Invent math (2010) 180: 589–610
DOI 10.1007/s00222-010-0238-y

Algebraic A-hypergeometric functions

Frits Beukers

Received: 19 December 2008 / Accepted: 19 February 2010 /
Published online: 13 March 2010

Abstract We formulate and prove a combinatorial criterion to decide if an A-hypergeometric system of differential equations has a full set of algebraic solutions or not. This criterion generalises the so-called interlacing criterion in the case of hypergeometric functions of one variable.

1 Introduction

The classically known hypergeometric functions of Euler-Gauss (${}_2F_1$), its one-variable generalisations ${}_{p+1}F_p$ and the many variable generalisations, such as Appell's functions, the Lauricella functions and Horn series are all examples of the so-called A-hypergeometric functions introduced by Gel'fand, Kapranov, Zelevinsky in [6–8]. We like to add that completely independently B. Dwork developed a theory of generalised hypergeometric functions in [4] which is in many aspects parallel to the theory of A-hypergeometric functions. The connection between the theories has been investigated in [1] and [5].

The definition of A-hypergeometric functions begins with a finite subset $A \subset \mathbb{Z}^r$ (hence their name) consisting of N vectors $\mathbf{a}_1, \dots, \mathbf{a}_N$ such that

(i) The $\mathbb{Z}$-span of $\mathbf{a}_1, \dots, \mathbf{a}_N$ equals $\mathbb{Z}^r$.
(ii) There exists a linear form h on $\mathbb{R}^r$ such that $h(\mathbf{a}_i) = 1$ for all i.

F. Beukers (✉)
Department of Mathematics, Universiteit Utrecht, P.O. Box 80010, 3508 TA, Utrecht, The Netherlands
e-mail: F.Beukers@uu.nl

The second condition ensures that we shall be working in the case of so-called Fuchsian systems. In a number of papers, e.g. [1], this condition is dropped to include the case of so-called confluent hypergeometric equations.

We are also given a vector of parameters $\alpha = (\alpha_1, \dots, \alpha_r)$ which could be chosen in $\mathbb{C}^r$, but we will usually take $\alpha \in \mathbb{Q}^r$. The lattice $L \subset \mathbb{Z}^N$ of relations consists of all $(l_1, \dots, l_N) \in \mathbb{Z}^N$ such that $\sum_{i=1}^N l_i \mathbf{a}_i = 0$.

The A-hypergeometric equations are a set of partial differential equations with independent variables $v_1, \dots, v_N$. This set consists of two groups. The first are the structure equations

$$\Box_{\mathbf{l}}\Phi := \prod_{l_i>0} \partial_i^{l_i}\Phi - \prod_{l_i<0} \partial_i^{|l_i|}\Phi = 0 \tag{A1}$$

for all $\mathbf{l} = (l_1, \dots, l_N) \in L$.

The operators $\Box_{\mathbf{l}}$ are called the box-operators. The second group consists of the homogeneity or Euler equations.

$$Z_i\Phi := (a_{i,1}v_1\partial_1 + a_{i,2}v_2\partial_2 + \cdots + a_{i,N}v_N\partial_N - \alpha_i)\Phi = 0, \quad i = 1, 2, \dots, r \tag{A2}$$

where $a_{i,k}$ denotes the i-th coordinate of $\mathbf{a}_k$.

We denote this system of equations by $H_A(\alpha)$. In general $H_A(\alpha)$ is a holonomic system of dimension equal to the $r-1$-dimensional volume of the so-called A-polytope $Q(A)$. This polytope is the convex hull of the endpoints of the $\mathbf{a}_i$. The volume-measure is normalised to 1 for a $r-1$-simplex of lattice-points in the plane $h(\mathbf{x}) = 1$ whose vertices are spanned by a set of r vectors with determinant 1. In the first days of the theory of A-hypergeometric systems there was some confusion as to what 'general' means, see [1]. To avoid these difficulties we make an additional assumption, which ensures that the dimension of the A-hypergeometric system indeed equals the volume of $Q(A)$.

(iii) The $\mathbb{R}_{\ge 0}$-span of A intersected with $\mathbb{Z}^r$ equals the $\mathbb{Z}_{\ge 0}$-span of A.

When A satisfies this condition we say that A is saturated. Under the saturation condition we have the following Theorem.

Theorem 1.1 (GKZ, Adolphson) *Let notations be as above. If condition* (iii) *is satisfied then the system of A-hypergeometric differential equations is holonomic of rank equal to the volume of the convex hull $Q(A)$ of A.*

For a complete story on the dimension of the solution space we refer to [15].

From now on we assume that all conditions (i), (ii), (iii) *are satisfied.*

In the present paper we shall use condition (iii) in the proof of the important Proposition 4.1.

To describe the standard hypergeometric solution of the A-hypergeometric system we define the projection map $\psi_L : \mathbb{R}^N \to \mathbb{R}^r$ given by $\psi_L : \mathbf{e}_i \mapsto \mathbf{a}_i$ for $i = 1, \ldots, N$. Here $\mathbf{e}_i$ denotes the i-th vector in the standardbasis of $\mathbb{R}^N$. Clearly the kernel of ψ_L is the space $L(\mathbb{R}) = L \otimes \mathbb{R}$. Choose a point $\gamma = (\gamma_1, \ldots, \gamma_N)$ in $\psi_L^{-1}(\alpha)$, in other words choose $\gamma_1, \ldots, \gamma_N$ such that $\alpha = \gamma_1 \mathbf{a}_1 + \cdots + \gamma_N \mathbf{a}_N$. Then a formal solution of the A-hypergeometric system can be given by

$$\Phi_{L,\gamma}(v_1, \ldots, v_N) = \sum_{\mathbf{l} \in L} \frac{\mathbf{v}^{\mathbf{l}+\gamma}}{\Gamma(\mathbf{l} + \gamma + \mathbf{1})}$$

where we use the short-hand notation

$$\frac{\mathbf{v}^{\mathbf{l}+\gamma}}{\Gamma(\mathbf{l} + \gamma + \mathbf{1})} = \frac{v_1^{l_1+\gamma_1} \cdots v_N^{l_N+\gamma_N}}{\Gamma(l_1 + \gamma_1 + 1) \cdots \Gamma(l_N + \gamma_N + 1)}.$$

By a proper choice of $\gamma \in \psi_L^{-1}(\alpha)$ this formal solution gives rise to actual powerseries solutions with a non-trivial region of convergence.

The real positive cone generated by the vectors $\mathbf{a}_i$ is denoted by $C(A)$. This is a polyhedral cone with a finite number of faces.

Definition 1.2 The hypergeometric system $H_A(\alpha)$ is called non-resonant if $\alpha + \mathbb{Z}^r$ contains no point in any face of $C(A)$. We call the system resonant otherwise.

Non-resonance is an important criterion in connection with (ir)reducibility of the hypergeometric system. We have

Theorem 1.3 *A non-resonant system of A-hypergeometric equations is irreducible.*

This theorem is proved in [9], Theorem 2.11 using perverse sheaves. It would be nice to have a more elementary proof however. The converse statement is almost true.

Remark 1.4 Let $H_A(\alpha)$ be a resonant system and suppose that for every $i \in \{1, 2, \ldots, N\}$ there exists $\mathbf{l} \in L$ such that $l_i \neq 0$. Then $H_A(\alpha)$ is reducible.

We could not find a proof of this in the literature and it will be subject of a paper to appear. In the statement of our Theorems we shall use the condition of non-resonance, although it would be more elegant to have the condition of irreducibility of the system. In many classical cases, such as one variable hypergeometric equations and Appell's equations, the equivalence can be found in the literature.

Let us now assume that $\alpha \in \mathbb{Q}^r$. We shall be interested in those non-resonant A-hypergeometric system that have a complete set of solutions algebraic over $\mathbb{C}(v_1, \dots, v_N)$. This question was first raised in the case of Euler-Gauss hypergeometric functions and the answer is provided by the famous list of H.A. Schwarz, see [17]. In 1989 this list was extended to general one-variable ${}_{p+1}F_p$ by Beukers and Heckman, see [3]. For the several variable cases, a characterization for Appell-Lauricella F_D was provided by Sasaki [16] in 1976 and Wolfart, Cohen [2] in 1992. The Appell systems F_4 and F_2 were classified by M. Kato in [10] (1997) and [11] (2000).

In the case of one-variable hypergeometric functions there is a simple combinatorial criterion to decide if they are algebraic or not. Consider the hypergeometric function

$$ {}_pF_{p-1}(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_{p-1}|z). $$

Define $\beta_p = 1$. We assume $\alpha_i - \beta_j \notin \mathbb{Z}$ for all $i, j = 1, \dots, p$, which ensures that the corresponding hypergeometric differential equation is irreducible. We shall say that the sets α_i and β_j interlace modulo 1 if the points of the sets $e^{2\pi i \alpha_j}$ and $e^{2\pi i \beta_k}$ occur alternatingly when running along the unit circle. The following Theorem is proved in [3].

Theorem 1.5 (Beukers, Heckman) *Suppose the one-variable hypergeometric equation with parameters $\alpha_i, \beta_j \in \mathbb{Q}$ $(i, j = 1, 2, \dots, p)$ with $\beta_p = 1$ is irreducible. Let D be the common denominator of the parameters. Then the solution set of the hypergeometric equation consists of algebraic functions (over $\mathbb{C}(z)$) if and only if the sets $k\alpha_i$ and $k\beta_j$ interlace modulo* 1 *for every integer k with $1 \le k < D$ and* $\gcd(k, D) = 1$.

It is the purpose of this paper to generalize the interlacing condition to a similar condition for A-hypergeometric systems.

Definition 1.6 A point $\mathbf{p} \in C(A)$ is called an *apexpoint* if $\mathbf{p} - \mathbf{q} \notin C(A)$ for every non-zero $\mathbf{q} \in C(A) \cap \mathbb{Z}^r$.

For practical purposes the following Proposition gives an easy criterion for an apexpoint.

Proposition 1.7 *A point $\mathbf{p} \in C(A)$ is an apexpoint if and only if $\mathbf{p} - \mathbf{a}_i \notin C(A)$ for $i = 1, \dots, n$.*

Clearly the condition $\mathbf{p} - \mathbf{a}_i \notin C(A)$ is a necessary one for an apexpoint. But since, by the saturation of A, every $\mathbf{q} \in C(A) \cap \mathbb{Z}^r$ can be written as a linear combination of the $\mathbf{a}_i$ with non-negative integer coefficients, the condition is also sufficient.

Definition 1.8 Let $\alpha \in \mathbb{R}^r$ and define $K_A(\alpha) = (\alpha + \mathbb{Z}^r) \cap C(A)$. We call the number of apexpoints in $K_A(\alpha)$ the signature of the polytope A and parameters α. Notation: $\sigma_A(\alpha)$.

One might be tempted to describe apexpoints as the vertices of the convex hull of $K_A(\alpha)$. That this is not so, can be seen from the example of the Horn G3 system in Sect. 2.

Proposition 1.9 *Let $\alpha \in \mathbb{R}^r$. Then $\sigma_A(\alpha)$ is less than or equal to the volume of the A-polytope $Q(A)$.*

We say that the signature is *maximal* if it equals the volume of $Q(A)$.

Theorem 1.10 *Let $\alpha \in \mathbb{Q}^r$ and suppose the A-hypergeometric system $H_A(\alpha)$ is non-resonant. Let D be the common denominator of the coordinates of α. Then the solution set of the A-hypergeometric system consists of algebraic solutions (over $\mathbb{C}(v_1, \ldots, v_N)$) if and only if $\sigma_A(k\alpha)$ is maximal for all integers k with $1 \le k < D$ and $\gcd(k, D) = 1$.*

As a first application of this criterion, Esther Bod, an Utrecht Ph.D.-student, managed to extend the Schwarz list to all four classes of Appell and Lauricella functions. This will be published in a forthcoming paper.

To compare Theorem 1.10 with the one-variable interlacing condition for ${}_2F_1$ we illustrate a connection. In the case of Euler-Gauss hypergeometric function we have $r = 3$, $N = 4$ and

$$\mathbf{a}_1 = (1, 0, 0), \qquad \mathbf{a}_2 = (0, 1, 0), \qquad \mathbf{a}_3 = (0, 0, 1), \qquad \mathbf{a}_4 = (1, 1, -1).$$

The faces of the cone generated by $\mathbf{a}_i$ $(i = 1, \ldots, 4)$ are given by $x = 0$, $y = 0$, $x + z = 0$, $y + z = 0$ (we use the coordinates x, y, z in $\mathbb{R}^3$. We define $\alpha = (-a, -b, c - 1)$. Theorem 1.3 implies that irreducibility comes down to the inequalities $-a, -b, -a + c, -b + c \notin \mathbb{Z}$. These are the familiar irreducibility conditions for the Euler-Gauss hypergeometric functions.

The lattice of relations has rank one and is generated by $(-1, -1, 1, 1)$. We choose $\gamma = (-a, -b, c - 1, 0)$. Then the formal solution $\Phi_{L,\gamma}$ reads

$$v_1^{-a} v_2^{-b} v_3^{c-1} \sum_{k \in \mathbb{Z}} \frac{v_1^{-k} v_2^{-k} v_3^k v_4^k}{\Gamma(-k - a + 1)\Gamma(-k - b + 1)\Gamma(c + k)\Gamma(k + 1)}.$$

Clearly $1/\Gamma(k + 1)$ vanishes for $k \in \mathbb{Z}_{<0}$, so our summation actually runs over $k \in \mathbb{Z}_{\ge 0}$. Apply the identity $1/\Gamma(1 - z) = \sin(\pi z)\Gamma(z)/\pi$ to $z = k + a$

and $z = k + b$ to obtain

$$\Phi_{L,\gamma} = v_1^{-a} v_2^{-b} v_3^{c-1} \frac{\sin(\pi a)\sin(\pi b)}{\pi^2} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)k!} \left(\frac{v_3 v_4}{v_1 v_2}\right)^k.$$

Setting $v_1 = v_2 = v_3 = 1$ and $v_4 = z$ we recognize the Euler-Gaussian hypergeometric series ${}_2F_1(a, b, c|z)$. By shifting over $\mathbb{Z}$ if necessary we can see to it that a, b, c are in the interval $(-1, 0)$. Suppose that the sets $\{a, b\}$ and $\{0, c\}$ interlace modulo 1. By interchange of a, b if necessary we can restrict ourselves to the case $-1 < a < c < b < 0$. It is straightforward to verify that $(-a, 1-b, c)$ and $(-a, -b, 1+c)$ are apexpoints of $K_A(\alpha)$. If the sets do not interlace then one checks that $(-a, -b, c)$ is the unique apexpoint if $a, b < c$ and $(-a, -b, 1+c)$ is the unique apexpoint if $c < a, b$.

A second example is Appell's hypergeometric equation F_2. The Appell F_2 hypergeometric function reads

$$F_2(a, b, b', c, c'|x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_m(c')_n m! n!} x^m y^n,$$

where $(x)_n$ denotes the Pochhammer symbol defined by $\Gamma(x+n)/\Gamma(x) = x(x+1)\cdots(x+n-1)$. The function F_2 satisfies a system of partial differential equations of rank 4. Algebraicity of these functions is completely described in [11].

The A-parameters are as follows. We have $r = 5$ and $N = 7$. The set A consists of the standard basisvectors $\mathbf{a}_1, \dots, \mathbf{a}_5$ in $\mathbb{R}^5$ and $\mathbf{a}_6 = (1, 1, 0, -1, 0)$, $\mathbf{a}_7 = (1, 0, 1, 0, -1)$. We take $\alpha = (-a, -b, -b', c, c')$. The lattice L of relations is generated by $(-1, -1, 0, 1, 0, 1, 0)$ and $(-1, 0, -1, 0, 1, 0, 1)$. Take $\gamma = (-a, -b, -b', c, c', 0, 0)$. In a similar way as we did for the Euler-Gauss functions we can now go from the formal expansion $\Phi_{L,\gamma}$ to the explicit Appell function F_2.

One can compute that the cone $C(A)$ has 8 faces and they are given by $x_1 = 0, x_2 = 0, x_3 = 0, x_2 + x_4 = 0, x_3 + x_5 = 0, x_1 + x_4 = 0, x_1 + x_5 = 0$ and $x_1 + x_4 + x_5 = 0$. Using Theorem 1.3 it follows that the A-hypergeometric system is irreducible if none of the following numbers is an integer,

$$a, b, b', -b + c, -b' + c', -a + c, -a + c', -a + c + c'.$$

These are precisely the irreducibility conditions for the F_2-system given in [11]. In that paper it is shown for example that with the choice

$$\alpha = (-a, -b, -b', c, c') = (1/10, 7/10, 9/10, 3/5, 1/5)$$

the solutions of the rank 4 Appell system are all algebraic. A small computer calculation shows that the apex points of $K_A(\alpha)$ are given by

$$(21/10, 7/10, 9/10, -2/5, -4/5), \qquad (11/10, 7/10, 9/10, 3/5, -4/5),$$
$$11/10, 7/10, 9/10, -2/5, 1/5), \qquad (1/10, 7/10, 9/10, 3/5, 1/5).$$

Similarly there are four apexpoints for the conjugate parameter 5-tuples $3\alpha, 7\alpha, 9\alpha$.

2 A simple example

In this section we show a more elaborate example of algebraic hypergeometric functions of Horn-type G3 which, to our knowledge, has not been dealt with before. The corresponding series with parameters a, b is given by

$$G_3(a, b, x, y) = \sum_{m\geq 0, n\geq 0} \frac{(a)_{2m-n}(b)_{2n-m}}{m!n!} x^m y^n.$$

Here again, $(x)_n$ denotes the Pochhammer symbol defined by $\Gamma(x+n)/\Gamma(x)$. However, now the index n may be negative, in which case the definition explicitly reads $\Gamma(x+n)/\Gamma(x) = 1/(x-1)\cdots(x-|n|)$. The system of differential equations is a rank 3 system. The set A can be chosen in $\mathbb{Z}^2$ for example as

$$\mathbf{a}_1 = (-1, 2), \qquad \mathbf{a}_2 = (0, 1), \qquad \mathbf{a}_3 = (1, 0), \qquad \mathbf{a}_4 = (2, -1).$$

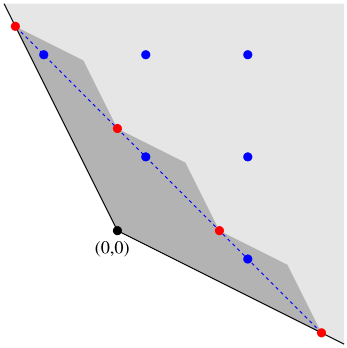
Below we show a picture of the cone $C(A)$ spanned by the elements of A, together with the points from A. In addition, the dark grey area indicates the set of apexpoints with respect to A. The parameter vector of the corresponding A-hypergeometric system is given by $(-a, -b)$.

Theorem 2.1 *Consider the A-hypergeometric system corresponding to the Horn G3 equations. In the following cases the system is irreducible and has only algebraic solutions.*

1. $a + b \in \mathbb{Z}$ *and* $a, b \notin \mathbb{Z}$.
2. $a \equiv 1/2 \pmod{\mathbb{Z}}, b \equiv 1/3, 2/3 \pmod{\mathbb{Z}}$.
3. $a \equiv 1/3, 2/3 \pmod{\mathbb{Z}}, b \equiv 1/2 \pmod{\mathbb{Z}}$.

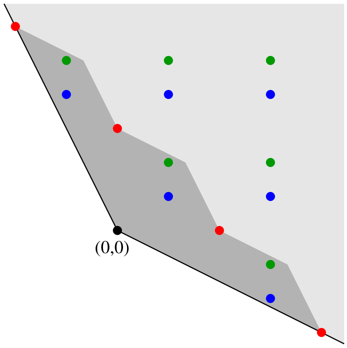
Proof This is an application of Theorem 1.10. In all cases we need only be interested in $a, b \pmod{\mathbb{Z}}$. In the following picture the light grey area is the cone $C(A)$, the dark grey area indicates the location of the apexpoints. If

$a + b \in \mathbb{Z}$ then we note that there are precisely 3 points of $(-a, -b) + \mathbb{Z}^2$ in the dark grey area, all lying on the line $x + y = 1$. Hence three apexpoints.



If $a+b \in \mathbb{Z}$ then also $ka+kb \in \mathbb{Z}$ for any integer. Irreducibility of the systems is ensured by Theorem 1.3 and the fact that $a, b \notin \mathbb{Z}$. Therefore, in the first case all conditions of Theorem 1.10 are fulfilled.

In the following picture we have drawn the sets $(1/2, 1/3) + \mathbb{Z}^2$ and $(1/2, 2/3) + \mathbb{Z}^2$ intersected with $C(A)$.



Clearly each set has three apexpoints and Theorem 1.10 can be applied to prove the second case. The third case runs similarly. □

It has been verified by J. Schipper, an Utrecht graduate student, that Theorem 2.1 gives the characterisation of all irreducible Horn G3-systems with algebraic solutions.

A fairly involved calculation reveals that a formula for $G_3(a, 1-a, x, y)$ can be given as follows

$$G_3(a, 1-a, x, y) = f(x, y)^a \sqrt{\frac{g(x, y)}{\Delta}}$$

where

$$\Delta = 1 + 4x + 4y + 18xy - 27x^2y^2$$

and

$$xf^3 - y = f - f^2, \qquad g(g - 1 - 3x)^2 = x^2\Delta.$$

For reference we display the series expansions of f and g.

$$f = 1 + (y - x) + \left(2x^2 - yx - y^2\right) + \left(-5x^3 + 3yx^2 + 2y^3\right)$$
$$+ \left(14x^4 - 10yx^3 + y^3x - 5y^4\right) + O(x, y)^5,$$
$$g = 1 + 2x - x(x + 2y) + 2x(x - y)^2 - x(x - y)^2(5x + 4y) + O(x, y)^5.$$

Moreover, $g = 1 + 4x - 2xf - 3x^2f^2$. In particular, f and g generate the same cubic extension of $\mathbb{Q}(x, y)$.

3 The signature

Proof of Proposition 1.9 We use the following property. Let $\mathbf{b}_1, \ldots, \mathbf{b}_r \in \mathbb{Z}^r$ be independent vectors. Let $\beta \in \mathbb{R}^r$. Then the number of points of $\beta + \mathbb{Z}^r$ inside the fundamental block $\{\sum_{i=1}^r \lambda_i \mathbf{b}_i \mid 0 \le \lambda_i < 1\}$ is equal to $|\det(\mathbf{b}_1, \ldots, \mathbf{b}_r)|$.

Write $Q(A)$ as a union of $r - 1$-simplices $\bigcup_{i=1}^m \sigma_i$ (a so-called triangulation of $Q(A)$). Every simplex σ_i is spanned by r independent vectors $\mathbf{a}_{i_j}$ ($j = 1, \ldots, r$). Let B_i be the fundamental block spanned by these vectors.

Let $\mathbf{a}$ be an apexpoint of $K_A(\alpha)$. Then $\mathbf{a}$ is contained in a positive cone spanned by one of the simplices σ_i. For every choice of $\mathbf{a}_{i_j}$ ($j = 1, \ldots, r$) the point $\mathbf{a} - \mathbf{a}_{i_j}$ falls outside this cone. If not, then $\mathbf{a}$ would be contained in $\mathbf{a}_{i_j} + C(A)$. Hence $\mathbf{a} \in B_i$. Since B_i contains at most $|\det(B_i)|$ point from $\alpha + \mathbb{Z}^r$, we see that the number of apexpoints is bounded above by $\sum_{i=1}^m |\det(B_i)|$. This equals precisely the $r - 1$-dimensional volume of the A-polytope $Q(A)$. $\square$

In the following recall the map $\psi : \mathbb{R}^N \to \mathbb{R}^r$ given by $\psi : \mathbf{e}_i \mapsto \mathbf{a}_i$ for $i = 1, \ldots, N$.

Proposition 3.1 *A point* $\mathbf{p} \in C(A)$ *is an apexpoint if and only if* $\psi^{-1}(\mathbf{p}) \cap \mathbb{R}^N_{\ge 0}$ *is contained in* $[0, 1)^N$.

Proof This statement follows immediately from the following equivalent statements

$$\begin{aligned}\mathbf{p} \text{ is an apexpoint} \quad &\iff \quad \mathbf{p}-\mathbf{a}_i \notin C(A) \quad \text{for } i=1,\dots,N\\ &\iff \quad (\psi^{-1}(\mathbf{p})-\mathbf{e}_i)\cap(\mathbb{R}^N_{\geq 0})=\emptyset\\ &\iff \quad \psi^{-1}(\mathbf{p})\cap\mathbb{R}^N_{\geq 0}\subset[0,1)^N. \qquad \square\end{aligned}$$

4 Mod p solutions

Let us assume that $\alpha \in \mathbb{Z}^r$ and let p be a prime. We describe the polynomial solutions in $\mathbb{F}_p[v_1,\dots,v_N]$ of the A-hypergeometric system with parameters A and α considered modulo p.

Let β_i/p $(i=1,\dots,\sigma)$ be the set of apexpoints of $(\alpha/p+\mathbb{Z}^r)\cap C(A)$. To any apexpoint we associate the set of lattice points

$$\Gamma_i=\psi^{-1}(\beta_i)\cap\mathbb{Z}^N_{\geq 0}$$

where $\psi:\mathbb{R}^N\to\mathbb{R}^N/L(\mathbb{R})$ is defined as in the previous section. For any $i=1,2,\dots,\sigma$ we define

$$\Psi_i:=\sum_{\mathbf{l}\in\Gamma_i}\frac{\mathbf{v}^{\mathbf{l}}}{\Gamma(\mathbf{l}+1)}.$$

This is a polynomial solution to the A-hypergeometric system with parameters A,β_i. Since β_i/p is an apexpoint of $(\alpha/p+\mathbb{Z}^r)\cap C(A)$, the preimage $\psi^{-1}(\beta_i/p)\cap\mathbb{R}^N_{\geq 0}$ is contained in the unit cube in $\mathbb{R}^N$ according to Proposition 3.1. Hence the points of $\Gamma_i=\psi^{-1}(\beta_i)\cap\mathbb{Z}^N_{\geq 0}$ are contained in the cube $0\leq x_i<p$ for $i=1,\dots,N$. In particular none of the positive coordinates of any $\mathbf{l}\in\Gamma_i$ is divisible by p, hence $\Gamma(\mathbf{l}+\mathbf{1})\not\equiv 0 \pmod p$ for all $\mathbf{l}\in\Gamma_i$. This means that Ψ_i can be reduced modulo p. Furthermore, since $\alpha\equiv\beta_i \pmod p$, the polynomial Ψ_i is a polynomial solution modulo p to the A-hypergeometric system with parameters A,α.

Since each of the sets Γ_i is contained in the cube $0\leq x_i<p$ any two shifts $\Gamma_i+p\mathbf{k}_i$ and $\Gamma_j+p\mathbf{k}_j$ for different i,j and $\mathbf{k}_i,\mathbf{k}_j\in\mathbb{Z}^N$ are disjoint. In particular the polynomials Ψ_i are linearly independent over $\mathbb{F}_p[v_1^p,\dots,v_N^p]$.

Proposition 4.1 *Every mod p polynomial solution of the A-hypergeometric system with parameters $A,\alpha\in\mathbb{Z}^r$ is an $\mathbb{F}_p[v_1^p,\dots,v_N^p]$-linear combination of the polynomials Ψ_i.*

Proof Let $P=\sum_{\mathbf{m}}p_{\mathbf{m}}\mathbf{v}^{\mathbf{m}}\in(\mathbb{Z}/p\mathbb{Z})[\mathbf{v}]$ be a polynomial solution of the A-hypergeometric system with parameters A,α. For formal reasons we extend the summation over all of $\mathbb{Z}^N$ but it should be understood that the set of multi-indices $\mathbf{m}$ with $p_{\mathbf{m}} \not\equiv 0 \pmod p$ is finite and contained in $\mathbb{Z}^N_{\geq 0}$.

For any $\mathbf{m} = (m_1, \ldots, m_N) \in \mathbb{Z}^N$ let us define

$$\lfloor \mathbf{m}/p \rfloor = (\lfloor m_1/p \rfloor, \ldots, \lfloor m_N/p \rfloor).$$

Define

$$\mathcal{P}_{\mathbf{0}} = \left\{ \sum_{0 \leq m_1, \ldots, m_N < p} c_{\mathbf{m}} \mathbf{v}^{\mathbf{m}} \mid c_{\mathbf{m}} \in \mathbb{Z}/p\mathbb{Z} \right\}.$$

Notice that $\Psi_i \in \mathcal{P}_{\mathbf{0}}$ for $i = 1, \ldots, \sigma$. The space $\mathcal{P}_{\mathbf{0}}$ is stable under the A-hypergeometric operators $\Box_{\mathbf{l}}$ and Z_i from equations (A1) and (A2). We can write P uniquely as a finite sum $P = \sum_{\mathbf{k}} \mathbf{v}^{p\mathbf{k}} P_{\mathbf{k}}$ where $P_{\mathbf{k}} \in \mathcal{P}_{\mathbf{0}}$ and $\mathbf{k}$ runs over a finite subset of $\mathbb{Z}^N_{\geq 0}$. Since $\mathcal{P}_0$ is stable under $\Box_{\mathbf{l}}$ and Z_i and $\mathbf{v}^p$ behaves like a constant under differentiation modulo p, the polynomials P_k must be solutions of the system $H_A(\alpha)$ modulo p.

Therefore, any mod p polynomial solution of $H_A(\alpha)$ is a $(\mathbb{Z}/p\mathbb{Z})[\mathbf{v}^p]$-linear combination of polynomials from $\mathcal{P}_{\mathbf{0}}$.

Let us, without loss of generality, assume that $P \in \mathcal{P}_{\mathbf{0}}$. Substitute P in the system (A1). Any $\mathbf{l} \in L$ can be decomposed as $\mathbf{l} = \mathbf{l}_+ - \mathbf{l}_-$ where $\mathbf{l}_+, \mathbf{l}_- \in \mathbb{Z}^N_{\geq 0}$ and we assume they have disjoint support. Denoting $[\mathbf{m}]_{\mathbf{r}} = \prod_{i=1}^N m_i \times (m_i - 1) \cdots (m_i - r_i + 1)$ for any $\mathbf{r} \in \mathbb{Z}^N_{\geq 0}$ we obtain

$$\begin{aligned} 0 &\equiv \sum_{\mathbf{m}} \left([\mathbf{m}]_{l_+} p_{\mathbf{m}} \mathbf{v}^{\mathbf{m}-l_+} - [\mathbf{m}]_{l_-} p_{\mathbf{m}} \mathbf{v}^{\mathbf{m}-l_-} \right) \pmod p \\ &\equiv \mathbf{v}^{-l_+} \sum_{\mathbf{m}} \left([\mathbf{m}]_{l_+} p_{\mathbf{m}} - [\mathbf{m} - \mathbf{l}]_{l_-} p_{\mathbf{m}-\mathbf{l}} \right) v^{\mathbf{m}} \pmod p. \end{aligned}$$

Hence

$$[\mathbf{m}]_{l_+} p_{\mathbf{m}} - [\mathbf{m} - \mathbf{l}]_{l_-} p_{\mathbf{m}-\mathbf{l}} \equiv 0 \pmod p \tag{1}$$

for every $\mathbf{m} \in \mathbb{Z}^N$ and every $\mathbf{l} \in L$. Substitution in (A2) gives

$$\sum_{\mathbf{m}} (-\alpha_i + a_{1i} m_1 + \cdots + a_{Ni} m_N) \mathbf{p}_{\mathbf{m}} v^{\mathbf{m}} \equiv 0 \pmod p$$

for $i = 1, 2, \ldots, r$. Hence $-\alpha_i + a_{1i} m_1 + \cdots + a_{Ni} m_N \equiv 0 \pmod p$ for every $\mathbf{m}$ with $p_{\mathbf{m}} \not\equiv 0 \pmod p$. Note that the system (A2) gives no extra relations between different $p_{\mathbf{m}}$. They only require that $\psi(\mathbf{m}) \equiv \alpha \pmod p$.

The system (A1) relates only those coefficients $p_{\mathbf{m}}$ for which the multi-indexes $\mathbf{m}$ differ by an element of L. Hence we can split P as a sum of terms of the form $P_\beta = \sum_{\psi(\mathbf{m}) = \beta} p_{\mathbf{m}} \mathbf{v}^{\mathbf{m}}$ with $\beta \in \mathbb{Z}^r$ and each summand P_β

satisfies modulo p the A-hypergeometric system $H_A(\alpha)$. The equations (A2) applied to P_β tell us that $\beta \equiv \alpha \pmod{p}$.

In recursion (1) let us replace $\mathbf{m} - \mathbf{l}$ by $\mathbf{m}'$. The new recursion reads

$$[\mathbf{m}]_{l_+} p_{\mathbf{m}} - [\mathbf{m}']_{l_-} p_{\mathbf{m}'} \equiv 0 \pmod{p} \tag{2}$$

where $l_+ = \mathbf{m} - \min(\mathbf{m}, \mathbf{m}')$ and $l_- = \mathbf{m}' - \min(\mathbf{m}, \mathbf{m}')$. Here, $\min(\mathbf{m}, \mathbf{m}')$ denotes the componentwise minimum of the coordinates of $\mathbf{m}, \mathbf{m}'$. In the new recursion $\mathbf{m}, \mathbf{m}'$ can be taken to be any two elements in $\psi^{-1}(\beta) \cap \mathbb{Z}^N$.

We define a partial ordering on $\mathbb{R}^N$. We say that $\mathbf{y} \geq \mathbf{x}$ if all components of $\mathbf{y}$ are larger or equal than the corresponding component of $\mathbf{x}$. In particular, when $\mathbf{y} \geq \mathbf{x}$ and $\mathbf{y} \neq \mathbf{x}$ we write $\mathbf{y} > \mathbf{x}$.

Lemma 4.2 *Let notations be as above. Then*

$$[\mathbf{m}]_{\mathbf{m}-\min(\mathbf{m},\mathbf{m}')} \not\equiv 0 \pmod{p} \quad \Longleftrightarrow \quad \lfloor \mathbf{m}'/p \rfloor - \lfloor \mathbf{m}/p \rfloor \geq \mathbf{0}.$$

Proof This follows from the following sequence of equivalences,

$$\begin{aligned}
[\mathbf{m}]_{\mathbf{m}-\min(\mathbf{m},\mathbf{m}')} \not\equiv 0 \pmod{p} \quad &\Longleftrightarrow \quad \forall i\colon \lfloor m_i/p \rfloor - \lfloor \min(m_i, m_i')/p \rfloor \leq 0 \\
&\Longleftrightarrow \quad \forall i\colon \lfloor m_i/p \rfloor - \lfloor m_i'/p \rfloor \leq 0 \\
&\Longleftrightarrow \quad \lfloor \mathbf{m}'/p \rfloor - \lfloor \mathbf{m}/p \rfloor \geq \mathbf{0}. \qquad \square
\end{aligned}$$

In particular it follows from this Lemma that if $\lfloor \mathbf{m}/p \rfloor = \mathbf{0}$ (i.e. $0 \leq m_i < p$ for all i) and $\mathbf{m}' \geq \mathbf{0}$, then the coefficient of $p_{\mathbf{m}}$ in the recursion (2) is non-zero modulo p. In order words if $\mathbf{m}' \in \psi^{-1}(\beta) \cap \mathbb{Z}^N_{\geq 0}$ then all values $p_{\mathbf{m}}$ with $\mathbf{m} \in [0, p)^N$ are determined by the value of $p_{\mathbf{m}'}$ through the recursion (2). So P_β is determined up to a constant factor by the value of β.

Suppose that the point β/p is not an apexpoint. Then there exists an index i such that $\beta/p - \mathbf{a}_i \in C(A)$. Hence $\beta - p\mathbf{a}_i \in C(A)$. By the saturatedness of A there exist non-negative integers $x_1, \ldots, x_N$ such that

$$\beta - p\mathbf{a}_i = x_1\mathbf{a}_1 + \cdots + x_N\mathbf{a}_N.$$

Hence $\psi^{-1}(\beta)$ contains the lattice point $\mathbf{m}' = p\mathbf{e}_i + x_1\mathbf{e}_1 + \cdots + x_N\mathbf{e}_N$. Note that $\lfloor \mathbf{m}'/p \rfloor > \mathbf{0}$. Hence $p_{\mathbf{m}'} = 0$ and via the recursion we conclude that $P_\beta \equiv 0$.

Thus we conclude that β/p is an apexpoint, i.e. there exists an i such that $\beta/p = \beta_i/p$. In that case we conclude that P_β is a scalar multiple of Ψ_i. $\square$

We now consider polynomial mod p solutions for A-hypergeometric systems with parameters $\alpha \in \mathbb{Q}^r$.

Proposition 4.3 *Let $\alpha \in \mathbb{Q}^r \setminus \mathbb{Z}^r$ and let $D > 1$ the common denominator of the coordinates of α. Let p be a prime not dividing D. Let $\rho \equiv -p^{-1} \pmod{D}$. Let s be the signature of A and $\rho\alpha$. Suppose that the A-hypergeometric system we consider is non-resonant. Then, when p is sufficiently large, the polynomial mod p solutions of the A-hypergeometric system with parameters A, α is a free $\mathbb{F}_p[v_i^p]$-module of rank s.*

Proof Let $\mathbf{k} = (1 + p\rho)\alpha$. Notice that $\mathbf{k} \in \mathbb{Z}^r$ and $\mathbf{k} \equiv \alpha \pmod{p}$. So it suffices to look at the mod p A-hypergeometric system with parameters $A, \mathbf{k}$. In Proposition 4.1 we saw that these solutions form a free module of rank s' where s' is the signature of A and $\mathbf{k}/p$. Let δ be the minimal distance of the points of $\rho\alpha + \mathbb{Z}^r$ to the faces of $C(A)$. Suppose $\delta = 0$. Then there is a point $\rho\alpha + \mathbf{k}$ with $\mathbf{k} \in \mathbb{Z}^r$ contained in a face of $C(A)$. Choose $\mu \in \mathbb{Z}$ such that $\mu\rho \equiv 1 \pmod{D}$. Then $\mu(\rho\alpha + \mathbf{k}) = \alpha + \mu\mathbf{k} + (\mu\rho - 1)\alpha$ is on a face of $C(A)$. This contradicts the irreducibility of our A-hypergeometric system by Theorem 1.3. So $\delta > 0$. Let us assume that p is so large that $|\alpha/p| < \delta$. Then the points of $(\rho\alpha + \mathbb{Z}^r) \cap C(A)$ and $(\mathbf{k}/p + \mathbb{Z}^r) \cap C(A)$ are in one-to-one correspondence given by $\mathbf{x} \sim \mathbf{y} \iff |\mathbf{x} - \mathbf{y}| < \delta$. In particular the number of apexpoints of both sets is equal, hence $s = s'$. This proves our assertion. □

5 Proof of the main theorem

This section is devoted to a proof of Theorem 1.10. Let notations be as in Theorem 1.10 and suppose we consider a non-resonant A-hypergeometric system with parameters $\alpha \in \mathbb{Q}^r$. Let p be a prime which is large enough in the sense of Proposition 4.3. Let D be the common denominator of the elements of α and $\rho \equiv -p^{-1} \pmod{D}$. Then the statement that $\sigma_A(\rho\alpha)$ is maximal is equivalent to the statement that the A-hypergeometric system modulo p has a maximal $\mathbb{F}(\mathbf{v}^p)$-independent set of polynomial solutions.

A fortiori the following two statements are equivalent:

(i) $\sigma_A(k\alpha)$ is maximal for every k with $1 \le k < D$ and $\gcd(k, D) = 1$.

(ii) Modulo almost every prime p the A-hypergeometric system modulo p has a maximal set of polynomial solutions modulo p.

A famous conjecture, attributed to Grothendieck implies that statement (ii) is equivalent to the following statement,

(iii) The A-hypergeometric system has a complete set of algebraic solutions.

If Grothendieck's conjecture were proven we would be done here. Fortunately, in two papers by N.M. Katz ([13] and [12]) Grothendieck's conjecture is proven in the case when the system of differential equations is (a factor of) a Picard-Fuchs system, i.e. a system of differential equations satisfied by the period integral on families of algebraic varieties. More precisely we refer to Theorem 8.1(5) of [12], which states

Theorem 5.1 (N.M. Katz, 1982) *Suppose we have a system of partial linear differential equations, as sketched above, whose p-curvature vanishes for almost all p. Then, if the system is a subsystem of a Picard-Fuchs system, the solution space consists of algebraic functions.*

The above theorem is formulated in terms of vanishing p-curvature for almost all p, but according to a Lemma by Cartier (Theorem 7.1 of [12]) this is equivalent to the system having a maximal set of independent polynomial solutions modulo p for almost all p.

To finish the proof of Theorem 1.10 it remains to show that the A-hypergeometric equations for $\alpha \in \mathbb{Q}^r$ do arise from algebraic geometry. We shall do so in Sects. 6 and 7, where we construct Euler type integrals for the solutions of the A-hypergeometric system.

In an attempt to maintain the lowtech nature of this paper we finish this section with a proof of the (easier) implication (iii) $\Rightarrow$ (ii). Before doing so we need a few introductory concepts from the theory of linear differential equations.

Let k be a field which, in our case, is usually $\mathbb{Q}$ or $\mathbb{F}_p$. Consider the differential field $K = k(v_1, \ldots, v_N) = k(\mathbf{v})$ with derivations $\partial_i = \frac{\partial}{\partial v_i}$ for $i = 1, \ldots, N$. The subfield $C_K \subset K$ of elements all of whose derivatives are zero, is called the field of constants. When the characteristic of k is zero we have $C_K = k$, when the characteristic is $p > 0$ we have $C_K = k(\mathbf{v}^p)$.

Throughout this section we let $\mathcal{L}$ be a finite set of linear partial differential operators with coefficients in K, like the A-hypergeometric system operators when $k = \mathbb{Q}$. Consider the differential ring $K[\partial_1, \ldots, \partial_N]$ and let $(\mathcal{L})$ be the left ideal generated by the differential operators of the system. We assume that the quotient $K[\partial_i]/(\mathcal{L})$ is a K-vector space of finite dimension d. Throughout this section we also fix a monomial K-basis $\partial^{\mathbf{b}} = \partial_1^{b_1} \cdots \partial_N^{b_N}$ with $\mathbf{b} \in B$ and where B is a finite set of N-tuples in $\mathbb{Z}_{\geq 0}^N$ of cardinality d.

Proposition 5.2 *Let $\mathcal{K}$ be some differential extension of K with field of constants C_K. Let $f_1, \ldots, f_m \in \mathcal{K}$ be a set of C_K-linear independent solutions of the system $L(f) = 0,\ L \in \mathcal{L}$. Then $m \leq d$. Moreover, if $m = d$ the determinant*

$$W_B(f_1, \ldots, f_d) = \det(\partial^{\mathbf{b}} f_i)_{\mathbf{b} \in B; i=1,\ldots,d}$$

is nonzero.

In case we have d independent solutions we call W_B the *Wronskian matrix* with respect to B and $f_1, \dots, f_d$. Obviously, if $g_1, \dots, g_d$ are C_K-linear dependent solutions then $W_B(g_1, \dots, g_d) = 0$.

Proof Suppose that either $m > d$ or $m = d$ and $W_B = 0$. In both cases there exists a $\mathcal{K}$-linear relation between the vectors $df_i := (\partial^{\mathbf{b}} f_i)_{\mathbf{b} \in B}$ for $i = 1, 2, \dots, m$. Choose $\mu < m$ maximal such that $df_i,\ i = 1, \dots, \mu$ are $\mathcal{K}$-linear independent. Then, up to a factor, the vectors $df_i,\ i = 1, \dots, \mu + 1$ satisfy a unique dependence relation $\sum_{i=1}^{\mu+1} A_i df_i = 0$ with $A_i \in \mathcal{K}$ not all zero. For any j we can apply the operator ∂_j to this relation to obtain

$$\sum_{i=1}^{\mu+1} \partial_j(A_i) df_i + A_i \partial_j(df_i) = 0.$$

Since $\partial_j \partial^{\mathbf{b}}$ is a K-linear combination of the elements $\partial^{\mathbf{b}}, \mathbf{b} \in B$ in $K[\partial_i]/(\mathcal{L})$ there exists a $d \times d$-matrix M_j with elements in K such that $\partial_j(df_i) = df_i \cdot M_j$. Consequently $\sum_{i=1}^{\mu+1} A_i \partial_j(df_i) = \sum_{i=1}^{\mu+1} A_i df_i \cdot M_j = 0$ and so we are left with $\sum_{i=1}^{\mu+1} \partial_j(A_i) df_i = 0$. Since the relation between $df_i, i = 1, \dots, \mu + 1$ is unique up to a scalar factor, there exists $\lambda_j \in \mathcal{K}$ such that $\partial_j(A_i) = \lambda_j A_i$ for all i. Suppose $A_1 \neq 0$. Then this implies that $\partial_j(A_i/A_1) = 0$ for all i and all j. We conclude that $A_i/A_1 \in C_K$ for all i. Hence there is a relation between the df_i with coefficients in C_K. A fortiori there is a C_K-linear relation between the f_i. This contradicts our assumption of independence of $f_1, \dots, f_m$.

So we conclude that $m \leq d$ and if $m = d$ then $W_B \neq 0$. □

Proposition 5.3 *Suppose the system of equations $L(y) = 0,\ L \in \mathcal{L}$ has only algebraic solutions and that they form a vector space of dimension d. Then for almost all p the system of equations modulo p has a $\mathbb{F}(\mathbf{v}^p)$-basis of d polynomial solutions in $\mathbb{F}(\mathbf{v})$.*

Proof Let $f_1, \dots, f_d$ be a basis of algebraic solutions. Choose a point $\mathbf{q} \in \mathbb{Q}^N$ such that f_i are all analytic near the point $\mathbf{q}$. Then $f_1, \dots, f_d$ can be considered as power series expansions in $\mathbf{v} - \mathbf{q}$. According to Eisenstein's theorem for powerseries of algebraic functions we have that the coefficients of the f_i can be reduced modulo p for almost all p. Moreover, let $\partial^{\mathbf{b}}, \mathbf{b} \in B$ be a monomial basis of $K[\partial_i]/(\mathcal{L})$. Then the Wronskian determinant $W_B(f_1, \dots, f_d)$ is non-zero. So for almost all p the powerseries f_i can be reduced modulo p and moreover, $W_B(f_1, \dots, f_d) \not\equiv 0 \pmod p$. Hence, for almost all p the powerseries $f_i \pmod p$ are linearly independent over the quotient field of $\mathbb{F}[[(\mathbf{v} - \mathbf{q})^p]]$, the power series in $(\mathbf{v} - \mathbf{q})^p$.

Fix one such prime p. Let P be the set $\{(b_1,\dots,b_N)\in\mathbb{Z}^N \mid 0\le b_i < p \text{ for } i=1,\dots,N\}$. Every solution f can be written in the form

$$f\equiv\sum_{\mathbf{b}\in P} a_{\mathbf{b}}(\mathbf{v}-\mathbf{q})^{\mathbf{b}} \pmod p,$$

where $a_{\mathbf{b}}\in\mathbb{F}[[(\mathbf{v}-\mathbf{q})^p]]$. For every $L\in\mathcal{L}$ we have that

$$\sum_{\mathbf{b}\in P} a_{\mathbf{b}}L(\mathbf{v}-\mathbf{q})^{\mathbf{b}}\equiv 0 \pmod p.$$

Let Q be the quotient field of $\mathbb{F}[[(\mathbf{v}-\mathbf{q})^p]]$. The Q-linear relations between the polynomials $L(\mathbf{v}-\mathbf{q})^{\mathbf{b}}$ for every L form a vector space of dimension d since the space of solutions mod p has this dimension. Moreover the space of Q-linear relations between the polynomials $L(\mathbf{v}-\mathbf{q})^{\mathbf{b}}$ is generated by $\mathbb{F}((\mathbf{v}-\mathbf{q})^p)$-linear relations or, what amounts to the same, $\overline{\mathbb{F}}(\mathbf{v}^p)$-linear relations. $\square$

6 Pochhammer cycles

In the construction of Euler integrals one often uses so-called twisted homology cycles. In [9] this is done on an abstract level, in [14] it is done more explicitly. In this paper we prefer to follow a more concrete approach by constructing a closed cycle of integration such that the (multivalued) integrand can be chosen in a continuous manner and the resulting integral is non-zero. For the ordinary Euler-Gauss function this is realised by integration over the so-called Pochhammer contour. Here we construct its n-dimensional generalisation. In Sect. 7 we use it to define an Euler integral for A-hypergeometric functions.

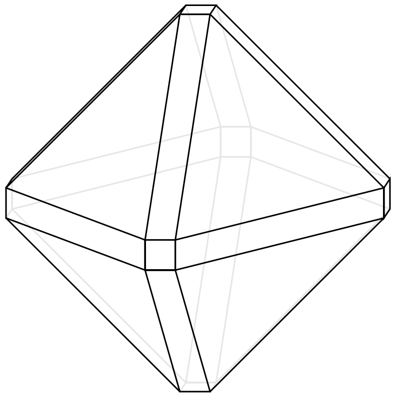
Consider the hyperplane H given by $t_0+t_1+\cdots+t_n=1$ in $\mathbb{C}^{n+1}$ and the affine subspaces H_i given by $t_i=0$ for $(i=0,1,2,\dots,n)$. Let H^o be the complement in H of all H_i. We construct an n-dimensional real cycle P_n in H^o which is a generalisation of the ordinary 1-dimensional Pochhammer cycle (the case $n=1$). When $n>1$ it has the property that its homotopy class in H^o is non-trivial, but that its fundamental group is trivial. One can find a sketchy discussion of such cycles in [18, Sect. 3.5].

Let ϵ be a positive but sufficiently small real number. We start with a polytope F in $\mathbb{R}^{n+1}$ given by the inequalities

$$|x_{i_1}|+|x_{i_2}|+\cdots+|x_{i_k}|\le 1-(n+1-k)\epsilon$$

for all $k=1,\dots,n+1$ and all $0\le i_1<i_2<\cdots<i_k\le n$. Geometrically this is an $n+1$-dimensional octahedron with the faces of codimension ≥ 2 sheared

off. For example in the case $n = 2$ it looks like



The faces of F can be enumerated by vectors $\mu = (\mu_0, \mu_1, \ldots, \mu_n) \in \{0, \pm 1\}^{n+1}$, not all μ_i equal to 0, as follows. Denote $|\mu| = \sum_{i=0}^{n} |\mu_i|$. The face corresponding to μ is defined by

$$
\begin{aligned}
F_\mu\colon \quad & \mu_0 x_0 + \mu_1 x_1 + \cdots + \mu_n x_n = 1 - (n + 1 - |\mu|)\epsilon, \\
& \mu_j x_j \geq \epsilon \text{ whenever } \mu_j \neq 0, \quad |x_j| \leq \epsilon \text{ whenever } \mu_j = 0.
\end{aligned}
$$

Notice that as a polytope F_μ is isomorphic to $\Delta_{|\mu|-1} \times I^{n+1-|\mu|}$ where Δ_r is the standard r-dimensional simplex and I the unit real interval. Notice in particular that we have $3^n - 1$ faces.

The $n - 1$-dimensional side-cells of F_μ are easily described. Choose an index j with $0 \leq j \leq n$. If $\mu_j \neq 0$ we set $\mu_j x_j = \epsilon$, if $\mu_j = 0$ we set either $x_j = \epsilon$ or $x_j = -\epsilon$. As a corollary we see that two faces F_μ and $F_{\mu'}$ meet in an $n - 1$-cell if and only if there exists an index j such that $|\mu_j| \neq |\mu'_j|$ and $\mu_i = \mu'_i$ for all $i \neq j$.

The vertices of F are the points with one coordinate equal to $\pm(1 - n\epsilon)$ and all other coordinates $\pm\epsilon$.

We now define a continuous and piecewise smooth map $P : \bigcup_\mu F_\mu \to H$ by

$$
P : (x_0, x_1, \ldots, x_n) \mapsto \frac{1}{y_0 + y_1 + \cdots + y_n} (y_0, y_1, \ldots, y_n) \tag{3}
$$

where $y_j = |x_j|$ if $|x_j| \geq \epsilon$ and $y_j = \epsilon e^{\pi i (1 - x_j/\epsilon)}$ if $|x_j| \leq \epsilon$. When ϵ is sufficiently small we easily check that P is injective. We define our n-dimensional Pochhammer cycle P_n to be its image.

Proposition 6.1 *Let $\beta_0, \beta_1, \dots, \beta_n$ be complex numbers. Consider the integral*

$$B(\beta_0, \beta_1, \dots, \beta_n) = \int_{P_n} \omega(\beta_0, \dots, \beta_n)$$

where

$$\omega(\beta_0, \dots, \beta_n) = t_0^{\beta_0 - 1} t_1^{\beta_1 - 1} \cdots t_n^{\beta_n - 1}\, dt_1 \wedge dt_2 \wedge \cdots \wedge dt_n.$$

Then, for a suitable choice of the multivalued integrand, we have

$$B(\beta_0, \dots, \beta_n) = \frac{1}{\Gamma(\beta_0 + \beta_1 + \cdots + \beta_n)} \prod_{j=0}^{n} (1 - e^{-2\pi i \beta_j}) \Gamma(\beta_j).$$

Proof The problem with ω is its multivaluedness. This is precisely the reason for constructing the Pochhammer cycle P_n. Now that we have our cycle we solve the problem by making a choice for the pulled back differential form $P^*\omega$ and integrating it over ∂F. Furthermore, the integral will not depend on the choice of ϵ. Therefore we let $\epsilon \to 0$. In doing so we assume that the real parts of all β_i are positive. The Proposition then follows by analytic continuation of the β_j.

On the face F_μ we define $T : F_\mu \to \mathbb{C}$ by

$$T : (x_0, x_1, \dots, x_n) = \prod_{\mu_j \neq 0} |x_j|^{\beta_j - 1} e^{\pi i (\mu_j - 1)\beta_j} \prod_{\mu_k = 0} \epsilon^{\beta_j - 1} e^{\pi i (x_j/\epsilon - 1)(\beta_j - 1)}.$$

This gives us a continuous function on ∂F. For real positive λ we define the complex power λ^z by $\exp(z \log \lambda)$. With the notations as in (3) we have $t_i = y_i/(y_0 + \cdots + y_n)$ and, as a result,

$$dt_1 \wedge dt_2 \wedge \cdots \wedge dt_n = \sum_{j=0}^{n} (-1)^j y_j dy_0 \wedge \cdots \wedge \check{dy_j} \wedge \cdots dy_n$$

where $\check{dy_j}$ denotes suppression of dy_j. It is straightforward to see that integration of $T(x_0, \dots, x_n)$ over F_μ with $|\mu| < n + 1$ gives us an integral of order $O(\epsilon^\beta)$ where β is the minimum of the real parts of all β_j. Hence they tend to 0 as $\epsilon \to 0$. It remains to consider the cases $|\mu| = n + 1$. Notice that T restricted to such an F_μ has the form

$$T(x_0, \dots, x_n) = \prod_{j=0}^{n} e^{\pi i (\mu_j - 1)\beta_j} |x_j|^{\beta_j - 1}.$$

Furthermore, restricted to F_μ we have

$$\sum_{j=0}^{n}(-1)^j y_j dy_0 \wedge \cdots \wedge d\check{y}_j \wedge \cdots dy_n = dy_1 \wedge dy_2 \wedge \cdots \wedge dy_n$$

and $y_0 + y_1 + \cdots + y_n = 1$. Our integral over F_μ now reads

$$\prod_{j=0}^{n} \mu_j e^{\pi i(\mu_j-1)\beta_j} \int_\Delta (1 - y_1 - \cdots - y_n)^{\beta_0-1} y_1^{\beta_1-1} \cdots y_n^{\beta_n-1} dy_1 \wedge \cdots \wedge dy_n$$

where Δ is the domain given by the inequalities $y_i \geq \epsilon$ for $i = 1, 2, \ldots, n$ and $y_1 + \cdots + y_n \leq 1 - \epsilon$. The extra factor $\prod_j \mu_j$ accounts for the orientation of the integration domains. The latter integral is a generalisation of the Euler beta-function integral. Its value is $\Gamma(\beta_0)\cdots\Gamma(\beta_n)/\Gamma(\beta_0 + \cdots + \beta_n)$. Adding these evaluation over all F_μ gives us our assertion. $\square$

For the next section we notice that if $\beta_0 = 0$ the subfactor $(1 - e^{-2\pi i\beta_0}) \times \Gamma(\beta_0)$ becomes $2\pi i$.

7 An Euler integral for A-hypergeometric functions

We now adopt the usual notation from A-hypergeometric functions. Define

$$I(A, \alpha, v_1, \ldots, v_N) = \int_\Gamma \frac{\mathbf{t}^\alpha}{1 - \sum_{i=1}^N v_i \mathbf{t}^{\mathbf{a}_i}} \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2} \wedge \cdots \wedge \frac{dt_r}{t_r},$$

where Γ is an r-cycle which doesn't intersect the hyperplane $1 - \sum_{i=1}^N v_i \mathbf{t}^{\mathbf{a}_i} = 0$ for an open subset of $\mathbf{v} \in \mathbb{C}^N$ and such that the multivalued integrand can be defined on Γ continuously and such that the integral is not identically zero. We shall specify Γ in the course of this section.

First note that an integral such as this satisfies the A-hypergeometric equations easily. The substitution $t_i \to \lambda_i t_i$ shows that

$$I(A, \alpha, \lambda^{\mathbf{a}_1} v_1, \ldots, \lambda^{\mathbf{a}_n} v_N) = \lambda^\alpha I(A, \alpha, v_1, \ldots, v_N).$$

This accounts for the homogeneity equations. For the "box"-equations, write $\mathbf{l} \in L$ as $\mathbf{u} - \mathbf{w}$ where $\mathbf{u}, \mathbf{w} \in \mathbb{Z}_{\geq 0}^N$ have disjoint supports. Then

$$\Box_{\mathbf{l}} I(A, \alpha, \mathbf{v}) = |\mathbf{u}|! \int_\Gamma \frac{\mathbf{t}^{\alpha + \sum_i u_i \mathbf{a}_i} - t^{\alpha + \sum_i w_i \mathbf{a}_i}}{(1 - \sum_{i=1}^N v_i \mathbf{t}^{\mathbf{a}_i})^{|\mathbf{u}|+1}} \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2} \wedge \cdots \wedge \frac{dt_r}{t_r}$$

where $|\mathbf{u}|$ is the sum of the coordinates of $\mathbf{u}$, which is equal to $|\mathbf{w}|$ since $|\mathbf{u}|-|\mathbf{w}|=|\mathbf{l}|=\sum_{i=1}^N l_i h(\mathbf{a}_i)=h(\sum_i l_i\mathbf{a}_i)=0$. Notice that the numerator in the last integrand vanishes because $\sum_i u_i\mathbf{a}_i=\sum_i w_i\mathbf{a}_i$. So $\Box_{\mathbf{l}} I(A,\alpha,\mathbf{v})$ vanishes.

We now specify our cycle of integration Γ. Choose r vectors in A such that their determinant is 1. After permutation of indices and change of coordinates if necessary we can assume that $\mathbf{a}_i=\mathbf{e}_i$ for $i=1,\dots,r$ (the standard basis of $\mathbb{R}^r$). Our integral now acquires the form

$$\int_\Gamma \frac{\mathbf{t}^\alpha}{1-v_1t_1-\cdots-v_rt_r-\sum_{i=r+1}^N v_i\mathbf{t}^{\mathbf{a}_i}}\frac{dt_1}{t_1}\wedge\frac{dt_2}{t_2}\wedge\cdots\wedge\frac{dt_r}{t_r}.$$

Perform the change of variables $t_i\to t_i/v_i$ for $i=1,\dots,r$. Up to a factor $v_1^{\alpha_1}\cdots v_r^{\alpha_r}$ we get the integral

$$\int_\Gamma \frac{\mathbf{t}^\alpha}{1-t_1-\cdots-t_r-\sum_{i=r+1}^N u_i\mathbf{t}^{\mathbf{a}_i}}\frac{dt_1}{t_1}\wedge\frac{dt_2}{t_2}\wedge\cdots\wedge\frac{dt_r}{t_r},$$

where the u_i are Laurent monomials in $v_1,\dots,v_N$. Without loss of generality we might as well assume that $v_1=\cdots=v_r=1$ so that we get the integral

$$\int_\Gamma \frac{\mathbf{t}^\alpha}{1-t_1-\cdots-t_r-\sum_{i=r+1}^N v_i\mathbf{t}^{\mathbf{a}_i}}\frac{dt_1}{t_1}\wedge\frac{dt_2}{t_2}\wedge\cdots\wedge\frac{dt_r}{t_r}.$$

For the r-cycle Γ we choose the projection of the Pochhammer r-cycle on $t_0+t_1+\cdots+t_r=1$ to $t_1,\dots,t_r$ space. Denote it by Γ_r. By keeping the v_i sufficiently small the hypersurface $1-t_1-\cdots-t_r-\sum_{i=r+1}^N v_i\mathbf{t}^{\mathbf{a}_i}=0$ does not intersect Γ_r.

To show that we get a non-zero integral we set $\mathbf{v}=\mathbf{0}$ and use the evaluation in Proposition 6.1. We see that it is non-zero if all α_i have non-integral values. When one of the α_i is integral we need to proceed with more care.

We develop the integrand in a geometric series and integrate it over Γ_r. We have

$$\begin{aligned}&\frac{\mathbf{t}^\alpha}{1-t_1-\cdots-t_r-\sum_{i=r+1}^N v_i\mathbf{t}^{\mathbf{a}_i}}\\&\quad=\sum_{m_{r+1},\dots,m_N\ge 0}\binom{|m|}{m_{r+1},\dots,m_N}\frac{\mathbf{t}^{\alpha+m_{r+1}\mathbf{a}_{r+1}+\cdots+m_N\mathbf{a}_N}}{(1-t_1-\cdots-t_r)^{|m|+1}}v_{r+1}^{m_{r+1}}\cdots v_N^{m_N}\end{aligned}$$

where $|m|=m_{r+1}+\cdots+m_N$. We now integrate over Γ_r term by term. For this we use Proposition 6.1. We infer that all terms are zero if and only if there exists i such that the i-th coordinate of α is integral and positive and the i-th coordinate of each of $\mathbf{a}_{r+1}, \ldots, \mathbf{a}_N$ is non-negative. In particular this means that the cone $C(A)$ is contained in the halfspace $x_i \geq 0$. Moreover, the points $\mathbf{a}_j = \mathbf{e}_j$ with $j \neq i$ and $1 \leq j \leq r$ are contained in the subspace $x_i = 0$, so they span (part of) a face of $C(A)$. The set $\alpha + \mathbb{Z}^r$ has non-trivial intersection with this face because $\alpha_i \in \mathbb{Z}$. From Theorem 1.3 it follows that our system is reducible, contradicting our assumption of irreducibility.

So in all cases we have that the Euler integral is non-trivial. By irreducibility of the A-hypergeometric system all solutions of the hypergeometric system can be given by linear combinations of period integrals of the type $I(A, \alpha, \mathbf{v})$ (but with different integration cycles).

Acknowledgements It is a pleasure to thank Laura Matusevich for several interesting conversations and remarks on a preliminary version of the paper. I am very grateful to the referee of this paper who suggested some significant improvements in the important Sects. 3 and 4 of the paper.

References

1. Adolphson, A.: Hypergeometric functions and rings generated by monomials. Duke Math. J. **73**, 269–290 (1994)
2. Beazley Cohen, P., Wolfart, J.: Algebraic Appell-Lauricella functions. Analysis **12**, 359–376 (1992)
3. Beukers, F., Heckman, G.: Monodromy for the hypergeometric function ${}_nF_{n-1}$. Invent. Math. **95**, 325–354 (1989)
4. Dwork, B.: Generalized Hypergeometric Functions. Oxford Mathematical Monographs. Oxford University Press, London (1990)
5. Dwork, B., Loeser, F.: Hypergeometric series. Jpn. J. Math. (N.S.) **19**, 81–129 (1993)
6. Gelfand, I.M., Graev, M.I., Zelevinsky, A.V.: Holonomic systems of equations and series of hypergeometric type. Dokl. Akad. Nauk SSSR **295**, 14–19 (1987) (in Russian)
7. Gelfand, I.M., Zelevinsky, A.V., Kapranov, M.M.: Equations of hypergeometric type and Newton polytopes. Dokl. Akad. Nauk SSSR **300**, 529–534 (1988) (in Russian)
8. Gelfand, I.M., Zelevinsky, A.V., Kapranov, M.M.: Hypergeometric functions and toric varieties. Funkt. Anal. Prilozhen. **23**, 12–26 (1989) (in Russian)
9. Gelfand, I.M., Kapranov, M.M., Zelevinsky, A.V.: Generalized Euler integrals and A-hypergeometric functions. Adv. Math. **84**, 255–271 (1990)
10. Kato, M.: Appell's F_4 with finite irreducible monodromy group. Kyushu J. Math. **51**, 125–147 (1997)
11. Kato, M.: Appell's hypergeometric systems F_2 with finite irreducible monodromy groups. Kyushu J. Math. **54**, 279–305 (2000)
12. Katz, N.M.: Algebraic solutions of differential equations (p-curvature and the Hodge filtration). Invent. Math. **18**, 1–118 (1972)
13. Katz, N.M.: A conjecture in the arithmetic theory of differential equations. Bull. SMF **110**, 203–239 (1982). Corrections on pp. 347–348
14. Kita, M.: On hypergeometric functions in several variables. I. New integral representations of Euler type. Jpn. J. Math. (N.S.) **18**, 25–74 (1992)
15. Matusevich, L.F., Miller, E., Walther, U.: Homological methods for hypergeometric families. J. Am. Math. Soc. **18**, 919–941 (2005)

16. Sasaki, T.: On the finiteness of the monodromy group of the system of hypergeometric differential equations (F_D). J. Fac. Sci. Univ. Tokyo Sect. IA Math. **24**, 565–573 (1977)
17. Schwarz, H.A.: Über diejenigen Fälle, in welchen die Gaussische hypergeometrische reihe eine algebraische Funktion ihres vierten Elements darstellt. J. Reine Angew. Math. **75**, 292–335 (1873)
18. Yoshida, M.: Hypergeometric Functions, My Love. Aspects of Mathematics, vol. 32. Vieweg, Wiesbaden (1997)