

The Batalin-Vilkovisky formalism on fermionic Kähler manifolds

S. Aoyama ¹ and S. Vandoren ²

Instituut voor Theoretische Fysica

Katholieke Universiteit Leuven

Celestijnenlaan 200D

B-3001 Leuven, Belgium

Abstract

We show that the Kähler structure can be naturally incorporated in the Batalin-Vilkovisky formalism. The phase space of the BV formalism becomes a fermionic Kähler manifold. By introducing an isometry we explicitly construct the fermionic irreducible hermitian symmetric space. We then give some solutions of the master equation in the BV formalism.

¹ E-mail : Shogo%tf%fys@cc3.kuleuven.ac.be

² E-mail : Stefan%tf%fys@cc3.kuleuven.ac.be

1. Introduction

The anti-bracket formalism already appeared in the beginning of the history of the BRST quantization. It was a useful tool to discuss renormalization of the non-abelian gauge theory in the BRST quantization^[1]. The formalism has been fairly elaborated by Batalin and Vilkovisky. It became a viable formalism for the BRST quantization of general gauge theories^[2]. For instance, it has been successfully applied^[3] to gauge theories with open algebras^[4]. The geometrical meaning of this BV formalism has been considerably clarified by Witten^[5]. More recently the BV formalism has been set up on a curved supermanifold of fields and anti-fields with a fermionic symplectic structure^[6]. It has been applied to study quantization of the string field theory^[7,8]. The application went far beyond the original motivation of the BRST quantization. As such an example we would also like to mention the work by Verlinde^[9].

In whatever circumstance it is used, the ultimate goal of the BV formalism is to determine the fermionic symplectic structure of the supermanifold and solve the master equation. Therefore it is important to understand the geometry of the fermionic symplectic structure.

In this note we introduce the Kähler structure to the supermanifold, and show that the symplectic structure is reduced to that given by a fermionic Kähler 2-form ω . The Kähler potential, which is the hallmark of such a supermanifold, is then fermionic. Secondly we introduce an isometry to the supermanifold. It is done similarly to the case of the bosonic Kähler manifold^[10]. The only complication is due to sign factors coming from ordering fermionic coordinates. The isometry is realized by the Killing vectors. They are given by a set of real potentials, which we call the Killing potentials. Our main message in this regard is that for a class of supermanifolds the fermionic Kähler 2-form ω can be explicitly constructed out of the Killing vectors of the (bosonic) irreducible hermitian symmetric space^[11]. The supermanifold with a symplectic structure given by this 2-form is called the fermionic irreducible hermitian symmetric space. Finally we are interested in solving the master equation with the fermionic symplectic structure given above. In the first place it is solved for the fermionic \mathbb{CP}^1 space by assuming that the coordinates of the space are space-time independent. If they are not, the master equation requires some appropriate quantum consideration^[12]. We are not going

to be involved in this problem. We only investigate the classical master equation, which is still important to study before quantization. Since it is a non-linear functional equation, many solutions are expected in principle. We find one solution for the fermionic irreducible hermitian symmetric space . It is given in terms of the Killing potentials.

Fermionic Kähler manifolds in the BV formalism have been discussed in the recent paper^[13]. However they did not study the isometry to compute the metric and the Killing potentials of the manifold. These quantities are important to find solutions of the master equation, as we will see.

2. The BV formalism

Let us start with a short review on the BV formalism. Consider a $2D$ manifold parametrized by real coordinates $y^i = (x^1, x^2, \dots, x^D, \xi^1, \xi^2, \dots, \xi^D)$ with x 's and ξ 's bosonic and fermionic respectively. Suppose that it has a symplectic structure given by a non-degenerate 2-form

$$\omega = dy^j \wedge dy^i \omega_{ij}, \quad (2.1)$$

which is closed

$$d\omega = 0. \quad (2.2)$$

These equations read in components

$$(-)^{ik} \partial_i \omega_{jk} + (-)^{ji} \partial_j \omega_{ki} + (-)^{kj} \partial_k \omega_{ij} = 0, \quad (2.3)$$

$$\omega_{ij} = -(-)^{ij} \omega_{ji}. \quad (2.4)$$

Here one should understand the short-hand notation for the grassmannian parity of the coordinates $\varepsilon(y^i) = i$ in the sign factor. By this notation we have $\varepsilon(\omega_{ij}) = i + j + 1$. We define the anti-bracket by

$$\{A, B\} = A \overleftarrow{\partial}_i \omega^{ij} \partial_j B, \quad (2.5)$$

in which ω^{ij} is the inverse matrix of ω_{ij} such that

$$\omega_{ij} \omega^{jk} = \omega^{kj} \omega_{ji} = \delta_i^k. \quad (2.6)$$

Note that the right-derivative $\overleftarrow{\partial}_i$ is related with the right-one by

$$A\overleftarrow{\partial}_i = (-)^{i(\varepsilon(A)+1)}\partial_i A.$$

In terms of ω^{ij} eqs (2.3) and (2.4) become respectively

$$(-)^{(i+1)(k+1)}\omega^{il}\partial_l\omega^{jk} + (-)^{(j+1)(i+1)}\omega^{jl}\partial_l\omega^{ki} + (-)^{(k+1)(j+1)}\omega^{kl}\partial_l\omega^{ij} = 0, \quad (2.7)$$

$$\omega^{ij} = -(-)^{(i+1)(j+1)}\omega^{ji}. \quad (2.8)$$

Owing to eq. (2.7) the anti-bracket (2.5) satisfies the Jacobi identity.

We define a second order differential operator by

$$\Delta \equiv \frac{1}{\rho}(-)^i\partial_i(\rho\omega^{ij}\partial_j), \quad (2.9)$$

with a bosonic function ρ . It satisfies the following properties:

$$\begin{aligned} \Delta\{A, B\} &= \{\Delta A, B\} + (-)^{\varepsilon(A)+1}\{A, \Delta B\}, \\ \Delta(AB) &= \Delta A \cdot B + (-)^{\varepsilon(A)}A\Delta B + (-)^{\varepsilon(A)}\{A, B\}. \end{aligned}$$

Moreover the operator Δ is nilpotent $\Delta^2 = 0$ if ρ obeys the equation

$$\Delta\left[\frac{1}{\rho}(-)^i\partial_i(\rho\omega^{ij})\right] = 0. \quad (2.11)$$

Finally the master equation in the BV formalism is given by

$$\Delta e^S = 0, \quad \text{or} \quad \Delta S + \frac{1}{2}\{S, S\} = 0. \quad (2.12)$$

3. The fermionic Kähler geometry

So far we have discussed the symplectic form of the $2D$ supermanifold. When $D = 2d$, the metric of the manifold γ_{ij} may be defined by

$$\gamma_{ij} = \omega_{ik}J_j^k \quad (3.1)$$

in which

$$J_j^k = \begin{pmatrix} 0 & \mathbb{1} & 0 & 0 \\ -\mathbb{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{1} \\ 0 & 0 & -\mathbb{1} & 0 \end{pmatrix}$$

with the $d \times d$ unit matrix $\mathbb{1}$. We shall now impose the condition on ω_{ij}

$$\omega_{kl} J_i^k J_j^l = \omega_{ij} \quad (3.2)$$

or equivalently

$$\gamma_{kl} J_i^k J_j^l = \gamma_{ij} \quad (3.3)$$

Since J_j^k is a bosonic matrix, the metric γ_{ij} is fermionic, $\varepsilon(\gamma_{ij}) = i + j + 1$. From eqs (2.4), (2.6) and (2.8) we have

$$\begin{aligned} \gamma_{ij} &= (-)^{ij} \gamma_{ji}, \\ \gamma_{ij} \gamma^{jk} &= \gamma^{kj} \gamma_{ji} = \delta_i^k, \\ \gamma^{ij} &= (-)^{(i+1)(j+1)} \gamma^{ji}. \end{aligned} \quad (3.4)$$

We also find the relation

$$\gamma_{ij} \omega^{jk} \gamma_{kl} = -\omega_{il}. \quad (3.5)$$

The affine connection may be defined by postulating

$$D_k \gamma_{ij} \equiv \partial_k \gamma_{ij} - \Gamma_{ki}^l \gamma_{lj} - (-)^{ij} \Gamma_{kj}^l \gamma_{li} = 0.$$

By solving this we obtain the connection

$$\Gamma_{ij}^k = \frac{1}{2} [\partial_i \gamma_{jl} + (-)^{ij} \partial_j \gamma_{il} - \gamma_{ij} \overleftarrow{\partial}_l] \gamma^{lk}, \quad (3.6)$$

which is bosonic, $\varepsilon(\Gamma_{ij}^k) = i + j + k$.

We shall go to the complex coordinate basis $y^i \rightarrow (\mathbf{z}^a, \bar{\mathbf{z}}^a)$ with

$$\mathbf{z}^a = (z^\alpha, \zeta^\alpha), \quad \bar{\mathbf{z}}^a = (\bar{z}^\alpha, \bar{\zeta}^\alpha), \quad \alpha = 1, 2, \dots, d,$$

defined by

$$z^\alpha = x^\alpha + ix^{d+\alpha}, \quad \zeta^\alpha = \xi^\alpha + i\xi^{d+\alpha}, \quad \text{c.c.} \quad .$$

The condition (3.2) or (3.3) reduces to

$$\begin{aligned}\omega_{ab} &= \omega_{\underline{a}\underline{b}} = 0, & \gamma_{ab} &= \gamma_{\underline{a}\underline{b}} = 0, \\ \omega_{a\underline{b}} &= i\gamma_{a\underline{b}}, & \omega_{\underline{a}b} &= -i\gamma_{\underline{a}b},\end{aligned}\tag{3.7}$$

together with \dagger

$$\begin{aligned}\omega_{a\underline{b}} &= -(-)^{ab}\omega_{\underline{b}a}, & \omega_{a\underline{b}}^* &= \omega_{\underline{a}b}, \\ \gamma_{a\underline{b}} &= (-)^{ab}\gamma_{\underline{b}a}, & \gamma_{a\underline{b}}^* &= \gamma_{\underline{a}b}.\end{aligned}\tag{3.8}$$

By means of these equations the symplectic form (2.1) takes the Kähler 2-form

$$\omega = 2id\overline{\mathbf{z}}^{\underline{b}} \wedge d\mathbf{z}^{\underline{a}} \gamma_{a\underline{b}}.\tag{3.9}$$

Then eq. (2.2) or equivalently (2.3) is solved by

$$\gamma_{a\underline{b}} = \partial_a \partial_{\underline{b}} K,\tag{3.10}$$

in which K is a fermionic Kähler potential. Thus the supermanifold acquires a fermionic Kähler geometry as the consequence of eqs (2.4) and (3.2). The affine connection (3.6) is simplified as

$$\Gamma_{ab}^c = \partial_a \gamma_{b\underline{d}} \cdot \gamma^{\underline{d}c}, \quad \Gamma_{\underline{a}\underline{b}}^c = \partial_{\underline{a}} \gamma_{\underline{b}d} \cdot \gamma^{dc},$$

and all the other components are vanishing. The covariant derivative of a holomorphic vector is defined by

$$D_a A_b \equiv \partial_a A_b - \Gamma_{ab}^c A_c, \quad D_a A^b \equiv \partial_a A^b + (-)^{a\varepsilon(A)} A^c \Gamma_{ca}^b.$$

4. The isometry

The fermionic Kähler manifold admits an isometry. It is realized by a set of Killing vectors $V^{Ai}(y)$, $A = 1, 2, \dots, N$, with $\varepsilon(V^{Ai}) = i$ in the real coordinates. They satisfy the Lie algebra of a group G

$$V^{Ai} \partial_i V^{Bj} - V^{Bi} \partial_i V^{Aj} = f^{ABC} V^{Cj},\tag{4.1}$$

\dagger Complex conjugation of fermion products is chosen as $(\zeta\eta)^* = \overline{\zeta}\overline{\eta}$.

with (real) structure constants f^{ABC} . The metric γ_{ij} obeys the Killing condition

$$\mathcal{L}_{V^A} \gamma_{ij} \equiv V^{Ak} \partial_k \gamma_{ij} + \partial_i V^{Ak} \gamma_{kj} + (-)^{ij} \partial_j V^{Ak} \gamma_{ki} = 0, \quad (4.2)$$

$$\mathcal{L}_{V^A} \omega_{ij} \equiv V^{Ak} \partial_k \omega_{ij} + \partial_i V^{Ak} \omega_{kj} - (-)^{ij} \partial_j V^{Ak} \omega_{ki} = 0, \quad (4.3)$$

or equivalently

$$\mathcal{L}_{V^A} \gamma^{ij} \equiv V^{Ak} \partial_k \gamma^{ij} - \gamma^{ik} \partial_k V^{Aj} - (-)^{(i+1)(j+1)} \gamma^{jk} \partial_k V^{Ai} = 0, \quad (4.4)$$

$$\mathcal{L}_{V^A} \omega^{ij} \equiv V^{Ak} \partial_k \omega^{ij} - \omega^{ik} \partial_k V^{Aj} + (-)^{(i+1)(j+1)} \omega^{jk} \partial_k V^{Ai} = 0. \quad (4.5)$$

From consistency of eqs (4.2)~(4.5) and the condition (3.1) we find the constraints on V^{Ai}

$$\partial_k V^{Al} J_i^k J_l^j = -\partial_i V^{Aj}.$$

In the complex coordinates this equation implies that the Killing vectors V^{Ai} are holomorphic:

$$V^{Ai} = (\mathbf{R}^{Aa}(\mathbf{z}), \bar{\mathbf{R}}^{Aa}(\bar{\mathbf{z}})). \quad (4.6)$$

Due to this property eq. (4.2) reduces to the form

$$\partial_c (\mathbf{R}^{Aa} \gamma_{a\bar{b}}) + (-)^{cb} \partial_{\bar{b}} (\bar{\mathbf{R}}^{Aa} \gamma_{ac}) = 0 \quad (\text{Killing equation}). \quad (4.7)$$

It then follows that the Killing vectors \mathbf{R}^{Aa} and $\bar{\mathbf{R}}^{Aa}$ are given by a set of real potentials Σ^A such that

$$\mathbf{R}^{Aa} \gamma_{a\bar{b}} = i \partial_{\bar{b}} \Sigma^A, \quad \bar{\mathbf{R}}^{Aa} \gamma_{\bar{a}b} = -i \partial_b \Sigma^A. \quad (4.8)$$

Σ^A are fermionic and called the Killing potentials. It is worth noting that the isometry transformations given by the Killing vectors (4.6) can be put in the form

$$\begin{aligned} \delta \mathbf{z}^a &= \epsilon^A \mathbf{R}^{Aa} = \{\mathbf{z}^a, \epsilon^A \Sigma^A\}, \\ \delta \bar{\mathbf{z}}^{\bar{a}} &= \epsilon^A \bar{\mathbf{R}}^{Aa} = \{\bar{\mathbf{z}}^{\bar{a}}, \epsilon^A \Sigma^A\}, \end{aligned} \quad (4.9)$$

by eqs (2.5), (3.5), (3.7), (3.8) and (4.8). Here ϵ^A , $A = 1, 2, \dots, N$ are global (real) parameters of the transformations. It is easy to show that by these transformations the Kähler and Killing potentials respectively transform as

$$\delta \Sigma^B = \epsilon^A f^{ABC} \Sigma^C, \quad (4.10)$$

and

$$\delta K = \epsilon^A F^A(\mathbf{z}) + \epsilon^A \overline{F}^A(\overline{\mathbf{z}}), \quad (4.11)$$

with some holomorphic functions $F^A(\mathbf{z})$ and their complex conjugates. Eq. (4.10) can be written by means of the anti-bracket as

$$\{\Sigma^A, \Sigma^B\} = f^{ABC} \Sigma^C. \quad (4.12)$$

Eq. (4.10) can also be put in the form

$$\Sigma^A = i f^{ABC} \mathbf{R}^{Bb} \gamma_{b\underline{c}} \overline{\mathbf{R}}^{C\underline{c}}. \quad (4.13)$$

by multiplying eq. (4.8) by f^{ABC} and using $f^{ABC} f^{ABD} = 2\delta^{CD}$. This way of calculating the Killing potentials Σ^A is more practical than using eq. (4.8), if the metric $\gamma_{a\underline{b}}$ is known.

5. The metric of the fermionic irreducible hermitian symmetric space

The holomorphic Killing vectors \mathbf{R}^{Aa} and $\overline{\mathbf{R}}^{A\underline{a}}$ in eq. (4.6) independently satisfy the Lie-algebra (4.1), i.e.,

$$\mathbf{R}^{Ai} \partial_i \mathbf{R}^{Bj} - \mathbf{R}^{Bi} \partial_i \mathbf{R}^{Aj} = f^{ABC} \mathbf{R}^{Cj},$$

and the complex conjugate. These equations can be solved by

$$\mathbf{R}^{Aa} = (R^{A\alpha}(z), S^{A\alpha}(z, \zeta)), \quad \alpha = 1, 2, \dots, d,$$

with

$$S^{A\alpha} = \zeta^\beta \frac{\partial}{\partial z^\beta} R^{A\alpha},$$

and $\varepsilon(R^{A\alpha}) = 0 = \varepsilon(S^{A\alpha}) - 1$, in which $R^{A\alpha}$ satisfy the Lie-algebra

$$R^{A\alpha} \frac{\partial}{\partial z^\alpha} R^{B\beta} - R^{B\alpha} \frac{\partial}{\partial z^\alpha} R^{A\beta} = f^{ABC} R^{C\beta}. \quad (5.1)$$

These Killing vectors $R^{A\alpha}$ define a bosonic Kähler manifold. For a class of bosonic Kähler manifolds, called the irreducible hermitian symmetric spaces, they can be

explicitly constructed by extending the strategy developed in ref. 15. (The cases of $E_6/SO(10) \otimes U(1)$ and $SU(m+n)/SU(m) \otimes SU(n) \otimes U(1)$ have been worked out there.) They are related to the metric of these manifolds by

$$R^{A\alpha} \bar{R}^{A\beta} = g^{\alpha\beta},$$

with

$$R^{A\alpha} R^{A\beta} = 0 \quad \text{c.c..} \quad (5.2)$$

The fermionic metric γ^{ij} can be given in terms of these bosonic Killing vectors :

$$\gamma^{a\underline{b}} = \begin{pmatrix} \gamma^{z\underline{z}} & \gamma^{z\underline{\zeta}} \\ \gamma^{\zeta\underline{z}} & \gamma^{\zeta\underline{\zeta}} \end{pmatrix} = \begin{pmatrix} 0 & R^{A\alpha} \bar{R}^{A\beta} \\ R^{A\alpha} \bar{R}^{A\beta} & R^{A\alpha} \bar{S}^{A\beta} + S^{A\alpha} \bar{R}^{A\beta} \end{pmatrix} \quad (5.3)$$

$$\gamma^{ab} = \gamma^{\underline{a}\underline{b}} = 0$$

in which $\gamma^{z\underline{z}}, \gamma^{z\underline{\zeta}}, \gamma^{\zeta\underline{z}}$ and $\gamma^{\zeta\underline{\zeta}}$ are $d \times d$ matrices. It indeed satisfies the closure property (2.7) and the Killing condition (4.4) by means of the Lie-algebra (4.1) and the formulae

$$f^{ABC} R^{B\beta} R^{C\gamma} = 0, \quad \text{c.c..}$$

(Here recall that $\omega^{a\underline{b}} = i\gamma^{a\underline{b}}.$) The last formulae are consequences of the condition (5.2)^[14]. The metric (5.3) can be inverted merely by knowing the inverse of $R^{A\alpha} \bar{R}^{A\beta}$, denoted by $g_{\alpha\beta}$:

$$\begin{aligned} \gamma_{\underline{b}a} &= \begin{pmatrix} \gamma_{\underline{z}z} & \gamma_{\underline{z}\zeta} \\ \gamma_{\underline{\zeta}z} & \gamma_{\underline{\zeta}\zeta} \end{pmatrix} \\ &= \begin{pmatrix} -g_{\beta\gamma} g_{\alpha\underline{\delta}} [R^{A\gamma} \bar{S}^{A\underline{\delta}} + S^{A\gamma} \bar{R}^{A\underline{\delta}}] & g_{\beta\alpha} \\ g_{\beta\alpha} & 0 \end{pmatrix} \end{aligned} \quad (5.4)$$

$$\gamma_{ba} = \gamma_{\underline{b}\underline{a}} = 0$$

Thus we have explicitly constructed the closed 2-form (2.1). We call the manifold with a symplectic structure given by this 2-form a fermionic irreducible hermitian symmetric space. For this class of Kähler manifolds the Killing potentials can be explicitly calculated by eq. (4.13).

As an example we show the fermionic \mathbb{CP}^1 space. It is parametrized by the supercoordinates (z, ζ) and their complex conjugates. The $SU(2)$ transformations of the coordinates are given by the Killing vectors:

$$\begin{aligned}\delta z &= \epsilon^A R^A z = i[\epsilon^- + \epsilon^0 z - \frac{1}{2}\epsilon^+ z^2] \\ \delta \zeta &= \epsilon^A \zeta \frac{\partial}{\partial z^\alpha} R^A z = i[\epsilon^0 \zeta - \epsilon^+ z \zeta]\end{aligned}\tag{5.5}$$

and the complex conjugates[†]. We calculate the metric of the fermionic \mathbb{CP}^1 space from eq. (5.3)

$$\gamma^{ab} = \begin{pmatrix} \gamma^{zz} & \gamma^{z\zeta} \\ \gamma^{\zeta z} & \gamma^{\zeta\zeta} \end{pmatrix} = \begin{pmatrix} 0 & (1 + \frac{1}{2}z\bar{z})^2 \\ (1 + \frac{1}{2}z\bar{z})^2 & (1 + \frac{1}{2}z\bar{z})(z\bar{\zeta} + \bar{z}\zeta) \end{pmatrix}.\tag{5.6}$$

Its inverse metric is given by

$$\gamma_{ba} = \begin{pmatrix} \gamma_{zz} & \gamma_{z\zeta} \\ \gamma_{\zeta z} & \gamma_{\zeta\zeta} \end{pmatrix} = \begin{pmatrix} -\frac{z\bar{\zeta} + \bar{z}\zeta}{(1 + \frac{1}{2}z\bar{z})^3} & \frac{1}{(1 + \frac{1}{2}z\bar{z})^2} \\ \frac{1}{(1 + \frac{1}{2}z\bar{z})^2} & 0 \end{pmatrix}.\tag{5.7}$$

Plugging this metric together with the Killing vectors (5.5) in eq. (4.13) we obtain the Killing potentials

$$\begin{aligned}\Sigma^+ &= \frac{\bar{\zeta}}{1 + \frac{1}{2}z\bar{z}} - \frac{\bar{z}(z\bar{\zeta} + \bar{z}\zeta)}{2(1 + \frac{1}{2}z\bar{z})^2}, \\ \Sigma^- &= \frac{\zeta}{1 + \frac{1}{2}z\bar{z}} - \frac{z(z\bar{\zeta} + \bar{z}\zeta)}{2(1 + \frac{1}{2}z\bar{z})^2}, \\ \Sigma^0 &= \frac{z\bar{\zeta} + \bar{z}\zeta}{(1 + \frac{1}{2}z\bar{z})^2}.\end{aligned}\tag{5.8}$$

It is worth checking that eq. (4.8) is indeed satisfied by these quantities. The fermionic Kähler potential follows simply by integrating eq. (3.10) with the metric (5.7):

$$K = \frac{z\bar{\zeta} + \bar{z}\zeta}{1 + \frac{1}{2}z\bar{z}}.\tag{5.9}$$

It is a consistency check of our calculations to see that both potentials given by (5.8) and (5.9) satisfy the properties (4.10) and (4.11) respectively.

[†] We have chosen the structure constants to be $f^{+-0} = -i$. Then the scalar product of the adjoint vectors is given by $a^A b^A = a^0 b^0 + a^+ b^- + a^- b^+$.

6. The master equation

Now we discuss the BV formalism on the fermionic \mathbb{CP}^1 space. The closed symplectic form ω is known explicitly from the metric (5.7) by eq. (3.7). With this symplectic form we define the second order differential operator according to eq. (2.9). Then the function ρ is fixed by the nilpotency condition (2.11). We find the unique $U(1)$ -invariant solution

$$\rho = p + q \frac{i\zeta\bar{\zeta}}{(1 + \frac{1}{2}z\bar{z})^2},$$

with arbitrary constants $p(\neq 0)$ and q . To check this it is useful to note that the metric (5.3) in general satisfies

$$(-)^a \partial_a \gamma^{ab} = 0, \quad \text{c.c..}$$

We may be interested in solving the master equation (2.12) with these ω^{ij} and ρ . The solution is given by

$$S = S_0 - \left(\frac{q}{2p} + r e^{-S_0}\right) \frac{i\zeta\bar{\zeta}}{(1 + \frac{1}{2}z\bar{z})^2}, \quad (6.1)$$

in which S_0 is an arbitrary function of z and \bar{z} , and r is an integration constant. We have assumed that z and ζ have no space-time dependence. They can be interpreted as coupling parameters for physical variables. Then $Z(= e^S)$ looks like the partition function of matrix models or 2-dim. topological conformal field theories, being a function of the coupling space. The BV formalism in the coupling space has been discussed by Verlinde^[9].

One can search for the solutions (6.1) satisfying the classical equation

$$\{S, S\} = 0, \quad (6.2)$$

which implies that S is BRST invariant. Assuming reality of S we find that

$$S = S_0, \quad (6.3)$$

or

$$S = a + b \frac{i\zeta\bar{\zeta}}{(1 + \frac{1}{2}z\bar{z})^2}, \quad (6.4)$$

with some arbitrary constants a and b . In the language of the BRST quantization, the first solution can be taken as a classical limit of the full solution (6.1). Namely its BRST transformation is trivial. By requiring the $SU(2)$ -invariance S_0 is restricted to be constant. The second solution is invariant by the $SU(2)$ transformations.

The master equation (2.12) may be solved also by allowing z and ζ space-time dependence. In this case the first piece of the equation suffers from the singularity $\delta(0)$. An appropriate regularization is necessary. It is not the aim of this letter to discuss regularization of this singularity. Therefore we study the classical master equation (6.2), which is still of great interest. Remarkably there is a solution for the general Kähler group manifold discussed above, although it would not be the unique one. It is given by the action of a 2-dim. field theory

$$S = \int d^2x G \partial_- \Sigma^0. \quad (6.5)$$

Here G is an arbitrary $U(1)$ -invariant function of bosons $z^\alpha(x^+, x^-)$, chiral fermions $\zeta^\alpha(x^+, x^-)$ and their complex coordinates. Σ^0 is the $U(1)$ -component of the Killing potentials given by eq. (4.13). We can verify that this action satisfies eq. (6.2) by calculating

$$\begin{aligned} \{S, S\} &= i(-)^a \partial_a S \gamma^{ab} \partial_b S - i(-)^a \partial_a S \gamma^{ab} \partial_b S \\ &= 2 \int d^2x [-i \partial_a G \gamma^{ab} \partial_b \Sigma^0 + i \partial_a G \gamma^{ab} \partial_b \Sigma^0] \partial_- G \partial_- \Sigma^0 \\ &= -2 \int d^2x [\mathbf{R}^{0a} \partial_a G + \overline{\mathbf{R}}^{0a} \partial_a G] \partial_- G \partial_- \Sigma^0 \\ &= 0, \end{aligned}$$

using eq. (4.8) and $U(1)$ -invariance of G . In the CP^1 case the solution (6.5) can be written in the general form

$$S = \int d^2x [(z\bar{\zeta} + \bar{z}\zeta)f + i(z\bar{\zeta} - \bar{z}\zeta)g] \partial_- \left[\frac{z\bar{\zeta} + \bar{z}\zeta}{(1 + \frac{1}{2}z\bar{z})^2} \right],$$

in which f and g are arbitrary real functions of $z\bar{z}$. The BRST transformations of

the fields are given by $\{z, S\}$ and $\{\zeta, S\}$, so that

$$\begin{aligned}\delta_{BRST}(z\bar{\zeta} + \bar{z}\zeta) &= -2z\bar{z}(1 + \frac{1}{2}z\bar{z})(z\bar{\zeta} + \bar{z}\zeta)g\partial_-\Sigma^0, \\ \delta_{BRST}(z\bar{\zeta} - \bar{z}\zeta) &= -2i z\bar{z}(1 + \frac{1}{2}z\bar{z})[(z\bar{\zeta} + \bar{z}\zeta)f \\ &\quad + (1 + \frac{1}{2}z\bar{z})\{(z\bar{\zeta} + \bar{z}\zeta)f' + i(z\bar{\zeta} - \bar{z}\zeta)g'\}]\partial_-\Sigma^0, \\ \delta_{BRST}(z\bar{z}) &= -2z\bar{z}(1 + \frac{1}{2}z\bar{z})^2g\partial_-\Sigma^0.\end{aligned}$$

Indeed the action (6.5) is invariant by these BRST transformations.

7. Conclusions

An explicit expression of the closed fermionic 2-form ω is of primary interest to start the BV formalism. Hence the supermanifold with such a symplectic structure should be properly understood. In this note we have discussed how the Kähler structure can be incorporated in the supermanifold. The closed fermionic form ω is then reduced to the Kähler 2-form. It has been given by means of the fermionic Kähler potential. Thus a fermionic Kähler manifold appears as the natural phase space of the BV formalism. Secondly we have introduced an isometry to this manifold. The novelty obtained by doing this was that the fermionic Kähler 2-form ω is given by the Killing vectors of the isometry. The existence of the Killing potentials for the isometry is most characteristic in the Kähler group manifold. They are fermionic in our case. We have given the formula to construct them out of the Killing vectors, eq. (4.13). It holds for the general Kähler group manifold. Finally we have studied the master equation with the symplectic structure given above. The space-time independent solution has been given by eq. (6.1). The BRST invariant solution (6.4) happens to be $SU(2)$ -invariant as well. For space-time dependent solutions we have studied the classical master equation. It is solved by eq. (6.5) for the irreducible fermionic hermitian symmetric space. The solution has merely $U(1)$ symmetry. This could be just one of many other solutions. The authors have not yet succeeded in finding a solution which is invariant by the full isometry. Space-time dependent solutions are certainly more interesting from the physical point of view. It deserves further study in order to find new solutions.

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