conclude that $E^\#$ induces an essentially surjective functor between the operad categories. On the other hand, $E^\#$ is also fully faithful, hence the induced functor is also fully faithful.

We infer that $\mathcal{O}p_{\mathcal{F}in_*}$ and $\mathcal{O}p_\Sigma$ are equivalent. ♦

As a consequence, we will not distinguish any more between the two categories of operads, and use the one which is at the moment more convenient for our purposes.

1.3 Cyclic operads

Let $(\mathcal{E}, \otimes, I, a, s, l, r)$ be a symmetric monoidal category. Recall that we denote by $\mathcal{F}in$ the category of finite sets and isomorphisms.

Definition 1.3.1. A *cyclic collection* in the category $\mathcal{E}$ is a contravariant functor $P\colon \mathcal{F}in^{\mathrm{op}} \longrightarrow \mathcal{E}$.

Definition 1.3.2. A *cyclic operad* P is a cyclic collection $P : \mathcal{F}in^{\mathrm{op}} \longrightarrow \mathcal{E}$ with structure maps

$$_x\circ_y : P(X) \otimes P(Y) \longrightarrow P(X{}_x\sqcup_y Y)$$

for any pair $(x, y) \in X \times Y$, satisfying the following properties:

Associativity. The following diagrams commute

$$\begin{tikzcd}
P(X) \otimes P(Y) \otimes P(Z) \arrow[r, "{_x\circ_y} \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes {_{y'}\circ_z}"'] & P(X{}_x\sqcup_y Y) \otimes P(Z) \arrow[d, "{_{y'}\circ_z}"] \\
P(X) \otimes P(Y{}_{y'}\sqcup_z Z) \arrow[r, "{_x\circ_y}"] & P(X{}_x\sqcup_y Y{}_{y'}\sqcup_z Z)
\end{tikzcd}$$

$$\begin{tikzcd}
P(X) \otimes P(Y) \otimes P(Z) \arrow[r, "{_x\circ_y} \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes s"'] & P(X{}_x\sqcup_y Y) \otimes P(Z) \arrow[dd, "{_{x'}\circ_z}"] \\
P(X) \otimes P(Z) \otimes P(Y) \arrow[d, "{_{x'}\circ_z} \otimes \mathrm{id}"'] & \\
P(X{}_{x'}\sqcup_z Z) \otimes P(Y) \arrow[r, "{_x\circ_y}"] & P(X{}_x\sqcup_y Y{}_{x'}\sqcup_z Z)
\end{tikzcd}$$

for any $x, x' \in X$, $y, y' \in Y$, $z \in Z$ such that $x \neq x'$ and $y \neq y'$.

Equivariance. For any $x \in X$, $y \in Y$, any maps $\sigma \colon X \longrightarrow X'$ and $\tau \colon Y \longrightarrow Y'$ in $\mathcal{F}in$ such that $\sigma(x) = x'$ and $\tau(y) = y'$, the following diagram commutes:

$$\begin{tikzcd}
P(X') \otimes P(Y') \arrow[r, "{_{x'}\circ_{y'}}"] \arrow[d, "\sigma \otimes \tau"'] & P(X'{}_{x'}\sqcup_{y'} Y') \arrow[d, "\sigma\, {_x\circ_y}\, \tau"] \\
P(X) \otimes P(Y) \arrow[r, "{_x\circ_y}"] & P(X{}_x\sqcup_y Y)
\end{tikzcd}$$

where $\sigma\,{_x\circ_y}\,\tau : X{}_x\sqcup_y Y \longrightarrow X'{}_{x'}\sqcup_{y'} Y'$ is the map whose restriction to $X \setminus \{x\}$ is $\sigma|_{X\setminus\{x\}}$, and to $Y \setminus \{y\}$ is $\tau|_{Y\setminus\{y\}}$.

Unit. For any two-point set $U = \{u_1, u_2\}$ there are maps $\eta_U : I \longrightarrow P(U)$ for which the following compositions are the left and right identities, for any $i \in \{0, 1\}$ and any set X, $x \in X$:

$$I \otimes P(X) \xrightarrow{\eta_U \otimes \mathrm{id}} P(U) \otimes P(X) \xrightarrow{{_{u_i}\circ_x}} P(U{}_{u_i}\sqcup_x X) \xrightarrow{e_{u_i x}} P(X)$$

$$P(X) \otimes I \xrightarrow{\mathrm{id} \otimes \eta_U} P(X) \otimes P(U) \xrightarrow{{_x\circ_{u_i}}} P(X{}_x\sqcup_{u_i} U) \xrightarrow{e_{x u_i}} P(X)$$

where $e_{xu_i} = e_{u_i x} : X \longrightarrow U{}_u\sqcup_x X$, $e_{xu_i}(x) = u_{1-i} \in U$ and $e_{xu_i}|_{X\setminus\{x\}} = \mathrm{id}$. Moreover, the following diagrams should commute for any two-point sets U and

U' and any map $\alpha : U \longrightarrow U'$ in $\mathcal{F}in$:

$$\begin{tikzcd}
I \arrow[r, "\eta_{U'}"] \arrow[d, equal] & P(U') \arrow[d, "\alpha"] \\
I \arrow[r, "\eta_U"] & P(U)
\end{tikzcd}$$

One can define *morphisms of cyclic operads* similarly to morphisms of operads. We will denote the category of cyclic operads by $\mathcal{C}yc\mathcal{O}p$.

Remark 1.3.3. Our definition is again equivalent to the different classical ones (see [14, 30]). The proof of this can be carried out similarly to the one of Subsection 1.2.2.

It is an interesting feature of cyclic operads that the existence of units implies that the ${}_x\circ_y$ operations are symmetric:

Proposition 1.3.4. *If P is a cyclic operad, then the following diagram commutes:*

$$\begin{tikzcd}
P(X) \otimes P(Y) \arrow[r, "{}_x\circ_y"] \arrow[d, "s"'] & P(X{}_x{\sqcup}_y Y) \arrow[d, equal] \\
P(Y) \otimes P(X) \arrow[r, "{}_y\circ_x"] & P(Y{}_y{\sqcup}_x X)
\end{tikzcd}$$

Proof. Let $U = \{0, 1\}$. Beside the axioms of a cyclic operad we will use that symmetry and left identity in $\mathcal{E}$ are canonical.

First we observe that in the diagram

$$\begin{tikzcd}
P(U) \otimes P(X) \otimes P(Y) \arrow[r, "{}_1\circ_x \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes s"'] & P(U{}_1{\sqcup}_x X) \otimes P(Y) \arrow[r, "e \otimes \mathrm{id}"] \arrow[dd, "{}_2\circ_y"] & P(X) \otimes P(Y) \arrow[dd, "{}_x\circ_y"] \\
P(U) \otimes P(Y) \otimes P(X) \arrow[d, "{}_2\circ_y \otimes \mathrm{id}"'] & & \\
P(U{}_2{\sqcup}_y Y) \otimes P(X) \arrow[r, "{}_1\circ_x"] \arrow[d, "e' \otimes \mathrm{id}"'] & P(U{}_1{\sqcup}_x X{}_2{\sqcup}_y Y) \arrow[r, "e\ {}_x\circ_y \mathrm{id}"] \arrow[d, "e'\ {}_y\circ_x \mathrm{id}"] & P(X{}_x{\sqcup}_y Y) \\
P(Y) \otimes P(X) \arrow[r, "{}_y\circ_x"] & P(Y{}_y{\sqcup}_x X) &
\end{tikzcd}$$

all the 3 squares commute and $e{}_x\circ_y \mathrm{id} = e'{}_y\circ_x \mathrm{id} = \mathrm{id}$. If we augment this diagram with the commutative square

$$\begin{tikzcd}
I \otimes P(X) \otimes P(Y) \arrow[r, "\eta_U \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes s"'] & P(U) \otimes P(X) \otimes P(Y) \arrow[d, "\mathrm{id} \otimes s"] \\
I \otimes P(Y) \otimes P(X) \arrow[r, "\eta_U \otimes \mathrm{id}"] & P(U) \otimes P(Y) \otimes P(X)
\end{tikzcd}$$

then by the unit axiom we get the following, simpler commutative diagram:

$$\begin{tikzcd}
I \otimes P(X) \otimes P(Y) \arrow[rr] \arrow[d, "\mathrm{id} \otimes s"'] & & P(X) \otimes P(Y) \arrow[dd, "\circ_{xy}"] \\
I \otimes P(Y) \otimes P(X) \arrow[d] & & \\
P(Y) \otimes P(X) \arrow[r, "{}_y\circ_x"] & P(Y\, {}_y\sqcup_x X) \arrow[r, equal] & P(X\, {}_x\sqcup_y Y)
\end{tikzcd}$$

Using that the left identity in $\mathcal{E}$ is natural, we obtain the desired diagram. ♦

1.3.1 Some remarks and examples

There is a forgetful functor $u^* : \mathcal{C}yc\mathcal{O}p \longrightarrow \mathcal{O}p$, induced by the obvious functor $u : \mathcal{F}in_* \longrightarrow \mathcal{F}in$. Thus every cyclic operad *is* an operad. One of the possible classical definitions of cyclic operads ([30], pp. 247-248 or [14]) views cyclic operads as operads where the actions of the symmetric groups Σ_n extend in a compatible way to $\Sigma_n^+ := \Sigma_{n+1}$ actions. In this sense, somewhat ambiguously, in the literature an operad P is called *cyclic* if such an extension for P exists. Examples include the operads Com and As in $dg\mathcal{V}ect$ (or in any symmetric monoidal category). The operad governing Leibniz algebras on the other hand is not cyclic in this sense. For more details on this matter and a classification of some special *cyclic quadratic operads* with one generator we refer to [14].

Another interesting example was given in [8], where the author shows that the operad of framed discs $\mathfrak{f}D_2$ is homotopy equivalent to a cyclic operad.

There are two important variations of cyclic operads which will be relevant later: *reduced cyclic operads* are cyclic operads with $P(O) = I$ for any one-point-set O, and *pseudo cyclic operads* are reduced cyclic operads without unit.

In the preliminaries we mentioned that cyclic operads are operads with extra structure and we also indicated the definition of cyclic operads following this line of thought, with the classical approach. If we want to use the finite set version of (cyclic) operads then the approach above translates as follows:

An operad P is cyclic if the actions of the maps of $\mathcal{F}in_*$ on P extend to actions of not necessarily pointed bijections and, if we set

$$P^c(X) := \mathrm{colim}_{\tau:(X,x_0)\longrightarrow(X,x_0')}\, P, \quad \text{where } \tau \text{ swaps } x_0 \text{ with } x_0'$$

then $P^c(X)$ naturally becomes a cyclic operad.

1.4 Cooperads and cyclic cooperads

In Chapter 2 we will need to use cyclic cooperads for some constructions, hence in this section we briefly introduce them, together with cooperads. Intuitively, these dual notions arise when one reverses all arrows in the definition of operads and cyclic operads, respectively.

Definition 1.4.1. A *cooperad* in a symmetric monoidal category $\mathcal{E}$ is a cocollection $D\colon \mathcal{F}in_* \longrightarrow \mathcal{E}$ (that is, a covariant functor), together with structure maps

$$\circ_x\colon D(Z, z_0) \longrightarrow D(X, x_0) \otimes D(Y, y_0)$$

for any finite pointed set (Z, x_0) and any decomposition $(Z, z_0) = (X {\sqcup}_x Y, x_0)$ where $x \in X$, $x \neq x_0$. These structure maps satisfy the obvious coassociativity, coequivariance and counit axioms given by reversing all arrows in the respective conditions in the definition of an operad.

Similarly,

Definition 1.4.2. A *cyclic cooperad* in a symmetric monoidal category $\mathcal{E}$ is a cyclic cocollection $C\colon \mathcal{F}in \longrightarrow \mathcal{E}$, together with structure maps

$${}_x\circ_y\colon C(Z) \longrightarrow C(X) \otimes C(Y)$$

for any finite set Z and any decomposition $Z = (X\,{}_x{\sqcup}_y Y)$ where $x \in X$ and $y \in Y$. The structure maps of C satisfy the obvious coassociativity, coequivariance and counit conditions.

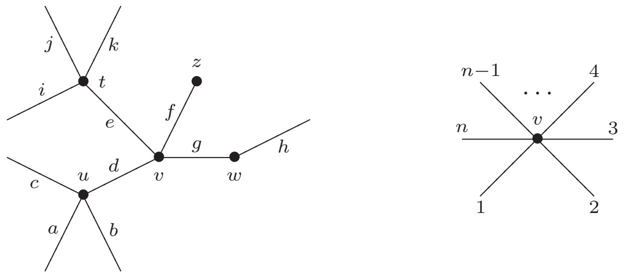
1.5 Free constructions

1.5.1 The free-forgetful adjunction $\mathcal{C}yc\mathcal{C}oll \longrightarrow \mathcal{C}yc\mathcal{O}p$

Similarly to the free operad construction, the free cyclic operad depends on a suitable groupoid of trees which we describe now. Before starting the description, we mention that there are more possibilities to start with, all of them giving the same result in the end, after the suitable choices of identifications: one of them is considering planar rooted trees and nonplanar, not necessarily root preserving isomorphisms of them; another possibility is the one we take.

Let us start with some conventions and notations we will use later in the thesis. A *tree* T for us is a finite contractible graph with at least one vertex and external edge. We will denote the set of edges by $\mathrm{Edg}(T)$, the set of vertices by $\mathrm{Vert}(T)$, the set of *external edges* or *leaves* (i.e. edges which have only one adjacent vertex) by $\mathrm{Leaves}(T)$ and the set of *internal edges* (those edges which are not external) by $\mathrm{InEdg}(T)$. For a vertex $v \in \mathrm{Vert}(T)$ the set of the edges adjacent to v will be denoted by $\mathrm{Edg}(v)$. A tree which has only one vertex and at least two leaves will be called a *corolla*.

The main ingredient of the free construction is the groupoid $\mathbb{T}$, which has trees as objects and isomorphisms of graphs as morphisms. The following picture displays two objects of $\mathbb{T}$: a typical tree and a corolla with n leaves (in the latter case we took $\{1, 2, \ldots, n\}$ for the set of edges).



For example, the tree T on the left has

$$\begin{aligned}
\mathrm{Edg}(T) &= \{a, b, c, d, e, f, g, h, i, j, k\},\\
\mathrm{Vert}(T) &= \{u, v, w, z, t\},\\
\mathrm{Leaves}(T) &= \{a, b, c, h, i, j, k\},\\
\mathrm{InEdg}(T) &= \{e, f, d, g\},\\
\mathrm{Edg}(u) &= \{a, b, c, d\}.
\end{aligned}$$

Note that the *stump* $|$ is not an object of $\mathbb{T}$ since it has no vertices. The intuition behind this choice is that we want to label the vertices of trees with corresponding elements of a cyclic collection to construct the free operad on that collection.

If T is a tree in $\mathbb{T}$ and X is a finite set then denote by $\lambda_T(X)$ the set of X-labelings of T, i.e. the set of bijections $X \longrightarrow \mathrm{Leaves}(T)$. The map λ_T is part of a functor $\lambda : \mathbb{T} \longrightarrow \mathcal{S}ets^{\mathcal{F}in^{\mathrm{op}}}$, defined in the obvious way. In fact, this functor defines the category of labeled trees by taking $\mathrm{colim}_{\mathbb{T}}\, \lambda$. Note that by composing with the strong symmetric monoidal functor $I[-] : \mathcal{S}ets \longrightarrow \mathcal{E}$, λ can be viewed as a functor $\mathbb{T} \longrightarrow \mathcal{E}^{\mathcal{F}in^{\mathrm{op}}}$.

Now we turn to the description of the free cyclic operad. Let $K : \mathcal{F}in^{\mathrm{op}} \longrightarrow \mathcal{E}$ be a cyclic collection in a symmetric monoidal category $\mathcal{E}$ with unit I. K induces a functor

$$\underline{K} \colon \mathbb{T}^{\mathrm{op}} \longrightarrow \mathcal{E},$$

defined by setting

$$\underline{K}(T) = \bigotimes_{v \in \mathrm{Vert}(T)} K(\mathrm{Edg}(v))$$

where the tensor product is the unordered one over the set of vertices of a given tree. Since any map $\phi : T \to T'$ in $\mathbb{T}$ is completely determined by its restrictions to the sub-corollas of T, the contravariant behavior of K determines $\underline{K}$ on the maps of $\mathbb{T}$. We can define the free pseudo cyclic operad generated by K as

$$\mathbb{F}_0 K = \underline{K} \otimes_{\mathbb{T}} \lambda.$$

Explicitly, $\mathbb{F}_0 K : \mathcal{F}in^{\mathrm{op}} \longrightarrow \mathcal{E}$, is the colimit of $\coprod_{T\in\mathbb{T}} \big(\underline{K}(T) \otimes \lambda_T(X)\big)$ over $\mathbb{T}$, that is

$$\mathbb{F}_0 K(X) = \coprod_{[T],T\in\mathbb{T}} \underline{K}(T) \otimes_{\mathrm{Aut}(T)} \lambda_T(X),$$

with the convention that if $\lambda_T(X) = \emptyset$ then we omit the corresponding component in the coproduct. Keeping in mind that each λ_T is a functor $\lambda_T : \mathcal{F}in^{\mathrm{op}} \longrightarrow \mathcal{E}$, there is an induced action of the maps of $\mathcal{F}in$ on the right of $\mathbb{F}_0 K$, hence $\mathbb{F}_0 K : \mathcal{F}in^{\mathrm{op}} \longrightarrow \mathcal{E}$ is well defined. The cyclic operad structure without unit on $\mathbb{F}_0 K$ is induced by the above described actions and the grafting operations of $\mathbb{T}$.

Indeed, an object $\underline{K}(T)\otimes_{\mathrm{Aut}(T)}\lambda_T(X)$ is determined by pairs $(T, \tau \in \lambda_T(X))$. If T and R are arbitrary objects of $\mathbb{T}$ with distinguished external edges t and r, X and Y are finite sets such that t and r are labeled by x and y respectively, the isomorphisms

$$\begin{aligned}
\underline{K}(T) \otimes \underline{K}(R) &= \left(\bigotimes_{v\in \mathrm{Vert}(T)} K(\mathrm{Edg}(v)) \right) \otimes \left(\bigotimes_{w\in \mathrm{Vert}(R)} K(\mathrm{Edg}(w)) \right) \\
&\simeq \bigotimes_{v,w} K(\mathrm{Edg}(v)) \otimes K(\mathrm{Edg}(w)) \\
&\simeq \bigotimes_{u\in \mathrm{Vert}(T_t\circ_r R)} K(\mathrm{Edg}(u)) \\
&= \underline{K}(T_t\circ_r R)
\end{aligned}$$

induce the structure maps ${}_x\circ_y : \mathbb{F}_0 K(X) \otimes \mathbb{F}_0 K(Y) \longrightarrow \mathbb{F}_0 K(X {}_x\sqcup_y Y)$.

Proposition 1.5.1. *The functor $\mathbb{F}_0$ from the category of cyclic collections to the category of pseudo cyclic operads is left adjoint to the forgetful functor.*

Proof. Suppose $\alpha : \mathbb{F}_0 K \longrightarrow P$ is a map of cyclic operads without unit. If X is a finite set, let Cor_X be a corolla with $\mathrm{Edg}(\mathrm{Cor}_X) = X$. Then $\mathbb{F}_0 K(X)$ has a summand $\underline{K}(\mathrm{Cor}_X) \otimes_{\mathrm{Aut}(\mathrm{Cor}_X)} \lambda_{\mathrm{Cor}_X}(X) \simeq K(X)$, thus α_X induces a map $\Psi(\alpha_X) : K(X) \longrightarrow P(X)$. The resulting $\Psi(\alpha) : K \longrightarrow P$ is a map of cyclic collections.

Suppose that $\beta : K \longrightarrow P$ is a map of cyclic collections. To define the inverse of Ψ, we have to define it on the components of each $\mathbb{F}K(X)$, i.e. the maps

$$\big(\Psi(\beta_X)\big)_T : \left(\bigotimes_{v\in \mathrm{Vert}(T)} K(\mathrm{Edg}(v)) \right) \otimes_{\mathrm{Aut}(T)} \lambda_T(X) \longrightarrow P(X)$$

need to be given. The $\big(\Psi(\beta_X)\big)_T$ above is induced by the map

$$\bigotimes_{v\in\mathrm{Vert}(T)} \beta: \bigotimes_{v\in\mathrm{Vert}(T)} K(\mathrm{Edg}(v)) \longrightarrow \bigotimes_{v\in\mathrm{Vert}(T)} P(\mathrm{Edg}(v)) \longrightarrow P(\mathrm{Leaves}(T))$$

where the right-hand side map is the iterated application of the structure maps of P, indicated by T. ♦

In view of the above construction, the free cyclic operad functor

$$\mathbb{F}\colon \mathcal{C}yc\mathcal{C}oll \longrightarrow \mathcal{C}yc\mathcal{O}p$$

is obtained from $\mathbb{F}_0$ by adding the unit freely:

Define $\mathbb{F}K(X) := \mathbb{F}_0K(X)$ for every set X with the property that $|X| \neq 2$, and let $\mathbb{F}K(E) := I \oplus \mathbb{F}_0(E)$ for any set E with two elements. Define the unit maps $I \longrightarrow \mathbb{F}K(E)$ to be the inclusions to the first component. The actions of the bijections $\sigma \in \mathcal{F}in$ between two-point sets are extended from F_0K to $\mathbb{F}K$ by $\mathrm{id}_I \oplus \sigma$. The new extra operations are defined in the obvious way: for example, if E and X are as above and $e \in E$, $x \in X$ then the composition map

$$\mathbb{F}K(E) \otimes \mathbb{F}K(X) \xrightarrow{\;{}_e\circ_x\;} \mathbb{F}K(E{}_e\sqcup_x X)$$

needs to be defined only on the component $I\otimes\mathbb{F}K(X)$ of $(I\oplus\mathbb{F}_0K(E))\otimes\mathbb{F}K(X)$. There is an obvious choice to define this map, namely $I \otimes \mathbb{F}K(X) \simeq \mathbb{F}K(X) \longrightarrow \mathbb{F}K(E{}_e\sqcup_x X)$ where the last map is the one induced by the identification of the sets $X \simeq E{}_e\sqcup_x X$.

By construction, $\mathbb{F}K$ is indeed a cyclic operad and it is the free one on the cyclic collection K. The proof of this is similar to that of Proposition 1.5.1, in this case keeping track of the units too.

Remark 1.5.2. The construction given above generalises the free operad construction to cyclic operads. An interesting fact about the free operad construction is that the free operad on a cyclic collection is cyclic in the sense of Subsection 1.3.1.

1.5.2 The free-forgetful adjunction $\mathcal{C}yc\mathcal{C}oll_* \longrightarrow \mathcal{C}yc\mathcal{O}p$

A *pointed cyclic collection* is a cyclic collection K together with maps $\eta_E : I \longrightarrow K(E)$ for any two-point set U. These maps (also called basepoints) need to satisfy the usual compatibility conditions, that is if $\tau : F \longrightarrow E$ is a bijection of sets with two elements then the following diagram commutes:

$$\begin{tikzcd} & I \arrow[dl, "\eta_E"'] \arrow[dr, "\eta_F"] & \\ K(E) \arrow[rr, "\tau"] & & K(F) \end{tikzcd}$$

There is a category of pointed cyclic collections which we denote by $\mathcal{C}yc\mathcal{C}oll_*$. There is an obvious forgetful functor $\mathcal{C}yc\mathcal{O}p \longrightarrow \mathcal{C}yc\mathcal{C}oll_*$ which has a left adjoint

$\mathbb{F}_* : \mathcal{C}yc\mathcal{C}oll_* \longrightarrow \mathcal{C}yc\mathcal{O}p$. We can give a formal definition of $\mathbb{F}_*$, using the already constructed free cyclic operad functor $\mathbb{F}$ of the previous subsection and the initial object of the category $\mathcal{C}yc\mathcal{O}p$.

First observe that the initial cyclic operad Θ is given by $\Theta(X) := I$ if X is a set with two elements and $\Theta(X) := 0$ otherwise, where 0 is the initial object of the category $\mathcal{E}$. Note that it is obvious how to define the equivariant structure and operations of Θ since the functor $A \otimes - : \mathcal{E} \longrightarrow \mathcal{E}$ for any object $A \in \mathcal{E}$ preserves colimits, hence also initial objects.

By the above data, one can define the functor $\mathbb{F}_*$ on the pointed cyclic collection K as the pushout

$$\begin{tikzcd} \mathbb{F}\Theta \arrow[r] \arrow[d] & \mathbb{F}K \arrow[d] \\ \Theta \arrow[r] & \mathbb{F}_*K \end{tikzcd}$$

where the upper horizontal map is induced by the basepoints of K.

Indeed, we can use that the above diagram is a pushout together with the fact that $\mathbb{F}$ is left adjoint to the forgetful functor $\mathcal{C}yc\mathcal{O}p \longrightarrow \mathcal{C}yc\mathcal{C}oll$ to conclude purely formally that F_* is left adjoint to the forgetful functor $\mathcal{C}yc\mathcal{O}p \longrightarrow \mathcal{C}yc\mathcal{C}oll_*$.

Later we will need an explicit construction of $\mathbb{F}_*$, hence we provide one in this subsection. Assume that K is a pointed collection and T is a tree in $\mathbb{T}$. Let V be a subset of the unary vertices of T. For an arbitrary vertex v of T define $\underline{K}^v_V(T) := I$ if $v \in V$ and $\underline{K}^v_V(T) := K(\mathrm{Edg}(v))$ if $v \notin V$, and let

$$\underline{K}_V(T) := \bigotimes_{v \in \mathrm{Vert}(T)} \underline{K}^v_V(T).$$

There is a category $\mathcal{C}_T$ whose objects are the nonempty subsets of the unary vertices of T and arrows are the inclusions of sets. Whenever $W \subseteq V$ is an arrow in $\mathcal{C}_T$, the basepoints $\eta_E : I \longrightarrow K(E)$ induce maps $\underline{K}_V(T) \longrightarrow \underline{K}_W(T)$, giving rise to a functor $\underline{K}_?(T) : \mathcal{C}_T^{\mathrm{op}} \longrightarrow \mathcal{E}$. Define

$$\underline{K}^-(T) := \mathrm{colim}_{V \in \mathcal{C}_T^{\mathrm{op}}} \underline{K}_V(T).$$

Intuitively, $\underline{K}^-(T)$ consists of copies of T, with vertices decorated by "elements" of the corresponding $K(X)$, such that if there is a unary vertex of T then it is allowed to leave it undecorated. The trees with undecorated vertices are identified with those trees where the corresponding vertex is decorated by the basepoint, keeping in mind that the latter tree still has to contain at least one undecorated vertex (we excluded $\emptyset$ from $\mathcal{C}_T$).

For an object $V \in \mathcal{C}_T$, denote by $T/_V$ the tree resulting from T by removing the vertices coming from V and gluing the corresponding edges. It follows that there is an evident isomorphism

$$\underline{K}_V(T) \xrightarrow{\simeq} \underline{K}(T), \tag{1.5.1}$$

and there is an induced map

$$\underline{K}^-(T) \longrightarrow \underline{K}(T). \tag{1.5.2}$$

Now we can proceed with the construction of $\mathbb{F}_*K$. For every finite set X, the object $\mathbb{F}_*K(X)$ is constructed as the sequential colimit of $\mathrm{Aut}(X)$-equivariant maps

$$\mathbb{F}_*K(X)_0 \longrightarrow \mathbb{F}_*K(X)_1 \longrightarrow \mathbb{F}_*K(X)_2 \longrightarrow \cdots,$$

where intuitively the index k stands for that part of $\mathbb{F}_*K(X)$ which is constructed only with trees with at most k unary vertices. Let $\mathbb{T}_k$ be the subcategory of $\mathbb{T}$ consisting of trees with exactly k unary vertices. Define

$$\mathbb{F}_*K(X)_0 := \coprod_{[T],T\in\mathbb{T}_0} \underline{K} \otimes_{\mathrm{Aut}(T)} \lambda_T(X).$$

Here again λ_T is viewed as a functor $\lambda_T : \mathcal{F}in^{\mathrm{op}} \longrightarrow \mathcal{E}$ and if $|X| \neq |\operatorname{Leaves}(T)|$ the corresponding component is omitted from the sum. Suppose that $\mathbb{F}_*K(X)_{k-1}$ was constructed and define $\mathbb{F}_*K(X)_k$ as the pushout

$$\begin{tikzcd}
\coprod_{[T],T\in\mathbb{T}_k} \underline{K}^-(T) \otimes_{\mathrm{Aut}(T)} \lambda_T(X) \arrow[r] \arrow[d] & \mathbb{F}_*K(X)_{k-1} \arrow[d] \\
\coprod_{[T],T\in\mathbb{T}_k} \underline{K}(T) \otimes_{\mathrm{Aut}(T)} \lambda_T(X) \arrow[r] & \mathbb{F}_*K(X)_k
\end{tikzcd},$$

where the horizontal map on the top is induced by the isomorphisms (1.5.1) and the vertical map on the left is induced by the maps (1.5.2). Note that the category $\mathcal{F}in$ acts on the objects on the left in this pushout diagram and the vertical map on the left is compatible with this action, thus by induction we can infer that $\mathbb{F}_*K(-)_k : \mathcal{F}in^{\mathrm{op}} \longrightarrow \mathcal{E}$ is a functor. Intuitively these pushouts make the necessary left identifications for the basepoints to become the units of the free cyclic operad.

To end the construction, define

$$\mathbb{F}_*K(X) := \operatorname{colim}_{k\in\mathbb{N}} \mathbb{F}_*K(X)_k,$$

and the cyclic operad structure on F_*K is again defined by the grafting operation of trees. We skip the proof of the following proposition, which can be carried out in a similar fashion to that of Proposition 1.5.1.

Proposition 1.5.3. *The functor $\mathbb{F}_* : \mathcal{C}yc\mathcal{C}oll_* \longrightarrow \mathcal{C}yc\mathcal{O}p$ is left adjoint to the forgetful functor.*

1.5.3 The free-forgetful adjunction $\mathcal{O}p \longrightarrow \mathcal{C}yc\mathcal{O}p$

Before we construct the free functor from operads to cyclic operads we need to describe the free functor from collections to cyclic collections. Note that the "forgetful functor" $u^* : \mathcal{C}yc\mathcal{C}oll \longrightarrow \mathcal{C}oll$ is defined as the pullback along the functor $u : \mathcal{F}in_* \longrightarrow \mathcal{F}in$, which forgets the basepoint of a pointed set (X, x_0).

For a collection $K : \mathcal{F}in_*^{\mathrm{op}} \longrightarrow \mathcal{E}$ define $\mathcal{F}K : \mathcal{F}in^{\mathrm{op}} \longrightarrow \mathcal{E}$ by

$$\mathcal{F}K(X) := \coprod_{x_0 \in X} K(X, x_0),$$
$$\mathcal{F}K(\sigma : X \longrightarrow Y) := \coprod_{x_0 \in X} K\big(\sigma : (X, x_0) \longrightarrow (Y, \sigma(x_0))\big).$$

We can infer that $\mathcal{F} : \mathcal{C}oll \longrightarrow \mathcal{C}yc\mathcal{C}oll$ is indeed a functor, since if $\alpha : K \longrightarrow L$, $\alpha = (\alpha_{(X,x_0)})$ is a map of collections then $\mathcal{F}\alpha = (\coprod_{x_0 \in X} \alpha_{(X,x_0)})$ is a map of cyclic collections. $\mathcal{F}$ is left adjoint to the forgetful functor $u^* : \mathcal{C}yc\mathcal{C}oll \longrightarrow \mathcal{C}oll$. Indeed, assume that $\alpha : \mathcal{F}K \longrightarrow L$ is a map of cyclic collections, so the maps $\alpha_X : \coprod_{x_0} K(X, x_0) \longrightarrow L(X)$ are given. Since $u^*L(X, x_0) = L(X)$ for any $x_0 \in X$, we can define a map of collections $\tilde{\alpha} : K \longrightarrow u^*L$ by composing with the canonical inclusions: indeed, if $\sigma : (X, x_0) \longrightarrow (Y, y_0)$ is a map in $\mathcal{F}in_*$ then commutativity of the diagram

$$\begin{tikzcd}
\mathcal{F}K(Y) \arrow[r, "\alpha_Y"] \arrow[d, "\coprod_{z_0 \in X} \sigma"'] & L(Y) \arrow[d, "\sigma"'] \\
\mathcal{F}K(X) \arrow[r, "\alpha_X"] & L(X)
\end{tikzcd}$$

implies for $z_0 = x_0$ the commutativity of

$$\begin{tikzcd}
K(Y, y_0) \arrow[r, "\alpha_{(Y,y_0)}"] \arrow[d, "\sigma"'] & u^*L(Y, y_0) \arrow[d, "\sigma"'] \\
K(X, x_0) \arrow[r, "\alpha_{(X,x_0)}"] & u^*L(X, x_0)
\end{tikzcd}$$

Similarly, if $\beta : K \longrightarrow u^*L$ is a map of collections then by universality of the coproduct we can define the maps $\hat{\beta}_X : \mathcal{F}K(X) \longrightarrow L(X)$ for every $X \in \mathcal{F}in$, and $\hat{\beta}$ is a map of cyclic collections – this follows again from the universality of the coproduct (of maps). The above two correspondences are natural inverses to each other.

Now we turn to the description of the free functor $\mathcal{F}^+ : \mathcal{O}p \longrightarrow \mathcal{C}yc\mathcal{O}p$ in categories with zero objects. Let P be an operad. The underlying cyclic collection of the cyclic operad $\mathcal{F}^+P$ is the free cyclic collection constructed above: $\mathcal{F}^+P = \mathcal{F}(UP)$ where U forgets the operad structure. Before defining the composition maps ${}_x\circ_y$, decompose

$$\mathcal{F}^+P(X)\otimes\mathcal{F}^+P(Y) = \left(\bigoplus_{\substack{x_0\in X\setminus\{x\}\\ y_0=y}} P(X,x_0)\otimes P(Y,y_0)\right) \oplus \left(\bigoplus_{\substack{x_0=x\\ y_0\in Y\setminus\{y\}}} P(X,x_0)\otimes P(Y,y_0)\right)\oplus \text{ rest},$$

and also

$$\mathcal{F}^+P(X{}_x\sqcup_y Y) = \left(\bigoplus_{z_0\in X\setminus\{x\}} P(X{}_x\sqcup_y Y, z_0)\right)\oplus\left(\bigoplus_{z_0\in Y\setminus\{y\}} P(X{}_x\sqcup_y Y, z_0)\right).$$

Then define ${}_x\circ_y := \alpha\oplus\beta\oplus 0$ where

$$\alpha : \bigoplus_{\substack{x_0\in X\setminus\{x\}\\ y_0=y}} P(X,x_0)\otimes P(Y,y_0)\longrightarrow \bigoplus_{x_0\in X\setminus\{x\}} P(X{}_x\sqcup_y Y, x_0),$$

$$\alpha := \oplus_{x_0}\circ_x$$

and

$$\beta : \bigoplus_{\substack{x_0=x\\ y_0\in Y\setminus\{y\}}} P(X,x_0)\otimes P(Y,y_0)\longrightarrow \bigoplus_{y_0\in Y\setminus\{y\}} P(X{}_x\sqcup_y Y, y_0),$$

$$\beta := (\oplus_{y_0}\circ_y)\circ\big(\text{symmetry}\big).$$

What we constructed so far is the free functor from pseudo operads to pseudo cyclic operads, so we still need to take care of the unit. To do so, we modify the current definition of $\mathcal{F}^+P(E) = P(E,e)\oplus P(E,f)$ where $E=\{e,f\}$ is any set with two elements: define the new $\mathcal{F}^+P(E)$ to be the coequalizer of the two inclusions $I\longrightarrow P(E,e)\longrightarrow P(E,e)\oplus P(E,f)$ and $I\longrightarrow P(E,f)\longrightarrow P(E,e)\oplus P(E,f)$, corresponding to the two possible unit maps of the operad P. One can check that the modified $\mathcal{F}^+$ functor is indeed left adjoint to the forgetful functor $\mathcal{C}yc\mathcal{O}p\longrightarrow\mathcal{O}p$. Let us state the relations between the different free constructions in the following

Proposition 1.5.4. *There is a commutative diagram of free-forgetful adjunctions*

$$\begin{tikzcd} \mathcal{C}oll \arrow[r, shift left, "\tilde{\mathbb{F}}"] \arrow[d, shift right, "\mathcal{F}"'] & \mathcal{O}p \arrow[l, shift left] \arrow[d, shift right, "\mathcal{F}^+"'] \\ \mathcal{C}yc\mathcal{C}oll \arrow[r, shift left, "\mathbb{F}"] \arrow[u, shift right] & \mathcal{C}yc\mathcal{O}p \arrow[l, shift left] \arrow[u, shift right] \end{tikzcd} \tag{1.5.3}$$

The proof follows from the fact that the diagram with only the forgetful functors is commutative.

1.6 Coloured operads and their algebras

Coloured operads are generalisations of operads in the sense that the inputs of the operations can have different "colour". They can also be viewed as generalisations of categories. While a coordinate-free definition of coloured operads is also possible (see [38]), we will work with the usual definition here since we use coloured operads only in this section.

A *coloured operad* P in a closed symmetric monoidal category $(\mathcal{E}, \otimes, I)$ consists of a set of colours or objects $ob(P)$ and for each ordered $n+1$-tuple of colours $(c_1, c_2, \ldots, c_n; c)$ an object of the category $\mathcal{E}$, denoted by $P(c_1, c_2, \ldots, c_n; c)$, which we think of as a space of operations. The operations can be composed in the way as the ones of classical operads do, and they satisfy the usual axioms of associativity, equivariance and unit. For example, a typical $\circ_i$ operation is a map in $\mathcal{E}$

$$\begin{array}{c} P(c_1, \ldots, c_i, \ldots, c_n; c) \otimes P(d_1, \ldots, d_m; c_i) \\ \Big\downarrow \circ_i \\ P(c_1, \ldots, c_{i-1}, d_1, \ldots, d_m, c_{i+1}, \ldots, c_n; c) \end{array}$$

One important example is the category $\mathcal{E}$ itself, which can be thought of as a coloured operad with set of colours the objects of the category; the spaces of operations being defined as

$$\mathcal{E}(A_1, A_2, \ldots, A_n; A) := \mathcal{E}(A_1 \otimes A_2 \otimes \cdots \otimes A_n, A).$$

For clarity, the second $\mathcal{E}$ in the line above refers to the internal hom functor of the category.

One can define maps of coloured operads similarly to the classical case. A map of coloured operads $P \longrightarrow \mathcal{E}$, where $\mathcal{E}$ refers to the coloured operad described above, is called a *P-algebra*. Note that classical operads are coloured operads with one colour, and a classical P-algebra has one underlying object (corresponding to the sole colour of P) while an algebra of a coloured operad has an underlying object for each colour. This variety allows us to construct a whole arsenal of coloured operads which govern different kinds of algebraic structures like categories, classical operads, maps of algebras over classical operads, etc. These constructions were not possible with classical operads.

For more details on coloured operads and their relations with categories the reader can consult [25]. For a variety of examples of algebraic structures which can be described as algebras of coloured operads see also [5].

Our goal in this section is to describe in detail coloured operads governing classical operads (symmetric and nonsymmetric) and cyclic operads (also symmetric and nonsymmetric). The example of a coloured operad P governing symmetric classical operads was already constructed in [5].

We will construct these coloured operads only in the category $\mathcal{S}ets$ since the strong symmetric monoidal functor $F : \mathcal{S}ets \longrightarrow \mathcal{E}$, $F(X) = \coprod_{x \in X} I$ allows us to transfer the constructions to any closed symmetric monoidal category $\mathcal{E}$. In all the cases mentioned the set of colours is the set of natural numbers $\mathbb{N}$.

The following subsections make use of the notations and conventions on trees which we stated in Subsection 1.5.1. We will restrict our attention to planar rooted trees in all cases. A *planar rooted tree* T is a tree with a distinguished external edge called *root* and an ordering on the set $\mathrm{Edg}(v)$ for every vertex v, compatible with the root. (The root of T induces a direction on the edges of T, hence every vertex has exactly one "outgoing" edge adjacent to it. We require this edge to be the last one in the order.) This time we will consider the stump $|$ also as a planar rooted tree.

1.6.1 A coloured operad governing non-symmetric operads

The first step to define the coloured operad C_1 is to give the sets of operations $C_1(n_1, \ldots, n_k; n)$. These sets consist of equivalence classes (T, σ), where T is a planar rooted tree with k vertices and n leaves, and $\sigma : \{1, \ldots, k\} \longrightarrow \mathrm{Vert}(T)$ is a bijection. Moreover, for all $i \in \{1, \ldots, k\}$ the vertex $\sigma(i)$ has valence n_i.* The classes are taken modulo the relations $(T, \sigma) \sim (T', \sigma')$ if there is a planar isomorphism $\Phi : T \longrightarrow T'$ such that $\Phi\sigma = \sigma'$.

Let us denote from now on the corolla $\mathrm{Cor}_{\{0,1,\ldots,n\}}$ by Cor_n and call it the n-corolla. Note that the definition of the sets of operations implies that $C_1(n; n) = \{\mathrm{Cor}_n\}$ and $C_1(\ ; 1) = \{|\}$. The latter will produce the unit of any algebra of C_1.

By the above, $C_1(n_1, \ldots, n_k; n)$ is defined only if $n_1 + \cdots + n_k = n - k + 1$. In all the other cases we set $C_1(n_1, \ldots, n_k; n) = \emptyset$. Every $\alpha \in \Sigma_k$ defines a right action

$$\alpha^* : C_1(n_1, \ldots, n_k; n) \longrightarrow C_1(n_{\alpha(1)}, n_{\alpha(2)}, \ldots, n_{\alpha(k)}; n)$$

by $(T, \sigma) \mapsto (T, \sigma\alpha)$.

The operad structure on C_1 is defined via the maps

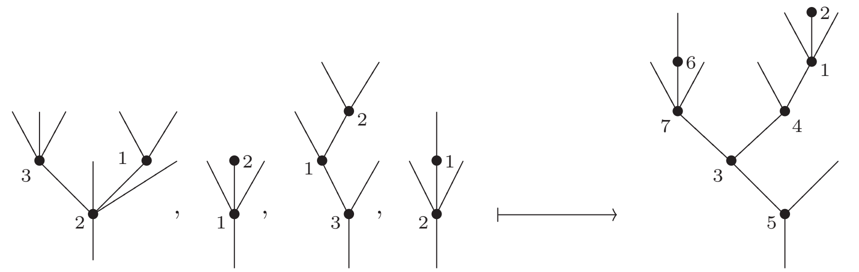
$$C_1(n_1, \ldots, n_k; n) \times \left(\prod_{i=1}^{k} C_1(n_i^1, \ldots, n_i^{l_i}; n_i) \right) \xrightarrow{\gamma} C_1(n_1^1, n_1^2, \ldots, n_k^{l_k}; n) \ ,$$

$$((T, \sigma), (T_1, \sigma_1), \ldots, (T_k, \sigma_k)) \mapsto (T', \sigma')$$

where T' is obtained from T by replacing every vertex $\sigma(i)$ with the tree T_i, identifying the output-edge of $\sigma(i)$ with the root of T_i and identifying the n_i input edges of $\sigma(i)$ with the n_i leaves of T_i, with respect to the planar order (from left to right). The vertices of T' are numbered as follows: first number the vertices of T_1 in the order given by σ_1, then continue with T_2 in the order given by σ_2, etc. (So

*Labeling the vertices in every possible way instead of using only one canonical order on them will be required to prove associativity of the algebras of C_1

just keep the pairs (T_i, σ_i) and shift the numberings accordingly whenever you go from T_i to T_{i+1}.) For example,



is obtained by an operation

$$\begin{array}{c} C_1(2,4,3;7)\times\Big(C_1(3,0;2)\times C_1(2,2,2;4)\times C_1(1,3;3)\Big) \\ \Big\downarrow\gamma \\ C_1(3,0,2,2,2,1,3;7), \end{array}$$

$$((T,\sigma),(T_1,\sigma_1),(T_2,\sigma_2),(T_3,\sigma_3)) \mapsto (T',\sigma')$$

An extra rule is valid for an operation of the type

$$\gamma : C_1(n_1,\ldots,1,\ldots,n_k;n)\times\cdots\times C_1(\ ;1)\times\cdots\longrightarrow C_1(\ldots;n).$$

In this case the univalent vertex is simply removed from the tree T and the ordering of the vertices continues in the obvious way.

Lemma 1.6.1. *C_1 is a symmetric coloured operad.*

Proof. It is immediate that the unique elements of $C_1(n;n)$ are the units of C_1. Next, we observe that if we neglect the vertex-labelings of the resulting tree after the performed operations, associativity and equivariance hold:

- applying a permutation does not affect the trees at all, so equivariance *up to vertex-labelings* holds;
- the order in which the vertex blow-ups are made is irrelevant, thus associativity *up to vertex-labelings* holds.

It remains to check that the labelings match in the associativity and equivariance diagrams. First notice that there is a commutative triangle

$$\begin{tikzcd}
C_1(n_1^1,\ldots,n_1^{l_1};n_1)\times\cdots\times C_1(n_k^1,\ldots,n_k^{l_k};n_k) \arrow[d, "U"'] \arrow[dr, "V"] & \\
\Sigma_k\times\Sigma_{l_1}\times\cdots\times\Sigma_{l_k} \arrow[r, "\gamma'"] & \Sigma_l
\end{tikzcd}$$

Here $l = l_1 + \cdots + l_k$; U is the product of the maps

$$\mathrm{can}^* : C_1(n_1, \ldots, n_k; n) \longrightarrow \Sigma_k$$

where $\mathrm{can}^*(T, \sigma) = \mathrm{can}^{-1} \circ \sigma$ and $\mathrm{can} : \{1, \ldots, k\} \longrightarrow \mathrm{Vert}(T)$ is any chosen canonical labeling of the vertices of T, induced by the planar structure (for example, *going always to the left*); γ' is the structure map of the associative operad in $\mathcal{S}ets$. The map V is defined as follows: if $\gamma((T, \sigma), (T_1, \sigma_1), \ldots, (T_k, \sigma_k)) = (T', \sigma')$ then

$$V((T, \sigma), (T_1, \sigma_1), \ldots, (T_k, \sigma_k)) = v \circ \sigma' \in \Sigma_l$$

where $v : \mathrm{Vert}(T') \longrightarrow \{1, \ldots, l\}$, v labels canonically all the trees T_i in T' and shifts the labels accordingly whenever we pass from T_i to T_{i+1}. This diagram and the fact that As is an operad in $\mathcal{S}ets$ imply that equivariance and associativity hold for the labels too. ♦

Proposition 1.6.2. *The algebras of C_1 are exactly the non-symmetric operads in $\mathcal{S}ets$.*

Proof. A C_1-algebra is given by a map of operads $\Psi : C_1 \longrightarrow \mathcal{S}ets$, that is maps

$$\Psi : C_1(n_1, \ldots, n_k; n) \longrightarrow \mathrm{Hom}(P(n_1) \times \cdots \times P(n_k), P(n))$$

which are compatible with the operadic structures.

We prove that the collection $(P(n))_{n \in \mathbb{N}}$ induced by Ψ is a non-symmetric operad in $\mathcal{S}ets$.

In particular $\Psi : C_1(\, ; 1) \longrightarrow \mathrm{Hom}(*, P(1))$ gives the identity of P, i.e. a map $\eta : * \longrightarrow P(1)$. Define the $\circ_i : P(a) \times P(b) \longrightarrow P(a + b - 1)$ maps by setting

$$\circ_i := \Psi((\mathrm{Cor}_a \sqcup_i \mathrm{Cor}_b, \mathrm{can}))$$

where $\mathrm{Cor}_a \sqcup_i \mathrm{Cor}_b$ is the tree with two vertices, obtained by grafting to a corolla with a leaves a corolla with b leaves, at the i-th leaf in the canonical order. In the remaining part of the proof we will denote these $\circ_i$ operations by f, g, etc. To see that the two associativity conditions hold for P, we observe that the diagram

$$\begin{tikzcd}
C_1(a, b+c-1; a+b+c-2) \times C_1(b, c, ; b+c-1) \arrow[r, "\gamma"] \arrow[d, "\Psi \times \Psi"'] & C_1(a, b, c; a+b+c-2) \arrow[d, "\Psi"] \\
{[P(a) \times P(b+c-1); P(a+b+c-2)] \times [P(b) \times P(c); P(b+c-1)]} \arrow[r, "\gamma"] & {[P(a) \times P(b) \times P(c); P(a+b+c-2)]}
\end{tikzcd}$$

commutes. In particular, if $(f, g) = (\Psi(T_1), \Psi(T_2))$, we deduce that

$$\gamma(f, g) = g \circ (\mathrm{id}_{P(a)} \times f)$$

and by the diagram $g \circ (\mathrm{id}_{P(a)} \times f)$ corresponds to $\Psi\gamma(T_1, T_2)$. On the other hand, there is another diagram,

$$\begin{tikzcd} C_1(a+b-1, c; a+b+c-2) \times C_1(a, b, ; a+b-1) \arrow[r, "\gamma"] \arrow[d, "\Psi \times \Psi"'] & C_1(a, b, c; a+b+c-2) \arrow[d, "\Psi"] \\ {[P(a+b-1) \times P(c); P(a+b+c-2)] \times [P(a) \times P(b); P(a+b-1)]} \arrow[r, "\gamma"] & {[P(a) \times P(b) \times P(c); P(a+b+c-2)]} \end{tikzcd}$$

where we can pick another pair T_1', T_2' such that $\gamma(T_1, T_2) = \gamma(T_1', T_2')$, and if $(f', g') = (\Psi(T_1'), \Psi(T_2'))$, we obtain

$$\gamma(f', g') = f' \circ (g' \times \mathrm{id}_{P(c)})$$

and $f' \circ (g' \times \mathrm{id}_{P(c)})$ corresponds to $\Psi\gamma(T_1', T_2')$. After decoding the T_i and T_i', by the above, the first associativity condition holds for P. To check the second, we approach it with similar diagrams. In this case we will make use of the labels of the vertices of the trees.

It remains to check that $\eta : * \longrightarrow P(1)$ defined above is indeed the unit of P. For example, the diagram

$$\begin{tikzcd} C_1(1, n; n) \times C_1(; 1) \arrow[r, "\gamma"] \arrow[d, "\Psi"'] & C_1(n; n) \arrow[d, "\Psi"'] \\ {[P(1) \times P(n); P(n)] \times [*; P(1)]} \arrow[r, "\gamma"] & {[P(n); P(n)]} \end{tikzcd}$$

commutes and tells us that

$$* \times P(n) \xrightarrow{\eta \times \mathrm{id}} P(1) \times P(n) \xrightarrow{f} P(n)$$

is the identity. Similarly, we can check that

$$P(n) \times * \xrightarrow{\mathrm{id} \times \eta} P(n) \times P(1) \xrightarrow{g} P(n)$$

is the identity.

So far we have proved that any C_1-algebra is a non-symmetric operad. To prove the reverse, let C_1 be a non-symmetric operad and define

$$\Psi : C_1(n_1, \dots, n_k; n) \longrightarrow \mathrm{Hom}(P(n_1) \times \cdots \times P(n_k), P(n))$$

in the obvious way. The associativity condition of the operad P implies that Ψ is well defined and compatible with the actions of the symmetric groups and compositions of the coloured operads C_1 and $\mathcal{S}ets$. ♦

1.6.2 A coloured operad governing symmetric operads

This coloured operad was defined in [5]. We only recall the definition here, to prove that C_2 is indeed an operad and C_2-algebras are symmetric operads in $\mathcal{S}ets$ one can proceed in a similar way to the case of C_1.

The set $C_2(n_1, \ldots, n_k; n)$ consists of equivalence classes (T, σ, τ) where T and σ are the same as in the definition of C_1 and $\tau : \{1, \ldots, n\} \longrightarrow \mathrm{In}(T)$ is a bijection (an ordering of the set of input edges of T). The classes are taken modulo planar isomorphisms of trees which respect both σ and τ. Any $\alpha \in \Sigma_k$ defines $\alpha^* : C_2(n_1, \ldots, n_k; n) \longrightarrow C_2(n_{\alpha(1)}, \ldots, n_{\alpha(k)}; n)$, $\alpha^*(T, \sigma, \tau) = (T, \sigma\alpha, \tau)$.

The structure maps of C_2 are defined in the same way as for C_1, until the point where we identified the n_i input edges of the vertex $\sigma(i)$ in the tree T with the n_i leaves of T_i, with respect to the planar orders. In this case the identifications are made with respect to the order given by τ_i: the l-th input edge of $\sigma(i)$ in the planar order is matched with the input edge $\tau_i(l)$ of T_i.

1.6.3 A coloured operad governing non-symmetric cyclic operads

The set $C_3(n_1, \ldots, n_k; n)$ consists of equivalence classes (T, σ, τ) as above with the difference that in this case $\tau : \{0, 1, \ldots, n\} \longrightarrow \mathrm{Edg}(T)$ labels clockwise all the external edges of T (including the root) in a cyclic way, with respect to the planar structure. In particular, $C_3(n; n)$ can be identified with the opposite group of $\mathbb{Z}_{n+1}$. We define $C_3(\ ; 1) := \{|\}$, so we do not distinguish between the two possible labelings of the stump.

The operations γ are defined in the same way as for C_2, with two conventions:

- when we have to make identifications of input edges of a vertex $\sigma(i)$ with external edges of a tree T_i, the outgoing edge of $\sigma(i)$ is the 0-th in the planar order;
- when blowing up the root vertex of a tree T, the new root will be the root of the tree T_i which is replacing the root vertex.

Note that in the case of C_1 and C_2 in blowing-ups we always identified the root vertex of T with another root and the outcome was an obvious choice for the new root. In this case, because the second root does not have to be labeled with 0, we have to make a choice. The justification of this one is immediate: the unit-condition for C_3 would not hold if we would pick the other possibility for the root.

Lemma 1.6.3. *C_3 is a symmetric coloured operad.*

Proof. Let us check that $(\mathrm{Cor}_n, 1, \mathrm{can}) \in C_3(n; n)$ are the units, where

$$\mathrm{can} : \{0, 1, \ldots, n\} \longrightarrow \mathrm{Edg}(\mathrm{Cor}_n)$$

labels the root of Cor_n with 0.

The fact that the composition

$$C_3(n_1,\ldots,n_k;n)\times * \xrightarrow{\mathrm{id}\times\eta} C_3(n_1,\ldots,n_k;n)\times C_3(n_i;n_i) \xrightarrow{\circ_i} C_3(n_1,\ldots,n_k;n)$$

is the projection onto the first component follows: if $i \neq 1$ then everything is the same as for the operad C_1; if $i = 1$ then we identify the two roots and nothing changes. For the composition

$$* \times C_3(n_1,\ldots,n_k;n) \xrightarrow{\eta\times\mathrm{id}} C_3(n,n) \times C_3(n_1,\ldots,n_k;n) \xrightarrow{\circ_1} C_3(n_1,\ldots,n_k;n)$$

we observe that the second convention ensures that this arrow is the projection onto the second component.

Equivariance holds because it holds for C_1 and it does not affect the edge-labelings. To prove the associativity, let us consider the $\circ_i$ operations instead of the γ-s. First note that associativity needs to be checked only for the root-vertex-operations: if the blowups are made in other vertices, then we can consider the trees T_i as having the root at the edge labeled with zero. Then any operation is the same as for C_1, with the T_i-s replaced by T_i', obtained by rotating each T_i until the 0-labeled external edge arrives to the roots position (and this edge will become the root of T_i'). On the other hand, we can do a similar rotation with the first tree T, if its root vertex is involved in a $\circ_i$ operation: if the label of the root of the blowing-up tree is l, rotate T until its l-th external edge in the canonical order reaches the roots position, etc. Combining these rotations with the blowups one can see that associativity holds in any case. ♦

Proposition 1.6.4. *The algebras of C_3 are exactly the non-symmetric cyclic operads in $\mathcal{S}ets$.*

Proof. Suppose P is a C_3-algebra, given by the map of operads $\Psi : C_3 \longrightarrow \mathcal{S}ets$. Since $C_3(n;n) = \mathbb{Z}_{n+1}^{\mathrm{op}}$, the algebra structure determines a right $\mathbb{Z}_{n+1}$ action on every $P(n)$. Define the unit of P in the same way as before, that is $\eta : * \longrightarrow P(1)$, $\eta = \Psi(|)$. Since $C_3(\ ;1)$ has only one element, the unique map $C_3(1;1) \times C_3(\ ;1) \longrightarrow C_3(\ ;1)$ forces the commutativity of the diagram

$$\begin{tikzcd} * \arrow[r, "\eta"] \arrow[d, equal] & P(1) \arrow[d, "t"] \\ * \arrow[r, "\eta"] & P(1) \end{tikzcd}$$

The operations are the same as for C_1: explicitly, $\mathrm{Cor}_a \sqcup_i \mathrm{Cor}_b \in C(a,b;a+b-1)$ with the canonical labeling corresponds to $\circ_i : P(a) \times P(b) \longrightarrow P(a+b-1)$. Thus P is a non-symmetric operad. To check that P is cyclic, we still have to see that the following two diagrams commute, according to [30], page 248:

$$\begin{tikzcd} P(a) \times P(b) \arrow[r, "\circ_i"] \arrow[d, "t_a \times \mathrm{id}"'] & P(a+b-1) \arrow[d, "t_{a+b-1}"] \\ P(a) \times P(b) \arrow[r, "\circ_{i-1}"] & P(a+b-1) \end{tikzcd}$$

for each $2 \leq i \leq a$ and

$$\begin{tikzcd} P(a) \times P(b) \arrow[r, "\circ_1"] \arrow[d, "t_a \times t_b"'] & P(a+b-1) \arrow[dd, "t_{a+b-1}"] \\ P(a) \times P(b) \arrow[d, "\mathrm{sym}"'] & \\ P(b) \times P(a) \arrow[r, "\circ_n"] & P(a+b-1) \end{tikzcd}$$

where t_a denotes the cycle $(0, 1, \ldots, a)$. The first one follows since the following two operations give the same output for the elements chosen as below:

$$C_3(a, b; a+b-1) \times C_3(a; a) \longrightarrow C_3(a, b; a+b-1)$$

for $(\mathrm{Cor}_a \sqcup_{i-1} \mathrm{Cor}_b, \mathrm{Cor}_a \cdot \tau)$, where $\tau : \{0, 1, \ldots, a\} \longrightarrow \mathrm{Cor}_a$ labels the root with a and

$$C_3(a+b-1; a+b-1) \times C_3(a, b; a+b-1) \longrightarrow C_3(a, b; a+b-1)$$

for $(\mathrm{Cor}_{a+b-1} \cdot \tau, \mathrm{Cor}_a \sqcup_i \mathrm{Cor}_b)$. A similar argument proves that the second diagram commutes too. This time we look at the images of well chosen elements under the operations

$$\begin{tikzcd} C_3(b, a; a+b-1) \times C_3(b; b) \times C_3(a; a) \arrow[d, "\sigma"] \\ C_3(a, b; a+b-1) \times C_3(a; a) \times C_3(b; b) \arrow[d] \\ C_3(a, b, ; a+b-1) \end{tikzcd}$$

and

$$C_3(a+b-1; a+b-1) \times C_3(a, b; a+b-1) \longrightarrow C_3(a, b; a+b-1)$$

Thus C_3-algebras are indeed non-symmetric cyclic operads. ♦

1.6.4 A coloured operad governing cyclic operads

It is clear that the only thing we have to change in the construction of C_3 is the range of the τ-s: in this case $C_4(n_1, \ldots, n_k; n)$ consists of equivalence classes (T, σ, τ) where $\tau : \{0, 1, \ldots, n\} \longrightarrow \mathrm{Edg}(T)$ is any possible ordering of the external edges of T.

2

Model structure and Boardman-Vogt resolution for cyclic operads

The goal of this chapter is to provide sufficient conditions for the existence of a model structure on cyclic operads and – once this question is solved – to extend the Boardman-Vogt resolution of operads to the cyclic case, providing thus a concrete functorial cofibrant resolution for cyclic operads. Our approach to attack these problems has its origins in [4] and [5] and it works in an arbitrary symmetric monoidal model category with some mild assumptions.

The first part of the chapter deals with the model structure and it starts with a discussion about Hopf objects –tools that in turn can be used to define cyclic cooperads in the underlying monoidal model category. The cyclic cooperads constructed in this way are used then to construct the desired model structure on cyclic operads.

The second part of the chapter covers the Boardman-Vogt resolution of cyclic operads. After the W-construction is introduced, we prove its cofibrancy and its functorial properties. The last subsection provides an example in the category of differential graded vector spaces, which is the extension of the bar-cobar resolution of [15] to the cyclic case.

2.1 Model structure on cyclic operads

2.1.1 Hopf objects and a cyclic cooperad structure on them

The key tool to prove the existence of a model structure on cyclic operads is a cyclic cooperad structure induced by some Hopf objects in the underlying symmetric monoidal model category. In this subsection we develop and study these notions. To fix notation, $(\mathcal{E}, \otimes, I, a, s, l, r)$ or briefly $\mathcal{E}$ will denote a symmetric monoidal category.

Hopf objects in $\mathcal{E}$ and their higher order properties

A Hopf object $(H, \mu, \eta, \Delta, \epsilon)$ in a symmetric monoidal category $\mathcal{E}$ is an object $H \in \mathcal{E}$ which is both a monoid (H, μ, η) and a comonoid (H, Δ, ϵ), such that μ and η are maps of comonoids (or equivalently Δ and ϵ are maps of monoids [37]). This definition makes sense because if (H, μ, η) is a monoid, then one can define a monoid structure $(H \otimes H, \mu', \eta')$ by setting μ' to be the composition

$$(H \otimes H) \otimes (H \otimes H) \xrightarrow{\mathrm{id} \otimes s \otimes \mathrm{id}} H \otimes H \otimes H \otimes H \xrightarrow{\mu \otimes \mu} H \otimes H$$

and η' to be the composition

$$I \xrightarrow{\simeq} I \otimes I \xrightarrow{\eta \otimes \eta} H \otimes H \ .$$

Similarly, there is a comonoid $(H \otimes H, \Delta', \epsilon')$ provided (H, Δ, ϵ) is a comonoid, and even more generally, one can define a comonoid $(H^{\otimes n+1}, \Delta_{n+1}, \epsilon_{n+1})$ for all $n \in \mathbb{N}$ by

$$H^{\otimes n+1} \xrightarrow{\Delta^{\otimes n+1}} (H \otimes H)^{\otimes n+1} \xrightarrow{\text{sym.}} H^{\otimes n+1} \otimes H^{\otimes n+1} \ .$$

The Δ_{n+1} above gives rise to maps $\Delta_X : H^{\otimes X} \longrightarrow H^{\otimes X} \otimes H^{\otimes X}$ for any finite set X, where $H^{\otimes X} := \bigotimes_{x \in X} H$ denotes the unordered tensor product.

Lemma 2.1.1. *The Δ_X maps are associative – the following diagram commutes for any finite set X:*

$$\begin{tikzcd}
H^{\otimes X} \arrow[r, "\Delta_X"] \arrow[d, "\Delta_X"'] & H^{\otimes X} \otimes H^{\otimes X} \arrow[d, "\Delta_X \otimes \mathrm{id}"] \\
H^{\otimes X} \otimes H^{\otimes X} \arrow[r, "\mathrm{id} \otimes \Delta_X"] & H^{\otimes X} \otimes H^{\otimes X} \otimes H^{\otimes X}
\end{tikzcd}$$

Proof. The claim is true for Δ_{n+1}, i.e. when $X = \{0, 1, \ldots, n\}$ with the obvious ordering. Then the general case follows because of the naturality of symmetry. ♦

The iterated use of the multiplication μ gives rise to maps $H^{\otimes n+1} \longrightarrow H$ for all $n \in \mathbb{N}$. If H is a commutative Hopf object (i.e. μ is commutative), for any finite set X we can define $\mu_X : H^{\otimes X} \longrightarrow H$, because the iterated multiplication does not depend on the chosen ordering $f : X \longrightarrow [n]$.

The following maps and their properties below will be relevant in the proof of coassociativity, when we define the cyclic cooperad structure:

$$H^{\otimes X_x \sqcup_y Y} \xrightarrow{\text{sym}} H^{\otimes X-\{x\}} \otimes H^{\otimes Y-\{y\}} \xrightarrow{\text{id} \otimes \mu} H^{\otimes X-\{x\}} \otimes H \xrightarrow{\text{sym}} H^{\otimes X}$$

which we will call $\mu_Y^x : H^{\otimes X_x \sqcup_y Y} \longrightarrow H^{\otimes X}$.

Lemma 2.1.2. *The following diagram commutes:*

$$\begin{tikzcd} H^{\otimes X_x \sqcup_y Y} \arrow[r, "\mu_Y^x"] \arrow[d, "\Delta_{X_x \sqcup_y Y}"'] & H^{\otimes X} \arrow[d, "\Delta_X"] \\ H^{\otimes X_x \sqcup_y Y} \otimes H^{\otimes X_x \sqcup_y Y} \arrow[r, "\mu_Y^x \otimes \mu_Y^x"] & H^{\otimes X} \otimes H^{\otimes X} \end{tikzcd}$$

Proof. Since μ is a map of comonoids,

$$\begin{tikzcd} H \otimes H \arrow[rr, "\mu"] \arrow[d, "\Delta \otimes \Delta"'] & & H \arrow[dd, "\Delta"] \\ H \otimes H \otimes H \otimes H \arrow[d, "\text{id} \otimes s \otimes \text{id}"'] & & \\ H \otimes H \otimes H \otimes H \arrow[rr, "\mu \otimes \mu"] & & H \otimes H \end{tikzcd}$$

commutes. This implies that the diagram

$$\begin{tikzcd} H^{\otimes X} \arrow[r, "\mu_X"] \arrow[d, "\Delta_X"'] & H \arrow[d, "\Delta"] \\ H^{\otimes X} \otimes H^{\otimes X} \arrow[r, "\mu_X \otimes \mu_X"] & H \otimes H \end{tikzcd}$$

commutes if X is any two-point-set. Then the last diagram commutes for any finite set X, which one can prove by induction on the cardinality of X: first tensor the diagram with the vertical arrow $\Delta : H \longrightarrow H \otimes H$ to obtain

$$\begin{tikzcd} H^{\otimes X} \otimes H \arrow[r, "\mu_X \otimes \text{id}"] \arrow[d, "\Delta_X \otimes \Delta"'] & H \arrow[d, "\Delta \otimes \Delta"] \\ H^{\otimes X} \otimes H^{\otimes X} \otimes H \otimes H \arrow[r, "\mu_X \otimes \mu_X \otimes \text{id}"] & H \otimes H \otimes H \otimes H \end{tikzcd}$$

Now glue to the right-hand side of the diagram above the one which says that μ is a map of comonoids, then use associativity of μ and naturality of s to conclude that the diagram commutes for the finite set $X \sqcup *$. To finish the proof, one has to tensor the resulting commutative diagram for the set $Y - \{y\}$ with the vertical arrow $\Delta_{X-\{x\}} : H^{\otimes X-\{x\}} \longrightarrow H^{\otimes X-\{x\}} \otimes H^{\otimes X-\{x\}}$ and use again naturality of s. ♦

To simplify notation, we introduce $XYZ := X_x\sqcup_y Y_{y'}\sqcup_z Z$, $XY := X_x\sqcup_y Y$ and $YZ := Y_{y'}\sqcup_z Z$.

Corollary 2.1.3. *The following diagram commutes:*

$$\begin{tikzcd}
H^{\otimes XYZ} \otimes H^{\otimes XYZ} \arrow[r, "\mu_Z^{y'} \otimes \mu_{XY}^z"] \arrow[d, "\Delta_{XYZ} \otimes \mathrm{id}"'] & H^{\otimes XY} \otimes H^{\otimes Z} \arrow[d, "\Delta_{XY} \otimes \mathrm{id}"] \\
H^{\otimes XYZ} \otimes H^{\otimes XYZ} \otimes H^{\otimes XYZ} \arrow[r, "\mu_Z^{y'} \otimes \mu_Z^{y'} \otimes \mu_{XY}^z"] & H^{\otimes XYZ} \otimes H^{\otimes XYZ} \otimes H^{\otimes Z}
\end{tikzcd}$$

Proof. It follows from Lemma 2.1.2 by tensoring the diagram for the finite set XYZ with the horizontal arrow $\mu_{XY}^z : H^{\otimes XYZ} \longrightarrow H^{\otimes Z}$. ♦

Lemma 2.1.4. *The following diagram commutes:*

$$\begin{tikzcd}
H^{\otimes XYZ} \otimes H^{\otimes XYZ} \otimes H^{\otimes XYZ} \arrow[r, "\mu_Z^{y'} \otimes \mu_Z^{y'} \otimes \mu_{XY}^z"] \arrow[d, "\mu_{YZ}^x \otimes \mu_X^y \otimes \mu_X^y"'] & H^{\otimes XY} \otimes H^{\otimes XY} \otimes H^{\otimes Z} \arrow[d, "\mu_Y^x \otimes \mu_X^y \otimes \mathrm{id}"] \\
H^{\otimes X} \otimes H^{\otimes YZ} \otimes H^{\otimes YZ} \arrow[r, "\mathrm{id} \otimes \mu_Z^{y'} \otimes \mu_Y^z"] & H^{\otimes X} \otimes H^{\otimes Y} \otimes H^{\otimes Z}
\end{tikzcd}$$

Proof. The diagram breaks into 3 small ones because of the functoriality of $\otimes$. Each of the small diagrams commutes since μ is commutative. ♦

The cyclic cooperad CH

Suppose H is a commutative Hopf object in $\mathcal{E}$. One can define a cyclic cooperad induced by H, following [3].

Let $CH(X) := H^{\otimes X}$ for any finite set X. The structure maps

$$CH(X_x\sqcup_y Y) \xrightarrow{{}_x\circ_y} CH(X) \otimes CH(Y)$$

are defined by

$$H^{\otimes X_x\sqcup_y Y} \xrightarrow{\Delta_{X_x\sqcup_y Y}} H^{\otimes X_x\sqcup_y Y} \otimes H^{\otimes X_x\sqcup_y Y} \xrightarrow{\mu_Y^x \otimes \mu_X^y} H^{\otimes X} \otimes H^{\otimes Y}$$

Theorem 2.1.5. *CH with the structure maps above forms a cyclic cooperad.*

Proof. Coassociativity. One of the coassociativity diagrams for CH is

$$\begin{tikzcd}
CH(X_x\sqcup_y Y_{y'}\sqcup_z Z) \arrow[r, "{}_{y'}\circ_z"] \arrow[d, "{}_x\circ_y"'] & CH(X_x\sqcup_y Y)\otimes CH(Z) \arrow[d, "{}_x\circ_y\otimes\mathrm{id}"] \\
CH(X)\otimes CH(Y_{y'}\sqcup_z Z) \arrow[r, "\mathrm{id}\otimes{}_{y'}\circ_z"] & CH(X)\otimes CH(Y)\otimes CH(Z)
\end{tikzcd}$$

Following the definition of the ${}_x\circ_y$ maps and the conventions on the finite sets stated before Corollary 2.1.3, this diagram is the big square

$$\begin{tikzcd}
H^{\otimes XYZ} \arrow[r, "\Delta_{XYZ}"] \arrow[d, "\Delta_{XYZ}"'] & H^{\otimes XYZ}\otimes H^{\otimes XYZ} \arrow[r, "\mu_Z^{y'}\otimes\mu_{XY}^z"] \arrow[d, "\Delta_{XYZ}\otimes\mathrm{id}"'] & H^{\otimes XY}\otimes H^{\otimes Z} \arrow[d, "\Delta_{XY}\otimes\mathrm{id}"] \\
H^{\otimes XYZ}\otimes H^{\otimes XYZ} \arrow[r, "\mathrm{id}\otimes\Delta_{XYZ}"] \arrow[d, "\mu_{YZ}^x\otimes\mu_X^y"'] & \left(H^{\otimes XYZ}\right)^{\otimes 3} \arrow[r, "\mu_Z^{y'}\otimes\mu_Z^{y'}\otimes\mu_{XY}^z"] \arrow[d, "\mu_{YZ}^x\otimes\mu_X^y\otimes\mu_X^y"'] & H^{\otimes XY}\otimes H^{\otimes XY}\otimes H^{\otimes Z} \arrow[d, "\mu_Y^x\otimes\mu_X^y\otimes\mathrm{id}"] \\
H^{\otimes X}\otimes H^{\otimes YZ} \arrow[r, "\mathrm{id}\otimes\Delta_{YZ}"] & H^{\otimes X}\otimes H^{\otimes YZ}\otimes H^{\otimes YZ} \arrow[r, "\mathrm{id}\otimes\mu_Z^{y'}\otimes\mu_Y^z"] & H^{\otimes X}\otimes H^{\otimes Y}\otimes H^{\otimes Z}
\end{tikzcd}$$

The small squares commute because of Lemma 2.1.1, Corollary 2.1.3 and Lemma 2.1.4, so the big square is commutative. One can similarly check that the other diagram for coassociativity commutes to:

$$\begin{tikzcd}
CH(X_x\sqcup_y Y_{x'}\sqcup_z Z) \arrow[r, "{}_{x'}\circ_z"] \arrow[d, "{}_x\circ_y"'] & CH(X_x\sqcup_y Y)\otimes CH(Z) \arrow[d, "{}_x\circ_y\otimes\mathrm{id}"] \\
CH(Y)\otimes CH(X_{x'}\sqcup_z Z) \arrow[r, "s\circ(\mathrm{id}\otimes{}_{x'}\circ_z)"] & CH(X)\otimes CH(Y)\otimes CH(Z)
\end{tikzcd}$$

Coequivariance. The diagram we are aiming at is

$$\begin{tikzcd}
CH(X_x\sqcup_y Y) \arrow[r, "{}_x\circ_y"] \arrow[d, "\sigma_x\circ_y\tau"'] & CH(X)\otimes CH(Y) \arrow[d, "\sigma\otimes\tau"] \\
CH(X'_{x'}\sqcup_{y'} Y') \arrow[r, "{}_{x'}\circ_{y'}"] & CH(X')\otimes CH(Y')
\end{tikzcd},$$

where $\sigma : X \longrightarrow X'$ and $\tau : Y \longrightarrow Y'$ are isomorphisms with $\sigma(x) = x'$ and $\tau(y) = y'$. (Recall that the underlying *cyclic cocollection* of a (cyclic) cooperad is a *covariant* functor $C : \mathcal{F}in \longrightarrow \mathcal{E}$). The diagram commutes because the following diagrams do so:

$$\begin{tikzcd}
H^{\otimes X_x \sqcup_y Y} \arrow[r, "\Delta"] \arrow[d, "\sigma_x \circ_y \tau"'] & H^{\otimes X_x \sqcup_y Y} \otimes H^{\otimes X_x \sqcup_y Y} \arrow[d, "\sigma_x \circ_y \tau \otimes \sigma_x \circ_y \tau"] \\
H^{\otimes X'_{x'} \sqcup_{y'} Y'} \arrow[r, "\Delta"] & H^{\otimes X'_{x'} \sqcup_{y'} Y'} \otimes H^{\otimes X'_{x'} \sqcup_{y'} Y'}
\end{tikzcd}$$

and

$$\begin{tikzcd}
H^{\otimes X_x \sqcup_y Y} \arrow[r, "\mathrm{sym}"] \arrow[d, "\sigma_x \circ_y \tau"'] & H^{\otimes X - \{x\}} \otimes H^{\otimes Y - \{y\}} \arrow[d, "\sigma \otimes \tau"] \\
H^{\otimes X'_{x'} \sqcup_{y'} Y'} \arrow[r, "\mathrm{sym}"] & H^{\otimes X' - \{x'\}} \otimes H^{\otimes Y' - \{y'\}}
\end{tikzcd}$$

commute since the symmetry of $\otimes$ is natural,

$$\begin{tikzcd}
H^{\otimes X - \{x\}} \otimes H^{\otimes Y - \{y\}} \arrow[r, "\mathrm{id} \otimes \mu_{Y - \{y\}}"] \arrow[d, "\sigma \otimes \tau"'] & H^{\otimes X - \{x\}} \otimes H \arrow[d, "\sigma \otimes \mathrm{id}"] \\
H^{\otimes X' - \{x'\}} \otimes H^{\otimes Y' - \{y'\}} \arrow[r, "\mathrm{id} \otimes \mu_{Y' - \{y'\}}"] & H^{\otimes X' - \{x'\}} \otimes H
\end{tikzcd}$$

and the other, similar diagram commute because μ is commutative. Finally,

$$\begin{tikzcd}
H^{\otimes X - \{x\}} \otimes H \arrow[r, "\mathrm{sym}"] \arrow[d, "\sigma \otimes \mathrm{id}"'] & H^{\otimes X} \arrow[d, "\sigma"] \\
H^{\otimes X' - \{x'\}} \otimes H \arrow[r, "\mathrm{sym}"] & H^{\otimes X'}
\end{tikzcd}$$

and the other, similar diagrams are commutative since $\otimes$ is natural. Patching together the above diagrams gives the coequivariance of CH.

Counit. The components of the counit, i.e. the maps $\varepsilon_E : CH(E) \longrightarrow I$ for any two-point-set E, are induced by

$$H \otimes H \xrightarrow{\varepsilon \otimes \varepsilon} I \otimes I \xrightarrow{\simeq} I\ . \tag{2.1.1}$$

Observe that the map defined in (2.1.1) is the same as

$$H \otimes H \xrightarrow{m} H \xrightarrow{\varepsilon} I \tag{2.1.2}$$

since ε is a map of monoids. This fact is needed to conclude that the counit condition of a cyclic cooperad is satisfied. Namely, (2.1.2) and commutativity of H imply that the diagram

$$\begin{tikzcd} CH(E) \arrow[r, "\varepsilon_E"] & I \arrow[d, equal] \\ CH(E') \arrow[u, leftarrow, "\tau"'] \arrow[r, "\varepsilon_{E'}"] & I \end{tikzcd}$$

commutes for any map $\tau : E' \longrightarrow E$, and (2.1.1) and (2.1.2) together imply the other two properties. For example, the following map is the identity:

$$CH(X) \xrightarrow{\text{sym}} CH(X {}_x\sqcup_e E) \xrightarrow{x \circ e} CH(X) \otimes CH(E) \xrightarrow{\mathrm{id} \otimes \varepsilon} CH(X) \otimes I \xrightarrow{\simeq} CH(X)$$

♦

Before we turn our attention to the abstract homotopy theory of cyclic operads, we need to recall the following definition.

Definition 2.1.6. A *monoidal model category* is a closed monoidal category $\mathcal{E}$ which is also a model category, and the two structures are compatible in the following sense:

If $f : A \longrightarrow B$ and $g : X \longrightarrow Y$ are cofibrations in $\mathcal{E}$ then the dotted arrow in the following pushout-diagram is a cofibration, which is trivial if one of f and g is.

$$\begin{tikzcd} A \otimes X \arrow[r, "f \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes g"'] & B \otimes X \arrow[d] \arrow[ddr, bend left] & \\ A \otimes Y \arrow[r] \arrow[drr, bend right] & \text{p.o.} \arrow[dr, dotted] & \\ & & B \otimes Y \end{tikzcd}$$

This compatibility condition is called the *pushout-product axiom*.

A monoidal model category is called *symmetric* whenever the closed monoidal category $\mathcal{E}$ is symmetric.

Since every cyclic operad is in particular an operad, proving the existence of a model structure for cyclic operads inherit the difficulties existing for operads, One of the problems is coming from the distinction between the reduced and unreduced case, as it was made apparent in [3]. We can use the techniques of [3] to remedy these problems. The first case we are studying is that of the *reduced cyclic operads*, i.e. cyclic operads with the property that $P(X) = I$ whenever X is a set with one element.

2.1.2 The model structure in the reduced case

In this subsection $\mathcal{C}yc\mathcal{O}p$ will denote the category of reduced cyclic operads (in a symmetric monoidal model category $\mathcal{E}$). We will consider only objects of this category and we will not mention the adjective "reduced" any more. The aim is to prove the following theorem:

Theorem 2.1.7. *Suppose that $\mathcal{E}$ is a symmetric monoidal model category with unit I and the following assumptions are satisfied:*

- *$\mathcal{E}$ is cofibrantly generated and I is cofibrant;*
- *$\mathcal{E}$ has a symmetric monoidal fibrant replacement functor;*
- *$\mathcal{E}$ admits a commutative Hopf interval.*

Then there is a cofibrantly generated model structure on the category $\mathcal{C}yc\mathcal{O}p$ of reduced cyclic operads in $\mathcal{E}$. This model structure is induced by the free-forgetful adjunction between the category $\mathcal{C}yc\mathcal{C}oll$ of reduced cyclic collections in $\mathcal{E}$ and the category $\mathcal{C}yc\mathcal{O}p$.

Proof. (1) $\mathcal{C}yc\mathcal{O}p$ is complete and cocomplete.
Since $\mathcal{E}$ is complete and cocomplete, $\mathcal{C}yc\mathcal{C}oll$ is also (limits and colimits in functor-categories are computed pointwise). The functor $U\mathbb{F} : \mathcal{C}yc\mathcal{C}oll \longrightarrow \mathcal{C}yc\mathcal{C}oll$ is naturally part of a monad. Moreover, U is monadic, that is the category of algebras of this monad $\mathcal{C}yc\mathcal{C}oll^{U\mathbb{F}}$ is isomorphic to $\mathcal{C}yc\mathcal{O}p$. (See [14] for a proof.)
We can use Proposition 4.3.6. of [7] to obtain immediately that $\mathcal{C}yc\mathcal{O}p$ is complete. For cocompleteness we only have to see that U preserves filtered colimits. But this is the case, since U creates filtered colimits.

(2) The functor $\mathbb{F} : \mathcal{C}yc\mathcal{C}oll \longrightarrow \mathcal{C}yc\mathcal{O}p$ preserves small objects.

(3) $\mathcal{C}yc\mathcal{O}p$ has a fibrant replacement functor.
Since $\mathcal{E}$ has a symmetric monoidal fibrant replacement functor $(-)^{\sim}$, if we define for a $P \in \mathcal{C}yc\mathcal{O}p$ the fibrant replacement $\tilde{P}$ pointwise: $\tilde{P}(X) := (P(X))^{\sim}$ then it inherits the structure of a cyclic operad from P and $(-)^{\sim}$.
Define the structure maps $\tilde{P}(X) \otimes \tilde{P}(Y) \xrightarrow{{}_x\circ_y} \tilde{P}(X\, {}_x\sqcup_y Y)$ by the composition of the structure maps of P and that of $(-)^{\sim}$, i.e.

$$(P(X))^{\sim} \otimes (P(Y))^{\sim} \xrightarrow{m} (P(X) \otimes P(Y))^{\sim} \xrightarrow{({}_x\circ_y)^{\sim}} (P(X\, {}_x\sqcup_y Y))^{\sim} = \tilde{P}(X\, {}_x\sqcup_y Y)$$

Then the axioms of a cyclic operad are satisfied, because m is associative, binatural and $(-)^{\sim}$ has a unit. For example equivariance reads as: for all $\sigma \colon X' \longrightarrow X$ and $\tau \colon Y' \longrightarrow Y$ such that $\sigma(x') = x$ and $\tau(y') = y$, the rectangle

$$\begin{tikzcd}
\tilde{P}(X) \otimes \tilde{P}(Y) \arrow[r, "{}_x\circ_y"] & \big(P(X) \otimes P(Y)\big)^{\sim} \arrow[r, "({}_x\circ_y)^{\sim}"] & \tilde{P}(X\, {}_x\sqcup_y Y) \\
\tilde{P}(X') \otimes \tilde{P}(Y') \arrow[r, "{}_{x'}\circ_{y'}"] \arrow[u, leftarrow, "\tilde{\sigma}\otimes\tilde{\tau}"'] & \big(P(X') \otimes P(Y')\big)^{\sim} \arrow[r, "({}_{x'}\circ_{y'})^{\sim}"] \arrow[u, leftarrow, dotted, "(\sigma\otimes\tau)^{\sim}"'] & \tilde{P}(X'\, {}_{x'}\sqcup_{y'} Y') \arrow[u, leftarrow, "(\sigma\, {}_{x'}\circ_{y'}\, \tau)^{\sim}"']
\end{tikzcd}$$

commutes because of binaturality and functoriality of $(-)\tilde{}$. So $\mathcal{C}yc\mathcal{O}p$ has a fibrant replacement functor.

(4) $\mathcal{C}yc\mathcal{O}p$ has functorial path objects for fibrant objects.
Suppose H is a commutative Hopf object in $\mathcal{E}$. Then CH defined above is a cyclic cooperad. Next, if C is a cyclic cooperad and P is a cyclic operad then $\underline{\mathrm{Hom}}(C,P) = P^C$ is a cyclic operad (see [3] and [14]): the action of a $\sigma : X' \longrightarrow X$ is defined by conjugation, giving a map $\sigma : P^C(X) \longrightarrow P^C(X')$ and the structure maps are defined by the compositions

$$P^C(X)\otimes P^C(Y) \to (P(X)\otimes P(Y))^{C(X)\otimes C(Y)} \to P(X\,{}_x{\sqcup}_y Y)^{C(X\,{}_x{\sqcup}_y Y)} \tag{2.1.3}$$

Note that the bifunctor $-^{-} : \mathcal{C}yc\mathcal{O}p \times \mathcal{C}yc\mathcal{C}oop \longrightarrow \mathcal{C}yc\mathcal{O}p$ is covariant in the first variable and contravariant in the second, so we indeed need a cyclic cooperad to define the right-hand side map in (2.1.3).
For the rest we can copy the proof of Theorem 3.1 in [3]:
Since $\mathcal{E}$ admits a commutative Hopf interval, the folding map $I \oplus I \longrightarrow I$ factors through a commutative Hopf object

$$I \oplus I \rightarrowtail H \xrightarrow{\sim} I. \tag{2.1.4}$$

Because I is cofibrant, so is $I \oplus I$ and H. By the pushout-product axiom $\alpha_X : (I \oplus I)^{\otimes X} \longrightarrow H^{\otimes X}$ is a cofibration: one can prove this by induction, for example in the following diagram

$$\begin{tikzcd}
(I \oplus I) \otimes (I \oplus I) \arrow[r, "\alpha\otimes \mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\alpha"'] & H \otimes (I \oplus I) \arrow[d] \arrow[ddr, bend left, "\mathrm{id}\otimes\alpha"] & \\
(I \oplus I) \otimes H \arrow[r] \arrow[drr, bend right, "\alpha\otimes\mathrm{id}"'] & \text{p.o.} \arrow[dr, dotted, "\beta"] & \\
 & & H \otimes H
\end{tikzcd}$$

all the maps in the pushout-square are cofibrations since the functors $(I \oplus I) \otimes -$ and $- \otimes (I \oplus I)$ are left Quillen functors (because $I \oplus I$ is cofibrant, see [19]) and pushouts preserve cofibrations. Then by the pushout-product axiom also β is a cofibration. Composing the respective maps we see that $\alpha_1 : (I \oplus I) \otimes (I \oplus I) \longrightarrow H \otimes H$ is a cofibration.
Now suppose that P is a *fibrant cyclic operad*, i.e. it is fibrant as a cyclic collection. By the above $P^{CH} \longrightarrow P^{C(I\oplus I)}$ is a fibration (since P^{-} is a contravariant right Quillen functor). The canonical map $P^{C(I\oplus I)} \longrightarrow P^I \times P^I$ induces for each set X with $|X| = n$ the projection to the first and last factors $P(X)^{2^{n+1}} \longrightarrow P(X) \times P(X)$ and this is a fibration since P is

fibrant and pullbacks preserve fibrations. (Note that this map is induced by $P^{C(I\oplus I)}(X) \simeq P(X)^{\coprod_{i=1}^{2^{n+1}} I} \simeq \prod_{i=1}^{2^{n+1}} P(X)^I \simeq P(X)^{2^{n+1}}$, where we used that the contravariant functor P^- is a right adjoint hence it preserves limits.) Thus $P^{CH} \longrightarrow P \times P$ is a fibration. Finally, $H \xrightarrow{\sim} I$ factors through a trivial cofibration as section:

$$\begin{tikzcd}
I \oplus I \arrow[d] & I \arrow[l, tail] \arrow[dl, tail, "\sim"'] \arrow[ddl, equal] \\
H \arrow[d, "\sim"'] & \\
I &
\end{tikzcd}$$

so $H^{\otimes X} \longrightarrow I^{\otimes X}$ is a weak equivalence between cofibrant objects.
We can conclude – using Theorem 2.3. of [3] – that $P \longrightarrow P^{CH}$ is a weak equivalence, and the proof is complete:

$$P = P^{CI} \xrightarrow{\sim} P^{CH} \twoheadrightarrow P^{C(I\oplus I)} \twoheadrightarrow P^I \times P^I = P \times P\,. \quad \blacklozenge$$

2.1.3 The unreduced case

Theorem 2.1.8. *Suppose that $\mathcal{E}$ is a cartesian closed model category, $\mathcal{E}$ is cofibrantly generated, has cofibrant terminal object and a symmetric monoidal fibrant replacement functor. Then there is a cofibrantly generated model structure on the category of cyclic operads, induced by the free-forgetful adjunction.*

Proof. The only difference between this and the previous proof is that we can put $\tilde{P}(O) = \widetilde{P(O)}$ for any one-point set O and in the last step we do not need a cyclic cooperad structure induced by various Hopf objects of $\mathcal{E}$ to define the cyclic operads P^C which helped us concluding the proof.

Indeed, exponentiation commutes with the product (since it is a right adjoint) and hence it is strong symmetric monoidal. This implies that we have a bifunctor

$$-^- : \mathcal{C}yc\mathcal{O}p \times \mathcal{E}^{\mathrm{op}} \longrightarrow \mathcal{C}yc\mathcal{O}p,$$

so for any interval $I \oplus I \rightarrowtail J \xrightarrow{\sim} I$ and a fibrant operad P we can define a path object in the category of operads by

$$P \xrightarrow{\sim} P^J \twoheadrightarrow P \times P\,. \qquad \blacklozenge$$

We can adapt this proof to a more general situation:

Theorem 2.1.9. *Let $\mathcal{E}$ be a symmetric monoidal model category with cofibrant unit, a symmetric monoidal fibrant replacement functor and an interval with a* *coassociative and cocommutative comultiplication. Then the transfer principle by the free-forgetful adjunction is applicable and yields a model structure on the category of cyclic operads.*

Proof. If J is a cocommutative comonoid in $\mathcal{E}$ and P is a cyclic operad then we can define a cyclic operad structure on the collection $P^J(X) = P(X)^J$, similarly to the convolution cyclic operad structure given by the bifunctor

$$-^- : \mathcal{C}yc\mathcal{O}p \times \mathcal{C}yc\mathcal{C}oop^{\mathrm{op}} \longrightarrow \mathcal{C}yc\mathcal{O}p. \qquad \blacklozenge$$

Remark 2.1.10. Theorem 2.1.9 also follows from Theorem 2.1 in [5], in view of the constructions of Section 1.6. Even more, under the circumstances of the theorem we have a model structure on non-symmetric cyclic operads, and since there are coloured operads governing reduced- and pseudo cyclic operads, their category also can be endowed with a model structure.

Corollary 2.1.11. *The adjunctions of the commutative square (1.5.3) are Quillen adjunctions.*

2.2 The Boardman-Vogt resolution of cyclic operads

In [4] the authors introduced a general Boardman-Vogt resolution for operads in arbitrary monoidal model categories, provided the underlying category satisfies some suitable conditions. In the familiar cases – like those of topological, simplicial, chain operads and many others – these conditions hold and are easy to check. As a consequence of the construction, a large number of well known resolutions of operads are special instances of the general Boardman-Vogt resolution: the topological Boardman-Vogt resolution, the Godement resolution, the cobar-bar resolution are examples.

The aim of this section is to provide a Boardman-Vogt resolution in the model category of cyclic operads. The ideas and proofs appearing in these notes are the ones from [4]. In fact everything carries through in the fashion of [4], the differences are only minor ones, caused by the presence of the free cyclic operad instead of the free operad of [3] and [4].

To fix notation, let $\mathcal{E}$ be a cofibrantly generated symmetric monoidal model category. We assume that the unit I of $\mathcal{E}$ is cofibrant and we are going to use the conventions and notations introduced in Section 1.5.

2.2.1 Equivariance in monoidal model categories

We will make frequent use of the following consequence of the pushout-product axiom in symmetric monoidal model categories:

Lemma 2.2.1 (Pushout-product lemma)**.** *Let $\{f_i : X_0^i \longrightarrow X_1^i\}_{i\in J}$ be a finite set of cofibrations in a symmetric monoidal model category $(\mathcal{E}, \otimes, I)$. Denote the partial order category of nonempty subsets of J by $\mathcal{C}$ and the characteristic* *function of a subset $S \subset J$ by χ_S. Define the functor $F : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{E}$ by setting for the objects $F(S) := \bigotimes_{i \in J} X^i_{\chi_S(i)}$ and for the maps the corresponding tensor product of the f_i together with the identities. Then the map*

$$\mathrm{colim}_{S \in \mathcal{C}} F(S) \longrightarrow \bigotimes_{i \in J} X^i_1$$

is a cofibration which is trivial if one of the f_i is trivial.

Proof. The colimit in the lemma is built by a finite number of iterated pushouts and the lemma follows from the iterated application of the pushout-product axiom. ♦

In the next part of the subsection we describe the used properties of G-objects in $\mathcal{E}$ where G is a discrete group. More concretely, G can be viewed as a category with one object where the arrows are the elements of G with the group-composition. Then the category of (right) G-objects in $\mathcal{E}$ is the functor category $\mathcal{E}^{G^{\mathrm{op}}}$. In the familiar cases of $\mathcal{S}ets$ and $\mathcal{T}op$ we get back respectively the usual notions of G-sets and topological spaces together with an action of G, where the arrows are those continuous maps which respect the action.

We are interested in the homotopy theory of these objects, in the following way.

First of all, the closed monoidal structure of $\mathcal{E}$ induces a closed monoidal structure on $\mathcal{E}^{G^{\mathrm{op}}}$: for arbitrary G-objects A and B the right action of G on $A \otimes B$ is the diagonal one. One can compute that this natural choice forces an action on the internal hom B^A by conjugation. The unit of this new closed monoidal category is I with the trivial G-action.

Second, we can also transfer the model structure of $\mathcal{E}$ to $\mathcal{E}^{G^{\mathrm{op}}}$ since $\mathcal{E}$ is cofibrantly generated: a map in $\mathcal{E}^{G^{\mathrm{op}}}$ is a fibration or weak equivalence if and only if the underlying map in $\mathcal{E}$ is a fibration or weak equivalence respectively. The (trivial) cofibrations in the model category $\mathcal{E}^{G^{\mathrm{op}}}$ are described via the lifting properties. We will call them (trivial) G-cofibrations.

Note that every G-cofibration is a cofibration. This is a consequence of Lemma 2.5.1 of [4], where we can take the group homomorphism to be the inclusion of the trivial group in G:

Lemma 2.2.2. *([4]) If $\mathcal{E}$ is a cofibrantly generated model category and $f\colon H \longrightarrow G$ is a group homomorphism then induction and restriction along f form a Quillen adjunction $\mathcal{E}^H \underset{\mathrm{Res}}{\overset{\mathrm{Ind}}{\rightleftarrows}} \mathcal{E}^G$. If f is an inclusion then restriction takes G-cofibrations to H-cofibrations.*

On the other hand not every cofibration is a G-cofibration: even if $f : A \rightarrowtail B$ is a G-equivariant cofibration – i.e. a map of G-objects which is a cofibration in $\mathcal{E}$ –, it does not have to be a G-cofibration: the existing lifts in a lifting diagram of G-objects, viewed in $\mathcal{E}$ do not have to respect the G-action!

After introducing the above monoidal- and model structures, the natural question is whether $\mathcal{E}^{G^{\mathrm{op}}}$ is a monoidal model category. The following lemma implies the pushout-product axiom in $\mathcal{E}^{G^{\mathrm{op}}}$ and it also gives rise to an equivariant pushout-product lemma.

Lemma 2.2.3. *([4]) Let $A \rightarrowtail B$ and $X \rightarrowtail Y$ be G-equivariant cofibrations. If one of them is a G-cofibration then the pushout-product map $(A \otimes Y) \cup_{A\otimes X} (B \otimes X) \longrightarrow B \otimes Y$ is a G-cofibration. Moreover, the latter is trivial if $A \rightarrowtail B$ or $X \rightarrowtail Y$ is.*

Proof. Suppose that we have a square in $\mathcal{E}^G$

$$\begin{tikzcd} (A \otimes Y) \cup_{A\otimes X} (B \otimes X) \arrow[r] \arrow[d] & Z \arrow[d] \\ B \otimes Y \arrow[r] & W \end{tikzcd}$$

where $Z \longrightarrow W$ is a trivial fibration in $\mathcal{E}^G$ – or equivalently, in $\mathcal{E}$ – and $A \longrightarrow B$ is a G-cofibration. To find a diagonal filler for this diagram, first transpose it to another diagram, using the tensor-exponential Quillen adjunction (in $\mathcal{E}$):

$$\Phi_P^{MN} : \ \mathcal{E}(M \otimes N, P) \xrightarrow{\simeq} \mathcal{E}(M, P^N) \ .$$

Φ is natural in all the three variables, explicitly if $\alpha : M \longrightarrow M'$, $\beta : N \longrightarrow N'$ and $\gamma : P \longrightarrow P'$ are maps in $\mathcal{E}$ then the following diagrams commute, meaning that the dotted arrows (compositions) on the right-hand side column of the table correspond to each other via Φ: We can deduce that the following commutative

$\begin{tikzcd} \mathcal{E}(M \otimes N, P) \arrow[r, "\Phi"] & \mathcal{E}(M, P^N) \\ \mathcal{E}(M' \otimes N, P) \arrow[r, "\Phi"] \arrow[u, "(\alpha\otimes\mathrm{id}_N)^*"] & \mathcal{E}(M', P^N) \arrow[u, "\alpha^*"'] \end{tikzcd}$	$\begin{tikzcd} M \otimes N \arrow[d, "\alpha\otimes\mathrm{id}_N"'] \arrow[dr, dotted] & & & M \arrow[d, "\alpha"'] \arrow[dr, dotted] & \\ M' \otimes N \arrow[r, "f"'] & P & \overset{\Phi}{\longleftrightarrow} & M' \arrow[r, "\Phi f"'] & P^N \end{tikzcd}$
$\begin{tikzcd} \mathcal{E}(M \otimes N, P) \arrow[r, "\Phi"] & \mathcal{E}(M, P^N) \\ \mathcal{E}(M \otimes N', P) \arrow[r, "\Phi"] \arrow[u, "(\mathrm{id}_M\otimes\beta)^*"] & \mathcal{E}(M, P^{N'}) \arrow[u, "(P^\beta)_*"'] \end{tikzcd}$	$\begin{tikzcd} M \otimes N \arrow[d, "\mathrm{id}_M\otimes\beta"'] \arrow[dr, dotted] & & & & P^N \\ M \otimes N' \arrow[r, "g"'] & P & \overset{\Phi}{\longleftrightarrow} & M \arrow[r, "\Phi g"'] \arrow[ur, dotted] & P^{N'} \arrow[u, "P^\beta"'] \end{tikzcd}$
$\begin{tikzcd} \mathcal{E}(M \otimes N, P) \arrow[r, "\Phi"] \arrow[d, "\gamma_*"'] & \mathcal{E}(M, P^N) \arrow[d, "(\gamma^N)_*"] \\ \mathcal{E}(M \otimes N, P') \arrow[r, "\Phi"] & \mathcal{E}(M, P'^N) \end{tikzcd}$	$\begin{tikzcd} M \otimes N \arrow[r, "h"] \arrow[dr, dotted] & P \arrow[d, "\gamma"] & \overset{\Phi}{\longleftrightarrow} & M \arrow[r, "\Phi h"] \arrow[dr, dotted] & P^N \arrow[d, "\gamma^N"] \\ & P' & & & P'^N \end{tikzcd}$

diagrams correspond to each other by the above adjointness, where in the left-hand side diagram the square above is a pushout square and in the right-hand side diagram the square below is a pullback square.

$$\begin{tikzcd} A\otimes X \arrow[r] \arrow[d] & B\otimes X \arrow[d] & \\ A\otimes Y \arrow[r] & \text{p.o.} \arrow[r] \arrow[d] & Z \arrow[d] \\ & B\otimes Y \arrow[r] & W \end{tikzcd} \quad \overset{\Phi}{\longleftrightarrow} \quad \begin{tikzcd} A \arrow[r] \arrow[d] & Z^Y \arrow[d] & \\ B \arrow[r] & \text{p.b.} \arrow[r] \arrow[d] & Z^X \arrow[d] \\ & W^Y \arrow[r] & W^X \end{tikzcd}$$

For example, to induce the map $B \longrightarrow$ p.b. one has to give two maps $B \longrightarrow Z^X$ and $B \longrightarrow W^Y$ such that their compositions with $Z^X \longrightarrow W^X$ and $W^Y \longrightarrow W^X$ respectively are the same. It is obvious that these maps are the corresponding ones to $B \otimes X \longrightarrow Z$ and $B \otimes Y \longrightarrow W$ of the left-hand side diagram under Φ^{-1}, as well as $(Z \longrightarrow W)^X = Z^X \longrightarrow W^X$ and $W^{X\longrightarrow Y} = W^Y \longrightarrow W^X$. The required equality of the compositions follows then from the adjointness and the commutativity of the diagram on the left.

Looking at these two diagrams we realize that the map p.o. $\longrightarrow B \otimes Y$ (the pushout-product map) is a G-cofibration if and only if for any trivial fibration $Z \longrightarrow W$ in $\mathcal{E}^{G^{\mathrm{op}}}$ the square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & Z^Y \arrow[d] \\ B \arrow[r] & W^Y \times_{W^X} Z^X \end{tikzcd}$$

admits a diagonal filler. If $A \longrightarrow B$ is a G-cofibration, such a filler exists whenever $Z^Y \longrightarrow W^Y \times_{W^X} Z^X$ is a trivial fibration in $\mathcal{E}^{G^{\mathrm{op}}}$ or equivalently in $\mathcal{E}$. On the other hand the pushout-product axiom has an equivalent "pullback-exponential" form via the Quillen adjunction between tensor and internal hom. In this case we can apply it: since $X \longrightarrow Y$ is a G-equivariant cofibration, the map $Z^Y \longrightarrow W^Y \times_{W^X} Z^X$ is a trivial fibration. A similar argument works for the case when $A \longrightarrow B$ is a G-equivariant cofibration and $X \longrightarrow Y$ is a G-cofibration and the second statement can be proved by following a similar pattern. ♦

Remark 2.2.4. Note that by the first part of the above proof the correspondence between the diagrams with the pullback and pushout can be stated as an adjointness between two endofunctors defined on the category of maps of $\mathcal{E}$. Indeed, let $f : A \longrightarrow B$, $g : X \longrightarrow Y$ and $h : Z \longrightarrow W$ be three maps in $\mathcal{E}$. We can define two new maps by setting $f \odot g : (A \otimes Y) \sqcup_{A\otimes X} (B \otimes X) \longrightarrow (B \otimes Y)$ and $h^g : Z^Y \longrightarrow W^Y \times_{W^X} Z^X$. If we denote the category of maps by $\mathrm{Maps}_{\mathcal{E}}$ then we just have defined two functors $-\odot g, (-)^g : \mathrm{Maps}_{\mathcal{E}} \longrightarrow \mathrm{Maps}_{\mathcal{E}}$, and by the first part of the above proof $-\odot g$ is left adjoint to $(-)^g$:

$$\mathrm{Maps}_{\mathcal{E}}(f \odot g, h) \simeq \mathrm{Maps}_{\mathcal{E}}(f, h^g).$$

An important consequence of the monoidal model structure on $\mathcal{E}^G$ is that it gives rise to a (monoidal) model structure on $\mathcal{C}yc\mathcal{C}oll$. We state this in the following proposition, where Σ_X denotes the symmetric group associated to the finite set X.

Proposition 2.2.5. *There is a symmetric monoidal model structure on the category $CycColl$, where*

- *a map $\alpha : K \longrightarrow L$ is a fibration if and only if $\alpha_X : K(X) \longrightarrow L(X)$ is a fibration for every $X \in \mathcal{F}in$;*
- *a map $\alpha : K \longrightarrow L$ is a weak equivalence if and only if $\alpha_X : K(X) \longrightarrow L(X)$ is a weak equivalence for every $X \in \mathcal{F}in$;*
- *a map $\alpha : K \longrightarrow L$ is a cofibration if and only if $\alpha_X : K(X) \longrightarrow L(X)$ is a Σ_X-cofibration for every $X \in \mathcal{F}in$.*

Proof. The model structure follows immediately from the fact that the direct product $\prod_{i\geq 0} \Sigma_i$ of the permutation groups is a skeleton of the groupoid $\mathcal{F}in$. It can be made symmetric monoidal once we define $(K \otimes L)(X) := K(X) \otimes L(X)$ for every $X \in Fin$. ♦

In view of the above model structure, we will call an operad $\mathcal{F}in$-cofibrant if it is cofibrant as a cyclic collection.

2.2.2 Segments and intervals

Definition 2.2.6. A *segment* in $\mathcal{E}$ is a factorization $I \sqcup I \xrightarrow{(0,1)} H \xrightarrow{\epsilon} I$ of the folding map, together with an associative and commutative operation $\vee : H \otimes H \longrightarrow H$ which has 0 as neutral element, 1 as absorbing element and ϵ as counit. The axioms satisfied by such a segment can be visualized with the following commutative diagrams, where all the occurring isomorphisms are the respective canonical ones of the symmetric monoidal structure of $\mathcal{E}$:

$$\begin{tikzcd}
(H\otimes H)\otimes H \arrow[rr, "\simeq"] \arrow[d, "\vee\otimes\mathrm{id}"'] & & H\otimes(H\otimes H) \arrow[d, "\mathrm{id}\otimes\vee"] \\
H\otimes H \arrow[r, "\vee"] & H & H\otimes H \arrow[l, "\vee"']
\end{tikzcd}
\qquad
\begin{tikzcd}
I\otimes H \arrow[r, "0\otimes\mathrm{id}"] \arrow[dr, "\simeq"'] & H\otimes H \arrow[d, "\vee"] & H\otimes I \arrow[l, "\mathrm{id}\otimes 0"'] \arrow[dl, "\simeq"] \\
& H &
\end{tikzcd}$$

$$\begin{tikzcd}
& I\otimes H \arrow[r, "1\otimes\mathrm{id}"] \arrow[dl, "\mathrm{id}\otimes\epsilon"'] \arrow[d, "\simeq"] & H\otimes H \arrow[d, "\vee"] & H\otimes I \arrow[l, "\mathrm{id}\otimes 1"'] \arrow[d, "\simeq"'] \arrow[dr, "\epsilon\otimes\mathrm{id}"] & \\
I\otimes I \arrow[r, "\simeq"] & I \arrow[r, "1"] & H & I \arrow[l, "1"'] & I\otimes I \arrow[l, "\simeq"']
\end{tikzcd}$$

$$\begin{tikzcd}
H\otimes H \arrow[rr, "\simeq"] \arrow[dr, "\vee"'] & & H\otimes H \arrow[dl, "\vee"] \\
& H &
\end{tikzcd}
\qquad
\begin{tikzcd}
H\otimes H \arrow[r, "\epsilon\otimes\epsilon"] \arrow[d, "\vee"'] & I\otimes I \arrow[d, "\simeq"] \\
H \arrow[r, "\epsilon"] & I
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[r, "0"] \arrow[d, "1"'] \arrow[dr, "\mathrm{id}"] & H \arrow[d, "\epsilon"] \\
H \arrow[r, "\epsilon"] & I
\end{tikzcd}$$

An *interval* in $\mathcal{E}$ is a segment in $\mathcal{E}$ such that $(0,1) : I \sqcup I \rightarrowtail H$ is a cofibration and $\epsilon : H \xrightarrow{\sim} I$ is a weak equivalence.

Note that if H is an interval in $\mathcal{E}$ then the maps $0, 1 : I \overset{\sim}{\rightarrowtail} H$ are trivial cofibrations and thus H is cofibrant since I is.

In the following by a tree we mean a non-planar non-rooted tree. Suppose that H is an interval in $\mathcal{E}$ and T is a tree. Denote the set of internal edges of T with $\mathrm{InEdg}(T)$. Define

$$H(T) := \bigotimes_{e \in \mathrm{InEdg}(T)} H,$$

where $\otimes$ refers to the unordered tensor product. $\mathrm{Aut}(T)$ – the automorphism group of T – acts on $H(T)$ on the right by the canonical symmetries of $\mathcal{E}$. For any subset $D \subseteq \mathrm{InEdg}(T)$ define

$$H_D(T) := \bigotimes_{e \in \mathrm{InEdg}(T)} H_e, \text{ where } H_e := \begin{cases} I & \text{if} \quad e \in D, \\ H & \text{if} \quad e \notin D. \end{cases}$$

In particular, $H_\emptyset(T) = H(T)$ and $H_{\mathrm{InEdg}(T)}(T) \simeq I$.
The trivial cofibration $0 : I \longrightarrow H$ and id_H induce an obvious canonical trivial cofibration for every $D \subseteq \mathrm{InEdg}(T)$:

$$H_D(T) \overset{\sim}{\rightarrowtail} H(T) . \tag{2.2.1}$$

Here we used a consequence of the pushout-product axiom: $- \otimes H$ and $- \otimes I$ are left-Quillen functors since H and I are cofibrant. Even more, let $\mathcal{C}$ denote the partial-order-category on the non-empty subsets of $\mathrm{InEdg}(T)$. It follows that $H_?(T) : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{E}$ is a functor which sends the inclusion $C \subseteq D$ to the trivial cofibration $H_D(T) \overset{\sim}{\rightarrowtail} H_C(T)$. To simplify notation, we introduce

$$H^-(T) := \mathrm{colim}_{D \in \mathcal{C}}\, H_D(T).$$

Because the maps in the diagram $H_?(T)$ are trivial cofibrations (they are iterated tensors of id_H and $0 : I \overset{\sim}{\rightarrowtail} H$), the iterated use of the pushout-product axiom yields a trivial cofibration

$$H^-(T) \overset{\sim}{\rightarrowtail} H(T) . \tag{2.2.2}$$

Another map we are going to use in the *W-construction* is

$$H(T) \longrightarrow H(T/_V) \tag{2.2.3}$$

for any non-empty set V of unary vertices in T and is defined as follows. Let us distinguish between two types of unary vertices of T and associate maps to each type:

– in the case of a vertex u connecting an internal edge with an external one render to u the counit $\epsilon : H \longrightarrow I$;
– in the case of a vertex u connecting two internal edges render to u the operation $\vee : H \otimes H \longrightarrow H$.

These two types of maps, together with id_H induce (2.2.3) which is well defined since $\vee$ is associative, commutative and ϵ is a counit for $\vee$.

Observe also that there is a natural isomorphism

$$H_D(T) \xrightarrow{\simeq} H(T/_D) , \tag{2.2.4}$$

where for any set of internal edges D of T, $T/_D$ denotes the tree resulting by contracting the edges in D.

2.2.3 The W-construction

Suppose that H is an interval in $\mathcal{E}$ and P is a *well-pointed $\mathcal{F}in$-cofibrant cyclic operad*. (Recall that – by definition – P is well-pointed if the unit of P is a cofibration.) We will construct a cyclic operad $W(H, P)$ as a sequential colimit of trivial cofibrations of cyclic collections:

$$W_0(H, P) \overset{\sim}{\rightarrowtail} W_1(H, P) \overset{\sim}{\rightarrowtail} W_2(H, P) \overset{\sim}{\rightarrowtail} \cdots$$

where $W_k(H, P)$ stands for the part of $W(H, P)$ which can be constructed with trees with at most k internal edges. For the sake of the construction only a segment and a cyclic operad would suffice, but to prove that $W(H, P)$ is indeed a cofibrant resolution we need the assumed extra structure.

The construction is carried out inductively: out of $W_{k-1}(H, P)$ and natural maps

$$\alpha_T : (H(T) \otimes \underline{P}(T)) \otimes_{\mathrm{Aut}(T)} \lambda_T(X) \longrightarrow W_{k-1}(H, P)(X) \tag{2.2.5}$$

defined for every tree T with at most $k-1$ internal edges and external edges labeled by the finite set X we construct $W_k(H, P)$ and the canonical maps α_T of the next level.

For $k = 0$ let $W_0(H, P)(X) := P(X)$ and define $\alpha_|$ to be the unit $I \longrightarrow P(X)$ – note that in this case X necessarily has two elements so $\alpha_|$ is well defined – and $\alpha_{\mathrm{Cor}(X)}$ to be $\mathrm{id}_{P(X)}$ for every X. (Here $\mathrm{Cor}(X)$ denotes the corolla with external edges labeled by X.)

For the next step consider for any X-labeled tree T with k internal edges the maps of $\mathrm{Aut}(T)$-objects $H^-(T) \longrightarrow H(T)$ and $\underline{P}^-(T) \longrightarrow \underline{P}(T)$. The first map is a trivial cofibration in $\mathcal{E}$ as stated in equation (2.2.2), the second one is an $\mathrm{Aut}(T)$-cofibration (this follows from the construction of the map (1.5.2) and the equivariant pushout-product lemma). It follows by Lemma 2.2.3 that the dotted arrow in

$$\begin{tikzcd}
H^-(T)\otimes \underline{P}^-(T) \arrow[r] \arrow[d] & H^-(T)\otimes \underline{P}(T) \arrow[d] \arrow[ddr, bend left] & \\
H(T)\otimes \underline{P}^-(T) \arrow[r] \arrow[drr, bend right] & \text{p.o.} \arrow[dr, dotted] & \\
 & & H(T)\otimes \underline{P}(T)
\end{tikzcd}$$

is a trivial $\mathrm{Aut}(T)$-cofibration. For simplicity denote the pushout in the above diagram by $(H\otimes P)^-(T)$ and the dotted map by

$$(H\otimes P)^-(T) \overset{\sim}{\rightarrowtail} H(T)\otimes \underline{P}(T)\ . \tag{2.2.6}$$

The maps α_T in (2.2.5) induce a Σ_X-equivariant map for every finite set X

$$\alpha_T^- : (H\otimes P)^-(T)\otimes_{\mathrm{Aut}(T)} \lambda_T(X) \longrightarrow W_{k-1}(H,P)(X). \tag{2.2.7}$$

Indeed, if $\emptyset \neq D \subseteq \mathrm{InEdg}(T)$ then the map in (2.2.4), the cyclic-operad composition $\underline{P}(T) \longrightarrow \underline{P}(T/D)$ and $\alpha_{T/D}$ define a Σ_X-equivariant map

$$(H_D(T)\otimes \underline{P}(T))\otimes_{\mathrm{Aut}(T)} \lambda_T(X) \longrightarrow W_{k-1}(H,P)(X). \tag{2.2.8}$$

In more detail, the map $H_D(T)\otimes \underline{P}(T)\otimes \lambda_T(X) \longrightarrow H(T/D)\otimes \underline{P}(T/D)\otimes \lambda_T(X)$ induced by the ones mentioned above and $\mathrm{id}_{\lambda_T(X)}$ is natural in the finite set X. These objects are $\mathrm{Aut}(T)$- and $\mathrm{Aut}(T/D)$-objects, respectively. There is a group map $\mathrm{Aut}(T) \longrightarrow \mathrm{Aut}(T/D)$, defined by "jumping over" the edges in D and the components of T affected by them. It follows that the induced map

$$(H_D(T)\otimes\underline{P}(T))\otimes_{\mathrm{Aut}(T)}\lambda_T(X) \longrightarrow (H(T/D)\otimes\underline{P}(T/D))\otimes_{\mathrm{Aut}(T/D)}\lambda_{T/D}(X)$$

is natural in X and in turn gives the map in (2.2.8) by composition with $\alpha_{T/D}$.

Taking the colimit induces a well-defined map

$$\big(H^-(T)\otimes \underline{P}(T)\big)\otimes_{\mathrm{Aut}(T)} \lambda_T(X) \longrightarrow W_{k-1}(H,P)(X), \tag{2.2.9}$$

which is again natural in X. Similarly, if $\emptyset \neq c \subseteq \mathrm{Vert}(T)$ is a subset of unary vertices in T, the maps $\underline{P}_c(T) \longrightarrow \underline{P}(T/c)$ and (2.2.3) define

$$(H(T)\otimes \underline{P}^-)\otimes_{\mathrm{Aut}(T)} \lambda_T(X) \longrightarrow W_{k-1}(H,P)(X), \tag{2.2.10}$$

and finally (2.2.9) and (2.2.10) together give the α_T^- maps of (2.2.7). Now take the coproduct of the maps in (2.2.7) over isomorphism classes of trees T (with external edges labeled by X and k internal edges) and also the coproduct of the trivial $\mathrm{Aut}(T)$ cofibrations in (2.2.6). The resulting maps fit in a pushout-diagram which defines $W_k(H,P)(X)$ and also the new maps α_T:

$$\begin{tikzcd}
\coprod\limits_{[T],T\in\mathbb{T}(X,k)} (H\otimes P)^{-}(T)\otimes_{\mathrm{Aut}(T)}\lambda_T(X) \arrow[r, "\coprod \alpha_T^{-}"] \arrow[d] & W_{k-1}(H,P)(X) \arrow[d] \\
\coprod\limits_{[T],T\in\mathbb{T}(X,k)} (H(T)\otimes \underline{P}(T))\otimes_{\mathrm{Aut}(T)}\lambda_T(X) \arrow[r, "\coprod \alpha_T"] & W_k(H,T)(X)
\end{tikzcd}$$

Note that the left-hand side vertical arrow in the above pushout diagram is a trivial Σ_X-cofibration (coproducts preserve trivial cofibrations) which implies that the map

$$W_{k-1}(H,P)(X) \overset{\sim}{\rightarrowtail} W_k(H,P)(X)$$

is again a trivial Σ_X-cofibration and it is natural in X. Moreover, it also follows from the construction that if we define

$$W(H,P)(X) = \mathrm{colim}_{k\in\mathbb{N}}\, W_k(H,P)(X)$$

then the inclusions $W_k(H,P)(X) \longrightarrow W(H,P)(X)$ are trivial Σ_X-cofibrations. In particular, for $k=0$ we get a trivial Σ_X-cofibration

$$P(X) \longrightarrow W(H,P)(X).$$

By the above, $WP(H,P) : \mathcal{F}in^{\mathrm{op}} \longrightarrow \mathcal{E}$ is a cyclic collection. We can give a cyclic operad structure to $W(H,P)$. Intuitively this structure comes from the grafting operations of trees with lengths on the internal edges, with the convention that after each grafting the resulting new internal edge gets length 1. The rigorous definition is given in terms of the α_T maps of (2.2.5), as follows below.

If the leaves of the trees T and R are labeled by X and Y respectively, and $x\in X$, $y\in Y$ then $T{}_x{\circ}_y R$ is the $X{}_x{\sqcup}_y Y$-labeled tree resulting from grafting T and R along the edges indicated by x and y. There is a map

$$H(T)\otimes H(R) \longrightarrow H(T{}_x{\circ}_y R)$$

which gives length 1 to the new internal edge in $T{}_x{\circ}_y R$ and there is another map

$$\underline{P}(T)\otimes \underline{P}(R) \longrightarrow \underline{P}(T{}_x{\circ}_y R)$$

coming from the free operad structure on $\underline{P}$. These two maps induce

$${}_x{\circ}_y : \big((H(T)\otimes\underline{P}(T)\big)_\lambda \otimes \big((H(R)\otimes\underline{P}(R))\big)_\lambda \longrightarrow \big((H(T{}_x{\circ}_y R)\otimes\underline{P}(T{}_x{\circ}_y R))\big)_\lambda,$$

where the index λ stands for $\otimes_{\mathrm{Aut}(T)}\lambda_T(X)$, etc. These ${}_x\circ_y$ maps induce the cyclic operad structure on $W(H,P)$ by requiring that the following diagram commutes:

$$\begin{tikzcd}
\big((H(T)\otimes \underline{P}(T)\big)_\lambda \otimes \big((H(R)\otimes \underline{P}(R))\big)_\lambda \arrow[r, "\alpha\otimes\alpha"] \arrow[d, "{}_x\circ_y"'] & W(H,P)(X)\otimes W(H,P)(Y) \arrow[d] \\
\big((H(T{}_x\circ_y R)\otimes \underline{P}(T{}_x\circ_y R))\big)_\lambda \arrow[r, "\alpha"] & W(H,P)(X{}_x\sqcup_y Y)
\end{tikzcd}$$

Remark 2.2.7. There is a W-construction for reduced cyclic operads, which is defined in the same way as in Subsection 2.2.3, but dropping all the trees with vertices of valence 0 from the groupoid $\mathbb{T}$. If one is interested in pseudo cyclic operads, then the W-construction is even easier since there is no need for the unit identifications in the construction, thus we can omit the binary operation on H from the construction of the previous section. We will denote the W-constructions corresponding to these cases by W^{red} and W^{ps} respectively.

Remark 2.2.8. Since the free operad on a cyclic collection is cyclic, we can conclude that the operadic Boardman-Vogt resolution of a cyclic operad is also a cyclic operad.

2.2.4 Cofibrancy and functoriality of the W-construction

The W-construction we gave in the previous subsection provides a cofibrant resolution of the cyclic operad P. In details,

Theorem 2.2.9. *If $\mathcal{E}$ is a cofibrantly generated symmetric monoidal model category with cofibrant unit I and an interval H then for any well pointed $\mathcal{F}in$-cofibrant cyclic operad, the counit $\mathbb{F}_*(P) \longrightarrow P$ of the adjunction of Subsection 1.5.2 factors as*

$$\mathbb{F}_*(P) \xrightarrow{\delta} W(H,P) \xrightarrow{\gamma} P$$

where δ is a cofibration and γ is a weak equivalence. Moreover, $\mathbb{F}_*(P)$ is a cofibrant cyclic operad, hence $W(H,P)$ is a cofibrant resolution of P.

Proof. The proof is carried out similarly to that of Theorem 5.1. in [4]. First of all, $F_*(P)$ is a cofibrant cyclic operad. Indeed, in the pushout diagram

$$\begin{tikzcd}
\mathbb{F}\Theta \arrow[r] \arrow[d] & \mathbb{F}P \arrow[d] \\
\Theta \arrow[r] & \mathbb{F}_*P
\end{tikzcd}$$

of Subsection 1.5.2 the upper horizontal arrow is a cofibration since P is cofibrant as a collection and $\mathbb{F} : \mathcal{C}yc\mathcal{C}oll \longrightarrow \mathcal{C}yc\mathcal{O}p$ preserves cofibrations.

The map γ intuitively makes all the compositions indicated by an X-labeled tree T with lengths on the internal edges until it gives an element of $P(X)$. Rigourously, this is obtained by the map $H(T) \otimes \underline{P}(T) \longrightarrow P(X)$, induced by the counit of H and the operad-structure of P. One can see that this gives rise to a map γ : $W(H, P)(X) \longrightarrow P(X)$, and in particular $W_0(H, P)(X) \xrightarrow{\sim} W(H, P)(X) \xrightarrow{\gamma} P(X)$ is the identity, hence we can conclude that γ is a weak equivalence.

The map $\delta : F_*(P) \longrightarrow W(H, P)$ intuitively includes the "elements" of $F_*(P)$ in $W(H, P)$ by colouring the internal edges of a P-decorated tree $\underline{P}(T)$ all with length 1. The rigorous definition can be given by the map $\underline{P}(T) \longrightarrow H(T) \otimes \underline{P}(T)$, induced by the absorbing element $1 : I \longrightarrow H$.

We can infer that $\gamma\delta$ is the counit of the pointed-free adjunction and the only thing left to prove is that δ is a cofibration. The diagram of cyclic operads

$$\begin{tikzcd} \mathbb{F}_*(P) \arrow[r] \arrow[d] & Y \arrow[d] \\ W(H,P) \arrow[r] & X \end{tikzcd}$$

transfers by adjunction to the diagram of cyclic collections

$$\begin{tikzcd} P = W_0(H,P) \arrow[r] \arrow[d] & Y \arrow[d] \\ W(H,P) \arrow[r] & X \end{tikzcd}$$

and a filler for the latter can be constructed by induction on the degree of filtration

$$W_0(H, P) \longrightarrow W_1(H, P) \longrightarrow W_2(H, P) \longrightarrow \cdots \longrightarrow W(H, P)$$

This induction uses the notion of *cyclic k-homomorphisms* and is essentially the same as in Lemma 5.4 of [4]. ♦

The functorial properties of the W-construction can be carried out in the same way as in [4] and we are only going to list them here. Note that W is a functor in two variables, and we are interested in the homotopical behavior of this functor. Using the terminology of [4], we call a segment H cofibrant if the map $I \sqcup I \longrightarrow H$ is a cofibration.

Proposition 2.2.10. *For any (trivial) cofibration of cofibrant segments $H \longrightarrow K$ ad any well pointed $\mathcal{F}in$-cofibrant cyclic operad P the induced map $W(H, P) \longrightarrow W(K, P)$ is a (trivial) cofibration of cyclic operads.*

Proposition 2.2.11. *For any cofibrant segment H and any (trivial) $\mathcal{F}in$-cofibration of well pointed $\mathcal{F}in$-cofibrant cyclic operads $P \longrightarrow Q$, the induced map $W(H, P) \longrightarrow W(H, Q)$ is a (trivial) cofibration of cyclic operads.*

2.2.5 The bar-cobar resolution of a cyclic operad

In this subsection we are going to extend the bar- and cobar constructions of operads and cooperads in differential graded vector spaces ([13, 15]) to cyclic (co)operads and prove that in the case of *reduced cyclic operads* the resulting cobar-bar functor $\Omega B : \mathcal{C}yc\mathcal{O}p \longrightarrow \mathcal{C}yc\mathcal{O}p$ is a special case of the W-construction of cyclic operads, thus it provides a cofibrant resolution for a cyclic operad in $dg\mathcal{V}ect$.

To fix the terminology, let $dg\mathcal{V}ect$ denote the symmetric monoidal model category of chain complexes of positively graded chain complexes over a field $\mathbb{k}$ of characteristic 0. Usually we will consider the ground field $\mathbb{k}$ as a chain complex concentrated in degree 0: this complex is the unit of the monoidal category.

In $dg\mathcal{V}ect$ the initial cyclic operad and terminal cyclic cooperad both consist of one copy of $\mathbb{k}$ in finite sets with two elements and they are 0 elsewhere. We will denote both of them by $\mathbb{k}$.

An *augmented cyclic operad* P is a cyclic operad together with a map of operads $\epsilon : P \longrightarrow \mathbb{k}$. The category of augmented cyclic operads is denoted by $\mathcal{C}yc\mathcal{O}p_a$. There is a free-forgetful adjunction

$$\mathcal{C}yc\mathcal{C}oll \underset{I}{\overset{\mathbb{F}}{\rightleftarrows}} \mathcal{C}yc\mathcal{O}p_a \,,$$

where $\mathbb{F}$ is the same free functor as in Section 1.5 since every free cyclic operad has a trivial augmentation. On the other hand, any augmented cyclic operad P splits as $P = \mathbb{k} \oplus \ker(\epsilon)$ and $\ker(\epsilon)$ is the underlying pseudo cyclic operad. We can conclude that the right adjoint I can be defined as $IP = \ker(\epsilon)$. The cyclic collection IP is called the *augmentation ideal* of P. The unit of the adjunction is the inclusion $u_K : K \longrightarrow I\mathbb{F}K$ for any cyclic collection K, via identifying a finite set X with the isomorphism class of a corolla labeled by X. If we denote the natural isomorphism of the adjunction by $\varphi : \mathcal{C}yc\mathcal{O}p_a(\mathbb{F}K, P) \longrightarrow \mathcal{C}yc\mathcal{C}oll(K, IP)$, it follows that any map of cyclic collections $g : K \longrightarrow I\mathbb{F}K$ extends uniquely to a map of augmented cyclic operads $\varphi^{-1}(g) : \mathbb{F}K \longrightarrow \mathbb{F}K$ such that $I(\varphi^{-1}(g))u_K = g$.

Later on we will need a similar property of degree -1 maps of cyclic collections. First recall that for any $m \in \mathbb{Z}$ the m-fold suspension $(X[m], d[m])$ of a chain complex (X, d) is defined by $X[m]_k := X_{k-m}$, $d[m] := (-1)^m d$; a degree $-m$ chain map $X \longrightarrow Y$ is a map of complexes $X \longrightarrow Y[m]$. With this terminology, a degree -1 map of cyclic collections $K \longrightarrow L$ is simply a map $K \longrightarrow L[1]$ in the category $\mathcal{C}yc\mathcal{C}oll$. By following the free construction, one can see that the adjunction between the functors I and $\mathbb{F}$ provides an extension for these maps too:

Lemma 2.2.12. *Suppose that $g : K \longrightarrow I\mathbb{F}K$ is a degree -1 map of cyclic collections. Then there is a unique extension of g to a degree -1 map of augmented cyclic operads $d_g : \mathbb{F}K \longrightarrow \mathbb{F}K$ such that $I(d_g)u_K = g$.*

A *coaugmented cyclic cooperad* is a cyclic cooperad C together with a map of cyclic cooperads $\eta : \mathbb{k} \longrightarrow C$. Such a cyclic cooperad splits as $C = \mathbb{k} \oplus \operatorname{coker}(\eta)$

and there is an adjunction between cyclic cocollections and coaugmented cyclic cooperads

$$\mathcal{C}yc\mathcal{C}oop_c \underset{\mathbb{F}^c}{\overset{J}{\rightleftarrows}} \mathcal{C}yc\mathcal{C}oll^{\mathrm{op}}$$

where $\mathbb{F}^c$ is the right adjoint. Note that for any $K \in \mathcal{C}yc\mathcal{C}oll^{\mathrm{op}}$ the underlying cyclic (co)collection of the cofree functor is the same as for the free cyclic operad functor, the difference comes from the operations: in this case, instead of grafting labeled trees, the operations are induced by decomposing labeled trees into graftable components. The functor J is defined by $JC := \mathrm{coker}(\eta)$. The counit of this adjunction is the canonical projection $v_K : J\mathbb{F}^c K \longrightarrow K$ and the following lemma is dual to Lemma 2.2.12:

Lemma 2.2.13. *Suppose that $f : J\mathbb{F}^c K \longrightarrow K$ is a degree -1 map of cyclic cocollections. Then there is a unique extension of f to a degree -1 map of coaugmented cyclic cooperads $d_f : \mathbb{F}^c K \longrightarrow \mathbb{F}^c K$ such that $v_K J(d_f) = f$.*

The bar construction on an augmented cyclic operad

Suppose that (P, ϵ) is an augmented cyclic operad and Z is a finite set. For any decomposition $Z = X{}_x\sqcup_y Y$, the operation ${}_x\circ_y$ of P induces a degree -1 map of chain complexes $IP(X)[1] \otimes IP(Y)[1] \longrightarrow IP(Z)[1]$. If we compose this map with the natural projection $J\mathbb{F}^c(IP[1])(Z) \longrightarrow IP(X)[1] \otimes IP(Y)[1]$, and sum over all the possible decompositions of Z into $X{}_x\sqcup_y Y$, we obtain a degree -1 map $J\mathbb{F}^c(IP[1])(Z) \longrightarrow IP(Z)[1]$ for every Z, which fits into a degree -1 map of cyclic cocollections. We can apply then Lemma 2.2.13 to obtain a degree -1 map of coaugmented cyclic cooperads

$$d_\mu : \mathbb{F}^c(IP[1]) \longrightarrow \mathbb{F}^c(IP[1]).$$

One can give this map explicitly as follows:

The homogeneous elements of $\mathbb{F}^c(IP[1])$ are described by trees whose vertices are labeled by corresponding elements of the operad P. If T is such a tree, write $p_v \in P(\mathrm{Edg}(v))$ for the label of vertex v. Then

$$d_\mu\left(\bigotimes_{v\in\mathrm{Vert}(T)} p_v\right) = \sum_{e\in\mathrm{InEdg}(T)} \mathrm{sgn}(p_{v_1}, p_{v_2})(p_{v_1}\circ p_{v_2})\otimes\left(\bigotimes_{v\in\mathrm{Vert}(T/e)\setminus v_1\circ v_2} p_v\right)$$

where v_1 and v_2 are the vertices adjacent to the inner edge e, $p_{v_1} \circ p_{v_2}$ is the composition in P indicated by e, and T/e denotes the tree resulting by contracting e in T. The sign factor comes from the degree of p_{v_1} and p_{v_2}. Since the composition maps of the cyclic operad P satisfy associativity, $d_\mu d_\mu = 0$. In view of Lemma 2.2.13, d_μ is the unique differential on $\mathbb{F}^c(IP[1])$ which *reflects* the cyclic operad structure on P. We will call d_μ the *bar differential*. Since d_μ is also a chain map of degree -1, it follows that it anticommutes with the internal differential d, hence $\delta := d + d_\mu$ is a differential on $\mathbb{F}^c(IP[1])$. The coaugmented cyclic cooperad $(\mathbb{F}^c(IP[1]), \delta)$ together with this differential is called *the bar construction on P* and it is denoted by $B(P)$.

The cobar construction on a coaugmented cyclic cooperad

Suppose that (C, η) is a coaugmented cyclic cooperad where all the chain complexes $C(X)$ have no degree 0 components. Suppose also that Z is a finite set and $Z = X_x \sqcup_y Y$ a decomposition of it. The structure map ${}_x\circ_y$ of C induces a degree -1 map of chain complexes $JC(Z)[-1] \longrightarrow JC(X)[-1] \otimes JC(Y)[-1]$. We can compose this map with the natural inclusion $JC(X)[-1] \otimes JC(Y)[-1] \longrightarrow I\mathbb{F}(JC[-1])(Z)$ and sum over all the decompositions of Z to obtain a degree -1 map of cyclic collections $JC[-1] \longrightarrow I\mathbb{F}(JC[-1])$. By Lemma 2.2.12 we get a unique degree -1 map of augmented cyclic operads

$$d_\Delta : \mathbb{F}(JC[-1]) \longrightarrow \mathbb{F}(JC[-1]).$$

Similarly to the bar construction, we can define a new differential $\delta := d + d_\Delta$ on $\mathbb{F}(JC[-1])$. This construction yields a new augmented cyclic operad – *the cobar construction on* C – which we denote by $\Omega(C)$.

The cobar-bar adjunction

The functor Ω is left adjoint to B when we restrict ourselves to the category of *connected* coaugmented cyclic cooperads (those ones which have no degree 0 terms). To prove this, one needs to introduce *twisting cochains*. To have the right intuition, first observe that we need to find a natural isomorphism

$$\Phi : \mathcal{C}yc\mathcal{O}p_a(\Omega(C), P) \longrightarrow \mathcal{C}yc\mathcal{C}oop_c(C, B(P)).$$

To construct such a map, one needs to examine for a connected coaugmented cyclic cooperad (C, η) and an augmented cyclic operad (P, ϵ) those maps of cyclic collections $f : C \longrightarrow P$ which induce maps $\Omega(C) \longrightarrow P$ and $C \longrightarrow B(P)$ in the respective categories. These maps will be called twisting cochains.

The first thing to observe is that we need to start with maps $JC[-1] \longrightarrow P$ and $C \longrightarrow IP[-1]$, thus f has to be a degree -1 map such that $f\eta = \epsilon f = 0$. Now let us focus on $\Omega(C) \longrightarrow P$. We have already seen that we can view f as $f : (JC)[-1] \longrightarrow IP$, hence the adjunction φ between $\mathbb{F}$ and I induces a map of augmented cyclic operads $\varphi^{-1}(f) : \bar{\Omega}(C) \longrightarrow P$ where $\bar{\Omega}$ indicates that the differential is the one without d_Δ. It follows that the only thing to check is that $\varphi^{-1}(f)$ commutes with $d_C + d_\Delta$. This descends to checking the commutativity of the diagram

$$\begin{tikzcd}
JC(Z)[-1] \arrow[rr, "f_Z"] \arrow[d, "d_C+\Delta"'] & & P(Z) \arrow[d, "d_P"] \\
JC(Z)[-1] \oplus M \arrow[r, "f_Z+\tilde{f}"] & P(Z) \oplus N \arrow[r, "\mathrm{id}+\mu"] & P(Z)
\end{tikzcd}$$

where $M = \bigoplus(JC(X)[-1] \otimes JC(Y)[-1])$, $N = \bigoplus(P(X) \otimes P(Y))$ and the sums are taken over all possible decompositions $Z = X_x \sqcup_y Y$; Δ, $\tilde{f}$ and μ are the obvious maps. This diagram gives the following condition:

$$d_P f + f d_C = f \cup f,$$

where $f \cup f$ is the composition

$$C \xrightarrow{\Delta} M \xrightarrow{\tilde{f}} N \xrightarrow{\mu} P .$$

One can check that if f induces a map of coaugmented cyclic cooperads $f : C \longrightarrow B(P)$ then it has to satisfy the same condition and vice versa. We can conclude that the following definition is plausible:

A *twisting cochain* from a connected coaugmented cyclic cooperad (C, η) to an augmented cyclic operad (P, ϵ) is a degree -1 map of cyclic collections $f : C \longrightarrow P$ satisfying the properties $f\eta = 0 = \epsilon f$ and $d_P f + f d_C = f \cup f$. We denote the set of all twisting cochains from C to P by $\mathcal{T}(C, P)$.

Proposition 2.2.14. *There exist natural bijections*

$$\mathcal{C}yc\mathcal{O}p_a(\Omega(C), P) \longrightarrow \mathcal{T}(C, P) \longrightarrow \mathcal{C}yc\mathcal{C}oop_c(C, B(P)),$$

hence Ω *is left adjoint to* B.

Proof. The following two maps are twisting cochains:

$$\tau^P : B(P) = \mathbb{F}^c(IP[1]) \longrightarrow IP[1] \longrightarrow IP \longrightarrow P,$$

$$\tau_C : C \longrightarrow JC \longrightarrow JC[-1] \longrightarrow \mathbb{F}(JC[-1]) = \Omega(C).$$

They are natural in P and C respectively, and the maps involved in the statement of the proposition are

$$\mathcal{C}yc\mathcal{O}p_a(\Omega(C), P) \xrightarrow{\tau_C^*} \mathcal{T}(C, P) \xrightarrow{(\tau_*^P)^{-1}} \mathcal{C}yc\mathcal{C}oop_c(C, B(P)) . \qquad \blacklozenge$$

Moreover, the counit $\Omega B(P) \longrightarrow P$ and unit $C \longrightarrow B\Omega(C)$ of this adjunction are weak equivalences (they induce isomorphism on homology), see the proof of Theorem 3.2.16 of [15].

In the case of reduced cyclic operads $\Omega B(P)$ is isomorphic to the reduced W-construction $W^{red}(H, P)$ where H is the interval induced by the representable simplicial set Δ^1 via the normalized chain complex functor. Indeed, in this case the reduced W-construction splits as $W^{red}(H, P) = \overline{\mathbb{k}} \oplus W^{ps}(H, \overline{P})$, where $\overline{P}$ is the pseudo cyclic operad part of P and $\mathbb{k}$ is the unit part. ($\overline{P}(O) = 0$ for any one-point-set O, $P(U) = \mathbb{k} \oplus \overline{P}(U)$ for any two-point-set U and $\overline{P}(X) = P(X)$ otherwise on the one hand, $\overline{\mathbb{k}}(X) = \mathbb{k}$ for sets X with one or two elements and $\overline{\mathbb{k}}(X) = 0$ otherwise on the other.)

Since W^{ps} is defined in the same way as W, but with dropping the vertices with valence 0 and the binary operation on H which was needed only for the unit-identifications, we can conclude that

$$W^{ps}(H, \overline{P}) = \mathbb{F}_0(\mathbb{F}(\overline{P}[1])[-1]).$$

An analysis of the differential of $W^{red}(H,P)$ shows that it has two parts $\partial = \partial_P + \partial_H$ and $W^{red}(H,P) \simeq \Omega B(P)$, where ∂_P corresponds to the part of the differential on $\Omega B(P)$ coming from the free and cofree constructions, while ∂_H corresponds to the cobar-bar part of the differential.

3

Dendroidal sets

In the second part of the thesis we are interested in some questions related to dendroidal sets. This chapter is dedicated to fix in short the definitions, notations and conventions we are going to use further on, when we discuss the Dold-Kan correspondence for planar dendroidal abelian groups and the dendroidal definition of weak n-categories. The interested reader may find more details about the concepts appearing here in [32, 33, 40].

3.1 Terminology and basic facts about dendroidal sets

Dendroidal sets generalise simplicial sets in a suitable way for studying the homotopy theory of (coloured) operads and their algebras. They were introduced in the papers of I. Moerdijk and I. Weiss [32, 33]. The idea behind the notion of dendroidal sets is that in the same way as simplicial sets help us understanding categories via the nerve functor, there should be an analogous notion for studying coloured operads as a generalisation of categories. Our goal here is to write a self-contained introduction to dendroidal sets, including all the terminology necessary for the next chapters.

Let us start with the notion of trees. A *tree* is a finite non-planar contractible graph with a distinguished leaf called *root*. A tree thus has many planar representatives, when we draw a picture of it we actually pick one. We will use some of the terminology on trees already introduced in Chapter 1, like $\mathrm{Cor}_n, \mathrm{Vert}(T), \mathrm{Edg}(T)$. We will say that a vertex $v \in \mathrm{Vert}(T)$ of a tree is an *outer vertex* if v is adjacent to at most one inner edge of T.

We will make frequent use of symmetric coloured operads (both in $\mathcal{S}ets$ and enriched in a monoidal category $\mathcal{E}$) and we will refer to them as operads from now on. Our point of view differs from that of Section 1.6: here we think of operads as generalisations of categories, in the sense of [25]. Recall that if P is an operad in $\mathcal{S}ets$, then it comes equipped with a set of objects or colours $ob(P)$ and for each ordered sequence of objects $\sigma = (e_1, \ldots, e_n; e)$, a set of operations $P(e_1, \ldots, e_n; e) = P(\sigma)$. We will use the $\circ_i$-definition for the composition of operations, i.e. if σ is as before and $\rho = (f_1, \ldots, f_m, e_i)$ for a fixed $1 \leq i \leq n$ then

$$\sigma \circ_i \rho = (e_1, \ldots, e_{i-1}, f_1, \ldots, f_m, e_{i+1}, \ldots, e_n; e)$$

and there is a given composition map

$$\circ_i \colon P(\sigma) \times P(\rho) \longrightarrow P(\sigma \circ_i \rho).$$

The category of operads in $\mathcal{S}ets$ will be denoted by $\mathcal{O}p$, and the category of planar- or non symmetric operads in $\mathcal{S}ets$ by $\mathcal{O}p^\pi$. Sometimes it will be useful to construct operads from planar ones, via the free symmetrization functor $\mathrm{Symm} \colon \mathcal{O}p^\pi \longrightarrow \mathcal{O}p$, the left adjoint to the forgetful functor $U \colon \mathcal{O}p \longrightarrow \mathcal{O}p^\pi$. This feature already appears in the definition of dendroidal sets.

The category Ω^π consists of planar trees as objects and planar operad maps as arrows. To be more precise, any planar tree T gives rise to a planar operad $\Omega^\pi(T)$. The objects of this non symmetric operad are the edges of T, and the operations are freely generated by the vertices of T, i.e. if $\sigma = (e_1, e_2, \ldots, e_n; e)$ is an ordered sequence of edges of T and there is a vertex v with incoming edges $e_1, \ldots, e_n$ in this order and outgoing edge e, then $\Omega^\pi(T)(\sigma) = \{v\}$. One can then "compose" vertices, indicated by the tree T. Hence a map $R \longrightarrow T$ in Ω^π is simply a planar operad map $\Omega^\pi(R) \longrightarrow \Omega^\pi(T)$. We observe that if $f \colon R \longrightarrow T$ is an isomorphism, then the planar operad structures imply that R and T have the same planar shape and they differ only on the names of their edges and vertices. To avoid dealing with these irrelevant isomorphisms, further on we will replace Ω^π by a skeleton of it, and call this new category Ω^π as well. With this new convention, we observe that all the maps of Ω^π are generated by two types, *faces* and *degeneracies*. These types of maps generalise the corresponding notions in the category Δ defining simplicial sets, in the following way. Let L_n denote the linear tree with n vertices, $n \geq 0$:

If we consider the categorical definition of Δ, we observe that the category

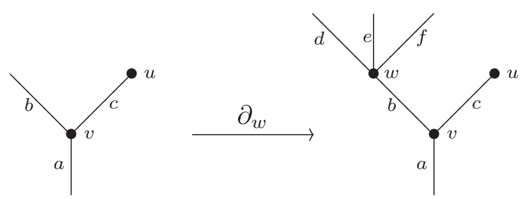
$$[n] = 0 \longleftarrow 1 \longleftarrow 2 \longleftarrow \cdots \longleftarrow n$$

is in fact $[n] = \Omega(L_n)$, hence Δ is fully faithfully embedded into Ω^π by $[n] \mapsto L_n$.

The face maps in Ω^π are all those monic operad maps $\partial\colon \Omega^\pi(R) \longrightarrow \Omega^\pi(T)$ which increase the number of vertices by one (i.e. $|\operatorname{Vert}(T)| = |\operatorname{Vert}(R)| + 1$) and the degeneracies are all those epic operad maps $\sigma\colon \Omega^\pi(T) \longrightarrow \Omega^\pi(R)$ which decrease the number of vertices by one.

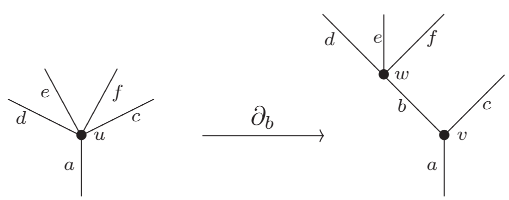
It follows that face maps can be of the following types (see Proposition 3.2.4):

(a) the following picture is an example of an *outer face*



which is just an inclusion of operads;

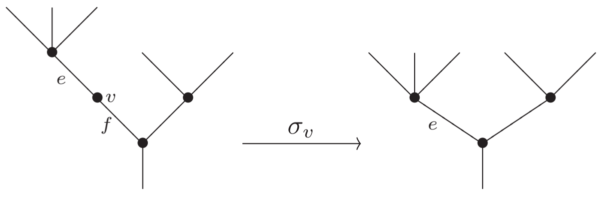
(b) the following picture is an example of an *inner face*



where $\partial_b\colon \Omega^\pi(R) \longrightarrow \Omega^\pi(T)$ is the identity on the objects (edges), and sends the operation $u \in \Omega^\pi(R)(d, e, f, c; a)$ to the composite operation $v \circ_1 w \in \Omega^\pi(T)(d, e, f, c; a)$, which we can denote without ambiguity by $v \circ_b w$.

Note that the seemingly special cases of face maps into the corolla Cor_n, $n \geq 2$ fall under case (a): these face maps are all the $n+1$ possible edge inclusions of the trivial tree | to Cor_n.

On the other hand, a degeneracy always looks like



where both of the objects e, f are sent to e, the operation v to the identity operation id_e and σ_v is the identity elsewhere.

We will use the following terminology with respect to faces and degeneracies:

- If e is an inner edge of a tree T, then T/e denotes the tree resulting from T by contracting e. The inner face corresponding to this contraction is usually denoted by $\partial_e \colon T/e \longrightarrow T$.
- If v is an outer vertex of a tree T (that is, it has exactly one inner edge adjacent to it), then T/v denotes the tree resulting from T by removing v and all the external edges adjacent to it (with the obvious choice for the root of T/v when one of these external edges happens to be the root of T). We call this procedure "cutting vertex v". The outer face correspondig to cutting v is usually denoted by $\partial_v \colon T/v \longrightarrow T$.
- If v is a vertex of valence one of a tree T then $T\backslash v$ denotes the tree resulting from T by removing v. The degeneracy corresponding to this removal is usually denoted by $\sigma_v \colon T \longrightarrow T\backslash v$.

The category Ω is obtained from Ω^π via the functor Symm. The objects of Ω are (non planar) trees and the arrows $R \longrightarrow T$ are operad maps $\mathrm{Symm}(\Omega(\bar{R})) \longrightarrow \mathrm{Symm}(\Omega(\bar{T}))$, where $\bar{T}$ denotes a planar representative of T. One can check that the resulting operad does not depend on the chosen representatives, hence the definition makes sense. Later on we will use this independence from chosen representatives: we often describe the operad $\Omega(T)$ by picking a representative $\bar{T}$ and giving only the description of the planar operad $\Omega^\pi(\bar{T})$.

The definition given above implies that there is an extra type of generator for the maps in Ω, namely the isomorphisms.

The category of dendroidal sets is the presheaf category on Ω:

$$d\mathcal{S}ets := \mathcal{S}ets^{\Omega^{\mathrm{op}}} = \mathrm{Funct}(\Omega^{\mathrm{op}}, \mathcal{S}ets).$$

If X is a dendroidal set, the elements of X_T are called *dendrices of shape* T. The *representable dendroidal set* associated to a tree T is the functor

$$\Omega[T] := \Omega(-, T) \colon \Omega^{\mathrm{op}} \longrightarrow \mathcal{S}ets.$$

By the Yoneda lemma, a dendrex $t \in X_T$ is the same thing as a map of dendroidal sets $\Omega[T] \longrightarrow X$. The Yoneda lemma in general is a very useful tool in the theory of simplicial- and dendroidal sets, allowing us to swap between maps and dendrices whenever needed. We are going to exploit this property in the following without mentioning it. A first application of the Yoneda lemma in the dendroidal setting proves that every dendroidal set is a colimit of representable ones, a property generalising the well known fact for simplicial sets.

For any given tree T one can define some dendroidal subsets of the representable $\Omega[T]$, like the boundary $\partial\Omega[T]$ or the inner horn $\Lambda^e[T]$ with respect to the inner edge e. Dendroidal sets are analogous to simplicial sets in many ways. For example, inner horns can be used to define *inner Kan complexes* in the category of dendroidal sets: we say that a dendroidal set X *satisfies the inner Kan condition* if for any inner horn $h\colon \Lambda^e[T] \longrightarrow X$ there exists a dendrex $t\colon \Omega[T] \longrightarrow X$ such that the following diagram commutes:

$$\begin{tikzcd} \Lambda^e[T] \arrow[r, "h"] \arrow[d, tail] & X \\ \Omega[T] \arrow[ur, "t"'] & \end{tikzcd}$$

In this case X is called an *inner Kan complex* or a *quasi-operad*, analogously to the simplicial case where an inner Kan complex was called by A. Joyal a *quasi-category*.

Another notion that generalises from simplicial sets and categories to dendroidal sets and operads is the nerve functor. The *dendroidal nerve* $N_d\colon \mathcal{O}p \longrightarrow d\mathcal{S}ets$ can be defined by setting for any operad P

$$\big(N_d(P)\big)_T := \mathcal{O}p(\Omega(T), P).$$

In the next few lines we introduce the notions of k*-skeleton* and k*-coskeleton* of a dendroidal set. For every $k \in \mathbb{N}$ let Ω_k denote the full subcategory of Ω consisting of trees with at most k vertices. The presheaf category on Ω_k is called the category of k*-truncated dendroidal sets* and is denoted by $d\mathcal{S}ets_k$. The inclusion $i_k\colon \Omega_k \longrightarrow \Omega$ induces the truncation functor $i_k^*\colon d\mathcal{S}ets \longrightarrow d\mathcal{S}ets_k$ which has a left adjoint $(i_k)_!$ and a right adjoint $(i_k)_*$. It follows that the composites

$$(i_k)_! i_k^*, (i_k)_* i_k^*\colon d\mathcal{S}ets \longrightarrow d\mathcal{S}ets$$

form an adjoint pair of endofunctors. The left adjoint $(i_k)_! i_k^*$ is usually denoted by Sk_k and is called the k-skeleton functor. The right adjoint is denoted by coSk_k and is called the k-coskeleton functor.

A dendroidal set X is said to be k*-coskeletal* if the canonical map $X \longrightarrow \mathrm{coSk}_k(X)$ is an isomorphism. Another way to state this is that for every dendroidal set Y and every map of dendroidal sets $\phi\colon \mathrm{Sk}_k(Y) \longrightarrow X$ there exists a unique extension

$$\begin{tikzcd} \mathrm{Sk}_k(Y) \arrow[r, "\phi"] \arrow[d, tail] & X \\ Y \arrow[ur, dotted, "\exists!"'] & \end{tikzcd}$$

Since any $Y \in dSets$ is a colimit of representables, we can infer that if the previous statement holds for all $Y = \Omega[T]$ where T is any tree with $k+1$ vertices, then it holds in general. In this case $\mathrm{Sk}_k(Y) = \mathrm{Sk}_k(\Omega[T]) = \partial\Omega[T]$. Note that in view of the Yoneda lemma we can think of the composite

$$Sk_k(\Omega[T]) \rightarrowtail \Omega[T] \xrightarrow{t} X$$

as the boundary- or k-skeleton of the dendrex t. To emphasize this point of view, sometimes we will denote this composite by $\mathrm{Sk}_k(t)$.

One can define the dual notion of k-skeletal dendroidal sets similarly.

Remark 3.1.1. Note that the dendroidal definition of the functors Sk_k and coSk_k uses a filtration of the objects of Ω by the number of the vertices of trees. Later on, we will use the term *degree* to refer to this natural number.

3.2 Dendroidal identities

One can describe the maps in the category Ω^π in terms of generators (faces and degeneracies) and relations. We present the relations in two different ways in this section, since both of these descriptions can be useful when reasoning with generators, as we will see later.

The first description uses the already familiar notation for faces and degeneracies: the maps are indexed by edges and vertices of trees. The second way is based on natural linear orders defined on the set of faces, and the set of degeneracies of a given tree. This description is presented in Subsection 3.2.4.

The relations generalise the simplicial identities in the category Δ, henceforth we will call them dendroidal identities. The unique epi-mono factorization theorem for maps in the category Δ extends to the category Ω^π, thus the relations we consider below cover indeed all the cases: one only has to look at all possible compositions $f = g_1 \circ g_2$ of two generators of Ω^π and see what are the other ways to decompose f into two generators. The result is summarized in Lemma 3.2.1.

We do not include in our first description the special case involving faces of the n-corolla, $n \geq 2$, although a statement similar to Lemma 3.2.1 can be given.

There is a little ambiguity in the language that follows, for example ∂_e can refer to two different face maps, but it is always clear from the context which one we are talking about.

3.2.1 Elementary face relations

Let $\partial_a \colon T/a \longrightarrow T$ and $\partial_b \colon T/b \longrightarrow T$ be two inner faces of T. It follows that the inner faces $\partial_a \colon (T/b)/a \longrightarrow T/b$ and $\partial_b \colon (T/a)/b \longrightarrow T/a$ exist, $(T/a)/b = (T/b)/a$ and the following diagram commutes:

$$\begin{tikzcd}
(T/a)/b \arrow[r, "\partial_b"] \arrow[d, "\partial_a"'] & T/a \arrow[d, "\partial_a"] \\
T/b \arrow[r, "\partial_b"] & T
\end{tikzcd}$$

Let $\partial_v \colon T/v \longrightarrow T$ and $\partial_w \colon T/w \longrightarrow T$ be two outer faces of T. Then the outer faces $\partial_w \colon (T/v)/w \longrightarrow T(v)$ and $\partial_v \colon (T/w)/v \longrightarrow T/w$ also exist, $(T/v)/w = (T/w)/v$ and the following diagram commutes:

$$\begin{tikzcd}
(T/v)/w \arrow[r, "\partial_w"] \arrow[d, "\partial_v"'] & T/v \arrow[d, "\partial_v"] \\
T/w \arrow[r, "\partial_w"] & T
\end{tikzcd}$$

The last remaining case is when we compose an inner face with an outer one in any order. There are several possibilities, in all of them suppose that $\partial_v \colon T/v \longrightarrow T$ is an outer face and $\partial_e \colon T/e \longrightarrow T$ is an inner face.

- If in the tree T the edge e is not adjacent to the vertex v then the outer face $\partial_v \colon (T/e)/v \longrightarrow T/e$ and inner face $\partial_e \colon (T/v)/e \longrightarrow T/v$ exist, $(T/e)/v = (T/v)/e$ and the following diagram commutes:

$$\begin{tikzcd}
(T/v)/e \arrow[r, "\partial_e"] \arrow[d, "\partial_v"'] & T/v \arrow[d, "\partial_v"] \\
T/e \arrow[r, "\partial_e"] & T
\end{tikzcd}$$

- Suppose that in T the inner edge e is adjacent to the vertex v and denote the other adjacent vertex to e by w. Following the terminology introduced in Section 3.1, v and w contribute to T/e a vertex $v \circ_e w$ or $w \circ_e v$. Let us denote this vertex by z. Notice that the outer face $\partial_z \colon (T/e)/z \longrightarrow T/e$ exists if and only if the outer face $\partial_w \colon (T/v)/w \longrightarrow T/v$ exists and in this case $(T/e)/z = (T/v)/w$. Moreover, the following diagram commutes:

$$\begin{tikzcd}
(T/v)/w \arrow[r, equal] \arrow[d, "\partial_w"'] & (T/e)/z \arrow[r, "\partial_z"] & T/e \arrow[d, "\partial_e"] \\
T/v \arrow[rr, "\partial_v"] & & T
\end{tikzcd}$$

  It follows that we can write $\partial_v\partial_w = \partial_e\partial_z$ where $z = v \circ_e w$ if v is "closer" to the root of T or $z = w \circ_e v$ if w is "closer" to the root of T.

3.2.2 Elementary degeneracy relations

Let $\sigma_v\colon T \longrightarrow T\backslash v$ and $\sigma_w\colon T \longrightarrow T\backslash w$ be two degeneracies of T. Then the degeneracies $\sigma_v\colon T\backslash w \longrightarrow (T\backslash w)\backslash v$ and $\sigma_w\colon T\backslash v \longrightarrow (T\backslash v)\backslash w$ exist, $(T\backslash v)\backslash w = (T\backslash w)\backslash v$ and the following diagram commutes:

$$\begin{tikzcd} T \arrow[r, "\sigma_v"] \arrow[d, "\sigma_w"'] & T\backslash v \arrow[d, "\sigma_w"] \\ T\backslash w \arrow[r, "\sigma_v"] & (T\backslash v)\backslash w \end{tikzcd}$$

3.2.3 Elementary combined relations

Let $\sigma_v\colon T \longrightarrow T\backslash v$ be a degeneracy and $\partial\colon T' \longrightarrow T$ a face map such that $\sigma_v\colon T' \longrightarrow T'\backslash v$ makes sense (i.e. T' still contains v and its two adjacent edges as a subtree). Then, there exists an induced face map $\partial\colon T'\backslash v \longrightarrow T\backslash v$, determined by the same vertex or edge as $\partial\colon T' \longrightarrow T$. Moreover, the following diagram commutes:

$$\begin{tikzcd} T \arrow[r, "\sigma_v"] & T\backslash v \\ T' \arrow[u, "\partial"] \arrow[r, "\sigma_v"] & T'\backslash v \arrow[u, "\partial"'] \end{tikzcd}$$

Let $\sigma_v\colon T \longrightarrow T\backslash v$ be a degeneracy and $\partial\colon T' \longrightarrow T$ be any of the two face maps induced by the adjacent edges to v or the removal of v, if that is possible. It follows that $T' = T\backslash v$ and the composition

$$T\backslash v \xrightarrow{\partial} T \xrightarrow{\sigma_v} T\backslash v$$

is the identity map $\mathrm{id}_{T\backslash v}$.

All these relations between the generators of the maps in Ω^π are summarized in the following lemma.

Lemma 3.2.1. *Let $f\colon R \longrightarrow T$ be the composite of two generators, $f = g_1 \circ g_2$ where both R and T have at least one vertex.*

(1) If $f \neq \mathrm{id}$ then there is exactly one more way to decompose f into two generators $f = g_1' \circ g_2'$, where $\{g_1, g_2\} \neq \{g_1', g_2'\}$ as sets. It follows that we obtain a commutative diagram

$$\begin{tikzcd} R \arrow[r, "g_2"] \arrow[d, "g_2'"'] & S \arrow[d, "g_1"] \\ S' \arrow[r, "g_1'"] & T \end{tikzcd}$$

which is a special case of one of the diagrams with the dendroidal identities listed above.

(2) *If $f = \mathrm{id}$ then $g_1 = \sigma_v$ for some vertex v and $g_2 = \partial$ is one of the two possible face maps, induced by an edge adjacent to v (or v itself in some cases).* ♦

We close this subsection by proving the following decomposition result, which is also a consequence of Theorem 2.3.27 of [40].

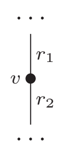
Lemma 3.2.2. *Every map $f \colon R \longrightarrow T$ in Ω^π is either the identity or it decomposes uniquely as $f = d \circ s$, where d is a composite of face maps and s is a composite of degeneracies.*

Proof. To prove the existence of the decomposition, we proceed by induction on $n = |\operatorname{Vert}(R)| + |\operatorname{Vert}(T)|$, the total number of vertices of R and T. If $n = 0$ then $f \colon | \longrightarrow |$ is the identity map and the statement is obvious. In general, f is a map of coloured operads and a part of it consists of a set-map between the colours, i.e., a map $E(f) \colon \operatorname{Edg}(R) \longrightarrow \operatorname{Edg}(T)$ between the edges. This map between the sets of colours has a unique epi-mono factorization

$$\operatorname{Edg}(R) \xrightarrow{s_0} \!\!\!\!\twoheadrightarrow X \overset{d_0}{\rightarrowtail} \operatorname{Edg}(T).$$

First suppose that there exist $r_1 \neq r_2 \in \operatorname{Edg}(R)$ such that $s_0(r_1) = s_0(r_2)$. Since f is a map of operads, r_1 and r_2 must be situated one above the other in a linear branch of R:

such that any edge r between them satisfies $s_0(r) = s_0(r_1) = s_0(r_2)$. Hence we can suppose that r_1 and r_2 are adjacent, joined by the vertex v:



It follows that f decomposes as $R \overset{\sigma_v}{\twoheadrightarrow} R \backslash v \xrightarrow{f'} T$ and by the inductive hypothesis we already have a decomposition $R \backslash v \twoheadrightarrow S \rightarrowtail T$ of f'.

Second, suppose that s_0 is bijective, hence we can assume that s_0 is the identity map. If d_0 is also the identity, it follows that f has to be the identity. Indeed, since we are working with non-symmetric operads, f preserves the order of the incoming edges at every vertex, and if $v \in \Omega(R)$ is a generator (a vertex of R) and $f(v)$ is not a generator of $\Omega(T)$ then there would be edges of T without preimage in R.

If d_0 is not the identity then let $e \in \mathrm{Edg}(T)$ be an edge skipped by d_0. We can distinguish two cases:

Suppose that e is an inner edge of T. It follows that f decomposes as

$$R \xrightarrow{f'} T/e \rightarrowtail^{\partial_e} T\,.$$

By the inductive hypothesis we obtain a decomposition of f'.

Now suppose that e is an outer edge of T, skipped by d_0. Since f is a map of operads, again any other outer edge, adjacent to e, has to be skipped by d_0. Denote the vertex adjacent to e by v. It follows that we can again decompose f as

$$R \xrightarrow{f'} T/v \rightarrowtail^{\partial_v} T$$

and obtain the desired factorization of f by induction.

To prove the uniqueness of the decomposition we proceed as follows. Suppose that there are two factorizations of f:

$$R \overset{s}{\twoheadrightarrow} S \overset{d}{\rightarrowtail} T \quad \text{and} \quad R \overset{s'}{\twoheadrightarrow} S' \overset{d'}{\rightarrowtail} T\,.$$

Looking at the decompositions only on the level of the edges, it follows that $\mathrm{Edg}(S) = \mathrm{Edg}(S')$, $ob(s) = ob(s')$ and $ob(d) = ob(d')$. Moreover, since s is a composite of degeneracies, it follows that if v is a generator of $\Omega(R)$ then $s(v) = v$ or $s(v)$ is the identity on some edge, in which case it is completely determined by $ob(s)$. Hence $s = s'$ and also $S = S'$. Similarly, d is also completely determined by what it does on the colours, thus $d = d'$. ♦

Remark 3.2.3. The lemma above does not imply that there are no other relations between the generators (faces and degeneracies) of Ω^π. For this result one needs an extension of the decomposition theorem, which in turn states that also the d and s maps decompose uniquely into some naturally ordered sequences of faces and degeneracies, respectively. Section 3.2.4 indicates such an order for faces and degeneracies, hence also the required extension of Lemma 3.2.2.

Proposition 3.2.4. *Let T be a tree in Ω^π. Then the faces of T are exactly those monic operad maps $\Omega^\pi(R) \longrightarrow \Omega^\pi(T)$ for which $|\,\mathrm{Vert}(T)| = |\,\mathrm{Vert}(R)| + 1$ and the degeneracies of T are exactly those epic operad maps $\Omega^\pi(T) \longrightarrow \Omega^\pi(S)$ for which $|\,\mathrm{Vert}(T)| = |\,\mathrm{Vert}(S)| + 1$.*

Proof. We prove only the assertion for the faces, the other statement can be proven similarly. Every face map is obviously a monic operad map with the required property, so we need to prove the converse only.

Suppose that $f\colon R \longrightarrow T$ is monic with the required property, but not a face. Lemma 3.2.2 implies that f can be decomposed as

$$R \xrightarrow{s} R' \xrightarrow{d} T$$

where s is a composite of degeneracies and d is a composite of faces. We infer that s cannot be the identity, since counting the vertices would imply that $f = d$ is a face in that case. It follows that s – and hence f as well – is not monic on the edges. This is a contradiction, since a monic operad map has to be monic on the colours. ♦

3.2.4 Linear orders on the faces and degeneracies of a tree

Since the trees we consider are planar, we can define canonically a linear order on the set of all the faces of any chosen tree. Similarly, a canonical linear order can be defined on the set of all degeneracies of a tree as well. We treat the case of the corollas separately. There are a number of different possibilities to start with if one wants to obtain such orders, we choose the following.

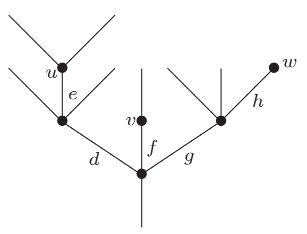
Let T be a tree in Ω^π such that $|\operatorname{Vert}(T)| \geq 2$. Assign to each face of T a natural number, respecting the following rules:

- If the root vertex $r \in \operatorname{Vert}(T)$ is outer then assign the number 0 to ∂_r.
- Starting from the root vertex, walk through all the edges and vertices of T by going always first to the left and upwards. When this is not possible any more, turn back to the closest, already visited vertex and choose the next, not yet covered edge left and upwards.
- Whenever an inner edge or an outer vertex is visited, assign the smallest, not yet used natural number to the corresponding face of T.

Suppose that T has n face maps. The process described above defines a bijection

$$\phi\colon \{\partial \mid \partial \text{ is a face of } T\} \longrightarrow \{0, 1, \ldots, n-1\},$$

hence also an order on the set of face maps of T. We define the i-th face of T by $\partial_i := \phi^{-1}(i)$. For example, if T is the tree



(3.2.1)

then T has 8 faces and $\partial_0 = \partial_d$, $\partial_1 = \partial_e$, $\partial_2 = \partial_u$, $\partial_3 = \partial_f$, $\partial_4 = \partial_v$, $\partial_5 = \partial_g$, $\partial_6 = \partial_h$ and $\partial_7 = \partial_w$.

We can use this convention on traversing the tree T to obtain another bijection

$$\rho\colon \{\sigma \mid \sigma \text{ is a degeneracy of } T\} \longrightarrow \{0, 1, \ldots, m-1\},$$

provided T has $m \geq 1$ degeneracies. For example, in the case of the tree drawn above $m = 1$ and $\sigma_0 = \sigma_v$. In case T is the n-corolla, the process of traversing T

"from left to right" induces a linear order on the set of faces of T, as well. After renaming these faces accordingly, we observe that ∂_0 is the inclusion of the trivial tree $|$ to the root of T, ∂_1 is the inclusion of $|$ to the leftmost leaf of T, etc.

Remark 3.2.5. The linear orders defined above extend the linear orders resulting from the usual numbering of faces and degeneracies of $[n] \in \Delta$. Indeed, if $T = L_n$ is the linear tree with n vertices, ∂_i and σ_i defined above correspond to the simplicial ones with the same index.

One can ask whether the dendroidal identities "remain the same" as the simplicial ones with respect to the linear orders. This is certainly true for the elementary degeneracy relations. Indeed, after renaming the maps of any commutative diagram with degeneracies as in Subsection 3.2.2, the relation becomes $\sigma_j\sigma_i = \sigma_i\sigma_{j+1}$ for some $i \leq j$.

On the other hand, the other types of elementary relations do not remain valid:

(a) In case of the elementary combined relations this fails because there can be fewer degeneracies of a tree than faces.

(b) In case of the elementary face relations, relations with the faces of the n-corolla give counterexamples on the one hand, but also the tree T drawn in (3.2.1) provides a counterexample since the relation $\partial_f\partial_g = \partial_g\partial_f$ translates as $\partial_3\partial_3 = \partial_5\partial_3$. We observe that such a situation can occur since some trees T have the property that the domain R of a face $\partial\colon R \longrightarrow T$ has two less faces than T.

In general, there are two possibilities for the elementary face relations:

$$\partial_i\partial_{j-1} = \partial_j\partial_i \qquad \text{or} \qquad \partial_i\partial_{j-2} = \partial_j\partial_i \quad \text{when} \quad i < j.$$

3.3 A closed symmetric monoidal category structure on dendroidal sets

Since $dSets$ is a presheaf category, it can be endowed with the usual cartesian closed category structure present in any presheaf category. There is another interesting symmetric monoidal structure on $dSets$ that will prove to be useful in the definition of dendroidal weak n-categories of Chapter 5. Our goal is to recall this monoidal structure in the current section, together with those properties that will be used. For more details on this subject one can consult [32, 33, 40].

One way to define the mentioned monoidal structure on $dSets$ is by transferring the Boardman-Vogt monoidal structure on $\mathcal{O}p$, via the dendroidal nerve functor. We adopt this road, and we start by recalling the Boardman-Vogt tensor product for symmetric operads (a generalisation of the B-V tensor product for classical operads in [6]).

Let $P, Q \in \mathcal{O}p$. We define a new operad, $P \otimes Q$, as follows. The set of objects is $ob(P \otimes Q) := ob(P) \times ob(Q)$ and we denote the elements of this set as $a \otimes x := (a, x)$. We describe the operations of $P \otimes Q$ in terms of generators and relations. There are two types of generators,

(a) For any $p \in P(a_1, \dots, a_n; a)$ and any $x \in ob(Q)$,

$$p \otimes x \in P \otimes Q(a_1 \otimes x, \dots, a_n \otimes x; a \otimes x).$$

(b) For any $a \in ob(P)$ and any $q \in Q(x_1, \dots, x_m; x)$,

$$a \otimes q \in P \otimes Q(a \otimes x_1, \dots, a \otimes x_m; a \otimes x).$$

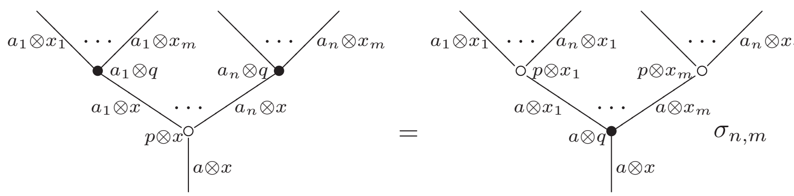
The relations also are of two types:

(a) Relations that imply precisely that the obvious maps

$$P \xrightarrow{\mathrm{id} \otimes x} P \otimes Q \text{ and } Q \xrightarrow{a \otimes \mathrm{id}} P \otimes Q$$

are maps of operads for any fixed $x \in ob(P), a \in ob(Q)$.

(b) For any $p \in P(a_1, \dots, a_n; a)$ and $q \in Q(x_1, \dots, x_m; x)$ the following two operations are the same in $P \otimes Q$



where $\sigma_{n,m} \in \Sigma_{n \cdot m}$ denotes the permutation that makes the order of the inputs on the right-hand side of the equation the same as the order of the inputs on the left-hand side.

The tensor product we defined is a bifunctor $- \otimes -: \mathcal{O}p \times \mathcal{O}p \longrightarrow \mathcal{O}p$ and it induces a symmetric closed monoidal category structure on $\mathcal{O}p$. The right adjoint of any functor $- \otimes Q$ is denoted by $\underline{\mathcal{O}p}(Q, -): \mathcal{O}p \longrightarrow \mathcal{O}p$. In particular, $\underline{\mathcal{O}p}(Q, \mathcal{S}ets)$ is *the operad of Q-algebras*. (For the definition of $\underline{\mathcal{O}p}(Q, -)$ see [40].)

We can now make use of the functor $N_d: \mathcal{O}p \longrightarrow d\mathcal{S}ets$ to transfer the Boardman-Vogt tensor product to dendroidal sets:

– For any two representable dendroidal sets $\Omega[T]$ and $\Omega[R]$, define

$$\Omega[T] \otimes \Omega[R] := N_d(\Omega(T) \otimes \Omega(R)).$$

– Extend the definition cocontinuously, i.e. for any $X, Y \in d\mathcal{S}ets$ write $X = \mathrm{colim}_T\, \Omega[T]$, $Y = \mathrm{colim}_R\, \Omega[R]$ as colimits of representables and define

$$X \otimes Y := \mathrm{colim}_{T,R}\, \Omega[T] \otimes \Omega[R].$$

The bifunctor $- \otimes -: d\mathcal{S}ets \times d\mathcal{S}ets \longrightarrow d\mathcal{S}ets$ induces a symmetric closed monoidal structure on $d\mathcal{S}ets$, the right adjoint of $- \otimes Y$ is the functor

$$\underline{d\mathcal{S}ets}(Y, -): d\mathcal{S}ets \longrightarrow d\mathcal{S}ets,$$

given on objects (by Yoneda lemma) by

$$\underline{d\mathcal{S}ets}(Y,Z)_T = d\mathcal{S}ets(Y \otimes \Omega[T], Z).$$

The following properties will prove to be useful in Chapter 5:

Proposition 3.3.1. *(Lemma 4.3.3 in [40]) For any operad $P \in \mathcal{O}p$ and for any tree $T \in \Omega$*

$$N_d(P) \otimes \Omega[T] \simeq N_d(P \otimes \Omega(T)).$$

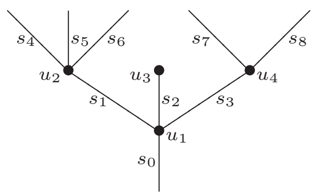
Proposition 3.3.2. *(Corollary 9.3 in [33]) For all operads $P, Q \in \mathcal{O}p$*

$$\underline{d\mathcal{S}ets}(N_d(P), N_d(Q)) \simeq N_d(\underline{\mathcal{O}p}(P,Q)).$$

3.4 The dendroidal Grothendieck construction

The aim of this section is to provide an ingredient we are going to use in the description of dendroidal weak higher categories. The data we start with is a functor $X\colon \mathbb{S}^{\mathrm{op}} \longrightarrow d\mathcal{S}ets$ where $\mathbb{S}$ is a cartesian category, and we are going to assign to X a new dendroidal set $\int_{\mathbb{S}} X$, called *the Grothendieck construction of X*.

To achieve this goal, we need some preliminary definitions. Since $\mathbb{S}$ is cartesian it is an operad, hence it makes sense to talk about the dendroidal set $N_d(\mathbb{S})$. Suppose that for a fixed tree T, $t \in N_d(\mathbb{S})$ is a dendrex of shape T. That is, t intuitively looks like the tree T decorated with objects and operations of the operad $\mathbb{S}$:



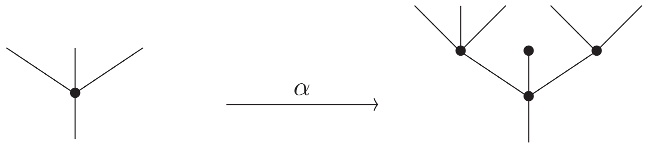
where the s_i are objects of $\mathbb{S}$, and – for example – $u_1\colon s_1 \times s_2 \times s_3 \longrightarrow s_0$ is a map in $\mathbb{S}$. To such a t we can assign an object of $\mathbb{S}$, called $\mathrm{in}(t)$, which is the cartesian product of the objects labeling the leaves of T: since $t \in \mathcal{O}p(\Omega(T), \mathbb{S})$,

$$\mathrm{in}(t) := \prod_{l \in \mathrm{Leaves}(T)} t(l).$$

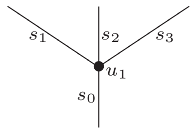
Furthermore, we can assign to a $t \in N_d(\mathbb{S})_T$ and a map $\alpha\colon R \longrightarrow T$ of Ω a map in $\mathbb{S}$

$$\mathrm{in}(\alpha)\colon\ \mathrm{in}(t) \longrightarrow \mathrm{in}(\alpha^* t)$$

by first composing the maps of $\mathbb{S}$, indicated by t and α, and then taking the product. For example, if $\alpha\colon R \longrightarrow T$ is the inclusion to the root vertex (in this case a composite of three outer faces)



and t is as above, then $\alpha^* t$ is



and $\mathrm{in}(\alpha) = u_2 \times u_3 \times u_4$. In particular, if α is an inner face or a degeneracy then $\mathrm{in}(\alpha)$ is the identity map of $\mathrm{in}(t) = \mathrm{in}(\alpha^* t)$, and if $R' \xrightarrow{\beta} R \xrightarrow{\alpha} T$ are maps of Ω then $\mathrm{in}(\alpha\beta) = \mathrm{in}(\beta)\,\mathrm{in}(\alpha)$.

In view of the definitions above we can define $\int_{\mathbb{S}} X$ as follows. The set $\left(\int_{\mathbb{S}} X\right)_T$ consists of pairs (t, x) where $t \in N_d(\mathbb{S})_T$ and

$$x \colon \Omega[T] \longrightarrow \coprod_{s \in \mathbb{S}} X(s)$$

is a degree preserving map such that $x(r) \in X(\mathrm{in}(r^* t))$ for any $r \in \Omega[T]_R$. There is one more condition on x: it has to be compatible with the dendroidal structure of the various dendroidal sets involved. Explicitly, for a chain of arrows $R' \xrightarrow{\alpha} R \xrightarrow{r} T$ in Ω we have $r \in \Omega[T]_R$ and $\alpha^* r = r\alpha \in \Omega[T]_{R'}$, hence

$$x(r) \in X\big(\mathrm{in}(r^* t)\big)_R \qquad \text{and} \qquad x(\alpha^* r) \in X\big(\mathrm{in}((r\alpha)^* t)\big)_{R'}.$$

The data above also induces two maps

$$\begin{tikzcd}
X\big(\mathrm{in}(r^* t)\big)_R \arrow[dr, "\alpha^*"'] & & X\big(\mathrm{in}((r\alpha)^* t)\big)_{R'} \arrow[dl, "X(\mathrm{in}\,\alpha)"] \\
& X\big(\mathrm{in}(r^* t)\big)_{R'} &
\end{tikzcd}$$

We require

$$\alpha^*\big(x(r)\big) = X(\mathrm{in}\,\alpha)\big(x(\alpha^* r)\big). \tag{3.4.1}$$

The dendroidal structure on $\int_{\mathbb{S}} X$ is defined as follows. Suppose that $\delta \colon R \longrightarrow T$ is a map in Ω and $(t, x) \in \left(\int_{\mathbb{S}} X\right)_T$ a dendrex of shape T. The map δ induces the map of dendroidal sets $\Omega[\delta] \colon \Omega[R] \longrightarrow \Omega[T]$. We define

$$\delta^*(t, x) := \big(\delta^* t, x \circ \Omega[\delta]\big). \tag{3.4.2}$$

One can check that with this structure $\int_{\mathbb{S}} X$ is indeed a dendroidal set. The following theorem and proposition collect two important properties of the dendroidal Grothendieck construction.

Theorem 3.4.1. *([32, 40]) Let $X\colon \mathbb{S}^{\mathrm{op}} \longrightarrow dSets$ be a diagram of dendroidal sets. If for all $s \in \mathbb{S}$ every $X(s)$ is an inner Kan complex then so is $\int_{\mathbb{S}} X$.*

Proposition 3.4.2. *Let $X\colon \mathbb{S}^{\mathrm{op}} \longrightarrow dSets$ be a diagram of dendroidal sets and $k \geq 2$ a natural number. If $X(s)$ is k-coskeletal for every $s \in \mathbb{S}$ then so is $\int_{\mathbb{S}} X$.*

Proof. Let us start with the remark that $k \geq 2$ is needed because dendroidal nerves of operads are 2-coskeletal (a generalisation of the well known fact for nerves of categories, proven in [32, 40]).

Our task is to prove that, for any tree T with $k+1$ vertices, every map of dendroidal sets $\phi\colon \partial\Omega[T] \longrightarrow \int_{\mathbb{S}} X$ extends uniquely as

$$\begin{tikzcd} \partial\Omega[T] \arrow[r, "\phi"] \arrow[d, tail] & \int_{\mathbb{S}} X \\ \Omega[T] \arrow[ur, dotted, "\exists!"'] & \end{tikzcd}$$

We suppose existence and prove uniqueness first. Let $(t_1, x_1), (t_2, x_2) \in \big(\int_{\mathbb{S}} X\big)_T$ be two dendrices filling the boundary ϕ. The dendroidal set $N_d(\mathbb{S})$ is k-coskeletal since $k \geq 2$. Hence by equation (3.4.2) we can infer that $t_1 = t_2 = t$. Let $u\colon R \longrightarrow T$ be a face. Since $u^*(t, x_1) = u^*(t, x_2)$, we obtain $x_1 \circ \Omega[u] = x_2 \circ \Omega[u]$. On the other hand,

$$x_i(u) = \big(x_i \circ \Omega[u]\big)(\mathrm{id}_R)$$

for $i = 1, 2$, implying $x_1(u) = x_2(u)$. We can use now equation (3.4.1) for $r = \mathrm{id}_T$ and $\alpha = u$ to conclude that $u^*(x_1(\mathrm{id}_T)) = u^*(x_2(\mathrm{id}_T))$ as dendrices of shape R in $X(\mathrm{in}(t))$. Since this is true for any face $u\colon R \longrightarrow T$ and $X(\mathrm{in}(t))$ is k-coskeletal, it follows that also $x_1(\mathrm{id}_T) = x_2(\mathrm{id}_T)$. We can infer that $x_1 = x_2$, thus the filler is unique.

The argument above also contains the information how to construct a filler (t, x) of ϕ, giving a proof of the existence of such an extension. ♦

Remark 3.4.3. If we restrict our attention to dendroidal sets where the only nontrivial dendrices are of linear shapes, Proposition 3.4.2 implies that the same property is true for simplicial sets and the simplicial Grothendieck construction.

3.5 The homotopy coherent dendroidal nerve of an operad

When $\mathcal{E}$ is a symmetric monoidal model category with an interval H (see Subsection 2.2.2 for the definition), one can modify the nerve construction for operads enriched in $\mathcal{E}$ in such a way that the resulting dendroidal set encodes also the homotopies in the operad.

An interesting example of such a situation is when $\mathcal{E}$ is the category of categories with the usual cartesian product, the folk model structure and the interval H is the category

$$0 \xrightarrow{\simeq} 1$$

with two objects and one isomorphism between them. The required structure on H is the obvious one: 0 is the neutral element, 1 is the absorbing one, and the rest of the interval structure on H is completely determined by the previous choices. Indeed, since the unit of the monoidal structure is the terminal object in $\mathcal{E}$ (the category $*$ with one object and no other morphisms than the identity), the counit $\epsilon\colon H \longrightarrow *$ is obvious. The various compatibility conditions imply that the monoid structure $\vee\colon H \times H \longrightarrow H$ is given by "the maximum operation": on the objects, $i \vee j = \max\{i, j\}$.

Since we are interested only in this example, from now on $\mathcal{E}$ denotes the category of categories with the structure mentioned above, although everything can be carried out similarly in the general case.

Let us denote the category of operads enriched in $\mathcal{E}$ by $\mathcal{O}p_{\mathcal{E}}$. The functor

$$\mathrm{hcN}_d\colon \mathcal{O}p_{\mathcal{E}} \longrightarrow d\mathcal{S}ets$$

is defined by

$$\mathrm{hcN}_d(P)_T := \mathcal{O}p_{\mathcal{E}}(W(\Omega(T)), P)$$

where W is the W-construction for coloured operads and $\Omega(T)$ is the discrete version in $\mathcal{E}$ of the operad induced by the tree T. We will need later an explicit description of $W(\Omega(T))$, hence we give it here.

Recall that for a tree T

$$\Omega(T) = \mathrm{Symm}(\Omega^{\pi}(\bar{T}))$$

where $\mathrm{Symm}\colon \mathcal{O}p^{\pi}_{\mathcal{E}} \longrightarrow \mathcal{O}p_{\mathcal{E}}$ is the $\mathcal{E}$-enriched version of the symmetrization functor from non symmetric operads to operads, and $\bar{T}$ is any planar representative of T. Moreover, the W-construction commutes with Symm, thus

$$W\Omega(T) = \mathrm{Symm}\left(W\Omega^{\pi}(\bar{T})\right).$$

This property allows us to describe $W\Omega(T)$ by using an arbitrary planar representative of T. The objects of $W\Omega^{\pi}(\bar{T})$ are the edges of T. Suppose that $\sigma = (e_1, e_2, \ldots, e_n; e)$ is an ordered sequence of objects. We can distinguish two cases for the category of operations corresponding to σ:

(1) If $\Omega^{\pi}(\bar{T})(\sigma) = \emptyset$ – the empty category – then also $W\Omega^{\pi}(\bar{T})(\sigma) = \emptyset$.
(2) If $\Omega^{\pi}(\bar{T})(\sigma) \neq \emptyset$, it follows that $\bar{T}$ has a subtree $\bar{T}_\sigma$ with leaves $e_1, \ldots, e_n$ and root e. The set of internal edges of $\bar{T}_\sigma$ is denoted by $\mathrm{InEdg}(\bar{T}_\sigma)$. From the W-construction it follows then that

$$W\Omega^{\pi}(\bar{T})(\sigma) = \prod_{f \in \mathrm{InEdg}(\bar{T}_\sigma)} H,$$

where in case the product is empty the result is the unit object of the monoidal structure, which is the category $*$ with one object and no other morphism than the identity.

We still need to define the composition maps in the operad $W\Omega^{\pi}(\bar{T})$. Suppose that $\sigma = (e_1, \ldots, e_n; e)$ and $\rho = (f_1, \ldots, f_m; e_i)$ are ordered sequences of edges of T, such that neither $\Omega^{\pi}(\bar{T})(\sigma)$, nor $\Omega^{\pi}(\bar{T})(\rho)$ is the empty category. It follows that $\bar{T}$ has subtrees $\bar{T}_\sigma$ and $\bar{T}_\rho$, the sets of internal edges of these trees are disjoint and the tree $\bar{T}_{\sigma \circ_i \rho}$ obtained by grafting along the edge e_i has one more internal edge than the previous two together. Let us denote these sets of internal edges by int(σ), int(ρ) and int($\sigma \circ_i \rho$) respectively. The composition map

$$\circ_i \colon W\Omega^{\pi}(\bar{T})(\sigma) \times W\Omega^{\pi}(\bar{T})(\rho) \longrightarrow W\Omega^{\pi}(\bar{T})(\sigma \circ_i \rho)$$

is given by

$$\left(\prod_{\text{int}(\sigma)} H\right) \times \left(\prod_{\text{int}(\rho)} H\right) \simeq \left(\prod_{\text{int}(\sigma)\cup\text{int}(\rho)} H\right) \times * \xrightarrow{\text{id} \times 1} \prod_{\text{int}(\sigma \circ_i \rho)} H ,$$

where the functor $1 \colon * \longrightarrow H$ is the absorbing element of H.

This concludes the description of the operad $W\Omega(T)$. Note that we still need to mention how the dendroidal structure on $\mathrm{hcN}_d(P)$ is defined. If $\delta \colon R \longrightarrow T$ is a face map in Ω then it induces a map of operads $\delta \colon W\Omega(R) \longrightarrow W\Omega(T)$ via the neutral element functor $0 \colon * \longrightarrow H$. In case δ is a degeneracy, the induced functor is obtained by the monoid structure $\vee \colon H \times H \longrightarrow H$. These definitions are functorial, hence they induce a dendroidal structure on $\mathrm{hcN}_d(P)$.

4

Dold-Kan correspondence for dendroidal abelian groups

The classical Dold-Kan correspondence [12, 22] establishes an equivalence between the category of simplicial abelian groups and the category of ($\mathbb{N}$-graded) chain complexes of abelian groups. In this chapter we generalise the correspondence to planar dendroidal abelian groups and a suitably constructed category of planar dendroidal complexes.

4.1 Planar dendroidal complexes

Our goal in this section is to introduce a category that – as we prove later – is equivalent to the category of planar dendroidal abelian groups. Let us start with some notation. We denote by $s\mathcal{A}b$ the category of simplicial abelian groups, by $\mathcal{C}h$ the category of chain complexes and by $pd\mathcal{A}b$ the category of planar dendroidal abelian groups (that is, functors $A\colon (\Omega^\pi)^{\mathrm{op}} \longrightarrow \mathcal{A}b$). Recall that if $A \in pd\mathcal{A}b$ then for any map $f\colon R \longrightarrow T$ of Ω^π the associated group homomorphism $A_T \longrightarrow A_R$ is denoted by f^*.

4.1.1 Normal faces

For any tree T a *maximal linear part* of T is an embedding $L_n \rightarrowtail T$ in Ω^π for some $n \geq 1$, such that whenever there is another such embedding that fits into a commutative diagram of inclusions

$$\begin{tikzcd} L_n \arrow[r, tail] \arrow[d, tail] & T \\ L_m \arrow[ur, tail] & \end{tikzcd}$$

then $m = n$. We say that a face map $\partial\colon R \longrightarrow T$ lives- or sits on a maximal linear part $L_n \rightarrowtail T$ when there exists a commutative diagram

$$\begin{tikzcd} L_{n-1} \arrow[r, tail] \arrow[d, "\partial_i"'] & R \arrow[d, "\partial"] \\ L_n \arrow[r, tail] & T \end{tikzcd} \tag{4.1.1}$$

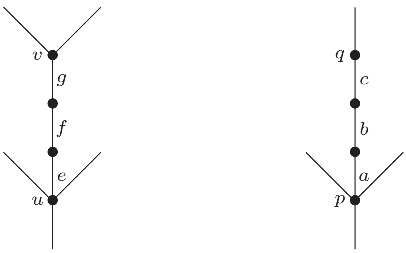
for some face $\partial_i\colon L_{n-1} \longrightarrow L_n$ where the index i is with respect to the linear order of Subsection 3.2.4. One can prove that if such a ∂_i exists, then it is unique. Moreover, if $\partial, \partial'\colon R \longrightarrow T$ sit on the same maximal linear part and ∂_i fills diagram (4.1.1) for both ∂ and ∂', then $\partial = \partial'$.

We infer that there are exactly $n+1$ faces sitting on a maximal linear part $\iota\colon L_n \longrightarrow T$, and with respect to the linear order defined in Subsection 3.2.4, they are the faces $\partial_k, \partial_{k+1}, \ldots, \partial_{k+n}$ for some k. To underline the similarity between the faces of $[n]$ in the category Δ, and the face maps sitting on a maximal linear part $\iota\colon L_n \longrightarrow T$, it proves to be convenient to shift their indices so they become $\partial_0^{(\iota)}, \ldots, \partial_n^{(\iota)}$.

We will say that the faces living on the same maximal linear part are *connected*.

Definition 4.1.1. A face map $\partial\colon R \longrightarrow T$ in Ω^π is *normal* whenever ∂ lives on a maximal linear part $\iota\colon L_n \longrightarrow T$ for some $n \geq 1$ and $\partial = \partial_i^{(\iota)}$ for some $0 \leq i < n$ in the associated order.

Remark 4.1.2. On the picture below T has one maximal linear part $L_2 \longrightarrow T$ and R has one maximal linear part $L_3 \longrightarrow R$.



T $\qquad\qquad$ R

The faces of T have the following properties: ∂_e and ∂_f are normal; ∂_g, ∂_u and ∂_v are not normal; ∂_e, ∂_f and ∂_g are connected to each other. The faces of R have the following properties: ∂_a, ∂_b and ∂_c are normal; ∂_p and ∂_q are not normal; ∂_a, ∂_b, ∂_c and ∂_q are connected to each other.

In general, if $\partial\colon R \longrightarrow T$ is a face that lives on a maximal linear part $L_n \longrightarrow T$ then ∂ is connected to precisely n other faces. Of these $n+1$ faces altogether, n are normal and one is not normal (the last one in the induced order). A special case of such is that the face $\partial_0^{(\iota)}\colon L_0 \longrightarrow L_1$ is normal, while $\partial_1^{(\iota)}\colon L_0 \longrightarrow L_1$ is not.

Remark 4.1.3. We made a choice in the definition of normal faces, since by our convention the last face on a maximal linear part is not normal, while all the other ones are. For our purposes with normal faces, the other possible choice (when the first face is not normal and the last one is, by definition) would be equally good. The reason for this is that the involution of categories $-^\circ\colon \Delta \longrightarrow \Delta$

$$[n]^\circ := [n], \text{ and for any } f\colon [n] \longrightarrow [m], \quad f^\circ(i) := m - f(n-i)$$

extends naturally to an involution $-^\circ\colon \Omega^\pi \longrightarrow \Omega^\pi$.

4.1.2 Planar dendroidal complexes

We say that an abelian group A is Ω^π-graded if there is a given direct sum decomposition $A = \bigoplus_{T\in\Omega^\pi} A_T$.

Definition 4.1.4. A *planar dendroidal complex* $(A, \mathcal{D})$ is an Ω^π-graded abelian group, together with given group homomorphisms $\delta^\sharp\colon A_T \longrightarrow A_R$ for every face map $\delta\colon R \longrightarrow T$. The data has to satisfy the following two conditions:

(1) $\delta^\sharp = 0$ if δ is a normal face;
(2) for any commutative diagram of elementary face relations

$$\begin{tikzcd} S \arrow[r, "\partial"] \arrow[d, "\delta"'] & R \arrow[d, "\partial_1"] \\ R' \arrow[r, "\delta_1"] & T \end{tikzcd}$$

the associated diagram

$$\begin{tikzcd} A_S & A_R \arrow[l, "\partial^\sharp"'] \\ A_{R'} \arrow[u, "\delta^\sharp"] & A_T \arrow[l, "\delta_1^\sharp"] \arrow[u, "\partial_1^\sharp"'] \end{tikzcd}$$

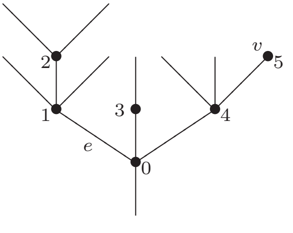
anticommutes.

Planar dendroidal complexes form a category with the obvious definition of maps. We denote this category by $pd\mathcal{C}omp$. Note that if in the definition of a planar dendroidal complex we replace the category Ω^π by its full subcategory Δ then we recover the notion of a chain complex.

4.1.3 A sign convention on faces of a tree

Our next goal is to extend the definition of the classical Moore complex functor to a functor $pd\mathcal{A}b \longrightarrow pd\mathcal{C}omp$. This extension relies on a suitable sign convention on the faces of any fixed tree T. Note that the linear order defined in Subsection 3.2.4 induces a sign convention in the obvious way. Unfortunately this will not be good for our purposes: we will need to modify it in such a way that Lemma 4.1.5 becomes true.

Let us begin by numbering the vertices of R from 0 to n, starting with the root-vertex and going always first to the left (see the next picture for an example). In this way we obtain a bijection $\sharp\colon \ \mathrm{Vert}(R) \longrightarrow \{0, \ldots, n\}$.



The sign of a face map ∂ is computed by the following rules:

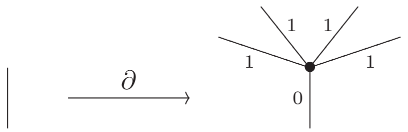
(i) If ∂ is an inner face map induced by the edge e and v is the upper vertex adjacent to e then $\mathrm{sgn}(\partial) = (-1)^{\sharp v}$.

(ii) If ∂ is an outer face map induced by the root-vertex r then $\mathrm{sgn}(\partial) = (-1)^{\sharp r} = 1$.

(iii) If ∂ is any other outer face map induced by the vertex v then $\mathrm{sgn}(\partial) = (-1)^{\sharp v+1}$.

For example, if ∂_e denotes the inner face associated to e on the tree above then $\mathrm{sgn}(\partial_e) = (-1)^1$. If on the same picture ∂_v denotes the outer face induced by v, then $\mathrm{sgn}(\partial_v) = (-1)^6$.

There is one exception to these rules, in the case of the inclusion of the tree with no vertices into a corolla:



In this case, if ∂ takes the sole edge of the stump to the root of the corolla then $\operatorname{sgn}(\partial) = 1$, otherwise $\operatorname{sgn}(\partial) = -1$.

The following result is an immediate consequence of the elementary face relations and our way of numbering the vertices.

Lemma 4.1.5. *Let f be a map in Ω^π such that it is a composition of two face maps $f = \partial_1 \circ \partial_2$. If $f = \partial_1' \circ \partial_2'$ is the other way to decompose f as a composition of two faces then* $\operatorname{sgn}(\partial_1)\operatorname{sgn}(\partial_2) = -\operatorname{sgn}(\partial_1')\operatorname{sgn}(\partial_2')$. ♦

4.1.4 The Moore complex associated to a planar dendroidal abelian group

Let A be a planar dendroidal abelian group. One can define a planar dendroidal complex $(MA, \mathcal{D})$ associated to A by setting $(MA)_T := A_T$ for every $T \in \Omega^\pi$, and for any face $\delta\colon R \longrightarrow T$ the map $\delta^\sharp$ is defined as follows.

(1) Let $\delta^\sharp := 0$ if δ is normal.
(2) Let $\delta^\sharp := \operatorname{sgn}(\delta) \cdot \delta^*$ if δ is not normal and not connected to any normal face.
(3) In the remaining case $\delta = \partial_n^{(\iota)}$ for a maximal linear part $L_n \overset{\iota}{\rightarrowtail} T$ and the induced order. Define in this case

$$\delta^\sharp := \sum_{i=0}^{n} \operatorname{sgn}(\partial_i) \cdot \partial_i^*.$$

Lemma 4.1.6. *The data $(MA, \mathcal{D})$ defines a planar dendroidal complex.*

Proof. Suppose that

$$\begin{tikzcd} S \arrow[r, "\delta'"] \arrow[d, "\delta"'] & R \arrow[d, "\delta_1'"] \\ R' \arrow[r, "\delta_1"] & T \end{tikzcd}$$

is a commutative diagram of elementary face relations. There are a couple of cases to distinguish.

If none of the four faces is normal or connected to a normal face then Lemma 4.1.5 ensures that the induced square anticommutes.

If each of the sets $\{\delta, \delta_1\}$, $\{\delta', \delta_1'\}$ contains at least a normal face then the induced square anticommutes trivially.

In case the set $\{\delta', \delta'_1\}$ contains a normal face while $\{\delta, \delta_1\}$ does not contain any, one has to prove that $\delta^\sharp \delta_1^\sharp = 0$. This can be done analogously to the simplicial case. First we observe that if δ' is normal, then δ_1 is always normal, hence we can suppose that δ'_1 is the sole normal face in the diagram. Moreover, it also follows that $\delta'_1 = \partial_{n-1}$ in the order induced by a maximal linear part $\iota\colon L_n \longrightarrow T$ and – since none of the elements of $\{\delta, \delta_1\}$ can be normal faces – δ' is connected to δ'_1, and both δ and δ_1 are connected to normal faces:

$$\delta^\sharp = \sum_{i=0}^{n} \mathrm{sgn}(\partial_i) \cdot \partial_i^*, \qquad \delta_1^\sharp = \sum_{i=0}^{n+1} \mathrm{sgn}((\partial_1)_i) \cdot (\partial_1)_i^*,$$

We conclude that $\delta^\sharp \delta_1^\sharp = 0$ now following the steps of the simplicial case, where one proves the differential property $d^2 = 0$ for the Moore chain complex associated to a simplicial abelian group.

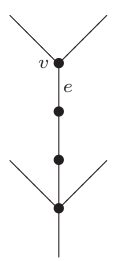
The remaining case, when none of the four faces are normal but some of them are connected to normal ones, breaks down into the following three cases.

(a) When both of δ' and δ'_1 are not normal, but connected to normal faces, they have to be induced on different maximal linear parts of the tree T. Hence δ and δ_1 are not normal, but connected to normal faces as well. Moreover, each summand in the definition of $\delta^\sharp$ and $\delta'^\sharp$ fits into a commutative diagram

$$\begin{tikzcd}
A_S & A_R \arrow[l, "(\partial')_j^*"'] \\
A_{R'} \arrow[u, "\partial_i^*"] & A_T \arrow[l, "(\partial_1)_j^*"] \arrow[u, "(\partial'_1)_i^*"']
\end{tikzcd}$$

hence Lemma 4.1.5 ensures that the required square anticommutes.

(b) Suppose that δ'_1 is connected to normal faces, while δ' is not. We analyse the case when δ' is "adjacent" to δ'_1 (the other cases are easier and left to the reader). This situation can be typically illustrated when T is the tree



and $\delta'_1 = \partial_e$, $\delta_1 = \partial_v$. Then it is clear what the other trees and face maps are in the diagram, and we see that $\delta\colon S \longrightarrow R'$ is connected to normal faces, but not normal. Again, a summand $(\partial'_1)_i^*$ of $(\delta'_1)^\sharp$ will correspond to the summand ∂_i^* of $\delta^\sharp$ such that the associated diagram

$$\begin{tikzcd} A_S & A_R \arrow[l, "(\delta')^*"'] \\ A_{R'} \arrow[u, "\partial_i^*"] & A_T \arrow[l, "(\delta_1)^*"] \arrow[u, "(\partial_1')_i^*"'] \end{tikzcd}$$

commutes for every i. We conclude that the required diagram is anticommutative.

(c) The remaining case, when δ' is connected to normal faces and δ_1' is not, is symmetric to case (b).

♦

The complex $(MA, \mathcal{D})$, or in short MA is called the *Moore complex* associated to A.

4.1.5 The normalized and degenerate subcomplexes

Let A be a planar dendroidal abelian group. We can construct a subcomplex NA of the Moore complex MA if we set

$$(NA)_T := \cap_{\tilde{\partial}} \ker(\tilde{\partial}^*) \leq A_T,$$

where $\tilde{\partial}$ runs through all normal faces with codomain T. Note that if T has no normal faces then we have an empty intersection and in that case we set $(NA)_T := A_T$. We can restrict the structure maps of $(MA, \mathcal{D})$ to get a planar dendroidal complex structure on NA. This is proven in the following proposition.

Another subcomplex DA of MA is defined by

$$(DA)_T := \sum_{\sigma\colon T \longrightarrow S} \sigma^*(A_S) \leq A_T,$$

where σ runs through all degeneracies with domain T. Again, the structure maps of $(MA, \mathcal{D})$ restrict to DA, thus we obtain a dendroidal subcomplex of MA which is called the *degenerate complex* associated to A.

Proposition 4.1.7. *The Ω^π-graded abelian groups NA and DA are dendroidal subcomplexes of MA.*

Proof. We deal first with the case of NA. Let $\delta\colon R \longrightarrow T$ be a face map in Ω^π. We need to prove that every $x \in (NA)_T$ satisfies $\delta^\sharp(x) \in (NA)_R$. There are three cases to distinguish.

(1) If δ is a normal face then $\delta^\sharp(x) = 0 \in (NA)_R$.

(2) If δ is neither normal nor connected to a normal face then $\delta^\sharp = \mathrm{sgn}(\delta) \cdot \delta^*$. Suppose that $\partial \colon S \longrightarrow R$ is a normal face. There is a commutative diagram of face-face dendroidal identities

$$\begin{tikzcd} S \arrow[r, "\partial"] \arrow[d, "\tilde{\gamma}"] & R \arrow[d, "\delta"] \\ R' \arrow[r, "\gamma"] & T \end{tikzcd}$$

by Lemma 3.2.1. It is easy to check that in such a case whenever ∂ is normal, γ is normal as well. We conclude that

$$\partial^* \delta^*(x) = \tilde{\gamma}^* \gamma^*(x) = 0$$

and thus $\partial^* \delta^\sharp(x) = 0$.

(3) In the remaining case $\delta = \partial_n$ for some maximal linear part $\iota \colon L_n \longrightarrow T$. In this case

$$\delta^\sharp = \sum_{i=0}^{n} \mathrm{sgn}(\partial_i) \cdot \partial_i^*.$$

Again let $\partial \colon S \longrightarrow R$ be a normal face. In the same way as in case (2), for every i we have $\partial_i \partial = \gamma_i \tilde{\gamma}_i$ for some normal face γ_i. Hence every summand of $\partial^* \delta^\sharp$ vanishes on x.

Next, we prove that DA is a subcomplex. Suppose that $\delta \colon R \longrightarrow T$ is a face map and $x \in (DA)_T$, so $x = \sigma_1^*(x_1) + \cdots + \sigma_k^*(x_k)$ for some degeneracies $\sigma_j \colon T \longrightarrow S_j$ and $x_j \in A_{S_j}$. There are again three cases to distinguish.

(1) If δ is a normal face then again $\delta^\sharp(x) = 0 \in (DA)_R$.

(2) If δ is neither normal nor connected to a normal face then there exists a commutative diagram of combined dendroidal identities

$$\begin{tikzcd} T \arrow[r, "\sigma_j"] & S_j \\ R \arrow[u, "\delta"] \arrow[r, "\sigma'_j"] & T_j \arrow[u, "\delta_j"'] \end{tikzcd},$$

for every j (otherwise δ would be a section of a degeneracy, hence normal or connected to a normal face). In this case

$$\delta^* \sigma_j^*(x_i) = \sigma_j'^* \delta_j^*(x_j) \in (DA)_R$$

for every j, thus $\delta^\sharp(x) \in (DA)_R$.

(3) In the remaining case $\delta^\sharp = \sum_{i=0}^{n} \mathrm{sgn}(\partial_i) \cdot \partial_i^*$ where $\partial_0, \ldots, \partial_{n-1}$ are normal faces sitting on the same maximal linear part $\iota \colon L_n \longrightarrow T$ where the

indices come from the induced order, and $\delta = \partial_n$ in this order. It follows that

$$\delta^\sharp(x) = \sum_{i,j} \operatorname{sgn}(\partial_i)\partial_i^*\sigma_j^*(x_j).$$

This sum can be divided into two parts.
The first part, Σ_1, consists of those components for which $\sigma_j\partial_i$ satisfy the combined dendroidal identities of the first type, that is $\sigma_j\partial_i = \partial_i'\sigma_j'$ for some face ∂_i' and degeneracy σ_j'. This part of the sum is clearly in $(DA)_R$.
The second part, Σ_2, consists of those summands for which $\sigma_j\partial_i$ satisfy the combined dendroidal identities of the second type, that is $\sigma_j\partial_i = \sigma_j\partial_i' = \mathrm{id}_R$ for some face ∂_i'. But in such a case $\operatorname{sgn}(\partial_i') = -\operatorname{sgn}(\partial_i)$ and one can form such pairs of the components of Σ_2 that cancel each other. Hence $\Sigma_2 = 0$.

♦

Proposition 4.1.8. *For any planar dendroidal abelian group A the associated Moore complex decomposes as $MA = NA \oplus DA$.*

Proof. The approach is similar to the one appearing in [17, 27, 28, 39], which establishes the same property for the classical Moore complex of a simplicial abelian group. We need to prove that $A_T = (NA)_T \oplus (DA)_T$ for every tree $T \in \Omega^\pi$. This follows from Lemma 4.1.9 and Lemma 4.1.10, below. ♦

Lemma 4.1.9. *The planar dendroidal complexes NA and DA satisfy*

$$(NA)_T \cap (DA)_T = 0$$

for every tree $T \in \Omega^\pi$.

Proof. Suppose that $0 \neq x \in (NA)_T \cap (DA)_T$ and write x as a finite sum of degeneracies

$$x = \sigma_1^*(x_1) + \cdots + \sigma_k^*(x_k),$$

such that the number of the summands is minimal. If $k = 1$ then $\sigma_1 \colon T \longrightarrow S$ has two right inverses in Ω^π and at least one of them, say $\partial \colon S \longrightarrow T$, is a normal face. It follows that $0 = \partial^*(x) = (\sigma_1\partial)^*(x_1) = x_1$ which contradicts $x \neq 0$.
If $k > 1$, we can use a similar pattern. Since k is minimal, $\sigma_i \neq \sigma_j$ for every $i \neq j$, hence σ_i and σ_j are induced by univalent vertices $v_i \neq v_j$. We can suppose that σ_1 sits on a linear part $L_n \longrightarrow T$ and satisfies that all the other σ_i are on a different linear part, or, if on the same one, that they come after σ_1 in the induced order. In other words, none of those vertices from $v_2, \ldots, v_k$ which are on the linear component of v_1 sit below v_1. Let ∂ be the normal right inverse to σ_k induced by the edge below v_1 or by cutting v_1. Then

$$\partial^*(x) - \big((\sigma_2\partial)^*(x_2) + \cdots + (\sigma_k\partial)^*(x_k)\big) = x_1,$$

$$x = \sigma_2^*(x_2) + \cdots + \sigma_k^*(x_k) - \big((\sigma_2\partial\sigma_1)^*(x_2) + \cdots + (\sigma_k\partial\sigma_1)^*(x_k)\big).$$

Let us look at the composite $\sigma_i \partial \sigma_1$ for all $i > 1$ and write it in another form with the help of the dendroidal identities. We observe that $\sigma_i \circ \partial \neq \mathrm{id}_T$ since we chose ∂ in a way that avoids this situation. It follows that we obtain commutative diagrams for all $1 < i \leq k$

$$\begin{tikzcd} T \arrow[r, "\sigma_1"] \arrow[d, dotted, "\sigma_i"'] & T\backslash v_1 \arrow[r, "\partial"] \arrow[d, "\sigma'"'] & T \arrow[d, "\sigma_i"] \\ T\backslash v_i \arrow[r, dotted, "\sigma''"] & (T\backslash v_1)\backslash v_i \arrow[r, "\partial'"] & T\backslash v_i \end{tikzcd}$$

where by the dendroidal identities the dotted vertical arrow can only be σ_i. We conclude that $(\sigma_i \partial \sigma_1)^*(u) = \sigma_i^*(u')$ for all $1 < i \leq k$ and

$$x = \sigma_2^*(y_2) + \cdots + \sigma_k^*(y_k).$$

This is a contradiction since k was chosen to be minimal. ♦

Lemma 4.1.10. *The planar dendroidal complexes NA and DA satisfy*

$$(NA)_T + (DA)_T = A_T$$

for every tree $T \in \Omega^\pi$.

Proof. Fix an $x \in A_T$ and define

$$N_x = \big\{\partial\colon S \longrightarrow T \;\big|\; \partial \text{ is a normal face such that } \partial^*(x) \neq 0\big\}.$$

We can suppose that N_x is not empty, otherwise $x \in (NA)_T$. Let $k \in \mathbb{N}$ be the number of the maximal linear parts of T. Partition N_x into subsets $N_x = N_x^{(\iota_1)} \cup \cdots \cup N_x^{(\iota_k)}$ for every maximal linear part $\iota_j\colon L_{n_j} \longrightarrow T$, where $N_x^{(\iota)}$ contains those elements of N_x which sit on ι.

The goal is to write x as $x = x_1 + y_1$, where $y_1 \in (DA)_T$ and $N_{x_1}^{(\iota_1)} = \emptyset$, while $|N_{x_1}^{\iota}| \leq |N_x^{\iota}|$ for the other linear parts $\iota \neq \iota_1$. If we succeed, we can iterate the process by finding a decomposition $x_1 = x_2 + y_2$ such that x_2 kills the set $N_{x_1}^{(\iota_2)}$ while does not increase the size of the other sets $N_{x_1}^{(\iota)}$. After k steps we would arrive at a decomposition

$$x = x_k + y_1 + y_2 + \cdots + y_k$$

where $N_{x_k} = \emptyset$ thus $x_k \in (NA)_T$, and $y_1 + \cdots + y_k \in (DA)_T$. Hence we could conclude that $x \in (NA)_T + (DA)_T$.

To obtain such a decomposition of x, we proceed as follows. Let $\partial \in N_x^{(\iota_1)}$ be the smallest element in the order induced by $\iota_1\colon L_{n_1} \longrightarrow T$. Let $\partial^*(x) = y$ and define $x' = x - \sigma^*(y)$ where $\sigma\colon T \to S$ is the biggest such degeneracy in the order induced by ι_1 for which $\sigma\partial = \mathrm{id}_S$. It follows that

$$\partial^*(x') = \partial^*(x) - (\sigma\partial)^*(y) = 0.$$

Now suppose that $\tilde{\partial}$ is any normal face such that $\tilde{\partial}^*(x) = 0$. We are going to prove that $\tilde{\partial}^*(x') \neq 0$ can happen only if $\tilde{\partial}$ is connected to ∂ and $\tilde{\partial} > \partial$ in the order induced by ι_1. Indeed, on the one hand if $\tilde{\partial}$ is not connected to ∂ then it is obvious that $\sigma\tilde{\partial} \neq \mathrm{id}$. On the other hand, the reason we chose ∂ to be minimal and σ maximal was that now $\sigma\tilde{\delta} \neq \mathrm{id}$ holds also whenever $\tilde{\partial} < \partial$ on the linear part ι_1. Hence if $\tilde{\partial}$ obeys one of these cases, we can fill the following diagram of dendroidal identities

$$\begin{tikzcd}
T \arrow[r, "\sigma"] & T\backslash v \arrow[r, "\partial"] & T \\
T' \arrow[u, "\tilde{\partial}"] \arrow[r, "\sigma'"] & T'\backslash v \arrow[u, "\partial'"] \arrow[r, dotted, "\partial''"] & T' \arrow[u, dotted, "\tilde{\partial}"']
\end{tikzcd}$$

It is immediate that in this diagram the dotted vertical arrow is $\tilde{\partial}$. Therefore

$$\tilde{\partial}^*(x') = -\tilde{\partial}^*\sigma^*(y) = -\tilde{\partial}^*\sigma^*\partial^*(x) = -\sigma'^*\partial''^*\tilde{\partial}^*(x) = 0.$$

Let us summarize what we managed to achieve with the $x = x' + \sigma(y)$ decomposition:

- The normal face ∂ satisfies $\partial(x) \neq 0$ and $\partial(x') = 0$.
- For any normal face $\tilde{\partial}$ such that $\tilde{\partial}^*(x) = 0$, we can have $\tilde{\partial}^*(x') \neq 0$ only if $\tilde{\partial} > \partial$ on the same linear part ι_1.

Now we can apply the same process for x' and the new smallest element in $N_{x'}^{(\iota_1)}$, etc. In a finite number of steps we arrive to the desired decomposition $x = x_1 + y_1$. ♦

4.2 The Dold-Kan correspondence

In this section we construct a right adjoint Γ to the normalized complex functor N and we prove that the pair (N, Γ) forms an equivalence of categories. This equivalence extends the classical Dold-Kan correspondence for simplicial abelian groups and chain complexes, in the sense that we can build a diagram

$$\begin{tikzcd}
s\mathcal{A}b \arrow[r, shift left, "i_!"] \arrow[d, shift right, "N_s"'] & pd\mathcal{A}b \arrow[l, shift left, "i^*"] \arrow[d, shift right, "N"'] \\
\mathcal{C}h \arrow[r, shift left, "j_!"] \arrow[u, shift right, "\Gamma_s"'] & pd\mathcal{C}omp \arrow[l, shift left, "j^*"] \arrow[u, shift right, "\Gamma"']
\end{tikzcd}$$

in which all the adjacent functors form adjunctions, and the desired relations between these functors hold.

Denote by Ω_{mono} the subcategory of Ω^π, consisting of all the trees as objects and only monomorphisms as maps. For every planar dendroidal complex C there is a functor $F_C\colon \Omega^{\mathrm{op}}_{mono} \longrightarrow \mathcal{A}b$ defined on objects by $F_C(T) := C_T$ and on face maps by

$$F_C(\partial) := \begin{cases} 0 & \text{if } \partial \text{ is normal,} \\ \operatorname{sgn}(\partial)\partial^\sharp & \text{oherwise.} \end{cases}$$

Note that F_C is indeed a functor since the sign convention on faces implies that commutative diagrams of dendroidal identities involving faces are taken via F_C to commutative diagrams of abelian groups. This functor will play a role in the construction of a right adjoint $\Gamma\colon pd\mathcal{C}omp \longrightarrow pd\mathcal{A}b$ to the normalized complex functor N.

The functor Γ is constructed similarly to the classical case. First suppose that such a right adjoint exists, thus there should be a one-to-one correspondence between hom sets

$$pd\mathcal{C}omp(NA, C) \simeq pd\mathcal{A}b(A, \Gamma C)$$

for any planar dendroidal complex C and planar dendroidal abelian group A. If one takes A to be the representable $\mathbb{Z}\Omega^\pi[T]$, one can conclude that

$$pd\mathcal{C}omp(N\mathbb{Z}\Omega^\pi[T], C) \simeq pd\mathcal{A}b(\mathbb{Z}\Omega[T], \Gamma C) \simeq (\Gamma C)_T \tag{4.2.1}$$

by the Yoneda lemma. Moreover, this correspondence has to be an isomorphism of groups, showing us a way to define $(\Gamma C)_T$ for every tree T. One can unpack the left-hand side of equation (4.2.1) to arrive at the definition

$$(\Gamma C)_T := \bigoplus_{r\colon T \twoheadrightarrow R} C_R,$$

where r runs through all epimorphisms in Ω^π with domain T. In the direct sum above we will denote by C^r_R the component C_R corresponding to an epimorphism r.

We still have to define ΓC on the maps of Ω^π. Suppose that $f\colon S \longrightarrow T$ is such a map and define $f^*\colon (\Gamma C)_T \longrightarrow (\Gamma C)_S$ in the following way. Let $r\colon T \twoheadrightarrow R$ be an epimorphism in Ω^π. The map $r \circ f\colon S \longrightarrow R$ has a unique factorization $d \circ s$ by Lemma 3.2.2:

$$\begin{tikzcd} S \arrow[r, two heads, "s"] \arrow[d, "f"'] & S' \arrow[d, tail, "d"] \\ T \arrow[r, two heads, "r"] & R \end{tikzcd}$$

We define f^* on the component C^r_R as the composite

$$(f^*)^r\colon C^r_R \xrightarrow{F_C(d)} C^s_{S'} \rightarrowtail (\Gamma C)_S \ .$$

We have finished the definition of Γ on objects. Let us check that ΓC is indeed a planar dendroidal abelian group for every $C \in pd\mathcal{C}omp$. It is easy to see that $\Gamma C(\mathrm{id}_T) = \mathrm{id}\colon \Gamma C_T \longrightarrow \Gamma C_T$. Suppose that $f\colon S \longrightarrow T$ and $g\colon U \longrightarrow S$ are two maps in Ω^π. It needs to be proven that for any epi $r\colon T \twoheadrightarrow R$ the components $(fg)^{*r}$ and $(g^*f^*)^r$ are the same. Indeed, since the epi-mono factorizations of rfg, and of rf followed by sg are unique, we can infer that $d = d_f d_g$ on the following diagram.

$$\begin{tikzcd}
U \arrow[r, two heads, "u"] \arrow[d, "g"'] & U' \arrow[d, tail, "d_g"] \arrow[dd, bend left, "d"] \\
S \arrow[r, two heads, "s"] \arrow[d, "f"'] & S' \arrow[d, tail, "d_f"] \\
T \arrow[r, two heads, "r"] & R
\end{tikzcd}$$

Since F_C is a functor, this implies the required equality.

It is easy to check that the obvious definition of Γ on maps of planar dendroidal complexes is functorial. Now we can prove the following propositions.

Proposition 4.2.1. *For every tree $T \in \Omega^\pi$ the abelian groups $(N\Gamma C)_T$ and C_T are equal.*

Proof. We have two decompositions of the abelian group $(\Gamma C)_T$ into a direct sum of subgroups. First, by definition

$$(\Gamma C)_T = C_T^{\mathrm{id}_T} \oplus \left(\bigoplus_{\substack{T \overset{r}{\twoheadrightarrow} R \\ r \neq \mathrm{id}_T}} C_R^r \right)$$

and second, by Proposition 4.1.8

$$(\Gamma C)_T = (N\Gamma C)_T \oplus (D\Gamma C)_T.$$

Hence it is enough to prove that $\left(\bigoplus_{r \neq \mathrm{id}_T} C_R^r \right) \leq (D\Gamma C)_T$ and $C_T^{\mathrm{id}_T} \leq (N\Gamma C)_T$.

To see the first assertion we pick an epimorphism $r\colon T \twoheadrightarrow R$, $r \neq \mathrm{id}_T$ and prove that the corresponding component $C_R^r \leq (\Gamma C)_T$ is in the image of a degeneracy. From our choice it follows that r decomposes as $r = \sigma \circ r'$ where $\sigma\colon T \longrightarrow S$ is a degeneracy and $r'\colon S \twoheadrightarrow R$ is another epimorphism (possibly the identity). Let us look at the image of $\sigma^*\colon (\Gamma C)_S \longrightarrow (\Gamma C)_T$ on the component $C_R^{r'}$. Since the unique epi-mono factorization of $r'\sigma$ is

$$\begin{tikzcd}
T \arrow[r, two heads, "r"] \arrow[d, "\sigma"'] & R \arrow[d, equal] \\
S \arrow[r, two heads, "r'"] & R
\end{tikzcd}$$

we can conclude that σ^* sends the component $C_R^{r'}$ to the component C_R^r.

The second assertion follows as well. Indeed, for an arbitrary normal face $\partial\colon S \longrightarrow T$ the induced map of abelian groups $\partial^*\colon (\Gamma C)_T \longrightarrow (\Gamma C)_S$ vanishes on $C_T^{\mathrm{id}_T}$ since $F_C(\partial) = 0$ by definition. ♦

Proposition 4.2.2. *Let A be a planar dendroidal abelian group and $r\colon T \twoheadrightarrow R$ an epimorphism in Ω^π. Define $(\Psi_A)^r_T$ to be the composite*

$$(NA)^r_R \rightarrowtail A_R \xrightarrow{r^*} A_T \ .$$

The induced map $(\Psi_A)_T\colon (\Gamma NA)_T \longrightarrow A_T$ is an isomorphism which is natural in both A and T.

Proof. We leave the check of naturality to the reader. In what follows we will write Ψ_T instead of $(\Psi_A)_T$ to simplify the notation. The first observation is that Ψ_T decomposes as a direct sum

$$N\Psi_T \oplus D\Psi_T\colon (N\Gamma(NA))_T \oplus (D\Gamma(NA))_T \longrightarrow (NA)_T \oplus (DA)_T,$$

and by Proposition 4.2.1 the $N\Psi_T$ component is $\mathrm{id}\colon (NA)_T \longrightarrow (NA)_T$. Hence in order to conclude that Ψ_T is an isomorphism, it is enough to prove that $D\Psi_T$ is surjective and injective.

We proceed by induction on the number of vertices of T. If $n = 0$ then $T = |$ and $(D\Gamma NA)_T = 0 = (DA)_T$. Suppose that $D\Psi_S$ is surjective and injective for every tree S with less than n vertices and let T be a tree with n vertices.

Let x be in the image of $\sigma^*\colon A_S \longrightarrow A_T$ for some degeneracy $\sigma\colon T \longrightarrow S$ and look at the commutative diagram

$$\begin{tikzcd}
(\Gamma NA)_S \arrow[r, "\Psi_S"] \arrow[d, "\sigma^*"'] & A_S \arrow[d, "\sigma^*"] \\
(\Gamma NA)_T \arrow[r, "\Psi_T"] & A_T
\end{tikzcd}$$

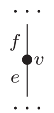
Since S has less vertices than T, $D\Psi_S$ is surjective and $x = D\Psi_T(y)$ for some $y \in (D\Gamma NA)_T$. Hence $D\Psi_T$ is surjective.

Let us prove that $D\Psi_T$ is injective. Suppose that $x \in \ker D\Psi_T$ and write

$$x = \sum_{T \overset{r}{\twoheadrightarrow} R} x^r.$$

For any tree R let $\mathcal{X}_{T,R} := \{r\colon T \twoheadrightarrow R \mid x^r \neq 0\}$. We are going to prove that $\mathcal{X}_{T,R} = \emptyset$ for every R, by contradiction. First we deal with the special case when $\mathcal{X}_{T,R}$ consists only of epimorphisms occurring in the same maximal linear part of T.

For each $r\colon T \twoheadrightarrow R$ we choose a specific section d_r, as follows. If $\sigma\colon T \to T\backslash v$ is a degeneracy and the vertex v has adjacent edges e, f with f situated above e in the gravitational order



then there is a unique face map $\partial\colon T\backslash v \longrightarrow T$ which omits the edge e. This face map satisfies $\sigma\partial = \mathrm{id}_{T\backslash v}$. If r decomposes as $r = \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_k$ into degeneracies then define the section $d_r := \partial_k \circ \cdots \circ \partial_2 \circ \partial_1$ of r where ∂_i is picked for σ_i in the way described above. Note that the dendroidal identities ensure that any other decomposition of r yields the same section.

Next we define a partial order on $\mathcal{X}_{T,R}$, as follows. If $r, s \in \mathcal{X}_{T,R}$ and for every edge e sitting on the relevant linear part of R the edge $d_r(e)$ is equal to- or below $d_s(e)$ in the gravitational order, then we say that $r \leq s$. We observe that

$$sd_r = \mathrm{id}_R \text{ for some } r, s \in \mathcal{X}_{T,R} \text{ implies } r \leq s. \tag{4.2.2}$$

Pick a maximal $r \in \mathcal{X}_{T,R}$ with respect to the order defined above. By equation (4.2.2), d_r satisfies

$$sd_r = \mathrm{id} \text{ implies } s = r,$$

hence we can conclude that in the commutative diagram

$$\begin{tikzcd}
(\Gamma NA)_T \arrow[r, "\Psi_T"] \arrow[d, "d_r^*"'] & A_T \arrow[d, "d_r^*"] \\
(\Gamma NA)_R \arrow[r, "\Psi_R"] & A_R
\end{tikzcd}$$

on the left-hand side $N(d_r^*(x)) = x^r$. Since $d_r^*\Psi_T = \mathrm{id}$ and $N\Psi_R = \mathrm{id}$, we infer that $x^r = 0$.

We still need to deal with the general case, when $\mathcal{X}_{T,R}$ contains epimorphisms which are not necessarily situated on a fixed maximal linear part of T. To do so, suppose that T has k different maximal linear parts $L_{n_i} \rightarrowtail T$, $i \in \{1, \ldots, k\}$. Decompose each $r \in \mathcal{X}_{T,R}$ to $r = r_1 \circ \cdots \circ r_k$ where $r_k\colon T \twoheadrightarrow R_{k-1}$ is situated on the maximal linear part $L_{n_k} \rightarrowtail T$, $r_{k-1}\colon R_{k-1} \twoheadrightarrow R_{k-2}$ sits on the obvious maximal linear part $L_{n_{k-1}} \rightarrowtail R_{k-1}$ of the intermediate tree R_{k-1}, etc. (Note that an epimorphism r_i that appears in such a decomposition can be the identity.) We can define a section d_r of r in the same way as in the special case above, moreover d_r decomposes as

$$d_r = d_{r_1} \circ \cdots \circ d_{r_k}$$

where d_{r_i} is a section of r_i. We can define a partial order on $\mathcal{X}_{T,R}$ as follows. Let $r, s \in \mathcal{X}_{T,R}$ have the associated decompositions

$$r = r_1 \circ \cdots \circ r_k \text{ and } s = s_1 \circ \cdots \circ s_k.$$

We say that $r \leq s$ if for every i there exist intermediate trees R_i, R_{i-1} such that $s_i, r_i \in \mathcal{X}_{R_i,R_{i-1}}$ and $r_i \leq s_i$ in the partial order defined in the special case above.

Again, if $sd_r = \mathrm{id}_R$ then $r \leq s$ and we can mimic the rest of the proof for the special case to conclude that $x^r = 0$ for a maximal $r \in \mathcal{X}_{T,R}$. ♦

Now we can prove the correspondence theorem.

Theorem 4.2.3. *The functors* $N\colon pd\mathcal{A}b \longrightarrow pd\mathcal{C}omp$ *and* $\Gamma\colon pd\mathcal{C}omp \longrightarrow pd\mathcal{A}b$ *form an equivalence of categories.*

Proof. Propositions 4.2.1 and 4.2.2 imply that N and Γ together form an adjoint equivalence where the unit of the adjunction is the natural isomorphism Ψ^{-1} of Proposition 4.2.2 and the counit is the identity. ♦

We finally summarize the relations between simplicial abelian groups, chain complexes, planar dendroidal abelian groups and planar dendroidal complexes.

The inclusion $i\colon \Delta \hookrightarrow \Omega^{\pi}$ induces an adjunction

$$\begin{tikzcd} s\mathcal{A}b \arrow[r, shift left, "i_!"] & pd\mathcal{A}b \arrow[l, shift left, "i^*"] \end{tikzcd},$$

where the left adjoint $i_!$ is given by

$$(i_!A)_T = \begin{cases} A_n & \text{if } T = L_n \\ 0 & \text{otherwise,} \end{cases}$$

and the right adjoint i^* is precomposition with i.

Similarly, there is a functor $j_!\colon \mathcal{C}h \longrightarrow pd\mathcal{C}omp$ which has a right adjoint j^*, that assigns to each planar dendroidal complex its linear part which is a chain complex. Let $N_s\colon s\mathcal{A}b \longrightarrow \mathcal{C}h$ and $\Gamma_s\colon \mathcal{C}h \longrightarrow s\mathcal{A}b$ denote the functors in the classical Dold-Kan correspondence. We can build the following diagram

$$\begin{tikzcd} s\mathcal{A}b \arrow[r, shift left, "i_!"] \arrow[d, shift right, "N_s"'] & pd\mathcal{A}b \arrow[l, shift left, "i^*"] \arrow[d, shift right, "N"'] \\ \mathcal{C}h \arrow[r, shift left, "j_!"] \arrow[u, shift right, "\Gamma_s"'] & pd\mathcal{C}omp \arrow[l, shift left, "j^*"] \arrow[u, shift right, "\Gamma"'] \end{tikzcd}$$

where all the adjacent functors form adjunctions with the left adjoints on the top or to the left. Moreover, the following relations hold:

$$\begin{array}{rclcrcl} N_s i^* & = & j^* N & \qquad & N i_! & = & j_! N_s \\ \Gamma_s j^* & = & i^* \Gamma & & \Gamma j_! & = & i_! \Gamma_s \\ i^* i_! & = & \mathrm{id} & & j^* j_! & = & \mathrm{id} \\ N\Gamma & = & \mathrm{id} & & N_s \Gamma_s & = & \mathrm{id} \\ \Gamma N & \cong & \mathrm{id} & & \Gamma_s N_s & \cong & \mathrm{id}\,. \end{array}$$

5

Dendroidal weak 2- and 3-categories

In this chapter we study the dendroidal definition of weak n-categories introduced in [32]. We are particularly interested in the $n = 2$ and $n = 3$ cases. In these cases the classical notions of weak n-categories are bicategories and tricategories, respectively. Our goal is to compare these classical notions with the corresponding dendroidal notions, in some suitable sense. Sections 5.1 and 5.2 deal with the preliminaries on dendroidal sets and the definition of weak n-categories via dendroidal sets. One important result in these sections is that weak n-categories are 3-coskeletal dendroidal sets, a result that allows us to restrict ourselves to studying only low degree dendrices when dealing with dendroidal weak 2- and 3-categories in Sections 5.3 and 5.4.

5.1 Categories enriched in dendroidal sets. Weak n-categories

For any set A there exists a planar operad As_A^π whose algebras are small categories with set of objects A. The objects of As_A^π are ordered pairs $(a_1, a_2) \in A \times A$, and the sets of operations are defined by

$$As_A^\pi\big(\quad ; (a, a)\big) = *,$$
$$As_A^\pi\big((a_1, a_2), (a_2, a_3), \ldots, (a_{n-1}, a_n); (a_1, a_n)\big) = *$$

and in all the other cases the set of operations is empty (those ordered sequences $\sigma = (c_1, c_2, \ldots, c_n; c)$ of objects of $A \times A$ for which $As_A^\pi(\sigma)$ is not empty will be called *admissible*).

Let $\alpha\colon As_A^\pi \longrightarrow \mathcal{S}ets$ be a map of operads. The data-part of such an α determines for any $(a_1, a_2) \in A \times A$ a set $\mathcal{A}(a_1, a_2)$ and for any admissible signature $\sigma = \big((a_1, a_2), (a_2, a_3), \ldots, (a_{n-1}, a_n); (a_1, a_n)\big)$ a function

$$comp_\sigma\colon \mathcal{A}(a_1, a_2) \times \mathcal{A}(a_2, a_3) \times \cdots \times \mathcal{A}(a_{n-1}, a_n) \longrightarrow \mathcal{A}(a_1, a_n)$$

which in the particular case of $\sigma = \big(\quad ; (a, a)\big)$ is a function $* \longrightarrow \mathcal{A}(a, a)$. The compatibility-part of such an α ensures that the various functions $comp_\sigma$ fit nicely to define units and compositions of arrows in a category $\mathcal{A}$ with object set A. Indeed, we arrive to the conclusion that the relevant signatures are of the type $\big((a_1, a_2), (a_2, a_3); (a_1, a_3)\big)$ and $\big(\quad ; (a, a)\big)$, etc.

Since the forgetful functor $U\colon \mathcal{O}p \longrightarrow \mathcal{O}p^\pi$ is right adjoint to the symmetrization functor, we infer that the algebras of the operad $As_A := \mathrm{Symm}(As_A^\pi)$ are categories with set of objects A as well.

Remark 5.1.1. Note that in the description of As_A^π-algebras given above we used the unconventional "left-to-right" composition order for arrows, i.e.

$$\mathcal{A}(a_1, a_2) \times \mathcal{A}(a_2, a_3) \longrightarrow \mathcal{A}(a_1, a_3)$$

instead of the conventional

$$\mathcal{A}(a_2, a_3) \times \mathcal{A}(a_1, a_2) \longrightarrow \mathcal{A}(a_1, a_3)$$

To avoid unnecessary complications in the future, arising only from notation, whenever we need to give such a composition map associated to some signature σ, we will always stick to the order determined by σ, thus the unconventional order. However, when it is required to give extra details with explicit composites of maps, we will use the conventional "right-to-left" order.

Let X be a dendroidal set and define the functor

$$\mathcal{C}at(X)_- : \mathcal{S}ets^{\mathrm{op}} \longrightarrow d\mathcal{S}ets, \quad \mathcal{C}at(X)_A := \underline{d\mathcal{S}ets}(N_d(As_A), X).$$

The dendroidal set of *categories enriched in* X is by definition the Grothendieck construction of $\mathcal{C}at(X)_-$. We denote it by

$$\mathcal{C}at(X) := \int_{\mathcal{S}ets} \mathcal{C}at(X)_- .$$

One can iterate the process above to obtain a definition of the dendroidal set of n*-categories enriched in* X:

$$\begin{aligned}\mathcal{C}at^0(X) &:= X,\\ \mathcal{C}at^n(X) &:= \mathcal{C}at(\mathcal{C}at^{n-1})(X).\end{aligned}$$

To see why this definition is plausible, one can try particular choices of X. For example if $X = N_d(\mathcal{S}ets)$, we can prove inductively that $\mathcal{C}at^n(X)$ is the dendroidal nerve of strict n-categories with the classical definition (see also Example 4.5.5 in [40]). Indeed, for $n = 1$

$$\begin{aligned}\mathcal{C}at(N_d(\mathcal{S}ets)) &= \int_{A\in\mathcal{S}ets} \underline{d\mathcal{S}ets}(N_d(As_A), N_d(\mathcal{S}ets))\\ &\simeq \int_{A\in\mathcal{S}ets} N_d(\underline{\mathcal{O}p}(As_A, \mathcal{S}ets))\\ &\simeq \int_{A\in\mathcal{S}ets} N_d(\mathcal{C}ateg_A)\\ &\simeq N_d(\mathcal{C}ateg),\end{aligned}$$

where $\mathcal{C}ateg$ denotes the usual monoidal category of small categories, viewed as an operad. The second part of the inductive proof is similar (one uses that for any monoidal category $\mathcal{M}$, $\underline{\mathcal{O}p}(As_A, \mathcal{M}) \simeq \mathcal{C}ateg_A(\mathcal{M})$, where the right-hand side denotes the monoidal category of categories enriched in $\mathcal{M}$, with set of object A).

We are interested here in another choice for X, which yields the dendroidal definition of weak n-categories: it is plausible to define $X := \mathrm{hcN}_d(\mathcal{C}tg)$ where $\mathcal{C}tg$ is the category of small categories enriched in $\mathcal{E}$. (Recall from Section 3.5 that for the rest of this thesis $\mathcal{E}$ denotes the symmetric monoidal model category of categories, together with the interval H. Hence the set of functors between two fixed categories is a category with natural transformations as maps.)

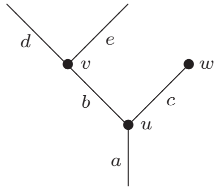
Definition 5.1.2. The dendroidal set of *weak* n*-categories* is defined as follows:

$$\begin{aligned}w\mathcal{C}at^0 &:= N_d(\mathcal{S}ets),\\ w\mathcal{C}at^n &:= \mathcal{C}at^{n-1}(\mathrm{hcN}_d(\mathcal{C}tg)) \quad \text{for } n > 0.\end{aligned}$$

The rest of this section is dedicated to the study of coskeletality of weak n-categories. The first result is

Lemma 5.1.3. *If* $T \in \Omega$ *is a tree with* 3 *vertices and* $t, s \in w\mathcal{C}at^1_T$ *satisfy* $\mathrm{Sk}_2(t) = \mathrm{Sk}_2(s)$ *then* $t = s$.

Proof. To illustrate our argument better, we will work with a chosen tree, the general case can be carried out in the same way. So let $T \in \Omega$ be the tree with a planar representative as below.



Let $x \in w\mathcal{C}at^1_T$ be a dendrex of shape T, that is a map of operads enriched in $\mathcal{E}$, $x : W\Omega(T) \longrightarrow \mathcal{C}tg$. Let us adopt the notations of Section 3.5. It follows that x consists of compatible functors $x_\sigma : H^{\mathrm{int}(\sigma)} \longrightarrow \mathcal{C}tg(\sigma)$ and the only functor we have to describe in terms of the 2-skeleton of x is the one corresponding to $\sigma = (d, e; a)$ (the other functors lie in the image of $\mathrm{Sk}_2(x)$). In this case the domain of x_σ is the groupoid $H^2 = H_c \times H_b$, represented as the square

$$\begin{tikzcd}
(0,0) \arrow[r, "(\mathrm{id}_0, b)"] \arrow[d, "(c, \mathrm{id}_0)"'] & (0,1) \arrow[d, dotted, "(c, \mathrm{id}_1)"] \\
(1,0) \arrow[r, dotted, "(\mathrm{id}_1, b)"] & (1,1)
\end{tikzcd}$$

where we think of the copy of H corresponding to an internal edge f as the groupoid $H_f = 0 \xrightarrow{f} 1$. Since the domain is a groupoid, we observe that if x_σ is already defined on a "connected part" of the "square" $H_c \times H_b$, then it is defined on the "convex hull" of that component. We conclude that in order to know x_σ, it is enough to know its image on the sets of arrows $Opd = \{(c, \mathrm{id}_1), (\mathrm{id}_1, b)\}$ and $Face = \{(c, \mathrm{id}_0), (\mathrm{id}_0, b)\}$.

To conclude the proof, first we show that since x is a map of operads, $x_\sigma(Opd)$ is determined by $\mathrm{Sk}_2(x)$. Indeed, the commutative square

$$\begin{tikzcd}
H_c \times \{v\} \arrow[r, "x \times x"] \arrow[d, "\circ_b"'] & \mathcal{C}tg(b; a) \times \mathcal{C}tg(d, e; b) \arrow[d, "\circ_b"] \\
H_c \times H_b \arrow[r, "x_\sigma"] & \mathcal{C}tg(d, e; a)
\end{tikzcd}$$

implies that we know x_σ on the arrow (c, id_1), and a similar square gives the image of (id_1, b).

Second, we show that $x_\sigma(Face)$ is in the image of $\mathrm{Sk}_2(x)$. We observe that the inner faces $\partial_b : R \longrightarrow T$ and $\partial_c : R' \longrightarrow T$, according to the definition of the dendroidal structure on $w\mathcal{C}at^1$, induce enriched operad maps $W\Omega(R) \longrightarrow W\Omega(T)$ and $W\Omega(R') \longrightarrow W\Omega(T)$ respectively. Each of these maps has in its image the corresponding element of $Dend$, hence $x(Face)$ is in the image of $\mathrm{Sk}_2(x)$. ♦

Proposition 5.1.4. *Let $T \in \Omega$ be a tree such that $|\operatorname{Vert}(T)| \geq 3$. If $t, s \in w\mathcal{C}at^1_T$ satisfy $\operatorname{Sk}_2(t) = \operatorname{Sk}_2(s)$ then $t = s$.*

Proof. We proceed by induction on $n = |\operatorname{Vert}(T)|$, the case $n = 3$ is covered in Lemma 5.1.3. Suppose that $x : W\Omega(T) \longrightarrow \mathcal{C}tg$ is a dendrex of shape T. First we notice that we only need to describe the functor $x_\sigma : H^{\operatorname{int}(\sigma)} \longrightarrow \mathcal{C}tg(\sigma)$ in terms of the $\operatorname{Sk}_{n-1}(x)$ where σ is the ordered sequence of colours $(\operatorname{Leaves}(\bar{T}); \operatorname{root}(\bar{T}))$ for a chosen planar representative $\bar{T}$. (The other components of x are already contained in the image of $\operatorname{Sk}_{n-1}(x)$.)

The domain of x_σ is a groupoid with the shape of an n-cube, having in its vertices the trivial categories $(\epsilon_1, \ldots, \epsilon_n)$, $\epsilon_i \in \{0, 1\}$. Denote by $\bar{H}_k$ the full subcategory of $H^{\operatorname{int}(\sigma)}$ spanned by the categories $(\epsilon_1, \ldots, \epsilon_{k-1}, 1, \epsilon_{k+1}, \ldots, \epsilon_n)$, $\epsilon_i \in \{0, 1\}$ (one of the hyperfaces of the n-cube, containing the vertex (1,1,…,1)). Denote by ϕ_k the arrow $(0, 0, \ldots, 0) \longrightarrow (0, \ldots, 0, 1, 0, \ldots, 0)$ of $H^{\operatorname{int}(\sigma)}$ (one of the edges of the n-cube, starting in $(0, 0, \ldots, 0)$). Define the sets

$$Opd := \{\bar{H}_k | k = 1, 2, \ldots n\} \quad \text{and} \quad Face := \{\phi_k | k = 1, 2, \ldots n\}.$$

Since the "convex hull" of $Opd \cup Face$ is the whole domain of x_σ, it is enough to prove that $x_\sigma(Opd)$ is completely determined by $\operatorname{Sk}_{n-1}(x)$ and $x_\sigma(Face)$ is in the image of $\operatorname{Sk}_k(x)$. Both of these assertions are true, by similar arguments to the ones in the proof of Lemma 5.1.3. ♦

The following proposition and theorem of [40] helps us in proving that $w\mathcal{C}at^1$ is 3-coskeletal.

Proposition 5.1.5. *(Proposition 3.2.5 in [40]) Let X be a dendroidal set and $k \geq 2$ an integer. If X satisfies the strict inner Kan condition for all trees T of degree at least k, then X is k-coskeletal.*

Theorem 5.1.6. *(Theorem 4.3.8 in [40]) Let P be a locally fibrant operad in $\mathcal{E}$ (that is, for any ordered sequence of objects $\sigma = (c_1, \ldots, c_n; c)$ the category $P(\sigma)$ is fibrant with respect to the folk model structure). Then $\operatorname{hcN}_d(P)$ is an inner Kan complex.*

Corollary 5.1.7. *([40]) The dendroidal set $w\mathcal{C}at^1$ is 3-coskeletal.*

Proof. In view of Proposition 5.1.5 it is enough to prove that $w\mathcal{C}at^1$ satisfies the strict inner Kan condition for all trees with $|\operatorname{Vert}(T)| \geq 3$. Let T be such a tree. Theorem 5.1.6 implies that $w\mathcal{C}at^1$ is an inner Kan complex, hence every inner horn $\Lambda^e[T] \longrightarrow w\mathcal{C}at^1$ has at least one filler t. Suppose that s is an other filler of the same horn. Since $|\operatorname{Vert}(T)| \geq 3$, it follows that $\operatorname{Sk}_2(t) = \operatorname{Sk}_2(s)$. We conclude thus by Proposition 5.1.4 that $t = s$. ♦

Corollary 5.1.7 implies that $w\mathcal{C}at^n$ is 3-coskeletal for every $n \geq 1$. (Note that $w\mathcal{C}at^0$ is already 2-coskeletal.) To prove this, the following lemma is needed.

Lemma 5.1.8. *If X is a k-coskeletal dendroidal set and Z is an arbitrary dendroidal set then $\underline{dSets}(Z, X)$ is k-coskeletal.*

Proof. The goal is to see that for any dendroidal set Y there exists a natural bijection

$$dSets(Y, \underline{dSets}(Z, X)) \simeq dSets(\mathrm{Sk}_k\, Y, \underline{dSets}(Z, X)).$$

Indeed, once one observes that $\mathrm{Sk}_k(Y \otimes Z) \subseteq (\mathrm{Sk}_k\, Y) \otimes Z$, one can conclude that there are natural one-to-one correspondences between the following Hom sets:

$$\begin{aligned} dSets(Y, \underline{dSets}(Z, X)) &\simeq dSets(Y \otimes Z, X) \\ &\simeq dSets(Y \otimes Z, \mathrm{coSk}_k\, X) \\ &\simeq dSets(\mathrm{Sk}_k(Y \otimes Z), X) \\ &\simeq dSets((\mathrm{Sk}_k\, Y) \otimes Z, X) \\ &\simeq dSets(\mathrm{Sk}_k\, Y, \underline{dSets}(Z, X)). \end{aligned}$$

♦

Theorem 5.1.9. *For every $n \geq 1$ the dendroidal set $w\mathcal{C}at^n$ is 3-coskeletal.*

Proof. We proceed by induction on n. It was proven in Corollary 5.1.7 that $w\mathcal{C}at^1$ is 3-coskeletal. Suppose that $w\mathcal{C}at^n$ is 3-coskeletal. It follows from Lemma 5.1.8 that for any set A, the dendroidal set $\mathcal{C}at(w\mathcal{C}at^n)_A = \underline{dSets}(N_d(As_A), w\mathcal{C}at^n)$ is 3-coskeletal. Hence Proposition 3.4.2 implies that

$$w\mathcal{C}at^{n+1} = \int_{\mathcal{S}ets} \mathcal{C}at(w\mathcal{C}at^n)_-$$

is 3-coskeletal. ♦

5.2 Weak 1-categories

In this section we are going to describe the dendroidal set $w\mathcal{C}at^1 = \mathrm{hcN}_d(\mathcal{C}tg)$. We can use Corollary 5.1.7 to come to the conclusion that it is enough to describe the sets $(w\mathcal{C}at^1)_T = \mathcal{O}p_{\mathcal{E}}(W\Omega(T), \mathcal{C}tg)$ for trees T with at most 3 vertices. Before we start with the description, let us make a useful notational convention: from now on, given n categories $X_1, \ldots, X_n$ and integers $1 \leq i \leq j \leq n$, $(X)_i^j$ will denote the category $X_i \times \cdots \times X_j$.

(1) The first choice of T is the tree $|$. In this case $W\Omega(T) = \Omega(|)$ is the operad on one object and only the identity operation, hence an element of $(w\mathcal{C}at^1)_|$ is the same as the choice of a category.

(2) Let $T = \mathrm{Cor}_n$, the n-corolla. In this case still $W\Omega(\mathrm{Cor}_n) = \Omega(\mathrm{Cor}_n)$, hence an element of $(w\mathcal{C}at^1)_{\mathrm{Cor}_n}$ is the same as the choice of $n+1$ categories $X_1, \ldots, X_n$ and X, together with a functor $F : (X)_1^n \longrightarrow X$. Note that in case $n = 0$, the $\mathcal{E}$-enriched operad structure on $\mathcal{C}tg$ implies that $(X)_1^n$ has to be considered the unit of the $\mathcal{E}$-enriched monoidal category $\mathcal{C}tg$. This unit is the category $*$ on one object and no other arrows than the identity. Hence we infer that a dendrex of shape Cor_0 amounts to the choice of a category X, together with an object of it.

(3) Let $T = \mathrm{Cor}_n \circ_i \mathrm{Cor}_m$. Let us give a detailed description of maps of operads $\alpha : W\Omega(T) \longrightarrow \mathcal{C}tg$ since this is the first time when the interval H plays a role in the definition of the operad $W\Omega(T)$. So far it is clear that, as in cases (1) and (2), such an α determines

(3a) a choice of $n+1$ categories $X_1, \ldots, X_n, X$ together with a functor $F_1 : (X)_1^n \longrightarrow X$;
(3b) a choice of m categories $Y_1, \ldots, Y_m$ and a functor $F_2 : (Y)_1^m \longrightarrow X_i$.

There is one more building part of such an α, which is a functor

$$H \longrightarrow \mathcal{C}tg\big((X)_1^{i-1} \times (Y)_1^m \times (X)_{i+1}^n, X\big).$$

But such a functor contains exactly the same data as the choice of two functors $G, G' : (X)_1^{i-1} \times (Y)_1^m \times (X)_{i+1}^n \longrightarrow X$ and a natural isomorphism $\phi : G \longrightarrow G'$.
The only thing we have not covered yet with the investigation of such a dendrex is that α is a map of operads, which means that the diagram of categories

$$\begin{tikzcd}
* \times * \arrow[r, "\alpha\times\alpha"] \arrow[d, "\circ_i"'] & \mathcal{C}tg\big((X)_1^n, X\big) \times \mathcal{C}tg\big((Y)_1^m, X_i\big) \arrow[d, "\circ_i"] \\
H \arrow[r, "\alpha"] & \mathcal{C}tg\big((X)_1^{i-1} \times (Y)_1^m \times (X)_{i+1}^n, X\big)
\end{tikzcd}$$

is commutative. One can spell out that this yields to $G' = F_1 \circ_i F_2$. We can conclude thus that the last bit of information α provides is

(3c) a choice of a functor $G : (X)_1^{i-1} \times (Y)_1^m \times (X)_{i+1}^n \longrightarrow X$ and a natural isomorphism $\phi : G \longrightarrow F_1 \circ_i F_2$.

For the remaining choices of the tree T we give only the result.

(4) Let $T = \mathrm{Cor}_n \circ_i (\mathrm{Cor}_m \circ_j \mathrm{Cor}_k)$. A map of operads $W\Omega(T) \longrightarrow \mathcal{C}tg$ is the same as

(4a) a choice of $n+1$ categories $X_1, \ldots, X_n, X$ together with a functor $F_1 : (X)_1^n \longrightarrow X$;
(4b) a choice of m categories $Y_1, \ldots, Y_m$ and a functor $F_2 : (Y)_1^m \longrightarrow X_i$;
(4c) a choice of k categories $Z_1, \ldots, Z_k$ and a functor $F_3 : (Z)_1^k \longrightarrow Y_j$;
(4d) a choice of a functor $G_1 : (X)_1^{i-1} \times (Y)_1^m \times (X)_{i+1}^n \longrightarrow X$ and a natural isomorphism $\phi_1 : G_1 \longrightarrow F_1 \circ_i F_2$;
(4e) a choice of a functor $G_2 : (Y)_1^{j-1} \times (Z)_1^k \times (Y)_{j+1}^m \longrightarrow X_i$ and a natural isomorphism $\phi_2 : G_2 \longrightarrow F_2 \circ_j F_3$;
(4f) a choice of a functor $K : (X)_1^{i-1} \times (Y)_1^{j-1} \times (Z)_1^k \times (Y)_{j+1}^m \times (X)_{i+1}^n \longrightarrow X$ and two natural isomorphisms

$$\psi_1 : K \longrightarrow F_1 \circ_i G_2, \quad \psi_2 : K \longrightarrow G_1 \circ_{\tilde{j}} F_3$$

where $\tilde{j} = i + j - 1$, such that the following diagram of natural isomorphisms is commutative:

$$\begin{tikzcd} K \arrow[r, "\psi_1"] \arrow[d, "\psi_2"'] & F_1 \circ_i G_2 \arrow[d, "F_1 \circ_i \phi_2"] \\ G_1 \circ_{\tilde{i}} F_3 \arrow[r, "\phi_1 \circ_{\tilde{j}} F_3"] & F_1 \circ_i F_2 \circ_j F_3 \end{tikzcd}$$

(5) Let $T = \mathrm{Cor}_n \circ_{i,j} (\mathrm{Cor}_m, \mathrm{Cor}_k)$ for some $1 \leq i < j \leq n$. A map of operads $W\Omega(T) \longrightarrow \mathcal{C}tg$ is the same as

(5a) a choice of $n + 1$ categories $X_1, \ldots, X_n, X$ together with a functor $F_1 : (X)_1^n \longrightarrow X$;

(5b) a choice of m categories $Y_1, \ldots, Y_m$ and a functor $F_2 : (Y)_1^m \longrightarrow X_i$;

(5c) a choice of k categories $Z_1, \ldots, Z_k$ and a functor $F_3 : (Z)_1^k \longrightarrow X_j$;

(5d) a choice of a functor $G_1 : (X)_1^{i-1} \times (Y)_1^m \times (X)_{i+1}^n \longrightarrow X$ and a natural isomorphism $\phi_1 : G_1 \longrightarrow F_1 \circ_i F_2$;

(5e) a choice of a functor $G_2 : (X)_1^{j-1} \times (Z)_1^k \times (X)_{j+1}^n \longrightarrow X$ and a natural isomorphism $\phi_2 : G_2 \longrightarrow F_2 \circ_j F_3$;

(5f) a choice of a functor $K : (X)_1^{i-1} \times (Y)_1^m \times (X)_{i+1}^{j-1} \times (Z)_1^k \times (X)_{j+1}^n \longrightarrow X$ and two natural isomorphisms

$$\psi_1 : K \longrightarrow F_1 \circ_i G_2, \quad \psi_2 : K \longrightarrow G_1 \circ_{\tilde{j}} F_3$$

where $\tilde{j} = i + j - 1$ (we suppose $j > i$), such that the following diagram of natural isomorphisms is commutative:

$$\begin{tikzcd} K \arrow[r, "\psi_1"] \arrow[d, "\psi_2"'] & F_1 \circ_i G_2 \arrow[d, "F_1 \circ_i \phi_2"] \\ G_1 \circ_{\tilde{i}} F_3 \arrow[r, "\phi_1 \circ_{\tilde{j}} F_3"] & F_1 \circ_{i,j} (F_2, F_3) \end{tikzcd}$$

We are going to illustrate with some examples the dendroidal structure of $w\mathcal{C}at^1$ in the context described above. Let $T = \mathrm{Cor}_n \circ_i \mathrm{Cor}_m$, hence a dendrex α of shape T is the same thing as the data described in (3) above. If $\partial : \mathrm{Cor}_n \longrightarrow T$ is the obvious outer face of T then $\partial^*(\alpha)$ corresponds to the choice of the categories $X_1, \ldots, X_n, X$ and the functor $F_1 : (X)_1^n \longrightarrow X$. If $\partial : \mathrm{Cor}_{n+m-1} \longrightarrow T$ is the inner face of T then $\partial^*(\alpha)$ corresponds to the choice of the categories $X_1, \ldots, X_{i-1}, Y_1, \ldots, Y_m, X_{i+1}, \ldots, X_n$, X and the functor $G : (X)_1^{i-1} \times (Y)_1^m \times (X)_{i+1}^n \longrightarrow X$. (The choice of G instead of $F_1 \circ_i F_2$ follows from the definition of the map of operads $\partial \colon W\Omega(\mathrm{Cor}_{n+m-1}) \longrightarrow W\Omega(T)$.)

One can similarly decipher what a degeneracy looks like. A simple case of such occurs when $R = \mathrm{Cor}_n \circ_i \mathrm{Cor}_1$, $T = \mathrm{Cor}_n$ and $\sigma : R \longrightarrow T$ is the degeneracy in question. If β is a dendrex of shape T, that is a choice of categories $X_1, \ldots, X_n, X$ and a functor $F_1 : (X)_1^n \longrightarrow X$, then $\sigma^*(\beta)$ adds to the information contained in β the identity functor $\mathrm{id} : X_i \longrightarrow X_i$, and the identity natural transformation $F_1 \longrightarrow F_1 \circ_i \mathrm{id}$.

5.3 Weak 2-categories

We turn our attention now to the dendroidal set $w\mathcal{C}at^2$. Our goal is to unpack the definition and compare the result with bicategories. It will become apparent later that the right notion to compare the data of $w\mathcal{C}at^2$ contained in lower degrees is unbiased bicategories and their homomorphisms. These notions were defined by Tom Leinster in [25].

The section is organized as follows: First we recall classical bicategories and their homomorphisms. Then we briefly discuss Leinster's unbiased bicategories and unbiased homomorphisms between them. After this we analyse the sets $w\mathcal{C}at^2_{|}$, $w\mathcal{C}at^2_{\mathrm{Cor}_1}$ and their relations to unbiased bicategories, and we prove that the category of unbiased bicategories is isomorphic to the homotopy category of dendroidal weak 2-categories. We conclude the section by a conjecture that predicts a stronger relation between bicategories and dendroidal weak 2-categories.

5.3.1 Notions of bicategories

The notion of bicategory first appeared explicitly in the paper of Bénabou [1]. Intuitively, bicategories are generalised categories where the composition of arrows is not strictly associative, only up to some coherent 2-cells which are part of the structure. The theory of bicategories had a quick development, due to the usefulness of the notion in different fashionable areas of mathematics. Amongst these areas we can find ordinary category theory: as Ross Street states in [36], many fundamental constructions of categories are bicategorical in nature. Bicategories can be considered as generalisations of monoidal categories as well, giving new insight to the theory of monoidal categories. Another area where bicategories were influential is algebraic topology, especially higher homotopy theory: bicategories are the first step in the build-up of higher categories and groupoids, which should provide algebraic models of homotopy n-types.

Classical bicategories

A bicategory $\mathbb{A}$ consists of the following data and axioms:

(D1) a set A, called the set of objects or 0-cells;

(D2) for every ordered pair of objects $(a_1, a_2) \in A \times A$ a category $\mathcal{A}(a_1, a_2)$. The objects of such a category are called arrows or 1-cells of $\mathbb{A}$, the maps are called 2-cells of $\mathbb{A}$. If $f, g \in \mathcal{A}(a_1, a_2)$ are 1-cells and $\phi \in \mathcal{A}(a_1, a_2)(f, g)$ is a 2-cell between them, we usually depict this situation as $f \xRightarrow{\phi} g$ or as

$$\begin{tikzcd} a_1 \arrow[r, bend left=50, "f"{name=U}] \arrow[r, bend right=50, "g"'{name=D}] & a_2 \arrow[from=U, to=D, phantom, "\Downarrow\phi"] \end{tikzcd}$$

The composition of 2-cells in a category $\mathcal{A}(a_1, a_2)$ is called vertical composition and for two composable 2-cells ϕ, ϕ' the composite is denoted by juxtaposition: $\phi\phi'$.

(D3) functors which define horizontal composition and units in $\mathbb{A}$:

(D3a) for all $(a_1, a_2, a_3) \in A^3$, $\psi\colon \mathcal{A}(a_1, a_2) \times \mathcal{A}(a_2, a_3) \longrightarrow \mathcal{A}(a_1, a_3)$. We denote by "$\cdot$" the horizontal composite of two 1-cells (and two 2-cells), thus $g \cdot f := \Psi(f, g)$ etc.

(D3b) for all $a \in A$, $\psi_0\colon * \longrightarrow \mathcal{A}(a, a)$, that is a 1-cell $\mathrm{Id}_a \in \mathcal{A}(a, a)$.

(D4a) natural isomorphisms, relating the two different ways of horizontal compositions of three 1-cells in $\mathbb{A}$: for all $a_1, a_2, a_3, a_4 \in A$

$$\begin{tikzcd}
\mathcal{A}(a_1, a_2) \times \mathcal{A}(a_2, a_3) \times \mathcal{A}(a_3, a_4) \arrow[r, "\mathrm{id} \times \psi"] \arrow[d, "\psi \times \mathrm{id}"'] & \mathcal{A}(a_1, a_2) \times \mathcal{A}(a_2, a_4) \arrow[d, "\psi"] \\
\mathcal{A}(a_1, a_3) \times \mathcal{A}(a_3, a_4) \arrow[r, "\psi"] \arrow[ur, phantom, "\Uparrow\alpha"] & \mathcal{A}(a_1, a_4)
\end{tikzcd}$$

that is, invertible 2-cells $(h \cdot g) \cdot f \stackrel{\alpha}{\Longrightarrow} h \cdot (g \cdot f)$ in $\mathbb{A}$ for any three composable 1-cells f, g, h.

(D4b) natural isomorphisms, relating composition with units to the identity: for all $a_1, a_2 \in A$,

$$\begin{tikzcd}
\mathcal{A}(a_1, a_2) \times * \arrow[r, "\mathrm{id} \times \psi_0"] \arrow[dr, equal] & \mathcal{A}(a_1, a_2) \times \mathcal{A}(a_2, a_2) \arrow[d, "\psi"] \arrow[dl, phantom, "\overset{\lambda}{\Leftarrow}"] \\
 & \mathcal{A}(a_1, a_2)
\end{tikzcd}$$

$$\begin{tikzcd}
* \times \mathcal{A}(a_1, a_2) \arrow[r, "\psi_0 \times \mathrm{id}"] \arrow[dr, equal] & \mathcal{A}(a_1, a_1) \times \mathcal{A}(a_1, a_2) \arrow[d, "\psi"] \arrow[dl, phantom, "\overset{\rho}{\Leftarrow}"] \\
 & \mathcal{A}(a_1, a_2)
\end{tikzcd}$$

that is, for any 1-cell $f \in \mathcal{A}(a_1, a_2)$ invertible 2-cells

$$\mathrm{Id}_{a_2} \cdot f \stackrel{\lambda}{\Longrightarrow} f \qquad \text{and} \qquad f \cdot \mathrm{Id}_{a_1} \stackrel{\rho}{\Longrightarrow} f.$$

The data given above is subject to two axioms that ensure that the various associativity and unit constraints α, ρ, λ compose coherently:

(A1) The following pentagon commutes for any involved composable 1-cells

$$
\begin{tikzcd}
 & ((k \cdot h) \cdot g) \cdot f \arrow[r, Rightarrow, "\alpha \cdot \mathrm{id}_f"] \arrow[dl, Rightarrow, "\alpha"'] & (k \cdot (h \cdot g)) \cdot f \arrow[dr, Rightarrow, "\alpha"] & \\
(k \cdot h) \cdot (g \cdot f) \arrow[drr, Rightarrow, "\alpha"'] & & & k \cdot ((h \cdot g) \cdot f) \arrow[dl, Rightarrow, "\mathrm{id}_k \cdot \alpha"] \\
 & & k \cdot (h \cdot (g \cdot f)) &
\end{tikzcd}
$$

(A2) The following triangle commutes for any involved composable 1-cells

$$
\begin{tikzcd}
(g \cdot \mathrm{Id}) \cdot f \arrow[rr, Rightarrow, "\alpha"] \arrow[dr, Rightarrow, "\rho \cdot \mathrm{id}_f"'] & & g \cdot (\mathrm{Id} \cdot f) \arrow[dl, Rightarrow, "\mathrm{id}_g \cdot \lambda"] \\
 & g \cdot f &
\end{tikzcd}
$$

Example 5.3.1. Any (strict) 2-category is a bicategory where the associativity and unit 2-cells α, ρ, λ are all identities.

Example 5.3.2. Any monoidal category $\mathcal{C}$ is a bicategory with one 0-cell. The 1-cells of this bicategory are the objects of $\mathcal{C}$ and the 2-cells are the arrows of $\mathcal{C}$. The other data and axioms of the bicategory are induced by the monoidal structure on $\mathcal{C}$, in the obvious way.

Example 5.3.3. There exists a bicategory $\mathbb{B}\mathrm{i}\mathbb{M}\mathrm{od}$, defined as follows:

– The 0-cells of $\mathbb{B}\mathrm{i}\mathbb{M}\mathrm{od}$ are rings with unit $A, B, \dots$
– The category of (A, B)-bimodules defines the 1- and 2-cells of $\mathbb{B}\mathrm{i}\mathbb{M}\mathrm{od}$.
– Horizontal composition, units, etc. are given by tensor product of bimodules.

Homomorphisms of classical bicategories

There exist a number of notions of homomorphisms of bicategories, the one we define here is that of weak homomorphisms in the literature. Thus, for us a homomorphism of bicategories $(F, f)\colon \mathbb{A} \longrightarrow \mathbb{B}$ consists of the following data and axioms:

(D1) A function $f\colon A \longrightarrow B$ from the set of 0-cells of $\mathbb{A}$ to the set of 0-cells of $\mathbb{B}$.
(D2) For every ordered pair of 0-cells $(a_1, a_2) \in A^2$ a functor

$$F_{a_1 a_2}\colon \mathcal{A}(a_1, a_2) \longrightarrow \mathcal{B}(f(a_1), f(a_2)).$$

(D3a) For every $(a_1, a_2, a_3) \in A^3$ natural isomorphisms relating horizontal compositions and F:

$$\begin{tikzcd}
\mathcal{A}(a_1,a_2) \times \mathcal{A}(a_2,a_3) \arrow[r, "\psi^{\mathbb{A}}"] \arrow[d, "F\times F"'] \arrow[dr, phantom, "\Uparrow\theta"] & \mathcal{A}(a_1,a_3) \arrow[d, "F"] \\
\mathcal{B}(f(a_1),f(a_2)) \times \mathcal{B}(f(a_2),f(a_3)) \arrow[r, "\psi^{\mathbb{B}}"'] & \mathcal{B}(f(a_1),f(a_3))
\end{tikzcd}$$

that is, invertible 2-cells $F(h)\cdot F(g) \xRightarrow{\theta} F(h\cdot g)$ for any composable 1-cells $h, g \in \mathbb{A}$.

(D3b) For every $a \in A$ natural isomorphisms relating units and F:

$$\begin{tikzcd}
* \arrow[r, "\psi_0^{\mathbb{A}}"] \arrow[d, equal] \arrow[dr, phantom, "\Uparrow\theta_0"] & \mathcal{A}(a,a) \arrow[d, "F"] \\
* \arrow[r, "\psi_0^{\mathbb{B}}"'] & \mathcal{B}(f(a),f(a))
\end{tikzcd}$$

that is, invertible 2-cells $\mathrm{Id}^{\mathbb{B}}_{f(a)} \xRightarrow{\theta_0} F(\mathrm{Id}^{\mathbb{A}}_a)$ for any $a \in A$.

The data described above is subject to axioms that ensure that F is coherent with the various associativity- and unit-constraints:

(A1) For every composable 1-cells $k, h, g \in \mathbb{A}$, the following hexagon of invertible 2-cells commutes

$$\begin{tikzcd}
 & F(k\cdot h)\cdot Fg \arrow[dr, Rightarrow, "\theta"] & \\
(Fk\cdot Fh)\cdot Fg \arrow[ur, Rightarrow, "\theta\cdot\mathrm{id}"] \arrow[d, Rightarrow, "\alpha^{\mathbb{B}}"'] & & F((k\cdot h)\cdot g) \arrow[d, Rightarrow, "F\alpha^{\mathbb{A}}"] \\
Fk\cdot(Fh\cdot Fg) \arrow[dr, Rightarrow, "\mathrm{id}\cdot\theta"'] & & F(k\cdot(h\cdot g)) \\
 & Fk\cdot F(h\cdot g) \arrow[ur, Rightarrow, "\theta"'] &
\end{tikzcd}$$

(A2) For any 1-cell $g \in \mathcal{A}(a,a')$ the following diagrams of invertible 2-cells commute:

$$\begin{tikzcd}
Fg\cdot \mathrm{Id}^{\mathbb{B}}_{fa} \arrow[r, Rightarrow, "\mathrm{id}\cdot\theta_0"] \arrow[d, Rightarrow, "\rho^{\mathbb{B}}"'] & Fg\cdot F(\mathrm{Id}^{\mathbb{A}}_a) \arrow[d, Rightarrow, "\theta"] \\
Fg & F(g\cdot \mathrm{Id}^{\mathbb{A}}_a) \arrow[l, Rightarrow, "F(\rho^{\mathbb{A}})"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Id}^{\mathbb{B}}_{fa'}\cdot Fg \arrow[r, Rightarrow, "\theta_0\cdot\mathrm{id}"] \arrow[d, Rightarrow, "\lambda^{\mathbb{B}}"'] & F(\mathrm{Id}^{\mathbb{A}}_{a'})\cdot Fg \arrow[d, Rightarrow, "\theta"] \\
Fg & F(\mathrm{Id}^{\mathbb{A}}_{a'}\cdot g) \arrow[l, Rightarrow, "F(\lambda^{\mathbb{A}})"]
\end{tikzcd}$$

Classical bicategories and their homomorphisms form a category that we denote by $bi\mathcal{C}tg$.

Unbiased bicategories

As we mentioned in the introductory part of Subsection 5.3.1, it is more natural to compare dendroidal bicategories (the lower degree terms of the dendroidal set $w\mathcal{C}at^2$) with the category of unbiased bicategories and their homomorphisms, notions that were defined by Tom Leinster in [25]. We will briefly discuss them here, the resulting category of unbiased bicategories will be denoted by $\overline{ubi\mathcal{C}tg}$. Since the categories $\overline{ubi\mathcal{C}tg}$ and $bi\mathcal{C}tg$ are equivalent, it is justified to compare unbiased bicategories instead of the classical ones with dendroidal bicategories.

The idea of unbiased bicategories comes from the observation that the definition of bicategories is "biased" towards a binary horizontal composition of 1-cells and a chosen associator between the two different ways to compose horizontally three 1-cells. One can eliminate this bias by considering a definition which resembles operads, as follows:

(a) for every $n \in \mathbb{N}$ give a horizontal composition of (composable) n-tuples of 1-cells;
(b) relate the n-ary compositions for various n-s by some given 2-cells (the associators);
(c) the associators should be coherent, thus they have to satisfy some obvious relations;
(d) take care of the unit 1-cells.

When one tries to work out the details of the points given above, one notices that step (b) can be fulfilled in two ways, depending on the preferred "operadic" approach one takes: the $\circ_i$-approach or the general γ-approach. These definitions are equivalent (the two resulting categories of unbiased bicategories are isomorphic). Leinster in his definition takes the second approach, we will take here the first one:

An unbiased bicategory $\mathbb{A}$ consists of the following data:

(D1) a set A;
(D2) for every $(a_1, a_2) \in A^2$ a category $\mathcal{A}(a_1, a_2)$;
(D3) for every integer $n \geq 0$ and every sequence of objects $(a_1, a_2, \ldots, a_{n+1}) \in A^{n+1}$ an associated functor of n-ary composition

$$\mathcal{A}(a_1, a_2) \times \mathcal{A}(a_2, a_3) \times \cdots \times \mathcal{A}(a_n, a_{n+1}) \xrightarrow{\Psi} \mathcal{A}(a_1, a_{n+1}),$$

we usually denote the n-fold horizontal composition of 1-cells by

$$(g_1 \cdot g_2 \cdot \ldots \cdot g_n) := \Psi(g_1, \ldots, g_n);$$

(D4) for all $n, m, i \in \mathbb{N}$ such that $1 \leq i \leq n$, $n \neq 0$ and composable sequences of 1-cells

$$(h_1, h_2, \ldots, h_n) \text{ and } (g_1, g_2, \ldots, g_m)$$

such that $(g_1 \cdot g_2 \cdot \ldots \cdot g_m) = h_i$, natural invertible 2-cells

$$(h_1 \cdot h_2 \cdot \ldots \cdot h_n) \xRightarrow{\phi} (h_1 \cdot \ldots \cdot h_{i-1} \cdot g_1 \cdot g_2 \cdot \ldots \cdot g_m \cdot h_{i+1} \cdot \ldots \cdot h_n)\,;$$

(D5) for every 1-cell g an invertible 2-cell $g \xRightarrow{\iota} (g)$.

This data has to satisfy some obvious axioms, ensuring coherence of compositions and units.

Homomorphisms of unbiased bicategories

Suppose that $\mathbb{A}$ and $\mathbb{B}$ are unbiased bicategories. A homomorphism $\mathbb{A} \xrightarrow{(F,f)} \mathbb{B}$ of unbiased bicategories consists of the following data and axioms:

(D1) a function $f\colon A \longrightarrow B$ between the 0-cells of $\mathbb{A}$ and $\mathbb{B}$;
(D2) for every ordered pair of 0-cells $(a_1, a_2) \in A^2$ a functor

$$F_{a_1 a_2}\colon \mathcal{A}(a_1, a_2) \longrightarrow \mathcal{B}(f(a_1), f(a_2));$$

(D3) for every $n \in \mathbb{N}$ and every ordered sequence of 0-cells of $\mathbb{A}$, $(a_1, \ldots, a_n) \in A^n$ natural isomorphisms

$$\begin{tikzcd}
\mathcal{A}(a_1,a_2) \times \cdots \times \mathcal{A}(a_{n-1},a_n) \arrow[r, "\psi^{\mathbb{A}}"] \arrow[d, "F^n"'] \arrow[dr, phantom, "\Uparrow\theta"] & \mathcal{A}(a_1,a_n) \arrow[d, "F"] \\
\mathcal{B}(f(a_1),f(a_2)) \times \cdots \times \mathcal{B}(f(a_{n-1}),f(a_n)) \arrow[r, "\psi^{\mathbb{B}}"'] & \mathcal{B}(f(a_1),f(a_n))
\end{tikzcd} \tag{5.3.1}$$

This data again is subject to some coherence axioms, ensuring the compatibility of F with the various associativity- and unit constraints of the involved bicategories.

Remark 5.3.4. The category of unbiased bicategories defined above is denoted by $\overline{ubi\mathcal{C}tg}$. It has a full subcategory $ubi\mathcal{C}tg$, whose objects are those unbiased bicategories for which the unit 2-cells of (D5) are all identities. We call them unbiased bicategories with strict unit, but note that this terminology is misleading since there is a chain of fully faithful embeddings of equivalent categories

$$bi\mathcal{C}tg \subseteq ubi\mathcal{C}tg \subseteq \overline{ubi\mathcal{C}tg}.$$

5.3.2 Dendroidal weak 2-categories

In this subsection we analyse those components of the dendroidal set $w\mathcal{C}at^2$ which will correspond to bicategories and homomorphisms of bicategories.

Dendrices of shape |

Since

$$(w\mathcal{C}at^2)_| = \left(\int_{\mathcal{S}ets} \mathcal{C}at(\mathrm{hcN}_d(\mathcal{C}tg))_-\right)_| ,$$

the definition of the Grothendieck construction implies that an element of $(w\mathcal{C}at^2)_|$ is a pair (A, x), where A is a set and x is a dendrex of shape $|$ in the dendroidal set $\mathcal{C}at(\mathrm{hcN}_d(\mathcal{C}tg))_A$. Hence

$$x \in d\mathcal{S}ets(N_d(As_A) \otimes \Omega[|], \mathrm{hcN}_d(\mathcal{C}tg)) = d\mathcal{S}ets(N_d(As_A), \mathrm{hcN}_d(\mathcal{C}tg)).$$

Since $\mathrm{hcN}_d(\mathcal{C}tg)$ is 3-coskeletal, it is enough to look at the degree 0, 1, 2 and 3 components of x.

(0) The degree 0 component of x is the map of sets

$$x_| : N_d(As_A)_| \longrightarrow \mathrm{hcN}_d(\mathcal{C}tg)_|.$$

Since $N_d(As_A)_|$ consists of the objects of the operad As_A and $\mathrm{hcN}_d(\mathcal{C}tg)_|$ consists of categories, it follows that $x_|$ is the same thing as the choice of a category $\mathcal{A}(a_1, a_2)$ for each ordered pair $(a_1, a_2) \in A \times A$.

(1) Let us look at the x_{Cor_n} component, $n \in \mathbb{N}$. There are three cases to distinguish.
First, an element in $N_d(As_A)_{\mathrm{Cor}_0}$ consists of a pair (a, a) where $a \in A$, and the operation $* \in As_A\big(\ ; (a, a)\big)$. We have seen in Section 5.2 that an element of $\mathrm{hcN}_d(\mathcal{C}tg)_{\mathrm{Cor}_0}$ is a category together with an object of it. Since x has to be compatible with the face map $| \longrightarrow \mathrm{Cor}_0$, it follows that x_{Cor_0} picks for each $a \in A$ a functor $\Psi_a : * \longrightarrow \mathcal{A}(a, a)$.
Second, an element in $N_d(As_A)_{\mathrm{Cor}_1}$ consists of a pair $(a_1, a_2) \in A^2$ and the operation $* \in As_A\big((a_1, a_2); (a_1, a_2)\big)$, which is also the corresponding unit operation in the operad As_A. An element of $\mathrm{hcN}_d(\mathcal{C}tg)_{\mathrm{Cor}_1}$ is a functor between two chosen categories. Again, since x has to be compatible with the various face and degeneracy maps, it follows that x_{Cor_1} amounts to choosing the identity functor on every already chosen category $\mathcal{A}(a_1, a_2)$. Hence x_{Cor_1} does not contribute with any new information.
Third, for $n \geq 2$ x_{Cor_n} picks for each admissible ordered sequence

$$\sigma = \big((a_1, a_2), (a_2, a_3), \ldots, (a_{n-1}, a_n); (a_1, a_n)\big)$$

of $n + 1$ objects of As_A a functor

$$\Psi_\sigma : \mathcal{A}(a_1, a_2) \times \cdots \times \mathcal{A}(a_{n-1}, a_n) \longrightarrow \mathcal{A}(a_1, a_n).$$

We can include the cases $n = 0, 1$ in the third one in the obvious way.

(2) The degree 2 component of x consists of x_T where $T = \mathrm{Cor}_n \circ_i \mathrm{Cor}_m$ for the various $n, m, i \in \mathbb{N}$, $n \neq 0$. There are face maps into the tree T from the m, n and $m + n - 1$ corollas, and in case $n = 1$ or $m = 1$ there are

also degeneracy maps with T as the domain. Since x has to be compatible with these faces and degeneracies, we can conclude that x_T provides the following bit of extra data:
For any pair of admissible ordered sequences

$$\sigma = \big((a_1, a_2), \ldots, (a_{n-1}, a_n); (a_1, a_n)\big),$$
$$\rho = \big((a_i, b_2), (b_2, b_3), \ldots, (b_{m-1}, a_{i+1}); (a_i, a_{i+1})\big)$$

and any $1 \leq i \leq n$ a natural isomorphism

$$\phi_{\sigma,\rho,i} \colon \Psi_{\sigma\circ_i\rho} \longrightarrow \Psi_\sigma \circ_i \Psi_\rho,$$

There is one condition on these natural isomorphisms: in case $n = 1$ or $m = 1$, the corresponding natural isomorphism has to be the identity (it follows from the compatibility with degeneracies again).

(3) The degree 3 components of x do not give rise to any extra data, but the dendroidal identities with face maps induce relations on the already existing one, corresponding to cases (4f) and (5f) of Section 5.2. Explicitly, the diagrams of functors

$$\begin{tikzcd}
\Psi_{\sigma\circ_i\rho\circ_j\tau} \arrow[r, "\phi"] \arrow[d, "\phi"'] & \Psi_{\sigma\circ_i\rho} \circ_j \Psi_\tau \arrow[d, "\phi\circ_j\Psi"] \\
\Psi_\sigma \circ_i \Psi_{\rho\circ_j\tau} \arrow[r, "\Psi\circ_i\phi"] & \Psi_\sigma \circ_i \Psi_\rho \circ_j \Psi_\tau
\end{tikzcd}$$

$$\begin{tikzcd}
\Psi_{\sigma\circ_{i,j}(\rho,\tau)} \arrow[r, "\phi"] \arrow[d, "\phi"'] & \Psi_{\sigma\circ_i\rho} \circ_j \Psi_\tau \arrow[d, "\phi\circ_j\Psi"] \\
\Psi_{\sigma\circ_j\tau} \circ_i \Psi_\rho \arrow[r, "\phi\circ_i\Psi"] & \Psi_\sigma \circ_{i,j} (\Psi_\rho, \Psi_\tau)
\end{tikzcd}$$

are commutative.

Dendrices of shape Cor_1

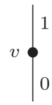
An element of $(w\mathcal{C}at^2)_{\mathrm{Cor}_1}$ consists of pairs (f, y) where $f : A \longrightarrow B$ is a map of sets and

$$y : \Omega[\mathrm{Cor}_1] \longrightarrow \coprod_{S\in\mathcal{S}ets} \underline{d\mathcal{S}ets}(N_d(As_S), w\mathcal{C}at^1)$$

has three relevant components:

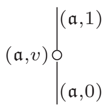
$$y_A \in d\mathcal{S}ets(N_d(As_A), w\mathcal{C}at^1),$$
$$y_B \in d\mathcal{S}ets(N_d(As_B), w\mathcal{C}at^1),$$
$$y_f \in d\mathcal{S}ets(N_d(As_A \otimes \Omega(\mathrm{Cor}_1)), w\mathcal{C}at^1).$$

These three components are related by the compatibility condition of the Grothendieck construction in the following way. Let Cor_1 be represented by the tree

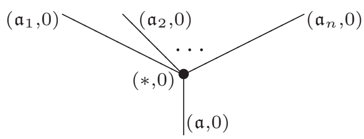


thus the set of colours of the operad $\Omega(\mathrm{Cor}_1)$ is $\{0,1\}$ and the only non-trivial operation is $v \in \Omega(\mathrm{Cor}_1)(1;0)$. If $\partial_1, \partial_0 \colon | \longrightarrow \mathrm{Cor}_1$ denote the face maps sending $|$ to the leaf and root of Cor_1 respectively then $\partial_1^*(y_f) = y_A$ and $\partial_0^*(y_f) = f^*(y_B)$. The components y_A and y_B were described in the first part of Subsection 5.3.2, hence we need to describe y_f only.

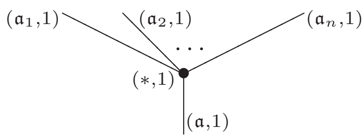
Let us recall first the operad $As_A \otimes \Omega(\mathrm{Cor}_1)$ in more detail. The set of colours of this operad contains all pairs $(\mathfrak{a}, l)$ where $\mathfrak{a} = (a_1, a_2) \in A^2$ is a colour of As_A and $l \in \{0,1\}$ is a colour of $\Omega(\mathrm{Cor}_1)$. The operations are generated by the following three types of basic ones:



is a picture of the basic operation $(\mathfrak{a}, v) \in \Omega(\mathrm{Cor}_1)((\mathfrak{a},1);(\mathfrak{a},0))$ induced by $v \in \Omega(\mathrm{Cor}_1)(1;0)$ for any $\mathfrak{a} = (a_1, a_2) \in A^2$, and



(5.3.2)

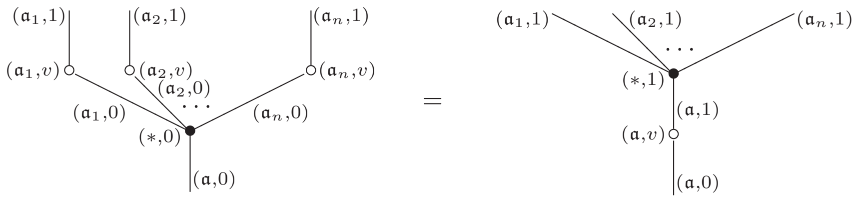


(5.3.3)

are pictures of the basic operations in $As_A \otimes \Omega(\mathrm{Cor}_1)((\mathfrak{a}_1,0),\dots,(\mathfrak{a}_n,0);(\mathfrak{a},0))$ and $As_A \otimes \Omega(\mathrm{Cor}_1)((\mathfrak{a}_1,1),\dots,(\mathfrak{a}_n,1);(\mathfrak{a},1))$ respectively, induced by the unique operation

$$* \in As_A((a_1,a_2),(a_2,a_3),\dots,(a_n,a_{n+1});(a_1,a_{n+1)})$$

where $\mathfrak{a}_i = (a_i, a_{i+1}) \in A^2$ and $\mathfrak{a} = (a_1, a_{n+1}) \in A^2$. The operations generated this way are subject to the relations which imply that the obvious projections $As_A \otimes \Omega(\mathrm{Cor}_1) \longrightarrow As_A$, $As_A \otimes \Omega(\mathrm{Cor}_1) \longrightarrow \Omega(\mathrm{Cor}_1)$ are maps of operads, and to the following relation (interchange law in the Boardman-Vogt tensor product for operads):

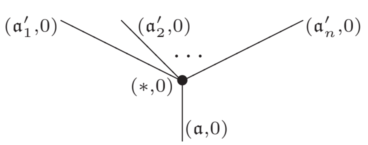


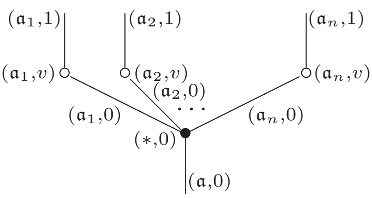
The properties of the operad $As_A \otimes \Omega(\mathrm{Cor}_1)$ imply

Lemma 5.3.5. *For any ordered sequence* $\sigma = \big((\mathfrak{a}_1, l_1), \ldots, (\mathfrak{a}_n, l_n); (\mathfrak{a}, l)\big)$*, the corresponding set of operations* $As_A \otimes \Omega(\mathrm{Cor}_1)(\sigma)$ *contains at most one element. Moreover, in case* $As_A \otimes \Omega(\mathrm{Cor}_1)(\sigma)$ *is not empty,* $\mathfrak{a}_i = (a_i, a_{i+1})$ *and* $\mathfrak{a} = (a_1, a_{n+1})$ *for some* $a_1, \ldots, a_{n+1} \in A$.

Proof. When $l = 1$, the only possibilities for the sequence σ that give nonempty $As_A \otimes \Omega(\mathrm{Cor}_1)(\sigma)$ are the ones corresponding to (5.3.3). Each such set of operations contains exactly one element: $(*, 1)$. If $l = 0$ and all $l_i = 0$, then again the only nonempty sets of operations are the ones corresponding to (5.3.2), with a unique operation $(*, 0)$ in each.

The remaining cases to study are the ones when σ is such that $l = 0$ and there exists i with $l_i = 1$. Suppose that in such a case $As_A \otimes \Omega(\mathrm{Cor}_1)(\sigma)$ is not empty. The interchange law implies then that these operations can always be reduced to a form, which can be visualized by a tree resulting from iterated gluing of the following two types of operations:



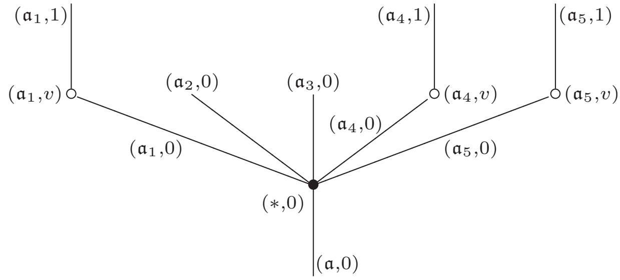


where "gluing" means identifying the two roots indexed by $(\mathfrak{a}, 0)$. We can conclude that $\mathfrak{a}_i = (a_i, a_{i+1})$, $\mathfrak{a} = (a_1, a_{n+1})$ for some $a_1, \ldots, a_{n+1} \in A$ and $As_A \otimes \Omega(\mathrm{Cor}_1)(\sigma)$ contains again exactly one element.

Let us illustrate the latter situation in a particular example, for instance when

$$\sigma = \big((\mathfrak{a}_1, 1), (\mathfrak{a}_2, 0), (\mathfrak{a}_3, 0), (\mathfrak{a}_4, 1), (\mathfrak{a}_5, 1); (\mathfrak{a}, 0)\big).$$

Any operation in $As_A \otimes \Omega(\mathrm{Cor}_1)(\sigma)$ can be reduced to the form



Since the set of operations is assumed to be nonempty, $\mathfrak{a}_i = (a_i, a_{i+1})$ for all $1 \leq i \leq 6$ and $\mathfrak{a} = (a_1, a_6)$, and there is exactly one element in the set $As_A \otimes \Omega(\mathrm{Cor}_1)(\sigma)$ (that is the composite illustrated by the tree above). ♦

We can use this discussion about the operad $As_A \otimes \Omega(\mathrm{Cor}_1)$ to understand the y_f component of an element $y \in w\mathcal{C}at^2_{\mathrm{Cor}_1}$. The degree 0 component of y_f consists of the map of sets

$$(y_f)_| : N_d(As_A \otimes \Omega(\mathrm{Cor}_1))_| \longrightarrow (w\mathcal{C}at^1)_|,$$

thus $(y_f)_|$ amounts to choosing categories $\mathcal{C}_1(a_1, a_2)$ and $\mathcal{C}_0(a_1, a_2)$ for every pair $(a_1, a_2) \in A^2$. The compatibility condition in the Grothendieck construction implies that these categories are the corresponding ones appearing in the definitions of y_A and y_B. Explicitly,

$$\begin{aligned} \mathcal{C}_1(a_1, a_2) &= \mathcal{A}(a_1, a_2) \text{ of } (y_A)_|, \\ \mathcal{C}_0(a_1, a_2) &= \mathcal{B}(f(a_1), f(a_2)) \text{ of } (y_B)_|. \end{aligned}$$

Lemma 5.3.5 implies that y_f in degree 1 amounts to the choice of a functor

$$\Psi_\sigma \colon \mathcal{C}_{l_1}(a_1, a_2) \times \cdots \times \mathcal{C}_{l_{n-1}}(a_{n-1}, a_n) \longrightarrow \mathcal{C}_l(a_1, a_n)$$

for every ordered sequence

$$\sigma = \big(((a_1, a_2), l_1), ((a_2, a_3), l_2), \ldots, ((a_{n-1}, a_n), l_{n-1}); ((a_1, a_n), l)\big)$$

of objects of $As_A \otimes \Omega(\mathrm{Cor}_1)$. A particular case of such a functor is

$$\Psi_\sigma \colon \mathcal{A}(a_1, a_2) \longrightarrow \mathcal{B}(f(a_1), f(a_2)).$$

As a consequence of the compatibility conditions of the Grothendieck construction, in case $l = l_i = 1$ or $l = l_i = 0$ for all i, one gets back the corresponding functors Ψ^A or Ψ^B respectively, resulting from y_A or y_B (we described them in Subsection 5.3.2).

The data contained in degree 2 of y_f amounts to choices of natural transformations $\phi \colon \Psi_\sigma \circ \Psi_\rho \longrightarrow \Psi_{\sigma \circ \rho}$, and similarly, the information contained in degree 3 can be described with the same semantics as in the description of $w\mathcal{C}at^2_|$.

5.3.3 The relation between bicategories and dendroidal weak 2-categories

In this subsection we aim to establish a relation between dendroidal weak 2-categories and (unbiased) bicategories. Let us recall first that the homotopy category of an inner Kan complex X in simplicial sets is the category $ho(X) := \tau(X)$, where τ is the left adjoint of the nerve functor $N\colon \mathcal{C}at \longrightarrow s\mathcal{S}ets$. The category $ho(X)$ is defined as follows:

- the objects of $ho(X)$ are the elements of X_0;
- the arrows of $ho(X)$ are equivalence classes of elements of X_1, where the equivalence relation is left homotopy: if $a, b \in X_0$ and $f, g\colon a \longrightarrow b$ are elements of $X(a,b) \subseteq X_1$, we say that f is left homotopic to g if there exists an element of X_2 that fills the triangle

$$\begin{tikzcd} & a \arrow[dr, "f"] & \\ a \arrow[ur, equal, "1_a"] \arrow[rr, "g"'] & & b \end{tikzcd}$$

Remark 5.3.6. We could have chosen to define the equivalence relation above with the dual notion of right homotopy, since X being an inner Kan complex implies that these two notions are the same. In fact, this is the only reason we renamed the category $\tau(X)$: the description of this category is much easier when X is an inner Kan complex.

The restriction functor $i^*\colon d\mathcal{S}ets \longrightarrow s\mathcal{S}ets$ preserves inner Kan complexes, hence it makes sense to talk about the category $ho(i^*(w\mathcal{C}at^2))$ (we will call it *the category of dendroidal weak 2-categories*). Our goal is to compare this category with $ubi\mathcal{C}tg$.

The description of the elements of $w\mathcal{C}at^2_{|}$ in Subsection 5.3.2 can be summarized in

Proposition 5.3.7. *The objects of the category $ho(i^*(w\mathcal{C}at^2))$ are in one-to-one correspondence with the objects of $ubi\mathcal{C}tg$.*

Let us turn to the comparison of morphisms of the categories in question. By Proposition 5.3.7 we have defined a functor $\Phi\colon ubi\mathcal{C}tg \longrightarrow ho(i^*(w\mathcal{C}at^2))$ on the objects. We complete the definition of Φ by

Proposition 5.3.8. *For any $\mathbb{A}, \mathbb{B} \in ubi\mathcal{C}tg$ there is a one-to-one correspondence between the hom-sets $ubi\mathcal{C}tg(\mathbb{A}, \mathbb{B})$ and $ho(i^*(w\mathcal{C}at^2))(\Phi(\mathbb{A}), \Phi(\mathbb{B}))$. This correspondence is functorial.*

Proof. Suppose that $(F, f)\colon \mathbb{A} \longrightarrow \mathbb{B}$ is a map of unbiased bicategories (with strict unit). With the use of the description of $w\mathcal{C}at^2_{\mathrm{Cor}_1}$ we gave in Subsection 5.3.2 first we define an $y^F \in w\mathcal{C}at^2_{\mathrm{Cor}_1}$, induced by (F, f). Recall, that such an y^F is determined by three components, and the components y^F_A, y^F_B are obvious.

Let us define the component $y_f^F \colon N_d(As_A)\otimes\Omega[\mathrm{Cor}_1] \longrightarrow w\mathcal{C}at^1$. In degree 0 again the definition of y_f^F is obvious, the first non-trivial choices we have to make arise in degree 1. We need to give the components $(y_f^F)_{\mathrm{Cor}_n}$ for every $n \geq 1$, i.e. a functor

$$\Psi_\sigma \colon \mathcal{C}_{l_1}(a_1,a_2)\times\mathcal{C}_{l_2}(a_2,a_3)\times\cdots\times\mathcal{C}_{l_{n-1}}(a_{n-1},a_n) \longrightarrow \mathcal{B}(f(a_1),f(a_n))$$

for every ordered sequence σ with $a_i \in A$, $l_i \in \{0,1\}$, where

$$\mathcal{C}_{l_i}(a_i,a_{i+1}) = \begin{cases} \mathcal{A}(a_i,a_{i+1}) & \text{if} \quad l_i = 1, \\ \mathcal{B}(f(a_i),f(a_{i+1})) & \text{if} \quad l_i = 0. \end{cases}$$

Define this functor to be the composite

$$\begin{tikzcd}
\mathcal{C}_{l_1}(a_1,a_2)\times\cdots\times\mathcal{C}_{l_{n-1}}(a_{n-1},a_n) \arrow[d, "G"'] \arrow[dr, dotted] & \\
\mathcal{B}(f(a_1),f(a_2))\times\cdots\times\mathcal{B}(f(a_{n-1}),f(a_n)) \arrow[r, "\Psi_\sigma^B"'] & \mathcal{B}(f(a_1),f(a_n))
\end{tikzcd}$$

where G is the product of the functors $\mathcal{C}_{l_i}(a_i,a_{i+1}) \longrightarrow \mathcal{B}(f(a_i),f(a_{i+1}))$ that are either $F_{(a_i,a_{i+1})}$ or identity, depending on l_i. These choices define y_f^F in degree 1.

The next step is to give the components of y_f^F in degree 2, that is we have to give natural isomorphisms $\Psi_\sigma \circ \Psi_\rho \longrightarrow \Psi_{\sigma\circ\rho}$. It is a straightforward computation to see that for any such natural isomorphism we have exactly one choice: some pasting of the invertible 2-cells in the data defining the map of bicategories (F,f), and any such pasting is unique due to the coherence conditions on (F,f). As a consequence, the relations imposed in degree 3 for y_f^F are also satisfied, hence y^F is indeed an element of $w\mathcal{C}at^2(\Phi(\mathbb{A}),\Phi(\mathbb{B})) \subseteq w\mathcal{C}at^2_{\mathrm{Cor}_1}$.

This far we have constructed a map of sets Φ:

$$ubi\mathcal{C}tg(\mathbb{A},\mathbb{B}) \longrightarrow w\mathcal{C}at^2(\Phi(\mathbb{A}),\Phi(\mathbb{B})) \longrightarrow ho(i^*(w\mathcal{C}at^2))(\Phi(\mathbb{A}),\Phi(\mathbb{B}))$$

that is clearly functorial. We still need to show that Φ is surjective and injective.

To treat surjectivity, for any $y \in w\mathcal{C}at^2(\Psi(\mathbb{A}),\Psi(\mathbb{B})) \subseteq w\mathcal{C}at^2_{\mathrm{Cor}_1}$ we construct a homomorphism of unbiased bicategories $(F,f)\colon \mathbb{A} \longrightarrow \mathbb{B}$ such that the associated y^F described above will be in the class of y in the homotopy category. For any such y it is straightforward how to get the map of sets $f\colon A \longrightarrow B$ and the functors $F_{(a_1,a_2)}\colon \mathcal{A}(a_1,a_2) \longrightarrow \mathcal{B}(f(a_1),f(a_2))$, hence we only need to construct the natural isomorphisms for the data defining (F,f), displayed in diagram (5.3.1). We will discuss only the case $n=2$, the general case can be treated analogously. These natural isomorphisms are obtained as the composite of two natural isomorphisms given by the degree 2- and 3 data in y_f:

(a) The first natural isomorphism is the one in in $(y_f)_{\mathrm{Cor}_1 \circ \mathrm{Cor}_2}$:

$$
\begin{tikzcd}
\mathcal{A}(a_1,a_2)\times\mathcal{A}(a_2,a_3) \arrow[r, "\Psi^A"] \arrow[dr, "K"'] & \mathcal{A}(a_1,a_3) \arrow[d, "F_{13}"] \arrow[dl, phantom, "\Uparrow\alpha"] \\
 & \mathcal{B}(f(a_1),f(a_2))
\end{tikzcd}
$$

(b) The second natural isomorphism comes from a pasting diagram, induced by the data in $(y_f)_{\mathrm{Cor}_2 \circ (\mathrm{Cor}_1, \mathrm{Cor}_1)}$:

$$
\begin{tikzcd}
 & & \mathcal{B}\times\mathcal{B} \arrow[dd, "\Psi^B"] \\
 & \mathcal{B}(f(a_1),f(a_2))\times\mathcal{A}(a_2,a_3) \arrow[ur, "\mathrm{id}\times F_{23}"] \arrow[dr, dotted, "H"] \arrow[r, phantom, "\Uparrow\alpha_2"] \arrow[d, phantom, "\Uparrow\beta_1"] & {} \\
\mathcal{A}(a_1,a_2)\times\mathcal{A}(a_2,a_3) \arrow[ur, "F_{12}\times\mathrm{id}"] \arrow[rr, "K"] \arrow[dr, "\mathrm{id}\times F_{23}"'] & {} & \mathcal{B}(f(a_1),f(a_3)) \\
 & \mathcal{A}(a_1,a_2)\times\mathcal{B}(f(a_2),f(a_3)) \arrow[ur, dotted, "G"'] \arrow[dr, "F_{12}\times\mathrm{id}"'] \arrow[r, phantom, "\Downarrow\alpha_1"] \arrow[u, phantom, "\Downarrow\beta_2"] & {} \\
 & & \mathcal{B}\times\mathcal{B} \arrow[uu, "\Psi^B"']
\end{tikzcd}
$$

Note that the coherence conditions in $(y_f)_{\mathrm{Cor}_2 \circ (\mathrm{Cor}_1, \mathrm{Cor}_1)}$ imply $\alpha_2 \cdot \beta_1 = \alpha_1 \cdot \beta_2$ as natural isomorphisms $K \Rightarrow \Psi^B \circ (F_{12} \times F_{23})$.

We can set the required natural isomorphism for the data in the homomorphism (F, f) to be $\alpha \circ (\alpha_2 \cdot \beta_1)$. The coherence conditions in the degree 3 components of y_f imply that $(F, f)\colon \mathbb{A} \longrightarrow \mathbb{B}$ is indeed a homomorphism of unbiased bicategories (with strict unit). The associated $y^F \in w\mathcal{C}at^2(\Phi(\mathbb{A}), \Phi(\mathbb{B}))$ is homotopic to y since we can construct an element in $w\mathcal{C}at^2_{\mathrm{Cor}_1 \circ \mathrm{Cor}_1}$ with faces $id_{\Phi(\mathbb{A})}, y$ and y^F. Hence the function $\Phi\colon ubi\mathcal{C}tg(\mathbb{A}, \mathbb{B}) \longrightarrow ho(i^* w\mathcal{C}at^2)(\Phi(\mathbb{A}, \mathbb{B})$ is surjective as well. This construction shows that Φ is injective as well and the proof is finished. ♦

Propositions 5.3.7 and 5.3.8 imply immediately

Theorem 5.3.9. *The categories $ubi\mathcal{C}tg$ and $ho(i^*(w\mathcal{C}at^2))$ are isomorphic. Hence the category of classical bicategories is equivalent to the category of dendroidal weak 2-categories.*

To conclude this section, we conjecture that the following, stronger statement is true:

Conjecture 5.3.10. *The inclusion of simplicial sets $N(ubi\mathcal{C}tg) \longrightarrow i^*(w\mathcal{C}at^2)$ is a weak equivalence in the Joyal model structure on $s\mathcal{S}ets$.*

5.4 Weak 3-categories

In this last section our aim is to compare dendroidal weak 3-categories with the classical notion of tricategories. To simplify our notation, we restrict ourselves only to one-object versions of these notions, the multi-object case can be treated similarly. Our aim is to compare classical tricategories with dendroidal ones in a similar way we did for bicategories: we would like to say that the category of tricategories is equivalent to the homotopy category of the quasi-category of dendroidal weak 3-categories. Unfortunately such a statement cannot be correct: on the one hand tricategories and their weak homomorphisms do not form a category (they are building parts of the weak 4-category of tricategories); on the other hand dendroidal weak tricategories will correspond only to some semistrict notion of tricategories. These obstacles can be remedied with Gordon, Power and Street's result on the strictification of tricategories ([16]): every tricategory is triequivalent to a Gray-category. We are going to use the following hierarchy of semistrict tricategories:

$$\text{Gray-categories} \subseteq \text{1-strict tricategories} \subseteq \text{Tricategories}$$

where 1-strict tricategories form a category that we prove to be equivalent to the homotopy category of dendroidal tricategories.

Let us begin with a brief review of tricategories.

5.4.1 One-object tricategories

Our references for this subsection are [16, 24]. A one-object tricategory $\mathbb{T}$ consists of the following 7 data and 3 axioms:

(D1) A bicategory $\mathbb{B}$. The objects or 0-cells of $\mathbb{B}$ are called the arrows or 1-cells of $\mathbb{T}$. The arrows or 1-cells of $\mathbb{B}$ are called the 2-cells of $\mathbb{T}$, (horizontal) composition of these is denoted by "$\cdot$". The 2-cells of $\mathbb{B}$ are called the 3-cells of $\mathbb{T}$, vertical composition of these is denoted by juxtaposition. For a pair (a, b) of objects of $\mathbb{B}$ we will denote the associated category of 1-cells and 2-cells of $\mathbb{B}$ by $\mathcal{A}(a, b)$.

(D2) A homomorphism of bicategories $\star\colon \mathbb{B} \times \mathbb{B} \longrightarrow \mathbb{B}$, that is

- A function between the sets of objects of the involved bicategories $\mathbb{B} \times \mathbb{B}$ and $\mathbb{B}$. The image of $(a, b) \in \mathbb{B} \times \mathbb{B}$ will be denoted by $a \star b$.
- For any pair of objects $\big((a_1, a_2), (b_1, b_2)\big) \in \mathbb{B} \times \mathbb{B}$, a functor

$$\mathcal{A}(a_1, a_2) \times \mathcal{A}(b_1, b_2) \xrightarrow{\star} \mathcal{A}(a_1 \star b_1, a_2 \star b_2)$$

- For every object (a, b) of $\mathbb{B} \times \mathbb{B}$, an invertible 2-cell of $\mathbb{B}$:

$$\mathrm{Id}_{a\star b} \xRightarrow{\phi_{ab}} \mathrm{Id}_a \star \mathrm{Id}_b$$

  which is natural. Here $\mathrm{Id}_a \in \mathcal{A}(a, a)$ denotes the unit 1-cell of $\mathbb{B}$, etc.
- Natural isomorphisms, relating the horizontal compositions in $\mathbb{B}$:

$$\begin{tikzcd}
\mathcal{A}(a_1,a_2)\times\mathcal{A}(b_1,b_2)\times\mathcal{A}(a_2,a_3)\times\mathcal{A}(b_2,b_3) \arrow[r, "\star"] \arrow[d] \arrow[dr, phantom, "\Uparrow\phi"] & \mathcal{A}(a_1\star b_1, a_2\star b_2)\times\mathcal{A}(a_2\star b_2, a_3\star b_3) \arrow[d] \\
\mathcal{A}(a_1,a_3)\times\mathcal{A}(b_1,b_3) \arrow[r, "\star"] & \mathcal{A}(a_1\star b_1, a_3\star b_3)
\end{tikzcd}$$

  that is, for any 4 arrows in $\mathbb{B}$: $a_1 \xrightarrow{f_1} a_2 \xrightarrow{f_2} a_3$, $b_1 \xrightarrow{g_1} b_2 \xrightarrow{g_2} b_3$ invertible 2-cells in $\mathbb{B}$

$$\begin{tikzcd}
a_1 \star b_1 \arrow[r, bend left, "(f_2\star g_2)\cdot(f_1\star g_1)"] \arrow[r, bend right, "(f_2\cdot f_1)\star(g_2\cdot g_1)"'] \arrow[r, phantom, "\Uparrow\phi"] & a_3 \star b_3
\end{tikzcd}$$

- Compatibility conditions, expressing the relations between the natural isomorphisms ϕ and the associativity- and unit constraints of $\mathbb{B}$.

(D3) A homomorphism of bicategories $* \xrightarrow{I} \mathbb{B}$. Explicitly, the relevant part of I breaks down to

- an object of $\mathbb{B}$: $I(*) = u \in \mathbb{B}$;
- an arrow of $\mathbb{B}$: $I_u \in \mathcal{A}(u, u)$;
- an invertible 2-cell of $\mathbb{B}$:

$$\begin{tikzcd}
u \arrow[r, bend left, "I_u"] \arrow[r, bend right, "\mathrm{Id}_u"'] \arrow[r, phantom, "\Uparrow\eta"] & u
\end{tikzcd}$$

(D4) A pseudo natural equivalence

$$\begin{tikzcd}
\mathbb{B}\times\mathbb{B}\times\mathbb{B} \arrow[r, "\star\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times\star"'] \arrow[dr, phantom, "\Downarrow\alpha"] & \mathbb{B}\times\mathbb{B} \arrow[d, "\star"] \\
\mathbb{B}\times\mathbb{B} \arrow[r, "\star"] & \mathbb{B}
\end{tikzcd}$$

that is

- For every object $(a,b,c) \in \mathbb{B}\times\mathbb{B}\times\mathbb{B}$, an invertible 1-cell in $\mathbb{B}$:

$$(a\star b)\star c \xrightarrow{\alpha} a\star(b\star c)$$

- For every 1-cell $\left(a \xrightarrow{f} a_1\,,\, b \xrightarrow{g} b_1\,,\, c \xrightarrow{h} c_1\right)$ in $\mathbb{B}\times\mathbb{B}\times\mathbb{B}$, an invertible 2-cell in $\mathbb{B}$:

$$\begin{tikzcd}
(a\star b)\star c \arrow[r, "\alpha"] \arrow[d, "(f\star g)\star h"'] \arrow[dr, phantom, "\Uparrow\alpha"] & a\star(b\star c) \arrow[d, "f\star(g\star h)"] \\
(a_1\star b_1)\star c_1 \arrow[r, "\alpha"] & a_1\star(b_1\star c_1)
\end{tikzcd}$$

that is natural. This natural 2-cell can be written also as a natural isomorphism, as follows:

$$\begin{tikzcd}
\mathcal{A}\times\mathcal{A}\times\mathcal{A}\big((a,b,c),(a_1,b_1,c_1)\big) \arrow[r, "\star(\star\times\mathrm{id})"] \arrow[d, "\star(\star\times\mathrm{id})"'] \arrow[dr, phantom, "\Uparrow\alpha"] & \mathcal{A}\big((a\star b)\star c,(a_1\star b_1)\star c_1\big) \arrow[d, "\alpha_*"] \\
\mathcal{A}\big(a\star(b\star c),a_1\star(b_1\star c_1)\big) \arrow[r, "\alpha^*"'] & \mathcal{A}\big((a\star b)\star c),a_1\star(b_1\star c_1)\big)
\end{tikzcd}$$

- Compatibility conditions for α, with respect to the associativity and unit constraints of $\mathbb{B}$.

(D5) Pseudo natural equivalences

$$\begin{tikzcd}
\mathbb{B}\times\mathbb{B} \arrow[dr, "\star"'] & \mathbb{B} \arrow[l, "I\times\mathrm{id}"'] \arrow[r, "\mathrm{id}\times I"] \arrow[d, equal] \arrow[dl, phantom, "\overset{\lambda}{\Leftarrow}"] \arrow[dr, phantom, "\overset{\rho}{\Rightarrow}"] & \mathbb{B}\times\mathbb{B} \arrow[dl, "\star"] \\
& \mathbb{B} &
\end{tikzcd}$$

that is,

- for every object $a \in \mathbb{B}$ invertible 1-cells in $\mathbb{B}$

$$u\star a \xleftarrow{\lambda} a \xrightarrow{\rho} a\star u$$

- for every 1-cell $f \in \mathcal{A}(a,a_1)$ invertible 2-cells in $\mathbb{B}$

$$\begin{tikzcd}
u\star a \arrow[d, "I_u\star f"'] \arrow[dr, phantom, "\Uparrow\lambda"] & a \arrow[l, "\lambda"'] \arrow[r, "\rho"] \arrow[d, "f"] \arrow[dr, phantom, "\Uparrow\rho"] & a\star u \arrow[d, "f\star I_u"] \\
u\star a_1 & a_1 \arrow[l, "\lambda"] \arrow[r, "\rho"'] & a\star u_1
\end{tikzcd}$$

- The cells mentioned above are natural and they obey some more compatibility conditions with the structure constraints of $\mathbb{B}$. Again, the 2-cells can be described as natural isomorphisms as well.

(D6) An invertible modification

$$
\begin{tikzcd}
& \mathbb{B}^4 \arrow[rr, "\star\times\mathrm{id}\times\mathrm{id}"] \arrow[dl, "\mathrm{id}\times\mathrm{id}\times\star"'] \arrow[dr, "\mathrm{id}\times\star\times\mathrm{id}"] & & \mathbb{B}^3 \arrow[dr, "\star\times\mathrm{id}"] & \\
\mathbb{B}^3 \arrow[dr, "\mathrm{id}\times\star"'] \arrow[rr, phantom, "\overset{\mathrm{id}\times\alpha}{\Leftarrow}"] & & \mathbb{B}^3 \arrow[rr, "\star\times\mathrm{id}"] \arrow[dl, "\mathrm{id}\times\star"] \arrow[ur, phantom, "\Downarrow\alpha\times\mathrm{id}"] & & \mathbb{B}^2 \arrow[dl, "\star"] \\
& \mathbb{B}^2 \arrow[rr, "\star"'] \arrow[urr, phantom, "\Downarrow\alpha"] & & \mathbb{B} &
\end{tikzcd}
\quad \overset{\pi}{\Rrightarrow} \quad
\begin{tikzcd}
& \mathbb{B}^4 \arrow[rr, "\star\times\mathrm{id}\times\mathrm{id}"] \arrow[dl, "\mathrm{id}\times\mathrm{id}\times\star"'] \arrow[rr, phantom, bend right, "="] & & \mathbb{B}^3 \arrow[dl, "\mathrm{id}\times\star"'] \arrow[dr, "\star\times\mathrm{id}"] & \\
\mathbb{B}^3 \arrow[dr, "\mathrm{id}\times\star"'] \arrow[rr, "\star\times\mathrm{id}"] & & \mathbb{B}^2 \arrow[dr, "\star"'] \arrow[rr, phantom, "\overset{\alpha}{\Leftarrow}"] & & \mathbb{B}^2 \arrow[dl, "\star"] \\
& \mathbb{B}^2 \arrow[rr, "\star"'] \arrow[u, phantom, "\Downarrow\alpha"] & & \mathbb{B} &
\end{tikzcd}
$$

(D7) Three more invertible modifications

$$
\begin{tikzcd}
\mathbb{B}^2 \arrow[dr, "\mathrm{id}\times I\times\mathrm{id}"] \arrow[drr, equal] \arrow[ddr, equal] & & \\
& \mathbb{B}^3 \arrow[r, "\star\times\mathrm{id}"'] \arrow[d, "\mathrm{id}\times\star"'] \arrow[dr, phantom, "\Downarrow\alpha"] \arrow[ur, phantom, "\Uparrow\rho\times\mathrm{id}"] \arrow[l, phantom, "\overset{\mathrm{id}\times\lambda}{\Leftarrow}"] & \mathbb{B}^2 \arrow[d, "\star"] \\
& \mathbb{B}^2 \arrow[r, "\star"'] & \mathbb{B}
\end{tikzcd}
\quad \overset{\mu}{\Rrightarrow} \quad
\begin{tikzcd}
\mathbb{B}^2 \arrow[drr, equal] \arrow[ddr, equal] \arrow[ddrr, "\star"] & & \\
& \arrow[ur, phantom, "="] \arrow[dl, phantom, "="] & \mathbb{B}^2 \arrow[d, "\star"] \\
& \mathbb{B}^2 \arrow[r, "\star"'] & \mathbb{B}
\end{tikzcd}
$$

$$
\begin{tikzcd}
& \mathbb{B}^3 \arrow[dr, "\star\times\mathrm{id}"] \arrow[d, phantom, "\Downarrow\lambda\times\mathrm{id}"] & \\
\mathbb{B}^2 \arrow[ur, "I\times\mathrm{id}\times\mathrm{id}"] \arrow[rr, equal] \arrow[d, "\star"'] & \arrow[d, phantom, "="] & \mathbb{B}^2 \arrow[d, "\star"] \\
\mathbb{B} \arrow[rr, equal] & \, & \mathbb{B}
\end{tikzcd}
\quad \overset{l}{\Rrightarrow} \quad
\begin{tikzcd}
& \mathbb{B}^3 \arrow[d, "\mathrm{id}\times\star"] \arrow[dr, "\star\times\mathrm{id}"] & \\
\mathbb{B}^2 \arrow[ur, "I\times\mathrm{id}\times\mathrm{id}"] \arrow[r, phantom, "="] \arrow[d, "\star"'] & \mathbb{B}^2 \arrow[r, phantom, "\Downarrow\alpha"] \arrow[dr, "\star"] \arrow[d, phantom, "\Downarrow\lambda"] & \mathbb{B}^2 \arrow[d, "\star"] \\
\mathbb{B} \arrow[ur, "I\times\mathrm{id}"] \arrow[rr, equal] & \, & \mathbb{B}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{B} \arrow[rr, equal] & \, & \mathbb{B} \\
\mathbb{B}^2 \arrow[u, "\star"] \arrow[rr, equal] \arrow[dr, "\mathrm{id}\times\mathrm{id}\times I"'] & \arrow[u, phantom, "="] \arrow[d, phantom, "\Uparrow\mathrm{id}\times\rho"] & \mathbb{B}^2 \arrow[u, "\star"'] \\
& \mathbb{B}^3 \arrow[ur, "\mathrm{id}\times\star"'] &
\end{tikzcd}
\quad \overset{r}{\Rrightarrow} \quad
\begin{tikzcd}
\mathbb{B} \arrow[rr, equal] \arrow[dr, "\mathrm{id}\times I"] & \, & \mathbb{B} \\
\mathbb{B}^2 \arrow[u, "\star"] \arrow[r, phantom, "="] \arrow[dr, "\mathrm{id}\times\mathrm{id}\times I"'] & \mathbb{B}^2 \arrow[ur, "\star"] \arrow[r, phantom, "\overset{\alpha}{\Rightarrow}"] \arrow[u, phantom, "\Uparrow\rho"] & \mathbb{B}^2 \arrow[u, "\star"'] \\
& \mathbb{B}^3 \arrow[u, "\star\times\mathrm{id}"] \arrow[ur, "\mathrm{id}\times\star"'] &
\end{tikzcd}
$$

So far we have enlisted all the data of the definition of one-object tricategories. There are still 3 more coherence conditions (axioms) that this data has to obey. For the sake of compactness, we are not going to enlist the rigourous diagrams here (they can be found in [16]), we only indicate the nature of these conditions below.

(A1) The *nonabelian 4-cycle condition* is a coherence condition for associativity, involving π.

(A2) The *left normalization condition* is a coherence condition for the left unit, involving l and μ.

(A3) The *right normalization condition* is a coherence condition for the right unit, involving r and μ.

This concludes the definition of tricategories. The complicated data and axioms make tricategories a rather difficult notion to work with, but fortunately we can simplify our task by means of the coherence theorem for tricategories, deduced in [16]. Let us recall this theorem here.

Definition 5.4.1. (1) Let $\mathbb{B}_1, \mathbb{B}_2, \dots, \mathbb{B}_n, \mathbb{B}$ be 2-categories. A homomorphism of bicategories $\mathbb{B}_1 \times \mathbb{B}_2 \times \cdots \times \mathbb{B}_n \xrightarrow{F} \mathbb{B}$ is called a *cubical functor* if for all pairs of 1-cells in $\mathbb{B}_1 \times \cdots \times \mathbb{B}_n$

$$(a_1, \dots, a_n) \xrightarrow{(f_1,\dots,f_n)} (a'_1, \dots, a'_n) \xrightarrow{(f'_1,\dots,f'_n)} (a''_1, \dots, a''_n)$$

satisfying

$$(i > j) \text{ implies } (f_i = \mathrm{id} \text{ or } f'_j = \mathrm{id}),$$

the comparison 2-cell of F

$$\begin{tikzcd} & F(a'_1, \dots, a'_n) \arrow[dr, "F(f'_1,\dots,f'_n)"] \arrow[d, phantom, "\Downarrow"] & \\ F(a_1, \dots, a_n) \arrow[ur, "F(f_1,\dots,f_n)"] \arrow[rr, "F(f'_1\cdot f_1,\dots,f'_n\cdot f_n)"'] & & F(a''_1, \dots, a''_n) \end{tikzcd}$$

is an identity.

(2) A one-object tricategory is called *cubical* if $\mathbb{B}$ is a 2-category and the functor $\star\colon \mathbb{B} \times \mathbb{B} \longrightarrow \mathbb{B}$ is a cubical functor, and it is called *strict cubical* if in addition the pseudo natural equivalences in (D4) and (D5) are identities. Note that in the latter case the invertible modifications of (D6) and (D7) are trivial as well and the axioms (A1)-(A3) hold automatically.

The relevance of cubical functors lies in

Theorem 5.4.2. *([16]) Every tricategory is triequivalent to a strict cubical tricategory.*

The next step in Gordon, Power and Street's work on the coherence of tricategories is to identify strict cubical tricategories with Gray-categories, that are defined as follows. If $\mathbb{A}$ and $\mathbb{B}$ are 2-categories, we can define their Gray tensor product $\mathbb{A} \otimes \mathbb{B}$ to be the following 2-category:

(D1) The 0-cells of $\mathbb{A} \otimes \mathbb{B}$ are pairs of 0-cells $(a, b) \in A \times B$. We will formally denote these pairs by $a \otimes b$.

(D2) The 1-cells are given by formal products $a \otimes g$, $f \otimes b$ where $a \in A$, $b \in B$, f is a 1-cell in $\mathbb{A}$ and g is a 1-cell in $\mathbb{A}$.

(D3) The 2-cells are generated by formal products $a \otimes \psi$, $f \otimes g$ and $\phi \otimes b$ where $a \in A$, $b \in B$, f is a 1-cell of $\mathbb{A}$, g is a 1-cell of $\mathbb{B}$, ϕ is a 2-cell in $\mathbb{A}$, ψ is a 2-cell in $\mathbb{B}$ and $f \otimes g$ denotes the following 2-cell

$$\begin{tikzcd} a \otimes b \arrow[r, "f\otimes b"] \arrow[d, "a\otimes g"'] \arrow[dr, phantom, "\Downarrow f\otimes g"] & a_1 \otimes b \arrow[d, "a_1\otimes g"] \\ a \otimes b_1 \arrow[r, "f\otimes b_1"'] & a_1 \otimes b_1 \end{tikzcd}$$

The generating 2-cells of $\mathbb{A} \otimes \mathbb{B}$ are subject to the following relations:

(A1) $a \otimes - \colon \mathbb{B} \longrightarrow \mathbb{A} \otimes \mathbb{B}$ and $- \otimes b \colon \mathbb{A} \longrightarrow \mathbb{A} \otimes \mathbb{B}$ are 2-functors.

(A2)

$$f_1 f \otimes g = (f_1 \otimes g) \circ (f \otimes g);$$

$$f \otimes g_1 g = (f \otimes g_1) \circ (f \otimes g);$$

$$f \otimes I_b = \mathrm{id}_f \otimes b = \mathrm{id}_{f \otimes b};$$

$$I_a \otimes g = a \otimes \mathrm{id}_g = \mathrm{id}_{a \otimes g}\,.$$

(A3) For any 2-cells $\phi\colon f \Rightarrow f_1$ in $\mathbb{A}$ and $\psi\colon g \to g_1$ in $\mathbb{B}$

$$(f_1 \otimes g) \circ (\phi \otimes b) = (\phi \otimes b_1) \circ (f \otimes g)$$

$$(f \otimes g_1) \circ (a \otimes \psi) = (a_1 \otimes \psi) \circ (f \otimes g)$$

The category of 2-categories is closed symmetric monoidal under the Gray tensor product. A Gray-category by definition is a category enriched in this symmetric monoidal category of 2-categories. There is a natural 1-1 correspondence between Gray-categories and strict cubical tricategories, hence

Theorem 5.4.3. *([16]) Every tricategory is triequivalent to a Gray-category.*

5.4.2 Trihomomorphisms

In this subsection we are going to describe (weak) homomorphisms between one-object tricategories. In the literature these notions are called trihomomorphisms. We adopt the notation of Subsection 5.4.1. Let $\mathbb{T}$ and $\mathbb{T}'$ be 1-object tricategories. A trihomomorphism $\mathbb{T} \longrightarrow \mathbb{T}'$ consists of the following data and axioms:

(D1) A homomorphism of bicategories $H\colon \mathbb{B} \longrightarrow \mathbb{B}'$;

(D2) A pseudo natural equivalence, relating H with the horizontal compositions

$$\begin{tikzcd} \mathbb{B} \times \mathbb{B} \arrow[r, "H\times H"] \arrow[d, "\star"'] \arrow[dr, phantom, "\Downarrow\chi"] & \mathbb{B}' \times \mathbb{B}' \arrow[d, "\star'"] \\ \mathbb{B} \arrow[r, "H"'] & \mathbb{B} \end{tikzcd}$$

(D3) A pseudo natural equivalence, relating H with the units

$$\begin{tikzcd} & * \arrow[dl, "I"'] \arrow[dr, "I'"] \arrow[d, phantom, "\overset{\iota}{\Leftarrow}"] & \\ \mathbb{B} \arrow[rr, "H"'] & & \mathbb{B}' \end{tikzcd}$$

(D4) Invertible modifications, analogous to (D6) in the definition of tricategories:

$$\begin{tikzcd}[column sep=small] & \mathbb{B}^3 \arrow[rr, "H^3"] \arrow[dl, "\mathrm{id}\times\star"'] \arrow[dr, "\star\times\mathrm{id}"] & & \mathbb{B}'^3 \arrow[dr, "\star'\times\mathrm{id}"] \arrow[dl, phantom, "\Downarrow\chi\times\mathrm{id}"] & & & & \mathbb{B}^3 \arrow[rr, "H^3"] \arrow[dl, "\mathrm{id}\times\star"'] \arrow[dr, phantom, "\mathrm{id}\times\chi"] & & \mathbb{B}'^3 \arrow[dl, "\mathrm{id}\times\star'"'] \arrow[dr, "\star'\times\mathrm{id}"] & \\ \mathbb{B}^2 \arrow[dr, "\star"'] \arrow[rr, phantom, "\overset{\alpha}{\Leftarrow}"] & & \mathbb{B}^2 \arrow[rr, "H^2"] \arrow[dl, "\star"] \arrow[dr, phantom, "\Downarrow\chi"] & & \mathbb{B}'^2 \arrow[dl, "\star'"] & \overset{\pi}{\Rrightarrow} & \mathbb{B}^2 \arrow[rr, "H^2"] \arrow[dr, "\star"'] & \arrow[d, phantom, "\Downarrow\chi"] & \mathbb{B}'^2 \arrow[dr, "\star'"'] \arrow[rr, phantom, "\overset{\alpha'}{\Leftarrow}"] & & \mathbb{B}'^2 \arrow[dl, "\star'"] \\ & \mathbb{B} \arrow[rr, "H"'] & & \mathbb{B}' & & & & \mathbb{B} \arrow[rr, "H"'] & & \mathbb{B}' & \end{tikzcd}$$

(D5) Two more invertible modifications, analogous to (D7) in the definition of tricategories.

The data is subject to some natural axioms, derived from axioms (A1-3) for tricategories. We will be interested in the following semistrict notion of tricategories:

Definition 5.4.4. A one-object tricategory $\mathbb{T}$ is called 1-*strict* if composition of 1-cells in $\mathbb{T}$ is strict in the following sense: the invertible pseudo natural transformations α, λ, ρ are identities and the invertible modifications π, μ, l, r have unique coherence isomorphisms as their components (we will call these modifications trivial). A trihomomorphism $H\colon \mathbb{T} \longrightarrow \mathbb{T}'$ is called strict, if H is locally strict, the pseudo-natural transformations χ and ι are identities and the invertible modifications of (D4) and (D5) are trivial. 1-strict one-object tricategories and strict trihomomorphisms form a category that we denote by $\mathcal{T}ricat_1$.

Let us conclude our brief introduction to tricategories with an important example from algebraic topology.

Example 5.4.5. Homotopy 3-types (or equivalently, simplicial 3-types) can be considered as examples of tricategories, in the following sense. Leroy in [26] constructed a 3-nerve functor from the category of Gray-categories to simplicial sets

$$N_3\colon GrCat \longrightarrow s\mathcal{S}ets.$$

Berger proved in [2] that the functor N_3 has a left adjoint t_3. Recall that a *simplicial 3-type* is (the homotopy class of) a simplicial set X such that $\pi_k(|X|) = 0$ for every $k > 3$. Let $GrGpd$ denote the full subcategory of $GrCat$, consisting of Gray-groupoids (i.e. Gray-categories with strictly invertible 1-, 2- and 3-cells). The adjunction mentioned above can be extended to an adjunction

$$\begin{tikzcd} GrGpd \arrow[r, shift left, "\hat{t}_3"] & s\mathcal{S}ets \arrow[l, shift left, "N_3"] \end{tikzcd}$$

via the free-forgetful adjunction $GrCat \rightleftarrows GrGpd$. This adjunction can be used to transfer the classical model structure on simplicial sets to the category of Gray-groupoids (see [2] for details), moreover

Theorem 5.4.6. *[2, 21, 26] The adjunction $(\hat{t}_3, N_3)$ induces an equivalence between the homotopy category $ho(GrGpd)$ of Gray-groupoids and the homotopy category $ho(sSets[3])$ of simplicial 3-types.*

5.4.3 Tricategories are dendroidal weak 3-categories

In conjunction with Definition 5.1.2 and Subsection 5.3.2, we can define a dendroidal weak 3-category as an element of the set $w\mathcal{C}at^3_{|}$. In particular, a one-object dendroidal weak 3-category is a map of dendroidal sets $N_d(As) \xrightarrow{y} w\mathcal{C}at^2$, where $As := As_*$ is the associative operad on one colour. Note that for any diagram of dendroidal sets $X\colon \mathbb{S}^{\mathrm{op}} \longrightarrow d\mathcal{S}ets$ there is an obvious projection

$$\int_{\mathbb{S}} X \xrightarrow{\pi} N_d(\mathbb{S}),$$

hence any one-object dendroidal weak 3-category determines a monoid M: indeed, the composite map of dendroidal sets

$$\begin{tikzcd} N_d(As) \arrow[r, "y"] \arrow[dr, dotted] & \displaystyle\int_{\mathcal{S}ets} \underline{d\mathcal{S}ets}\big(N_d(As_-), \mathrm{hcN}_d(\mathcal{C}tg)\big) \arrow[d, "\pi"] \\ & N_d(\mathcal{S}ets) \end{tikzcd}$$

is the same thing as a set M, together with a monoid structure on it. Let us denote the multiplication of this monoid by $\star\colon M \times M \longrightarrow M$. As we will see later, this monoid structure will be the composition of 1-cells in a one-object dendroidal weak 3-category, and this fact will also show that general one-object tricategories cannot be described by elements of $w\mathcal{C}at^3_{|}$ (since composition of 1-cells in a tricategory is not necessarily strict). On the other hand,

Proposition 5.4.7. *There is an inclusion of sets*

$$\iota\colon \{\text{1-strict tricategories}\} \hookrightarrow w\mathcal{C}at^3_{|}$$

Proof. We are going to prove this for the one-object case, the general case is entirely similar. The proof is basically an analysis of a map of dendroidal sets

$$N_d(As) \xrightarrow{y} w\mathcal{C}at^2,$$

where we will also make some particular choices for the building parts of y, and show that these choices are allowed. Again, by Theorem 5.1.9 it is enough to analyse the degree 0, 1, 2 and 3 terms of such a map.

In degree 0 we have

$$y_| : * \longrightarrow \left(\int_{\mathcal{S}ets} \underline{d\mathcal{S}ets}\big(N_d(As_-), \mathrm{hcN}_d(\mathcal{C}tg)\big)\right)_| ,$$

i.e. a map of dendroidal sets $y_| \colon N_d(As_M) \longrightarrow \mathrm{hcN}_d(\mathcal{C}tg)$, where M is the underlying set of the monoid mentioned above. In view of Subsection 5.3.2, such a map is a generalised (unbiased) bicategory, and in particular we can pick it in such a way that it represents a bicategory $\mathbb{B}$. Obviously the set of objects of $\mathbb{B}$ has to be M. We will denote the categories of 1- and 2-cells in $\mathbb{B}$ as before, by $\mathcal{A}(a_1, a_2)$, etc.

In degree 1 we have the corollas as indices of y. Out of these possibilities, y_{Cor_1} does not contribute any new data. Indeed, the dendroidal identities force the commutativity of the diagram

$$\begin{tikzcd}
* \arrow[r, "y_{\mathrm{Cor}\,1}"] \arrow[d, equal] & \displaystyle\int_{\mathcal{S}ets} \underline{d\mathcal{S}ets}\big(N_d(As_-), \mathrm{hcN}_d(\mathcal{C}tg)\big)_{\mathrm{Cor}_1} \arrow[r, "\pi"] \arrow[d, shift left] \arrow[d, shift right] & N_d(\mathcal{S}ets)_{\mathrm{Cor}_1} \arrow[d, shift left] \arrow[d, shift right] \\
* \arrow[r, "y_|"] & \displaystyle\int_{\mathcal{S}ets} \underline{d\mathcal{S}ets}\big(N_d(As_-), \mathrm{hcN}_d(\mathcal{C}tg)\big)_| \arrow[r, "\pi"] \arrow[u] & N_d(\mathcal{S}ets)_| \arrow[u]
\end{tikzcd}$$

hence y_{Cor_1} is the following data:

- a map of sets $A_1 \xrightarrow{f} A_2$ which has to be $M \xrightarrow{\mathrm{id}} M$ by the dendroidal identities (with degeneracies);
- a map of dendroidal sets $N_d(As_M) \otimes \Omega[\mathrm{Cor}_1] \longrightarrow \mathrm{hcN}_d(\mathcal{C}tg)$, i.e. an unbiased homomorphism of unbiased bicategories $\mathbb{B}_0 \xrightarrow{F} \mathbb{B}_1$ that has to be $\mathrm{id}_{\mathbb{B}}$, again by the dendroidal identities.

In degree 1, y_{Cor_2} contributes with the following data:

- the structure map of the monoid, $M \times M \xrightarrow{\star} M$;
- a map of dendroidal sets $N_d(As_M) \otimes \Omega[\mathrm{Cor}_2] \xrightarrow{x} \mathrm{hcN}_d(\mathcal{C}tg)$.

Such a map x can be thought of as an unbiased homomorphism $\mathbb{B}_0 \times \mathbb{B}_1 \longrightarrow \mathbb{B}_2$ for some unbiased bicategories, and as before, we observe that dendroidal identities force $\mathbb{B}_i = \mathbb{B}$. In particular, we can pick x in such a way that it contains the same information that a homomorphism of bicategories $\star \colon \mathbb{B} \times \mathbb{B} \longrightarrow \mathbb{B}$ does (that restricts to $\star$ on the objects of $\mathbb{B}$): indeed, as one can check, such a choice for x would obey all the requirements. For example, a relevant component of x is in degree Cor_2, which in the general case amounts to functors

$$\mathcal{A}(a_1, a_2) \times \mathcal{A}(b_1, b_2) \xrightarrow{F} \mathcal{A}(a_1 \star b_1, a_2 \star b_2)$$

for every pairs of objects $(a_1, a_2), (b_1, b_2) \in M^2$, etc.

Analizing y_{Cor_n} when $n > 2$, we observe that it amounts to the choice of the iterated structure map of the monoid, $M^n \xrightarrow{\star^{(n)}} M$ (which is unambiguous by strictness of $\star$), and a map of dendroidal sets

$$N_d(As_M) \otimes \Omega[\mathrm{Cor}_n] \longrightarrow \mathrm{hcN}_d(\mathcal{C}tg)\,.$$

Once again, we can pick this map in such a way that it contains the same information that a homomorphism of bicategories $\mathbb{B}^n \longrightarrow \mathbb{B}$ does, and more, we can choose it to be an iteration of $\star\colon \mathbb{B} \times \mathbb{B} \longrightarrow \mathbb{B}$, denoted by $\star^{(n)}$. (Here we have to make choices for bracketing!)

In degree 1 there is only y_{Cor_0} left to analyse. Again, one can see that this component of y can be chosen in such a way that it contains the same data as a homomorphism of bicategories $* \xrightarrow{I} \mathbb{B}$.

Let us turn our attention to the degree 2 parts of y. In indices $\mathrm{Cor}_1 \circ_i \mathrm{Cor}_n$ and $\mathrm{Cor}_n \circ_i \mathrm{Cor}_1$ we will not get any new data again, due to the relations dictated by degeneracies. Generally, for all $n, m \geq 2$ in indices $\mathrm{Cor}_n \circ_i \mathrm{Cor}_m$ we observe a behavior similar to the cases $\mathrm{Cor}_2 \circ_1 \mathrm{Cor}_2$ and $\mathrm{Cor}_2 \circ_2 \mathrm{Cor}_2$. Hence, we only study these two cases.

If $T = \mathrm{Cor}_2 \circ_1 \mathrm{Cor}_2$, y_T gives us the following data:

- the structure map of the monoid $M \times M \xrightarrow{\star} M$ twice, and also the composite $M \times M \times M \xrightarrow{\star \circ_1 \star} M$;
- a map of dendroidal sets $N_d(As_M) \otimes \Omega[T] \xrightarrow{z} \mathrm{hcN}_d(\mathcal{C}tg)$.

An important component of z is in degree T, which contains the following data:

- the already chosen homomorphism of bicategories $\mathbb{B} \times \mathbb{B} \xrightarrow{\star} \mathbb{B}$ twice;
- the composite $\star \circ_1 \star$;
- a third homomorphism of bicategories $F\colon \mathbb{B} \times \mathbb{B} \times \mathbb{B} \longrightarrow \mathbb{B}$ which has to be the homomorphism $\star^{(3)}$ chosen in degree 1 since Cor_3 has a face map to T;
- natural transformations

$$\begin{tikzcd}
\mathcal{A}(a,a_1) \times \mathcal{A}(b,b_1) \times \mathcal{A}(c,c_1) \arrow[r, "\star\circ_1\star"] \arrow[dr, "\star^{(3)}"'] & \mathcal{A}((a\star b)\star c, (a_1 \star b_1)\star c_1) \arrow[d, equal] \arrow[dl, phantom, "\Rightarrow"] \\
 & \mathcal{A}(a\star b\star c, a_1\star b_1\star c_1)
\end{tikzcd}$$

for every $a, b, c, a_1, b_1, c_1 \in M$.

If $T = \mathrm{Cor}_2 \circ_2 \mathrm{Cor}_2$, we get similar data, where the natural transformations of the last step are

$$\begin{tikzcd}
\mathcal{A}(a,a_1)\times\mathcal{A}(b,b_1)\times\mathcal{A}(c,c_1) \arrow[d, "\star\circ_2\star"'] \arrow[dr, "\star^{(3)}"] \arrow[dr, phantom, shift right=4ex, "\Leftarrow"] & \\
\mathcal{A}(a\star(b\star c), a_1\star(b_1\star c_1)) \arrow[r, equal] & \mathcal{A}(a\star b\star c, a_1\star b_1\star c_1)
\end{tikzcd}$$

The last two diagrams fit into a square as in (D4), Subsection 5.4.1 and we see that we can choose y in indices $\mathrm{Cor}_2 \circ_1 \mathrm{Cor}_2$ and $\mathrm{Cor}_2 \circ_2 \mathrm{Cor}_2$ in such a way that it contains the same data as a pseudo natural equivalence given in (D4) does, with the restriction that it has to be the identity on objects.

We can see also that y can be chosen in indices $\mathrm{Cor}_2 \circ_1 \mathrm{Cor}_0$ and $\mathrm{Cor}_2 \circ_2 \mathrm{Cor}_0$ in a way that it contains the same data as a pseudo natural equivalence in (D5) does.

Since (D6) and (D7) are trivial for 1-strict tricategories, it is a tedious, but straightforward check that the degree 3 components of y obey these axioms. (A1)-(A3) are satisfied as well by any y in degree 4. ♦

Theorem 5.4.8. *The category $ho(i^*(w\mathcal{C}at^3))$ is equivalent to $\mathcal{T}ricat_1$.*

Proof. The proof follows the same steps as the proof of Theorem 5.3.9. Suppose that $y \in w\mathcal{C}at^3$ is an arbitrary object. One needs to analyse first the elements of $w\mathcal{C}at^3_{\mathrm{Cor}_1}$.

In general, the elements of $w\mathcal{C}at^3_{\mathrm{Cor}_1}$ again look like $(f\colon A \longrightarrow B, x)$ where f is a map in $\mathcal{S}ets$ and x has 3 components.

$$\begin{aligned}
x_A &\in d\mathcal{S}ets(N_d(As_A), w\mathcal{C}at^2),\\
x_B &\in d\mathcal{S}ets(N_d(As_B), w\mathcal{C}at^2),\\
x_f &\in d\mathcal{S}ets(N_d(As_A \otimes \Omega(\mathrm{Cor}_1)), w\mathcal{C}at^2).
\end{aligned}$$

The components of x are related by the compatibility condition of the Grothendieck construction: if $\partial_{in}, \partial_{out}\colon | \longrightarrow \mathrm{Cor}_1$ denote the face maps sending $|$ to the leaf and root of Cor_1 respectively then $\partial_{in}^*(x_f) = x_A$ and $\partial_{out}^*(x_f) = f^*(x_B)$. The only relevant component of x is thus x_f. One can follow the description we gave in Subsection 5.3.2 to describe this x_f, and then see that strict trihomomorphisms embed as $\iota\colon \mathcal{T}ricat(\mathbb{T},\mathbb{T}') \longrightarrow w\mathcal{C}at^3(\iota(\mathbb{T}),\iota(\mathbb{T}'))$. Then the arguments in the proof of Proposition 5.3.8 can be carried out in the same way to see that ι induces a fully faithful and essentially surjective functor

$$\tilde{\iota}\colon \mathcal{T}ricat_1 \longrightarrow ho(i^* w\mathcal{C}at^3).$$

♦

We have identified dendroidal weak 3-categories with 1-strict tricategories, hence by Gordon, Power and Street's strictification theorem

Corollary 5.4.9. *Every dendroidal weak 3-category is triequivalent to a Gray-category.*

We believe that a stronger version of Theorem 5.4.8 holds:

Conjecture 5.4.10. *The inclusion of simplicial sets* $N(\mathcal{T}ricat_1) \longrightarrow i^*(w\mathcal{C}at^3)$ *is a weak equivalence in the Joyal model structure on* $s\mathcal{S}ets$.

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Samenvatting

Dit proefschrift voegt enkele nieuwe resultaten toe aan de theorie van operaden. Operaden worden al bijna een halve eeuw gebruikt en bestudeerd door wiskundigen. Ze blijken bijzonder geschikt te zijn voor het bestuderen van de homotopietheorie van bepaalde algebraïsche structuren in de topologie, waar ze op een systematische en volledige wijze de homotopie-informatie inherent aan deze structuren catalogiseren. Teneinde de homotopietheorie van de door operaden beschreven algebraïsche structuren te doorgronden, zal men eerst de homotopietheorie van operaden zelf moeten begrijpen. Het is bijvoorbeeld al lange tijd onder algebraïsch topologen bekend dat een algebra "op homotopie na" over een operade P hetzelfde is als een algebra over een "cofibrant replacement" van P. Deze bewering wordt pas precies als we de abstracte homotopietheorie van operaden begrijpen of, in andere woorden, als we een modelstructuur hebben op de categorie van operaden. Het bestaan van een dergelijke modelstructuur (voor een algemene klasse van categorieën van operaden) is aangetoond door I. Moerdijk en C. Berger.

In de laatste twee decennia hebben verschillende generalisaties van operaden het licht gezien. Een van deze nieuwe concepten, dat van cyclische operade, is door E. Getzler en M. Kapranov geïntroduceerd met het doel de cyclische cohomologie van verscheidene algebraïsche structuren eenduidig te beschrijven. Later leverden deze cyclische operaden nieuwe inzichten op voor de ontwikkeling van Kontsevich' theorie van formele symplectische meetkunde. Een natuurlijke vraag over cyclische operaden is of ze onder dezelfde voorwaarden als operaden een fatsoenlijke homotopietheorie toestaan. In de eerste twee hoofdstukken van dit proefschrift bestuderen we deze vraag, en geven we een bevestigend antwoord. Onder enkele milde aannamen bewijzen we dat cyclische operaden een modelstructuur toestaan als de onderliggende categorie een modelstructuur toestaat. Onder deze omstandigheden construeren we ook een Boardman-Vogt-resolutie voor cyclische operaden. We bewijzen tevens dat deze resolutie samenvalt met de klassieke "bar-cobar" resolutie in het geval van ketencomplexen van vectorruimten.

Een andere bekende eigenschap van (gekleurde) operaden is dat ze categorieën generaliseren. Ruwweg bestaat een categorie uit een verzameling objecten, en voor ieder geordend paar objecten een verzameling pijlen van het eerste object (de input) naar het tweede (de output). Evenzo bestaat een gekleurde operade uit een verzameling objecten en pijlen van een bepaalde input naar een output; het verschil met categorieën zit hem erin dat de input een eindige rij objecten mag zijn in plaats van een enkel object. Deze manier om tegen gekleurde operaden aan te kijken, geeft aanleiding tot enkele interessante vragen. Men kan zich bijvoorbeeld afvragen of er in de context van operaden een analogon is van simpliciale verzamelingen. Om precies te zijn: we weten dat simpliciale verzamelingen categorieën generaliseren door middel van de adjunctie tussen de "nerve functor" en de fundamentaalcategorie-functor. De vraag is dan of er een preschovencategorie bestaat die operaden generaliseert op dezelfde manier als simpliciale verzamelingen categorieën generaliseren. Het bevestigende antwoord op deze vraag is gegeven door I. Moerdijk en I. Weiss. Zij noemden de resulterende preschovencategorie "dendroïdale verzamelingen". Naar aanleiding van de analogie met simpliciale verzamelingen kan men een aantal vragen stellen over dendroïdale verzamelingen. In hoofdstuk vier geven we het bevestigend antwoord op één van die vragen: we bewijzen dat de klassieke Dold-Kan-correspondentie tussen simpliciale abelse groepen en ketencomplexen een dendroïdaal analogon bezit.

In hoofdstuk vijf bestuderen we een andere eigenschap van dendroïdale verzamelingen, namelijk het feit dat ze gebruikt kunnen worden om een nieuwe definitie van zwakke hogere categorieën te geven. Sinds het werk van o. a. J. M. Boardman, R. Vogt en A. Joyal is het bekend dat bepaalde simpliciale verzamelingen (door Boardman en Vogt "beperkte Kan-complexen" en door Joyal "quasicategorieën" genoemd) kunnen worden gezien als zwakke ω-categorieën. De dendroïdale analoga van quasicategorieën zijn quasioperaden. Men kan voor ieder natuurlijk getal n stapsgewijs een quasioperade construeren die zwakke n-categorieën beschrijft, zoals ontdekt door Moerdijk en Weiss. In het laatste hoofdstuk van het proefschrift bestuderen we twee zulke quasioperaden, corresponderend met klassieke bi- en tricategorieën. We bewijzen dat deze "dendroïdale zwakke 2- en 3-categorieën" op een bepaalde precies gedefinieerde wijze equivalent zijn aan hun klassieke analoga.

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Curriculum Vitae

Andor Lukács was born in Odorheiu Secuiesc, Romania on August 29th 1981. In the period 1995-1999 he was a student at the Tamási Áron Gimnázium in Odorheiu Secuiesc. During these years he took part in various mathematical, physics and chess competitions, both on the national and international level.

After obtaining his baccalaureate, in October 1999 he moved to Cluj Napoca where he studied mathematics at Babeş-Bolyai University. Between 2000 and 2003 he coauthored three lecture notes in mathematics for high school students and a book on Number Theory exercises for undergraduate students.

In September 2003 he joined the master class program on Noncommutative Geometry, organised by the Mathematical Research Institute at Utrecht University. He graduated the master class a year later, under the supervision of Prof. dr. Ieke Moerdijk. In September 2004 he also obtained his master's degree in mathematics under the supervision of Prof. dr. Andrei Mărcuş at Babeş-Bolyai University.

From February 2005 he has been a Ph.D. student at Utrecht University under the guidance of Prof. dr. Ieke Moerdijk.