Journal of Pure and Applied Algebra ∎ (∎∎∎∎) ∎∎∎–∎∎∎

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

Dold–Kan correspondence for dendroidal abelian groups

Javier J. Gutiérrez [a], Andor Lukacs [b], Ittay Weiss [b,*]

[a] Centre de Recerca Matemàtica, Apartat 50, 08193 Bellaterra (Barcelona), Spain

[b] Mathematisch Instituut, Postbus 80.010, 3508 TA Utrecht, The Netherlands

ARTICLE INFO

Article history:
Received 12 October 2009
Received in revised form 25 May 2010
Available online xxxx
Communicated by J. Adámek

MSC:
Primary: 55U05; 18G30
Secondary: 18D50

ABSTRACT

We prove a Dold–Kan type correspondence between the category of planar dendroidal abelian groups and a suitably constructed category of planar dendroidal chain complexes. Our result naturally extends the classical Dold–Kan correspondence between the category of simplicial abelian groups and the category of non-negatively graded chain complexes.

1. Introduction

The classical Dold–Kan correspondence [5,7] states that there is an equivalence between the category $s\mathcal{A}b$ of simplicial abelian groups and the category $\mathcal{C}h$ of non-negatively graded chain complexes of abelian groups. This equivalence is given by the normalization functor N_s that sends a simplicial abelian group A to the chain complex

$$N_s(A)(n) = \bigcap_{i=0}^{n-1} \ker(\partial_i^*),$$

where $\partial_i^* : A_n \longrightarrow A_{n-1}$ are the induced face maps, and the differential is defined by $d = (-1)^n \partial_n^*$. Moreover, the functor N_s has a right adjoint Γ_s such that the adjoint pair

$$N_s : s\mathcal{A}b \rightleftarrows \mathcal{C}h : \Gamma_s$$

is an adjoint equivalence. There exist several generalizations of this correspondence which characterize simplicial objects in more structured algebraic categories in terms of the appropriate algebraic chain objects [13,14].

In this paper, we extend this result to the framework of planar dendroidal sets. The category of planar dendroidal sets is a presheaf category on a certain category of trees Ω_p. Dendroidal sets generalize simplicial sets in a way suitable for studying the homotopy theory of coloured operads and their algebras [11]. In the context of non-symmetric coloured operads the role of dendroidal sets is replaced by planar dendroidal sets. The idea behind the notion of (planar) dendroidal sets is that in the same way as simplicial sets help us understanding categories via the nerve functor, there should be an analogous notion for studying (non-symmetric) coloured operads as generalization of categories. Indeed, much of the fundamentals of simplicial sets that relate to category theory extend to (planar) dendroidal sets. In [12], Moerdijk and the third-named author develop the theory of inner Kan complexes in the category of dendroidal sets. Inner Kan complexes in the category of

* Corresponding author.
E-mail addresses: JGutierrez@crm.cat (J.J. Gutiérrez), A.Lukacs@uu.nl (A. Lukacs), I.Weiss@uu.nl, weittay@gmail.com (I. Weiss).

0022-4049/$ – see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.jpaa.2010.10.003

simplicial sets were first introduced by Boardman and Vogt in [3]. In a later paper [6] Joyal began a reexamination of inner Kan complexes under the name quasi-categories. One of the results of his research is the establishment of a Quillen model category structure on simplicial sets in which the fibrant objects are precisely the inner Kan complexes. The same model structure was also obtained by Lurie in his work on higher topos theory [9]. In [4] Cisinski and Moerdijk construct a Quillen model structure on the category of dendroidal sets in which the fibrant objects are precisely the inner Kan complexes. The results in this paper add to the above mentioned theory by showing that the Dold–Kan correspondence similarly extends to planar dendroidal sets.

There is a fully faithful embedding $i\colon \Delta \longrightarrow \Omega_p$, where Δ denotes the simplicial category, inducing an adjoint pair

$$i_!\colon s\mathcal{A}b \rightleftarrows d\mathcal{A}b\colon i^*,$$

where $d\mathcal{A}b$ denotes the category of planar dendroidal abelian groups. From now on 'dendroidal' will mean 'planar dendroidal'.

The category of chain complexes is not big enough to define a Dold–Kan correspondence for the category of dendroidal abelian groups. In order to solve this problem, we introduce the category of dendroidal chain complexes $d\mathcal{C}h$. The category of dendroidal chain complexes is a category of Ω_p-graded abelian groups together with structure maps induced by the face maps in Ω_p and satisfying certain conditions. In the same way as the category of dendroidal abelian groups extends the category of simplicial abelian groups, the category of dendroidal chain complexes extends the category of chain complexes, i.e., there are adjoint functors

$$j_!\colon \mathcal{C}h \rightleftarrows d\mathcal{C}h\colon j^*,$$

where j^* is the restriction functor and its left adjoint $j_!$ is 'extension by zero'. However, there is an important difference between the categories Δ and Ω_p that leads to a more complicated definition of $d\mathcal{C}h$ than expected: roughly speaking, there are fewer degeneracies in Ω_p than in Δ. (We explain this subtlety in detail in Remark 6.2.)

We define the normalization functor $N\colon d\mathcal{A}b \longrightarrow d\mathcal{C}h$ and a right adjoint Γ and prove that they form an adjoint equivalence of categories. We also show that there is a commutative diagram of adjoint functors

$$\begin{array}{ccc} s\mathcal{A}b & \underset{i^*}{\overset{i_!}{\rightleftarrows}} & d\mathcal{A}b \\ N_s \downarrow\uparrow \Gamma_s & & N \downarrow\uparrow \Gamma \\ \mathcal{C}h & \underset{j^*}{\overset{j_!}{\rightleftarrows}} & d\mathcal{C}h, \end{array}$$

relating the classical Dold–Kan correspondence with the dendroidal one.

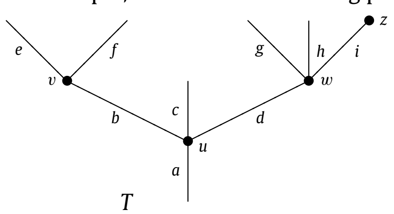
2. A formalism of trees

A *tree* is a connected finite graph with no loops. A vertex in a graph is called *outer* if it has only one edge attached to it. All the trees we will consider are *rooted trees*, i.e., equipped with a distinguished outer vertex called the *output* and a (possibly empty) set of outer vertices (not containing the output vertex) called the set of *inputs*.

When drawing trees, we will delete the output and input vertices from the picture. From now on, the term 'vertex' in a tree will always refer to a remaining vertex. Given a tree T, we denote by $V(T)$ the set of vertices of T and by $E(T)$ the set of edges of T.

The edges attached to the deleted input vertices are called *input edges* or *leaves*; the edge attached to the deleted output vertex is called *output edge* or *root*. The rest of the edges are called *inner edges*. The root induces an obvious direction in the tree, 'from the leaves towards the root'. If v is a vertex of a finite rooted tree, we denote by $\mathrm{out}(v)$ the unique outgoing edge and by $\mathrm{in}(v)$ the set of incoming edges (note that $\mathrm{in}(v)$ can be empty). The cardinality of $\mathrm{in}(v)$ is called the *valence of* v, the element of $\mathrm{out}(v)$ is the *output of* v and the elements of $\mathrm{in}(v)$ are the *inputs of* v.

As an example, consider the following picture of a tree T:



(1)

The output vertex at the edge a and the input vertices at e, f, c, g and h have been deleted. This tree has four vertices u, v, w and z of respective valences 3, 2, 3 and 0. It also has five input edges or leaves, namely e, f, c, g and h. The edges b and d are inner edges and the edge a is the root.

Please cite this article in press as: J.J. Gutiérrez, et al., Dold–Kan correspondence for dendroidal abelian groups, Journal of Pure and Applied Algebra (2010), doi:10.1016/j.jpaa.2010.10.003

Since every planar representation of a rooted tree comes naturally with an ordering of the inputs of any given vertex (from left to right), we can give the following definition:

Definition 2.1. A *planar rooted tree* is a rooted tree T together with a linear ordering of $\mathrm{in}(v)$ for each vertex v of T.

In the rest of the paper we will work with *planar rooted trees*, unless otherwise stated, and for the sake of bookkeeping whenever it is obvious from the context we will refer to them as trees.

3. Coloured operads and the dendroidal category

The dendroidal category Ω was introduced in [11,16] as an extension of the simplicial category Δ. The category Ω is a category of trees. Its objects are (non-planar) rooted trees and the set of morphisms between two trees is given by the set of maps between the symmetric coloured operads associated to each of them. The presheaves on Ω, called *dendroidal sets*, are very useful for the study of operads and their algebras in the framework of homotopy theory. Since the terminology of dendroidal sets is recent, we will recall all the needed parts of it here.

In this section, we are going to describe a variation on the category Ω, called the *planar dendroidal category*, which we denote by Ω_p, whose objects are the planar rooted trees. More concretely, let $P\colon \Omega^{\mathrm{op}} \longrightarrow \mathcal{S}ets$ be the presheaf on Ω that sends each tree to its set of planar structures. Then $P(T)$ is a torsor under $\mathrm{Aut}(T)$ for every tree T, where $\mathrm{Aut}(T)$ denotes the set of automorphisms of T, and Ω_p is equal to the category of elements Ω/P, whose objects are pairs (T, x), with $x \in P(T)$, and a morphism between two objects (T, x) and (S, y) is given by a morphism $f\colon T \longrightarrow S$ in Ω such that $P(f)(y) = x$.

In order to have a better understanding of the morphisms in Ω_p we need the notion of coloured operad. For us all coloured operads come *without* an action of the symmetric group. Usually they are referred to as non-symmetric coloured operads or non-symmetric multicategories in the literature (see, for example, [1,8]).

3.1. Coloured operads

A coloured operad P consists of a set of *colours*, denoted by $\mathrm{clr}(P)$, together with a set of *operations* $P(c_1, \ldots, c_n; c)$ for every $n \geq 0$ and each ordered $(n+1)$-tuple of colours $(c_1, \ldots, c_n; c)$, and a distinguished operation id_c in $P(c; c)$ for every colour c, called the *identity on c*. These operations are related by means of *composition product* maps $\circ_i$

$$\begin{array}{c} P(c_1, \ldots, c_i, \ldots, c_n; c) \times P(a_1, \ldots, a_m; c_i) \\ \downarrow{\scriptstyle \circ_i} \\ P(c_1, \ldots, c_{i-1}, a_1, \ldots, a_m, c_{i+1}, \ldots, c_n; c) \end{array}$$

for every $c_1, \ldots, c_n, c$ and $a_1, \ldots, a_m$ in $\mathrm{clr}(P)$, and every $1 \leq i \leq n$. The composition product maps are subject to the usual associativity and unitary compatibility relations; see, for example, [8]. If $u \in P(c_1, \ldots, c_n; c)$ is an operation then the sequence $(c_1, \ldots, c_n)$ is called the *input* of u and c is called the *output* of u.

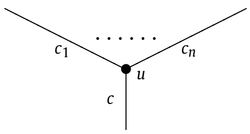
A *map of coloured operads* $\varphi\colon P \longrightarrow Q$ consists of a map $\mathrm{clr}(P) \longrightarrow \mathrm{clr}(Q)$ between the colours and maps $P(c_1, \ldots, c_n; c) \longrightarrow Q(\varphi(c_1), \ldots, \varphi(c_n); \varphi(c))$ on the operations, such that φ sends units to units and is compatible with the composition product of operations. We denote by $\mathcal{O}per$ the category of coloured operads.

Coloured operads generalize small categories. Any small category $\mathcal{C}$ can be viewed as a coloured operad $j_!(\mathcal{C})$, where $\mathrm{clr}(j_!(\mathcal{C}))$ is precisely the set of objects of $\mathcal{C}$ and the only operations are $j_!(\mathcal{C})(A; B) = \mathcal{C}(A, B)$. There is a pair of adjoint functors

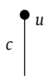
$$j_!\colon \mathcal{C}at \rightleftarrows \mathcal{O}per\colon j^*$$

where $\mathcal{C}at$ denotes the category of small categories. The left adjoint $j_!$ is full and faithful. The right adjoint j^* sends a coloured operad P to the category $j^*(P)$ whose objects are the colours of P and whose morphisms are the unary operations of P.

In a similar way we can think of operations as generalizations of the notion of morphisms in categories. If P is a coloured operad, a suitable intuitive way to depict an operation $u \in P(c_1, \ldots, c_n; c)$ is to draw the tree

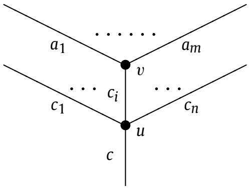


Note that the case $n = 0$ is allowed in the definition of operations, thus there are operations $u \in P(\,; c)$ with no input and one output. Operations of this type can be thought of as constants, and are represented pictorially as:

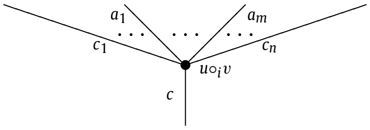


Please cite this article in press as: J.J. Gutiérrez, et al., Dold–Kan correspondence for dendroidal abelian groups, Journal of Pure and Applied Algebra (2010), doi:10.1016/j.jpaa.2010.10.003

To have a good intuition of the nature of the associativity relations of the composition product, we can use pictures of trees. A composition product map $\circ_i$ can be thought of as taking two operations u, v as its inputs, depicted like a tree with two vertices



and producing a new operation by 'grafting' the edge c_i (the sole inner edge of the tree) as follows



The associativity relation states that whenever we have three operations depicted on a tree with three vertices (thus with two inner edges) then grafting both of the inner edges does not depend on the chosen order.

The definition we gave for the composition product of a coloured operad is not the common one in the literature, but it is equivalent to it (see the same definition in [10] for the one-colour case, or an alternative definition for the general case in [1,8,11]).

3.2. The planar dendroidal category

Any tree T gives rise to a (non-symmetric) coloured operad which we denote by $\Omega_p(T)$. The set of colours is the set of edges of T and the operations of $\Omega_p(T)$ are freely generated by the vertices of T. That is, if v is a vertex with the ordered sequence of input edges $(c_1, c_2, \dots, c_n)$ and output edge c then $\Omega_p(T)(c_1, \dots, c_n; c) = \{v\}$. All the other operations are identities or compositions of the previous ones. As a consequence, any set of operations is either empty or it contains only one element.

For example, if T is the tree depicted in (1), then there are four generating operations: $\Omega_p(T)(e, f; b) = \{v\}$, $\Omega_p(T)(b, c, d; a) = \{u\}$, $\Omega_p(T)(g, h, i; d) = \{w\}$ and $\Omega_p(T)(; i) = \{z\}$. For the other operations, we have $\Omega_p(T)(e, f, c, d; a) = \{u \circ_b v\}$ and so on. Observe that the b in $\circ_b$ refers to the position of b in the ordered sequence of input edges of u, i.e., $\circ_b = \circ_1$, and in general this notation is not ambiguous since for $n \neq 1$, $\Omega_p(T)(c_1, \dots, c_n; c) = \emptyset$ if $c_i = c_j$ for some $i \neq j$ or $c_i = c$. We will keep using this notation in what follows.

Now, consider the category whose objects are planar rooted trees and whose morphisms $R \longrightarrow T$ are given by coloured operad maps $\Omega_p(R) \longrightarrow \Omega_p(T)$. Note that if $\Omega_p(R) \longrightarrow \Omega_p(T)$ is an isomorphism, then the non-symmetric operad structures imply that R and T have the same planar shape and they differ only by the names of their vertices and edges. To avoid set-theoretic difficulties we proceed as follows. We fix a countable set of symbols and assume that the edges and vertices of all our trees are in the set. We then consider the equivalence relation on such trees, in which $T \sim S$ precisely when there exists an isomorphism from $\Omega_p(T)$ to $\Omega_p(S)$. Notice that when two trees are in the same equivalence class there is a unique such isomorphism. This fact can be used in all of the constructions below to establish independence of representatives, a detail which we will further omit.

Definition 3.1. The *planar dendroidal category* Ω_p is the category whose objects are isomorphism classes of planar rooted trees and the morphisms are given as follows. For any class $[R]$ choose a representative of that class $R_0 \in [R]$. Define

$$\Omega_p([R], [T]) = \mathcal{O}per(\Omega_p(R_0), \Omega_p(T_0))$$

for every two classes $[R]$ and $[T]$ in Ω_p.

In order to simplify the notation, we will omit mentioning isomorphism classes and we will write T instead of $[T]$.

The simplicial category Δ can be viewed as the full subcategory of $\mathcal{C}at$, the category of small categories, spanned by $\{[n] \mid n \geq 0\}$, where $[n]$ is the category whose objects are $\{0, 1, \dots, n\}$ and for $0 \leq i, j \leq n$ there is only one arrow $i \longrightarrow j$ if $i \leq j$. The category Ω_p extends the category Δ. Indeed, if we denote by L_n the linear tree with n vertices and $n + 1$ edges, then the simplicial category can be identified with the full subcategory of Ω_p consisting of linear trees as objects by means of a functor

$$i\colon \Delta \longrightarrow \Omega_p \tag{2}$$

sending $[n]$ to L_n, which is a full and faithful embedding, since $j_!([n]) \cong \Omega_p(L_n)$.

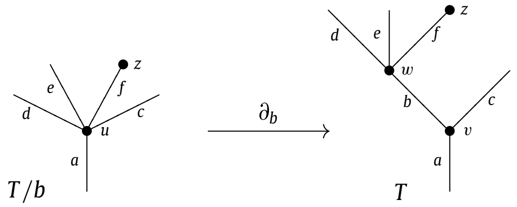
Observe that there is a canonical order on the edges of a linear tree by numbering them in increasing order from bottom to top. Whenever we speak of an order on the edges of a linear tree we will be referring to this order.

The morphisms in Ω_p are generated by two types of maps called *elementary faces* and *elementary degeneracies*, which we discuss in the following sections.

Please cite this article in press as: J.J. Gutiérrez, et al., Dold–Kan correspondence for dendroidal abelian groups, Journal of Pure and Applied Algebra (2010), doi:10.1016/j.jpaa.2010.10.003

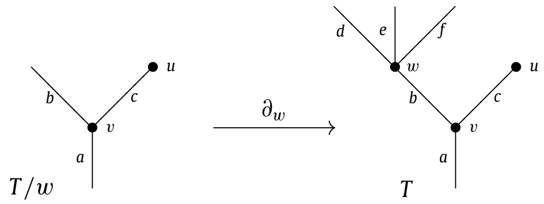
3.3. Elementary face maps

Suppose that T is (a representative of) an object of Ω_p and b is an inner edge of T, as in (3) below. Denote by T/b the tree obtained from T by contracting b. Corresponding to this operation the *elementary face map* $\partial_b : T/b \longrightarrow T$ is the inclusion on the colours of $\Omega_p(T/b)$ and on the generating operations of $\Omega_p(T/b)$, except for the operation u, which is sent to $v \circ_b w$. The elementary face maps ∂_b associated to inner edges in such a way are called *inner faces* of T.



(3)

Now suppose that T is (a representative of) an object of Ω_p and w is a vertex of T with exactly one inner edge attached to it as in (4). It follows that if we remove from T the vertex w and all the outer edges attached to it, we obtain a new tree T/w. Associated to this operation, the elementary face map $\partial_w : T/w \longrightarrow T$ is the inclusion both on the colours and on the generating operations of $\Omega_p(T/w)$. Maps of this type are called *outer faces* of T.

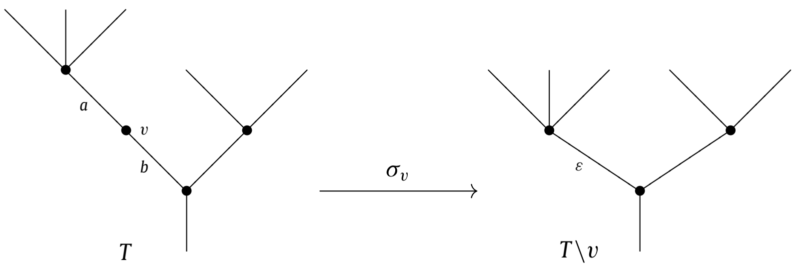


(4)

Note that the possibility of removing the root vertex of T is included in this definition. This situation can happen only if the root vertex is attached to exactly one inner edge, thus not every tree T has an outer face induced by removing its root. There is another particular situation which requires special attention, namely the inclusion of the tree with no vertices, denoted by η, into a tree with one vertex, called a *corolla*. In this case if the corolla has n leaves we get $n + 1$ elementary face maps. The operad $\Omega_p(\eta)$ consists of only one colour and the identity operation on it. Then, a map of operads $\Omega_p(\eta) \longrightarrow \Omega_p(T)$ is just a choice of an edge of T.

3.4. Elementary degeneracy maps

Suppose that T is (a representative of) an object of Ω_p and v is a vertex of T with valence one. Let a be the sole incoming edge and b the outgoing edge of v. We obtain a new tree $T\backslash v$ by removing v from T and identifying a with b (in (5) below we refer to this new edge as ϵ). There is a map $\sigma_v : T \longrightarrow T\backslash v$ in Ω_p associated to this operation which sends the colours a and b of $\Omega_p(T)$ to ϵ, sends the generating operation v to id_ϵ and is the identity on the other colours and operations. The maps of this type are called the *elementary degeneracies* of T.



(5)

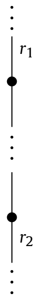
A non-empty composition of elementary face maps will be referred to as a *face map* and similarly, a non-empty composition of elementary degeneracies as a *degeneracy*. An important fact about elementary faces and elementary degeneracies is that they generate the maps in Ω_p, henceforth we will refer to these maps as *generators*. We have the following decomposition result, which is also a direct consequence of [16, Theorem 2.3.27].

Lemma 3.2. *Every map $f : R \longrightarrow T$ in Ω_p is either the identity or it decomposes uniquely as $f = d \circ s$, where d is a face map and s is a degeneracy.*

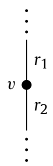
Proof. To prove the existence of the decomposition, we proceed by induction on $n = |V(R)| + |V(T)|$, the total number of vertices of R and T. If $n = 0$ then $f : | \longrightarrow |$ is the identity map and the statement is obvious. The function $E(f)\colon E(R) \longrightarrow E(T)$ has a unique factorization as an epimorphism followed by a monomorphism

$$E(R) \xrightarrow{s_0} \!\!\!\twoheadrightarrow X \rightarrowtail^{d_0} E(T) .$$

First, suppose that there exist $r_1 \neq r_2 \in E(R)$ such that $s_0(r_1) = s_0(r_2)$. Since f is a map of operads, r_1 and r_2 must be situated one above the other in a linear branch of R:



such that any edge r between them satisfies $s_0(r) = s_0(r_1) = s_0(r_2)$. Hence we can suppose that r_1 and r_2 are adjacent, joined at the vertex v:



It follows that f decomposes as $R \overset{\sigma_v}{\twoheadrightarrow} R\backslash v \overset{f'}{\rightarrowtail} T$ and by the inductive hypothesis we already have a decomposition $R\backslash v \twoheadrightarrow S \rightarrowtail T$ of f'.

Second, suppose that s_0 is bijective, hence we can assume that s_0 is the identity map. If d_0 is also the identity, it follows that f has to be the identity too. Indeed, since we are working with non-symmetric operads, f preserves the order of the incoming edges at every vertex, and if $v \in \Omega_p(R)$ is a generator (a vertex of R) and $f(v)$ is not a generator of $\Omega_p(T)$ then there would be edges of T with no preimage in R.

If d_0 is not the identity then let $e \in E(T)$ be an edge skipped by d_0. We can distinguish two cases:

If e is an inner edge of T, it follows that f decomposes as

$$R \xrightarrow{f'} T/e \overset{\partial_e}{\rightarrowtail} T .$$

By the inductive hypothesis we obtain a decomposition of f'.

Now suppose that e is an outer edge of T, skipped by d_0. Since f is a map of operads, again any other outer edge adjacent to e has to be skipped by d_0. Denote the vertex adjacent to e by v. It follows that we can again decompose f as

$$R \xrightarrow{f'} T/v \overset{\partial_v}{\rightarrowtail} T$$

and obtain the desired factorization of f by induction.

To prove the uniqueness of the decomposition we proceed in the following way. Suppose that there are two factorizations of f:

$$R \overset{s}{\twoheadrightarrow} S \overset{d}{\rightarrowtail} T \quad \text{and} \quad R \overset{s'}{\twoheadrightarrow} S' \overset{d'}{\rightarrowtail} T .$$

Looking at the decompositions only on the level of the edges, it follows that $E(S) = E(S')$, $\mathrm{clr}(s) = \mathrm{clr}(s')$ and $\mathrm{clr}(d) = \mathrm{clr}(d')$. (Here $\mathrm{clr}(-)$ denotes the map at the level of colours of the associated morphism of coloured operads.) Moreover, since s is a degeneracy, it follows that if v is a generator of $\Omega_p(R)$ then $s(v) = v$ or $s(v)$ is the identity on some edge, in which case it is completely determined by $\mathrm{clr}(s)$. Hence $s = s'$ and also $S = S'$. Similarly, d is also completely determined by what it does on the colours, thus $d = d'$. □

Remark 3.3. An extension of Lemma 3.2 also holds, namely that the maps d and s also decompose uniquely into some naturally ordered sequence of elementary faces and elementary degeneracies, respectively. In Section 4.4 we indicate how this is done. Note that this implies the existence of a Reedy structure on Ω_p, extending the standard one on Δ (see [2]).

Proposition 3.4. *Let T be a tree in Ω_p. Then, the elementary faces of T are exactly those monic operad maps $\Omega_p(R) \longrightarrow \Omega_p(T)$ for which $|V(T)| = |V(R)| + 1$ and the elementary degeneracies of T are exactly those epic operad maps $\Omega_p(T) \longrightarrow \Omega_p(S)$ for which $|V(T)| = |V(S)| + 1$.*

Proof. We prove only the assertion for the faces, the other statement can be proved similarly. Let $f: \Omega_p(R) \longrightarrow \Omega_p(T)$ be a monic operad map with the required property and suppose that it is not an elementary face. By Lemma 3.2 we know that f can be decomposed as

$$R \xrightarrow{\ s\ } R' \xrightarrow{\ d\ } T,$$

where s is a degeneracy and d is a face. Note that s cannot be the identity, since counting the vertices would imply that $f = d$ is an elementary face in that case. It follows that s, and hence f as well, are not injective on the edges. This is a contradiction, since a monic operad map has to be injective on the colours. □

4. Dendroidal identities

In this section we are going to make explicit the relations between the elementary faces and degeneracies of Ω_p. We present the relations in two different ways in this section, since both of these descriptions can be useful when reasoning with these maps as generators, as we will see later.

The first description uses the already familiar notation for elementary faces and degeneracies indexing the maps by edges and vertices of trees. The second way, described in Section 4.4 is based on natural linear orders defined on the set of elementary faces and the set of elementary degeneracies of a given tree.

The relations we present are called the *dendroidal relations* and they generalize the simplicial identities in the category Δ, henceforth we will call them *dendroidal identities*. The unique epi-mono factorization theorem for maps in the category Δ extends to the category Ω_p, thus the relations we consider below cover indeed all the cases: one only has to look at all possible decompositions $f = g_1 \circ g_2$ into two generators of Ω_p and see what are the other ways to decompose f into two generators. The result is summarized in Lemma 4.1.

We do not include in our first description the special case involving faces of the n-corolla, $n \geq 2$, although a statement similar to Lemma 4.1 can be given.

There is a little ambiguity in the language that follows. For example ∂_e can refer to two different face maps, but it is always clear from the context which one we are talking about.

4.1. Elementary face relations

Let $\partial_a : T/a \longrightarrow T$ and $\partial_b : T/b \longrightarrow T$ be two inner faces of T. It follows that the inner faces $\partial_a : (T/b)/a \longrightarrow T/b$ and $\partial_b : (T/a)/b \longrightarrow T/a$ exist, $(T/a)/b = (T/b)/a$ and that the following diagram commutes:

$$\begin{array}{ccc} (T/a)/b & \xrightarrow{\partial_b} & T/a \\ {\scriptstyle\partial_a}\downarrow & & \downarrow{\scriptstyle\partial_a} \\ T/b & \xrightarrow{\partial_b} & T. \end{array}$$

Let $\partial_v : T/v \longrightarrow T$ and $\partial_w : T/w \longrightarrow T$ be two outer faces of T. Then the outer faces $\partial_w : (T/v)/w \longrightarrow T/v$ and $\partial_v : (T/w)/v \longrightarrow T/w$ also exist, $(T/v)/w = (T/w)/v$ and the following diagram commutes:

$$\begin{array}{ccc} (T/v)/w & \xrightarrow{\partial_w} & T/v \\ {\scriptstyle\partial_v}\downarrow & & \downarrow{\scriptstyle\partial_v} \\ T/w & \xrightarrow{\partial_w} & T. \end{array}$$

The last remaining case is when we compose an inner face with an outer one in any order. There are several possibilities, in all of them suppose that $\partial_v : T/v \longrightarrow T$ is an outer face and $\partial_e : T/e \longrightarrow T$ is an inner face.

(i) If the inner edge e is not adjacent in T to the vertex v, then the outer face $\partial_v : (T/e)/v \longrightarrow T/e$ and inner face $\partial_e : (T/v)/e \longrightarrow T/v$ exist, $(T/e)/v = (T/v)/e$ and the following diagram commutes:

$$\begin{array}{ccc} (T/v)/e & \xrightarrow{\partial_e} & T/v \\ {\scriptstyle\partial_v}\downarrow & & \downarrow{\scriptstyle\partial_v} \\ T/e & \xrightarrow{\partial_e} & T. \end{array}$$

(ii) Suppose that the inner edge e is adjacent in T to the vertex v and denote the other adjacent vertex to e by w. Following the notation of Section 3.2, v and w contribute to T/e a vertex $v \circ_e w$ or $w \circ_e v$. Let us denote this vertex by z. Notice that the outer face $\partial_z : (T/e)/z \longrightarrow T/e$ exists if and only if the outer face $\partial_w : (T/v)/w \longrightarrow T/v$ exists and in this case $(T/e)/z = (T/v)/w$. Moreover, the following diagram commutes:

$$\begin{array}{ccccc} (T/v)/w & = & (T/e)/z & \xrightarrow{\partial_z} & T/e \\ \downarrow{\scriptstyle \partial_w} & & & & \downarrow{\scriptstyle \partial_e} \\ T/v & & \xrightarrow{\partial_v} & & T. \end{array}$$

It follows that we can write $\partial_v \partial_w = \partial_e \partial_z$ where $z = v \circ_e w$ if v is 'closer' to the root of T or $z = w \circ_e v$ if w is 'closer' to the root of T.

4.2. Elementary degeneracy relations

Let $\sigma_v : T \longrightarrow T\backslash v$ and $\sigma_w : T \longrightarrow T\backslash w$ be two degeneracies of T. Then the degeneracies $\sigma_v : T\backslash w \longrightarrow (T\backslash w)\backslash v$ and $\sigma_w : T\backslash v \longrightarrow (T\backslash v)\backslash w$ exist, $(T\backslash v)\backslash w = (T\backslash w)\backslash v$ and the following diagram commutes:

$$\begin{array}{ccc} T & \xrightarrow{\sigma_v} & T\backslash v \\ \downarrow{\scriptstyle \sigma_w} & & \downarrow{\scriptstyle \sigma_w} \\ T\backslash w & \xrightarrow{\sigma_v} & (T\backslash v)\backslash w. \end{array}$$

4.3. Combined relations

Let $\sigma_v : T \longrightarrow T\backslash v$ be a degeneracy and $\partial : T' \longrightarrow T$ a face map such that $\sigma_v : T' \longrightarrow T'\backslash v$ makes sense (i.e., T' still contains v and its two adjacent edges as a subtree). Then, there exists an induced face map $\partial : T'\backslash v \longrightarrow T\backslash v$, determined by the same vertex or edge as $\partial : T' \longrightarrow T$. Moreover, the following diagram commutes:

$$\begin{array}{ccc} T & \xrightarrow{\sigma_v} & T\backslash v \\ \uparrow{\scriptstyle \partial} & & \uparrow{\scriptstyle \partial} \\ T' & \xrightarrow{\sigma_v} & T'\backslash v. \end{array} \tag{6}$$

Let $\sigma_v : T \longrightarrow T\backslash v$ be a degeneracy and $\partial : T' \longrightarrow T$ be a face map induced by one of the adjacent edges to v or the removal of v, if that is possible. It follows that $T' = T\backslash v$ and the composition

$$T\backslash v \xrightarrow{\partial} T \xrightarrow{\sigma_v} T\backslash v \tag{7}$$

is the identity map $\mathrm{id}_{T\backslash v}$.

All these relations between the generators of the maps in Ω_p are summarized in the following lemma whose proof is a direct consequence of the dendroidal identities above.

Lemma 4.1. *Let $f : R \longrightarrow T$ be the composite of two generators $f = g_1 \circ g_2$, where both R and T have at least one vertex. If $f \neq \mathrm{id}$, then there is exactly one more way to write f as the composition of two generators $f = g_1' \circ g_2'$, where $\{g_1, g_2\} \neq \{g_1', g_2'\}$ as sets. It follows that we obtain a commutative diagram*

$$\begin{array}{ccc} R & \xrightarrow{g_2} & S \\ \downarrow{\scriptstyle g_2'} & & \downarrow{\scriptstyle g_1} \\ S' & \xrightarrow{g_1'} & T \end{array}$$

which is a special case of one of the diagrams of the dendroidal identities listed above.

If $f = \mathrm{id}$, then $g_1 = \sigma_v$ for some vertex v and $g_2 = \partial$ is one of the two possible face maps induced by an edge adjacent to v (or v itself in some cases). □

4.4. Linear orders on elementary faces and degeneracies

Since the trees we consider are planar, we can canonically define a linear order on the set of elementary faces of any chosen tree. Similarly, a canonical linear order can be defined on the set of elementary degeneracies of a tree. We will treat the case of the corollas separately. This order will allow one to translate the dendroidal identities discussed above into corresponding identities that are closely related to the classical simplicial relations, as we discuss below.

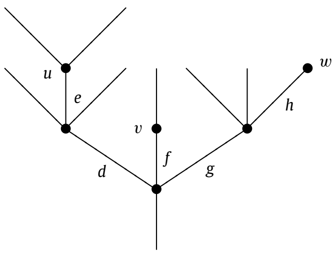
Let T be a tree in Ω_p such that $|V(T)| \geq 2$. Assign to each elementary face of T a natural number, respecting the following rules:

(i) If the vertex above the root $r \in V(T)$ is outer then assign the number 0 to ∂_r.
(ii) Starting from the root vertex, walk through all the edges and vertices of T by going always first to the left and upwards. When this is not possible any more, turn back to the closest, already visited vertex and choose the next, not yet covered edge left and upwards.
(iii) Whenever an inner edge or an outer vertex is visited, assign the smallest not yet used natural number to the corresponding elementary face of T.

Suppose that T has n elementary face maps. The process described above defines a bijection

$$\phi\colon \{\partial \mid \partial \text{ is an elemetary face of } T\} \longrightarrow \{0, 1, \ldots, n-1\},$$

hence also an order on the set of elementary face maps of T. We define the i-th elementary face of T by $\partial_i := \phi^{-1}(i)$. For example, if T is the tree



(8)

then T has eight elementary faces and $\partial_0 = \partial_d$, $\partial_1 = \partial_e$, $\partial_2 = \partial_u$, $\partial_3 = \partial_f$, $\partial_4 = \partial_v$, $\partial_5 = \partial_g$, $\partial_6 = \partial_h$ and $\partial_7 = \partial_w$.

We can use this convention on traversing the tree T to obtain another bijection

$$\rho\colon \{\sigma \mid \sigma \text{ is an elementary degeneracy of } T\} \longrightarrow \{0, 1, \ldots, m-1\},$$

provided T has $m \geq 1$ elementary degeneracies. For example, in the case of the tree (8) drawn above $m = 1$ and $\sigma_0 = \sigma_v$.

In case T is the n-corolla, the process of traversing T from left to right induces a linear order on the set of elementary faces of T as well. After renaming these faces accordingly, we observe that ∂_0 is the inclusion of the trivial tree η into the root of T, ∂_1 is the inclusion of η into the leftmost leaf of T, and so on.

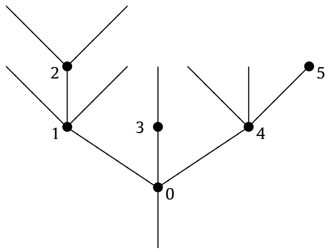
Remark 4.2. The linear orders defined above extend the linear orders obtained from the usual numbering of faces and degeneracies in the simplicial category Δ. Indeed, if $T = L_n$ is the linear tree with n vertices, then ∂_i and σ_i defined above correspond to the simplicial ones with the same index.

One can ask wether the dendroidal identities remain the same as the simplicial ones with respect to the linear orders. This is certainly true for the elementary degeneracy relations. Indeed, after renaming the maps of any commutative diagram with degeneracies as in Section 4.2, the relation becomes $\sigma_j\sigma_i = \sigma_i\sigma_{j+1}$ for some $i \leq j$.

On the other hand, the other types of elementary relations do not remain valid. In the case of combined relations this fails because there can be fewer degeneracies of a tree than faces. In the case of elementary face relations, the tree pictured in (8) provides a counterexample since the relation $\partial_f\partial_g = \partial_g\partial_f$ translates as $\partial_3\partial_3 = \partial_5\partial_3$. We can obtain other counterexamples by considering the faces of the n-corolla. We observe that such a situation can occur since some trees T have the property that the domain R of an elementary face $\partial\colon R \longrightarrow T$ has two elementary faces less than T. In general, there are two possibilities for the elementary face relations: $\partial_i\partial_{j-1} = \partial_j\partial_i$ or $\partial_i\partial_{j-2} = \partial_j\partial_i$ when $i < j$.

4.5. A sign convention for elementary faces

To any elementary face map $\partial : T \longrightarrow R$ in Ω_p we can associate a sign $\mathrm{sgn}(\partial) = \pm 1$. We begin by numbering the vertices of R from 0 to n, starting with the root-vertex and traversing the tree by going always first to the left. In this way we obtain a bijection $\sharp : V(R) \longrightarrow \{0, \ldots, n\}$ (see the next picture for an example).

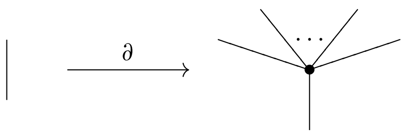


The sign of an elementary face map ∂ is computed by the following rules:

(i) If ∂ is an inner face map induced by an edge e and v is the upper vertex adjacent to e, then $\mathrm{sgn}(\partial) = (-1)^{\sharp v}$.
(ii) If ∂ is an outer face map induced by the root-vertex r, then $\mathrm{sgn}(\partial) = (-1)^{\sharp r}$ which equals 1.
(iii) If ∂ is any other outer face map induced by a vertex v, then $\mathrm{sgn}(\partial) = (-1)^{\sharp v+1}$.

For example, if ∂_b denotes the inner face map of Figure (3), then $\mathrm{sgn}(\partial_b) = (-1)^1 = -1$. If ∂_w denotes the outer face of Figure (4), then $\mathrm{sgn}(\partial_w) = (-1)^2 = 1$.

There is one exception to these rules in the case of the inclusion of the tree with no vertices into a corolla:



In this case, if ∂ takes the sole edge of the tree η to the root of the corolla then $\mathrm{sgn}(\partial) = 1$, otherwise $\mathrm{sgn}(\partial) = -1$.

The following result is an immediate consequence of the elementary face relations (see Section 4.1) and our way of numbering the vertices.

Lemma 4.3. *Let f be a map in Ω_p such that it is a composition of two elementary face maps $f = \partial_1 \circ \partial_2$. If $f = \partial'_1 \circ \partial'_2$ is the other way to decompose f as a composition of two elementary faces then* $\mathrm{sgn}(\partial_1)\mathrm{sgn}(\partial_2) = -\mathrm{sgn}(\partial'_1)\mathrm{sgn}(\partial'_2)$. □

Remark 4.4. The linear order described in Section 4.4 could also have been used to define a sign convention. While both conventions coincide when restricted to Δ, Lemma 4.3 holds only for the convention given.

5. Normal faces

For any tree T a *maximal linear part* of T is an embedding $L_n \rightarrowtail T$ in Ω_p for some $n \geq 1$, such that whenever there is another such embedding $L_m \rightarrowtail T$ for $n \leq m$ that fits into a commutative diagram of inclusions

$$\begin{array}{ccc} L_n & \rightarrowtail & T \\ \downarrow & \nearrow & \\ L_m & & \end{array}$$

then $m = n$. We say that an elementary face map $\partial : R \longrightarrow T$ *lives* or *sits on a maximal linear part* $L_n \rightarrowtail T$ when there exists another embedding $L_{n-1} \rightarrowtail R$ and a commutative diagram

$$\begin{array}{ccc} L_{n-1} & \rightarrowtail & R \\ {\scriptstyle \partial_i}\downarrow & & \downarrow{\scriptstyle \partial} \\ L_n & \rightarrowtail & T \end{array} \qquad (9)$$

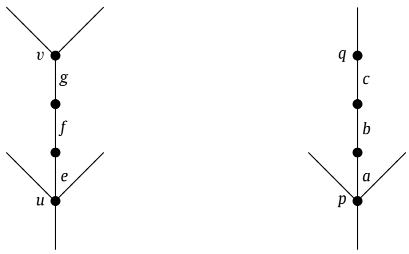
for some elementary face map $\partial_i : L_{n-1} \longrightarrow L_n$, where the index i is taken with respect to the order defined in Section 4.4. One can prove that if such a ∂_i exists, then it is unique. Moreover, if $\partial, \partial' : R \longrightarrow T$ sit on the same maximal linear part and ∂_i fills diagram (9) for both ∂ and ∂', then $\partial = \partial'$.

We can observe that there are exactly $n + 1$ elementary faces sitting on a maximal linear part $\iota : L_n \longrightarrow T$, and with respect to the linear order defined in Section 4.4 they are the faces $\partial_k, \partial_{k+1}, \ldots, \partial_{k+n}$ for some k. To underline the similarity between the faces of $[n]$ in the category Δ, and the face maps sitting on a maximal linear part $\iota : L_n \longrightarrow T$, it proves to be convenient to shift their indices so they become $\partial_0^{(\iota)}, \ldots, \partial_n^{(\iota)}$. We will also say that the faces living on the maximal linear part are *connected*.

Please cite this article in press as: J.J. Gutiérrez, et al., Dold–Kan correspondence for dendroidal abelian groups, Journal of Pure and Applied Algebra (2010), doi:10.1016/j.jpaa.2010.10.003

Definition 5.1. An elementary face map $\partial: R \longrightarrow T$ in Ω_p is *normal* if ∂ lives on a maximal linear part $\iota: L_n \longrightarrow T$ for some $n \geq 1$ and $\partial = \partial_i^{(\iota)}$ for some $0 \leq i < n$ in the associated order.

Example 5.2. Let T and R be the following trees:



T $\qquad\qquad$ R

The tree T has one maximal linear part $L_2 \longrightarrow T$ and R has one maximal linear part $L_3 \longrightarrow R$. The elementary faces of T have the following properties: ∂_e and ∂_f are normal; ∂_g, ∂_u and ∂_v are not normal; ∂_e, ∂_f and ∂_g are connected to each other. The elementary faces of R have the following properties: ∂_a, ∂_b and ∂_c are normal; ∂_p and ∂_q are not normal; ∂_a, ∂_b, ∂_c and ∂_q are connected to each other.

In general, if $\partial: R \longrightarrow T$ is an elementary face that lives on a maximal linear part $L_n \longrightarrow T$ then ∂ is connected to precisely n other elementary faces. Of these $n+1$ faces altogether, n are normal and exactly one is not normal (the last one in the induced order). A special case is that the face $\partial_0^{(\iota)}: L_0 \longrightarrow L_1$ is normal, while $\partial_1^{(\iota)}: L_0 \longrightarrow L_1$ is not.

Remark 5.3. An arbitrary choice is made here about which faces to treat as normal (i.e., exclude the case $i = n$). But we could have excluded the case $i = 0$ instead. For the general theory this does not make a difference since if one makes the other choice then all the results remain true with the obvious changes in the proofs.

6. Dendroidal chain complexes

In this section we introduce the category of dendroidal chain complexes $d\mathcal{C}h$. This category extends the category of chain complexes of abelian groups and, as we will see in the next section, it is equivalent to the category of dendroidal abelian groups $d\mathcal{A}b$, i.e., the category of functors from Ω_p^{op} to abelian groups. If A is a dendroidal abelian group and $f: R \longrightarrow T$ is any map in Ω_p, then the associated group homomorphism $A_T \longrightarrow A_R$ is denoted by f^*.

We say that an abelian group A is an Ω_p-graded abelian group if $A = \bigoplus_{T \in \Omega_p} A_T$.

Definition 6.1. A *dendroidal chain complex* A is an Ω_p-graded abelian group together with structure maps given by group homomorphisms $\delta^\sharp: A_T \longrightarrow A_R$, for every elementary face map $\delta: R \longrightarrow T$, satisfying the following two conditions:

(i) If δ is a normal face, then $\delta^\sharp = 0$.
(ii) For any commutative diagram of elementary face relations

$$\begin{array}{ccc} S & \xrightarrow{\partial} & R \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\partial_1} \\ R' & \xrightarrow{\delta_1} & T \end{array}$$

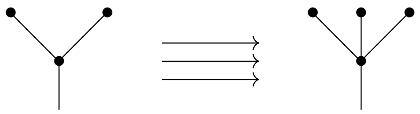
the associated diagram

$$\begin{array}{ccc} A_S & \xleftarrow{\partial^\sharp} & A_R \\ {\scriptstyle\delta^\sharp}\uparrow & & \uparrow{\scriptstyle\partial_1^\sharp} \\ A_{R'} & \xleftarrow{\delta_1^\sharp} & A_T \end{array}$$

anticommutes, i.e., $\delta^\sharp \delta_1^\sharp = -\partial^\sharp \partial_1^\sharp$.

Remark 6.2. The definition given above indeed generalizes chain complexes, since restricting it to linear trees gives exactly one normal face $[n] \longrightarrow [n+1]$ for every n. We mentioned in the introduction that the complicated definition of dendroidal chain complexes arises from a subtle difference between the categories Δ and Ω_p. Explicitly, there exist pairs of trees in Ω_p

with multiple face maps between them but no degeneracies. For example, on the picture below the three arrows represent the three different inner faces of the codomain, while it is clear that there are no degeneracies between the trees involved.



A map $\varphi: A \longrightarrow B$ between dendroidal chain complexes is given by a sequence of maps $\varphi_T: A_T \longrightarrow B_T$ compatible with the group homomorphisms. We denote the category of dendroidal chain complexes by $d\mathcal{C}h$. Note that if in the definition of a dendroidal chain complex we replace the category Ω_p by its full subcategory Δ, then we recover the notion of a chain complex. In fact, if we denote by $\mathcal{C}h$ the category of chain complexes, then there is a pair of adjoint functors

$$j_!: \mathcal{C}h \rightleftarrows d\mathcal{C}h: j^* \tag{10}$$

where j^* is the restriction functor, i.e., $j^*(A)_n = A_{L_n}$. Its left adjoint $j_!$ is 'extension by zero' and sends a chain complex A to the dendroidal chain complex

$$j_!(A)_T = \begin{cases} A_n & \text{if } T \cong L_n, \\ 0 & \text{otherwise.} \end{cases}$$

6.1. The unnormalized dendroidal chain complex

Let A be a dendroidal abelian group. One can define a dendroidal chain complex CA associated to A by setting $(CA)_T = A_T$ for every $T \in \Omega_p$. For any elementary face $\delta: R \longrightarrow T$ in Ω_p, the structure map $\delta^\sharp$ is defined as follows:

(i) If δ is a normal face, then $\delta^\sharp = 0$.
(ii) If δ is not a normal face and it is not connected to any normal face, then $\delta^\sharp = \mathrm{sgn}(\delta) \cdot \delta^*$.
(iii) In the remaining case $\delta = \partial_n^{(\iota)}$ for a maximal linear part $L_n \xrightarrow{\iota} T$ in the induced order. Define in this case

$$\delta^\sharp = \sum_{i=0}^{n} \mathrm{sgn}(\partial_i) \cdot \partial_i^*.$$

Lemma 6.3. *The Ω_p-graded abelian group CA defined above is in fact a dendroidal chain complex.*

Proof. Suppose that

$$\begin{array}{ccc} S & \xrightarrow{\delta'} & R \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\delta_1'} \\ R' & \xrightarrow{\delta_1} & T \end{array}$$

is a commutative diagram of elementary face relations. There are several cases to distinguish according to the type of faces involved.

If none of the four faces are normal or connected to a normal face then Lemma 4.3 ensures that the induced square anticommutes.

If each of the sets $\{\delta, \delta_1\}$, $\{\delta', \delta_1'\}$ contains at least one normal face then the induced square anticommutes trivially, since at least one of the induced maps in each set is the zero map.

In case the set $\{\delta', \delta_1'\}$ contains a normal face while $\{\delta, \delta_1\}$ does not contain any, one has to prove that $\delta^\sharp\delta_1^\sharp = 0$. First we observe that if δ' is normal, then δ_1 is also normal, hence we can assume that δ_1' is the only normal face in the diagram. Moreover, it also follows that $\delta_1' = \partial_{n-1}$ in the order induced by a maximal linear part $L_n \longrightarrow T$ and, since none of the elements of $\{\delta, \delta_1\}$ can be normal faces, δ' is connected to δ_1', and both δ and δ_1 are connected to normal faces. Hence

$$\delta^\sharp = \sum_{i=0}^{n} \mathrm{sgn}(\partial_i) \cdot \partial_i^* \quad \text{and} \quad \delta_1^\sharp = \sum_{i=0}^{n+1} \mathrm{sgn}((\partial_1)_i) \cdot (\partial_1)_i^*.$$

We now conclude that $\delta^\sharp\delta_1^\sharp = 0$ by following the proof for the simplicial case, when one proves the differential property $d^2 = 0$ for the unnormalized chain complex associated to a simplicial abelian group.

The remaining case, when none of the four faces are normal but some of them are connected to normal ones, breaks down into the following three cases.

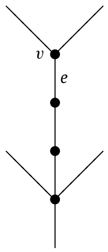
Please cite this article in press as: J.J. Gutiérrez, et al., Dold–Kan correspondence for dendroidal abelian groups, Journal of Pure and Applied Algebra (2010), doi:10.1016/j.jpaa.2010.10.003

(i) When both of δ' and δ'_1 are not normal, but connected to normal faces, they have to live on different maximal linear parts of the tree T. Hence δ and δ_1 are not normal, but connected to normal faces. Moreover, each summand in the definition of $\delta^\sharp$ and $\delta'^\sharp$ fits into a commutative diagram

$$\begin{array}{ccc} A_S & \xleftarrow{(\partial')^*_j} & A_R \\ {\scriptstyle \partial^*_i}\uparrow & & \uparrow{\scriptstyle (\partial'_1)^*_i} \\ A_{R'} & \xleftarrow[(\partial_1)^*_j]{} & A_T \end{array}$$

hence Lemma 4.3 ensures that the required square anticommutes.

(ii) Suppose that δ'_1 is connected to normal faces, while δ' is not. We analyze the case when δ' is 'adjacent' to δ'_1 (the other cases are easier and we omit them). This situation can typically be illustrated when T is a tree of the form



and $\delta'_1 = \partial_e, \delta_1 = \partial_v$. Then it is clear what the other trees and face maps are in the diagram, and we see that $\delta: S \longrightarrow R'$ is connected to normal faces, but is not normal. Again, a summand $(\partial'_1)^*_i$ of $(\delta'_1)^\sharp$ will correspond to the summand ∂^*_i of $\delta^\sharp$ such that the associated diagram

$$\begin{array}{ccc} A_S & \xleftarrow{(\delta')^*} & A_R \\ {\scriptstyle \partial^*_i}\uparrow & & \uparrow{\scriptstyle (\partial'_1)^*_i} \\ A_{R'} & \xleftarrow[(\delta_1)^*]{} & A_T \end{array}$$

commutes for every i. We conclude that the required diagram is anticommutative.

(iii) The remaining case, when δ' is connected to normal faces and δ'_1 is not, is symmetric to case (ii).

This exhausts all the possible combinations of faces and completes the proof. □

The dendroidal chain complex CA is called the *unnormalized complex* associated to A.

6.2. The normalized and degenerate dendroidal chain subcomplexes

Let A be a dendroidal abelian group. We can construct a subcomplex NA of the unnormalized complex CA by setting

$$(NA)_T = \cap_{\tilde{\partial}} \ker(\tilde{\partial}^*) \le A_T,$$

where $\tilde{\partial}$ runs through all normal faces with codomain T. Note that if T has no normal faces, then we have an empty intersection and in that case we set $(NA)_T = A_T$. We can restrict the structure maps of CA to get a dendroidal chain complex structure on NA. The dendroidal chain complex NA is called the *normalized complex* associated to A.

Another subcomplex DA of CA is defined by

$$(DA)_T = \sum_{\sigma: T \to S} \sigma^*(A_S) \le A_T,$$

where σ runs through all elementary degeneracies with domain T. Again, the structure maps of CA restrict to DA, thus we obtain a dendroidal chain subcomplex of CA which is called the *degenerate complex* associated to A.

Proposition 6.4. *The Ω_p-graded abelian groups NA and DA are dendroidal chain subcomplexes of CA.*

Proof. We first consider the case of NA. Let $\delta: R \longrightarrow T$ be an elementary face map in Ω_p. We need to prove that every $x \in (NA)_T$ satisfies $\delta^\sharp(x) \in (NA)_R$. There are three cases to distinguish:

(i) If δ is a normal face, then $\delta^\sharp(x) = 0 \in (NA)_R$.

(ii) If δ is neither normal nor connected to a normal face, then $\delta^\sharp = \mathrm{sgn}(\delta) \cdot \delta^*$. Suppose that $\partial: S \longrightarrow R$ is a normal face. There is a commutative diagram of elementary face relations

$$\begin{array}{ccc} S & \xrightarrow{\partial} & R \\ {\scriptstyle\tilde{\gamma}}\downarrow & & \downarrow{\scriptstyle\delta} \\ R' & \xrightarrow{\gamma} & T \end{array}$$

by Lemma 4.1. It is easy to check that in such a case whenever ∂ is normal, γ is normal as well. We conclude that

$$\partial^*\delta^*(x) = \tilde{\gamma}^*\gamma^*(x) = 0$$

and thus $\partial^*\delta^\sharp(x) = 0$.

(iii) In the remaining case, $\delta = \partial_n$ for some maximal linear part $\iota: L_n \longrightarrow T$. Therefore

$$\delta^\sharp = \sum_{i=0}^{n} \mathrm{sgn}(\partial_i) \cdot \partial_i^*.$$

Again let $\partial: S \longrightarrow R$ be a normal face. In the same way as in case (ii), for every i we have $\partial_i\partial = \gamma_i\tilde{\gamma}_i$ for some normal face γ_i. Hence every summand of $\partial^*\delta^\sharp$ vanishes on x.

Next, we prove that DA is a subcomplex. Suppose that $\delta: R \longrightarrow T$ is an elementary face map and let $x \in (DA)_T$ such that $x = \sigma_1^*(x_1) + \cdots + \sigma_k^*(x_k)$ for some elementary degeneracies $\sigma_j: T \longrightarrow S_j$ and $x_j \in A_{S_j}$. There are again three cases to distinguish:

(i) If δ is a normal face, then $\delta^\sharp(x) = 0 \in (DA)_R$ as before.

(ii) If δ is neither normal nor connected to a normal face, then there exists a commutative diagram of combined dendroidal relations

$$\begin{array}{ccc} T & \xrightarrow{\sigma_j} & S_j \\ {\scriptstyle\delta}\uparrow & & \uparrow{\scriptstyle\delta_j} \\ R & \xrightarrow{\sigma_j'} & T_j \end{array}$$

for every j (otherwise δ would be a section of a degeneracy, hence normal or connected to a normal face). In this case

$$\delta^*\sigma_j^*(x_i) = \sigma_j'^*\delta_j^*(x_j) \in (DA)_R$$

for every j, thus $\delta^\sharp(x) \in (DA)_R$.

(iii) In the remaining case, $\delta^\sharp = \sum_{i=0}^n \mathrm{sgn}(\partial_i) \cdot \partial_i^*$ where $\partial_0, \ldots, \partial_{n-1}$ are normal faces sitting on the same maximal linear part $\iota: L_n \longrightarrow T$ where the indices come from the induced order, and $\delta = \partial_n$ in this order. It follows that

$$\delta^\sharp(x) = \sum_{i,j} \mathrm{sgn}(\partial_i)\partial_i^*\sigma_j^*(x_j).$$

This sum can be divided into two parts. The first part consists of those components for which $\sigma_j\partial_i$ satisfies the combined dendroidal relation (6), i.e., $\sigma_j\partial_i = \partial_i'\sigma_j'$ for some face ∂_i' and degeneracy σ_j'. This part of the sum is clearly in $(DA)_R$. The second part consists of those summands for which $\sigma_j\partial_i$ satisfies the combined dendroidal relation (7) of the second type, that is $\sigma_j\partial_i = \sigma_j\partial_i' = \mathrm{id}_R$ for some face ∂_i'. But in such a case $\mathrm{sgn}(\partial_i') = -\mathrm{sgn}(\partial_i)$ and one can form such pairs from the components of this part of the sum cancelling each other. □

The unnormalized dendroidal chain complex associated to a dendroidal abelian group splits as a direct sum of the normalized part and the degenerate part. The approach is similar to the one appearing in [15], which establishes the same property for the classical unnormalized chain complex of a simplicial abelian group. We need to show that $A_T = (NA)_T \oplus (DA)_T$ for every tree $T \in \Omega_p$.

Lemma 6.5. *For any dendroidal abelian group A the dendroidal chain complexes NA and DA satisfy* $(NA)_T \cap (DA)_T = 0$ *for every tree* $T \in \Omega_p$.

Proof. Suppose that $0 \neq x \in (NA)_T \cap (DA)_T$ and write x as a finite sum of elementary degeneracies

$$x = \sigma_1^*(x_1) + \cdots + \sigma_k^*(x_k),$$

such that the number of the summands is minimal. If $k = 1$ then $\sigma_1: T \longrightarrow S$ has two right inverses in Ω_p and at least one of them, say $\partial: S \longrightarrow T$, is a normal face. It follows that $0 = \partial^*(x) = (\sigma_1\partial)^*(x_1) = x_1$ which contradicts $x \neq 0$.

If $k > 1$, we can use a similar argument. Since k is minimal, $\sigma_i \neq \sigma_j$ for every $i \neq j$, hence σ_i and σ_j are induced by univalent vertices $v_i \neq v_j$. We can suppose that σ_1 sits on a linear part $L_n \longrightarrow T$ and that all the other σ_i are on a different linear part or, if on the same one, that they come after σ_1 in the induced order. In other words, none of those vertices $v_2, \ldots, v_k$ which are on the linear component of v_1 sit below v_1. Let ∂ be the normal right inverse to σ_1 induced by the edge below v_1 or by cutting v_1. Then

$$-\big((\sigma_2\partial)^*(x_2) + \cdots + (\sigma_k\partial)^*(x_k)\big) = x_1,$$

$$x = \sigma_2^*(x_2) + \cdots + \sigma_k^*(x_k) - \big((\sigma_2\partial\sigma_1)^*(x_2) + \cdots + (\sigma_k\partial\sigma_1)^*(x_k)\big).$$

Let us look at the composite $\sigma_i\partial\sigma_1$ for all $i > 1$ and write it in another form with the help of the dendroidal identities. We observe that $\sigma_i \circ \partial \neq \mathrm{id}_T$ since we chose ∂ in a way that avoids this situation. It follows that we obtain commutative diagrams for all $1 < i \leq k$:

$$\begin{array}{ccccc} T & \xrightarrow{\sigma_1} & T\backslash v_1 & \xrightarrow{\partial} & T \\ \Big\downarrow{\scriptstyle \sigma_i} & & \Big\downarrow{\scriptstyle \sigma'} & & \Big\downarrow{\scriptstyle \sigma_i} \\ T\backslash v_i & \xrightarrow{\sigma''} & (T\backslash v_1)\backslash v_i & \xrightarrow{\partial'} & T\backslash v_i, \end{array}$$

where by the dendroidal identities the dotted vertical arrow can only be σ_i. We conclude that $(\sigma_i\partial\sigma_1)^*(u) = \sigma_i^*(u')$ for all $1 < i \leq k$ and

$$x = \sigma_2^*(y_2) + \cdots + \sigma_k^*(y_k).$$

This is a contradiction since k was chosen to be minimal. □

Lemma 6.6. *For any dendroidal abelian group A the dendroidal chain complexes NA and DA satisfy $(NA)_T + (DA)_T = A_T$ for every tree $T \in \Omega_p$.*

Proof. Fix an $x \in A_T$ and define

$$N_x = \big\{\partial\colon S \longrightarrow T \;\big|\; \partial \text{ is a normal face such that } \partial^*(x) \neq 0\big\}.$$

We can assume that N_x is not empty, otherwise $x \in (NA)_T$. Let $k \in \mathbb{N}$ be the number of the maximal linear parts of T. Partition N_x into subsets $N_x = N_x^{(\iota_1)} \cup \cdots \cup N_x^{(\iota_k)}$ for every maximal linear part $\iota_j\colon L_{n_j} \longrightarrow T$, where $N_x^{(\iota)}$ contains those elements of N_x which sit on ι.

The goal is to write x as $x = x_1 + y_1$, where $y_1 \in (DA)_T$ and $N_{x_1}^{(\iota_1)} = \emptyset$, while $|N_{x_1}^{(\iota)}| \leq |N_x^{(\iota)}|$ for the other linear parts $\iota \neq \iota_1$. If we succeed, we can iterate the process by finding a decomposition $x_1 = x_2 + y_2$ such that x_2 kills the set $N_{x_1}^{(\iota_2)}$ while it does not increase the size of the other sets $N_{x_1}^{(\iota)}$. After k steps we would arrive at a decomposition

$$x = x_k + y_1 + y_2 + \cdots + y_k,$$

where $N_{x_k} = \emptyset$, thus $x_k \in (NA)_T$ and $y_1 + \cdots + y_k \in (DA)_T$, thereby finishing the proof.

To obtain such a decomposition of x, we proceed as follows. Let $\partial \in N_x^{(\iota_1)}$ be the smallest element in the order induced by $\iota_1\colon L_{n_1} \longrightarrow T$. Let $\partial^*(x) = y$ and define $x' = x - \sigma^*(y)$ where $\sigma\colon T \to S$ is the biggest such degeneracy in the order induced by ι_1 for which $\sigma\partial = \mathrm{id}_S$. It follows that

$$\partial^*(x') = \partial^*(x) - (\sigma\partial)^*(y) = 0.$$

Now suppose that $\tilde{\partial}$ is any normal face such that $\tilde{\partial}^*(x) = 0$. We are going to prove that $\tilde{\partial}^*(x') \neq 0$ can happen only if $\tilde{\partial}$ is connected to ∂ and $\tilde{\partial} > \partial$ in the order induced by ι_1. Indeed, on one hand if $\tilde{\partial}$ is not connected to ∂ then it is obvious that $\sigma\tilde{\partial} \neq \mathrm{id}$. On the other hand, the reason we chose ∂ to be minimal and σ maximal was that now $\sigma\tilde{\delta} \neq \mathrm{id}$ holds also whenever $\tilde{\partial} < \partial$ on the linear part ι_1. Hence if $\tilde{\partial}$ obeys one of these cases, we can fill the following diagram of dendroidal identities

$$\begin{array}{ccccc} T & \xrightarrow{\sigma} & T\backslash v & \xrightarrow{\partial} & T \\ \Big\uparrow{\scriptstyle \tilde{\partial}} & & \Big\uparrow{\scriptstyle \partial'} & & \Big\uparrow{\scriptstyle \tilde{\partial}} \\ T' & \xrightarrow{\sigma'} & T'\backslash v & \xrightarrow{\partial''} & T'. \end{array}$$

It is immediate that in this diagram the dotted vertical arrow is $\tilde{\partial}$. Therefore

$$\tilde{\partial}^*(x') = -\tilde{\partial}^*\sigma^*(y) = -\tilde{\partial}^*\sigma^*\partial^*(x) = -\sigma'^*\partial''^*\tilde{\partial}^*(x) = 0.$$

Let us summarize what we managed to achieve with the $x = x' + \sigma(y)$ decomposition:

(i) The normal face ∂ satisfies $\partial(x) \neq 0$ and $\partial(x') = 0$.
(ii) For any normal face $\tilde{\partial}$ such that $\tilde{\partial}^*(x) = 0$, we can have $\tilde{\partial}^*(x') \neq 0$ only if $\tilde{\partial} > \partial$ on the same linear part ι_1.

Now we can apply the same process for x' and the new smallest element in $N_{x'}^{(\iota_1)}$, and so on. In a finite number of steps we arrive to the desired decomposition $x = x_1 + y_1$. □

Proposition 6.7. *For any dendroidal abelian group A the associated unnormalized dendroidal chain complex CA decomposes as $NA \oplus DA$.*

Proof. It follows directly from Lemmas 6.5 and 6.6. □

7. The Dold–Kan correspondence

The canonical inclusion $i\colon \Delta \longrightarrow \Omega_p$ induces a restriction functor $i^*\colon d\mathcal{A}b \longrightarrow s\mathcal{A}b$ which has a left adjoint $i_!$ given by Kan extension. The functor i^* sends a dendroidal abelian group A to the simplicial abelian group

$$i^*(A)_n = A_{i([n])}.$$

Its left adjoint $i_!\colon s\mathcal{A}b \longrightarrow d\mathcal{A}b$ is 'extension by zero', and sends a simplicial abelian group A to the dendroidal abelian group given by

$$i_!(A)_T = \begin{cases} A_n & \text{if } T \cong i([n]), \\ 0 & \text{otherwise.} \end{cases}$$

We will define a right adjoint Γ to the normalized dendroidal chain complex functor $N\colon d\mathcal{A}b \longrightarrow d\mathcal{C}h$ and we will prove that the pair (N, Γ) forms an equivalence of categories. This equivalence extends the classical Dold–Kan correspondence for simplicial abelian groups and chain complexes in the following precise sense. There is a commutative diagram of adjoint functors

$$\begin{array}{ccc} s\mathcal{A}b & \overset{i_!}{\underset{i^*}{\rightleftarrows}} & d\mathcal{A}b \\ N_s \downarrow \uparrow \Gamma_s & & N \downarrow \uparrow \Gamma \\ \mathcal{C}h & \overset{j_!}{\underset{j^*}{\rightleftarrows}} & d\mathcal{C}h, \end{array}$$

where $(j_!, j^*)$ is the adjunction described in (10). Moreover the following relations hold:

$$\begin{array}{ll} N_s i^* = j^* N, & N i_! = j_! N_s, \\ \Gamma_s j^* = i^* \Gamma, & \Gamma j_! = i_! \Gamma_s, \\ i^* i_! = \mathrm{id}, & j^* j_! = \mathrm{id}, \\ N\Gamma = \mathrm{id}, & N_s \Gamma_s = \mathrm{id}, \\ \Gamma N \cong \mathrm{id}, & \Gamma_s N_s \cong \mathrm{id}. \end{array}$$

The rest of this section is devoted to the construction of the functor Γ. We denote by Ω_{mono} the subcategory of Ω_p consisting of all the trees as objects and only monomorphisms as maps. For every dendroidal chain complex C there is a functor $F_C\colon \Omega_{mono}^{\mathrm{op}} \longrightarrow \mathcal{A}b$ defined on objects by $F_C(T) = C_T$ and on elementary face maps by

$$F_C(\partial) = \begin{cases} 0 & \text{if } \partial \text{ is normal}, \\ \mathrm{sgn}(\partial)\partial^\sharp & \text{oherwise.} \end{cases}$$

Observe that F_C is indeed a functor since the sign convention on faces implies that commutative diagrams of dendroidal identities involving faces are taken via F_C to commutative diagrams of abelian groups. This functor will play a role in the construction of a right adjoint $\Gamma\colon d\mathcal{C}h \longrightarrow d\mathcal{A}b$ to the normalized dendroidal chain complex functor N.

If we assume that such a right adjoint exists, then there should be a one-to-one correspondence between the sets of morphisms

$$d\mathcal{C}h(NA, C) \cong d\mathcal{A}b(A, \Gamma C)$$

for any dendroidal chain complex C and dendroidal abelian group A. If one takes A to be the representable $\mathbb{Z}\Omega_p[T]$, then

$$d\mathcal{C}h(N\mathbb{Z}\Omega_p[T], C) \cong d\mathcal{A}b(\mathbb{Z}\Omega_p[T], \Gamma C) \cong (\Gamma C)_T \tag{11}$$

by the Yoneda lemma. Moreover, this correspondence has to be an isomorphism of groups, showing us a way to define $(\Gamma C)_T$ for every tree T. One can unpack the left-hand side of Eq. (11) to arrive at the definition

$$(\Gamma C)_T = \bigoplus_{r\colon T \twoheadrightarrow R} C_R,$$

where r runs through all epimorphisms in Ω_p with domain T. In the direct sum above we will denote by C_R^r the component C_R corresponding to an epimorphism r.

Please cite this article in press as: J.J. Gutiérrez, et al., Dold–Kan correspondence for dendroidal abelian groups, Journal of Pure and Applied Algebra (2010), doi:10.1016/j.jpaa.2010.10.003

We still have to define ΓC on the maps of Ω_p. Suppose that $f: S \longrightarrow T$ is such a map and define $f^*: (\Gamma C)_T \longrightarrow (\Gamma C)_S$ in the following way. Let $r: T \twoheadrightarrow R$ be an epimorphism in Ω_p. The map $r \circ f: S \longrightarrow R$ has a unique factorization $d \circ s$ by Lemma 3.2:

$$\begin{array}{ccc} S & \xrightarrow{s} & S' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle d} \\ T & \xrightarrow{r} & R. \end{array}$$

We define f^* on the component C_R^r as the composite

$$(f^*)^r: C_R^r \xrightarrow{F_C(d)} C_{S'}^s \rightarrowtail (\Gamma C)_S.$$

We have finished the definition of Γ on objects. Let us check that ΓC is indeed a dendroidal abelian group for every $C \in d\mathcal{C}h$. It is easy to see that $\Gamma C(\mathrm{id}_T) = \mathrm{id}: \Gamma C_T \longrightarrow \Gamma C_T$. Suppose that $f: S \longrightarrow T$ and $g: U \longrightarrow S$ are two maps in Ω_p. We need to check that for any epimorphism $r: T \twoheadrightarrow R$, the components $((fg)^*)^r$ and $(g^*f^*)^r$ are the same. Indeed, since the epi-mono factorizations of rfg, and of rf followed by sg are unique, we infer that $d = d_f d_g$ in the following diagram:

$$\begin{array}{ccc} U & \xrightarrow{u} & U' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle d_g} \\ S & \xrightarrow{s} & S' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle d_f} \\ T & \xrightarrow{r} & R \end{array} \qquad d: U' \to R$$

Since F_C is a functor, this implies the required equality.

It is easy to check that the obvious definition of Γ on maps of dendroidal chain complexes is functorial. Now we can prove the following propositions.

Proposition 7.1. *For every tree $T \in \Omega_p$ the abelian groups $(N\Gamma C)_T$ and C_T are equal.*

Proof. We have two decompositions of the abelian group $(\Gamma C)_T$ into a direct sum of subgroups. First, by definition

$$(\Gamma C)_T = C_T^{\mathrm{id}_T} \oplus \bigoplus_{\substack{T \overset{r}{\twoheadrightarrow} R \\ r \neq \mathrm{id}_T}} C_R^r$$

and second, by Proposition 6.7

$$(\Gamma C)_T = (N\Gamma C)_T \oplus (D\Gamma C)_T.$$

Hence it is enough to prove that $\bigoplus_{r\neq \mathrm{id}_T} C_R^r \leq (D\Gamma C)_T$ and $C_T^{\mathrm{id}_T} \leq (N\Gamma C)_T$.

To see the first assertion we pick an epimorphism $r: T \twoheadrightarrow R$, $r \neq \mathrm{id}_T$ and prove that the corresponding component $C_R^r \leq (\Gamma C)_T$ is in the image of a degeneracy. It follows from our choice that r decomposes as $r = \sigma \circ r'$ where $\sigma: T \longrightarrow S$ is an elementary degeneracy and $r': S \twoheadrightarrow R$ is another epimorphism (possibly the identity). Let us look at the image of $\sigma^*: (\Gamma C)_S \longrightarrow (\Gamma C)_T$ on the component $C_R^{r'}$. Since the unique epi-mono factorization of $r'\sigma$ is

$$\begin{array}{ccc} T & \xrightarrow{r} & R \\ {\scriptstyle \sigma}\downarrow & & \| \\ S & \xrightarrow{r'} & R, \end{array}$$

we can conclude that σ^* sends the component $C_R^{r'}$ to the component C_R^r.

The second assertion follows as well. Indeed, for an arbitrary normal face $\partial: S \longrightarrow T$ the induced map of abelian groups $\partial^*: (\Gamma C)_T \longrightarrow (\Gamma C)_S$ vanishes on $C_T^{\mathrm{id}_T}$ since $F_C(\partial) = 0$ by definition. □

Proposition 7.2. *Let A be a dendroidal abelian group and $r: T \twoheadrightarrow R$ an epimorphism in Ω_p. If we define $(\Psi_A)_T^r$ to be the composite*

$$(NA)_R^r \rightarrowtail A_R \xrightarrow{r^*} A_T \;,$$

then the induced map $(\Psi_A)_T: (\Gamma NA)_T \longrightarrow A_T$ is an isomorphism which is natural in both A and T.

Please cite this article in press as: J.J. Gutiérrez, et al., Dold–Kan correspondence for dendroidal abelian groups, Journal of Pure and Applied Algebra (2010), doi:10.1016/j.jpaa.2010.10.003

Proof. The proof of naturality is routinely verified. In what follows we will write Ψ_T instead of $(\Psi_A)_T$ to simplify the notation. The first observation is that Ψ_T decomposes as a direct sum

$$N\Psi_T \oplus D\Psi_T \colon (N\Gamma(NA))_T \oplus (D\Gamma(NA))_T \longrightarrow (NA)_T \oplus (DA)_T,$$

and by Proposition 7.1 the $N\Psi_T$ component is id: $(NA)_T \longrightarrow (NA)_T$. Hence in order to conclude that Ψ_T is an isomorphism, it is enough to prove that $D\Psi_T$ is surjective and injective.

We proceed by induction on the number n of vertices of T. If $n = 0$ then $T = \eta$ and $(D\Gamma NA)_T = 0 = (DA)_T$. Suppose that $D\Psi_S$ is surjective and injective for every tree S with less than n vertices and let T be a tree with n vertices. Let x be in the image of $\sigma^* \colon A_S \longrightarrow A_T$ for some elementary degeneracy $\sigma \colon T \longrightarrow S$ and look at the commutative diagram

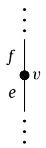
$$\begin{array}{ccc} (\Gamma NA)_S & \xrightarrow{\Psi_S} & A_S \\ {\scriptstyle \sigma^*}\downarrow & & \downarrow{\scriptstyle \sigma^*} \\ (\Gamma NA)_T & \xrightarrow{\Psi_T} & A_T. \end{array}$$

Since S has fewer vertices than T, we have that $D\Psi_S$ is surjective and $x = D\Psi_T(y)$ for some $y \in (D\Gamma NA)_T$. Hence $D\Psi_T$ is surjective.

Let us prove that $D\Psi_T$ is also injective. Suppose that $x \in \ker D\Psi_T$ and write

$$x = \sum_{T \overset{r}{\twoheadrightarrow} R} x^r.$$

For any tree R let $\Theta_{T,R} = \{r \colon T \twoheadrightarrow R \mid x^r \neq 0\}$ and suppose that $\Theta_{T,R} \neq \emptyset$ for every R. First we deal with the special case when $\Theta_{T,R}$ consists only of epimorphisms r occurring in the same maximal linear part of T. For each $r \colon T \twoheadrightarrow R$ we choose a specific section d_r, as follows. If $\sigma \colon T \to T\backslash v$ is a degeneracy and the vertex v has adjacent edges e, f with f situated above e



then there is a unique face map $\partial \colon T\backslash v \longrightarrow T$ which omits the edge e. This face map satisfies $\sigma\partial = \mathrm{id}_{T\backslash v}$. If r decomposes as $r = \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_k$ into elementary degeneracies then define the section $d_r = \partial_k \circ \cdots \circ \partial_2 \circ \partial_1$ of r where ∂_i is picked for σ_i in the way described above. Note that the dendroidal identities ensure that any other decomposition of r yields the same section.

Next we define a partial order on $\Theta_{T,R}$ as follows. If $r, s \in \Theta_{T,R}$ and for every edge e sitting on the relevant linear part of R the edge $d_r(e)$ is equal to or is below $d_s(e)$, then we say that $r \leq s$. We observe that

$$sd_r = \mathrm{id}_R \text{ for some } r, s \in \Theta_{T,R} \text{ implies } r \leq s. \tag{12}$$

Pick a maximal $r \in \Theta_{T,R}$ with respect to the order defined above. By (12) and the maximality of r, the map d_r satisfies that $sd_r = \mathrm{id}$ implies $s = r$, hence we can conclude that in the commutative diagram

$$\begin{array}{ccc} (\Gamma NA)_T & \xrightarrow{\Psi_T} & A_T \\ {\scriptstyle d_r^*}\downarrow & & \downarrow{\scriptstyle d_r^*} \\ (\Gamma NA)_R & \xrightarrow{\Psi_R} & A_R, \end{array}$$

on the left-hand side $N(d_r^*(x)) = x^r$. Since $d_r^*\Psi_T = \mathrm{id}$ and $N\Psi_R = \mathrm{id}$, we infer that $x^r = 0$, which is a contradiction.

We still need to deal with the general case, when $\Theta_{T,R}$ contains epimorphisms which are not necessarily situated on a fixed maximal linear part of T. To do so, suppose that T has k different maximal linear parts $L_{n_i} \rightarrowtail T$, $i \in \{1, \ldots, k\}$. Decompose each $r \in \Theta_{T,R}$ to $r = r_1 \circ \cdots \circ r_k$ where $r_k \colon T \twoheadrightarrow R_{k-1}$ is located on the maximal linear part $L_{n_k} \rightarrowtail T$, $r_{k-1} \colon R_{k-1} \twoheadrightarrow R_{k-2}$ sits on the obvious maximal linear part $L_{n_{k-1}} \rightarrowtail R_{k-1}$ of the intermediate tree R_{k-1}, and so on. (Note that an epimorphism r_i that appears in such a decomposition can be the identity.) We can define a section d_r of r in the same way as in the special case above, moreover d_r decomposes as

$$d_r = d_{r_1} \circ \cdots \circ d_{r_k},$$

where d_{r_i} is a section of r_i. We can define a partial order on $\Theta_{T,R}$ as follows. Let $r, s \in \Theta_{T,R}$ have the associated decompositions

$$r = r_1 \circ \cdots \circ r_k \quad \text{and} \quad s = s_1 \circ \cdots \circ s_k.$$

We say that $r \leq s$ if for every i there exist intermediate trees R_i, R_{i-1} such that $s_i, r_i \in \Theta_{R_i,R_{i-1}}$ and $r_i \leq s_i$ in the partial order defined in the special case above.

Again, if $sd_r = \mathrm{id}_R$ then $r \leq s$ and we can mimic the rest of the proof of the special case to conclude that $x^r = 0$ for a maximal $r \in \Theta_{T,R}$, arriving again to a contradiction. □

Please cite this article in press as: J.J. Gutiérrez, et al., Dold–Kan correspondence for dendroidal abelian groups, Journal of Pure and Applied Algebra (2010), doi:10.1016/j.jpaa.2010.10.003

Now we are ready to prove the Dold–Kan correspondence theorem.

Theorem 7.3. *The functors* $N: d\mathcal{A}b \longrightarrow d\mathcal{C}h$ *and* $\Gamma: d\mathcal{C}h \longrightarrow d\mathcal{A}b$ *form an equivalence of categories.*

Proof. Propositions 7.1 and 7.2 imply that N and Γ together form an adjoint equivalence where the unit of the adjunction is the natural isomorphism Ψ^{-1} of Proposition 7.2 and the counit is the identity. □

Acknowledgements

We would like to thank Ieke Moerdijk for helpful suggestions and comments and the referee's diligent reading and helpful remarks on an earlier version of the paper. The first-named author wishes to thank the Department of Mathematics at Utrecht University for its hospitality. The first-named author was supported by the Spanish Ministry of Education and Science under grant MTM2007-63277, and by the Catalan Government under grant 2006 BP-B1 00044.

References

[1] C. Berger, I. Moerdijk, Resolution of coloured operads and rectification of homotopy algebras, in: Categories in Algebra, Geometry and Mathematical Physics, in: Contemp. Math., vol. 431, AMS, Providence, RI, 2007.
[2] C. Berger, I. Moerdijk, On an extension of the notion of Reedy category, Math. Z. (in press).
[3] J.M. Boardman, R.M. Vogt, Homotopy invariant algebraic structures on topological spaces, in: Lecture Notes in Math., vol. 347, Springer-Verlag, Berlin, New York, 1973.
[4] D.-C. Cisinski, I. Moerdijk, Dendroidal sets as models for homotopy operads, J. Topol. (in press).
[5] A. Dold, Homology of symmetric products and other functors of complexes, Ann. of Math. (2) 68 (1958).
[6] A. Joyal, Quasi-categories and Kan complexes, J. Pure Appl. Algebra 175 (1–3) (2002) 207–222.
[7] D. Kan, Functors involving c.s.s. complexes, Trans. Amer. Math. Soc. 87 (1958).
[8] T. Leinster, Higher operads, higher categories, in: London Math. Soc. Lecture Note Ser., vol. 298, CUP, 2004.
[9] J. Lurie, Higher Topos Theory, in: Ann. of Math. Stud., vol. 170, Princeton University Press, Princeton, NJ, 2009.
[10] M. Markl, S. Shnider, J. Stasheff, Operads in Algebra, Topology and Physics, in: Math. Surveys Monogr., vol. 96, AMS, Providence, RI, 2002.
[11] I. Moerdijk, I. Weiss, Dendroidal sets, Algebr. Geom. Topol. 7 (2007) 1441–1470.
[12] I. Moerdijk, I. Weiss, On inner Kan complexes in the category of dendroidal sets, Adv. Math. 221 (2) (2009) 343–389.
[13] T. Pirashvili, Dold–Kan type theorem for Γ-groups, Math. Ann. 318 (2) (2000) 277–298.
[14] J. Słomińska, Dold–Kan type theorems and Morita equivalences of functor categories, J. Algebra 274 (1) (2004) 118–137.
[15] C. Weibel, An introduction to homological algebra, in: Cambridge Stud. Adv. Math., vol. 38, CUP, 1994.
[16] I. Weiss, Dendroidal sets, Ph.D. Thesis, University of Utrecht, 2007.

Please cite this article in press as: J.J. Gutiérrez, et al., Dold–Kan correspondence for dendroidal abelian groups, Journal of Pure and Applied Algebra (2010), doi:10.1016/j.jpaa.2010.10.003