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Simplifications in Lagrangian BV quantization exemplified by the anomalies of chiral W_3 gravity

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Abstract

The Batalin–Vilkovisky (BV) formalism is a useful framework to study gauge theories. We summarize a simple procedure to find a a gauge–fixed action in this language and a way to obtain one–loop anomalies. Calculations involving the antifields can be greatly simplified by using a theorem on the antibracket cohomology. The latter is based on properties of a 'Koszul–Tate differential', namely its acyclicity and nilpotency. We present a new proof for this acyclicity, respecting locality and covariance of the theory. This theorem then implies that consistent higher ghost terms in various expressions exist, and it avoids tedious calculations.

This is illustrated in chiral W_3 gravity. We compute the one–loop anomaly without terms of negative ghost number. Then the mentioned theorem and the consistency condition imply that the full anomaly is determined up to local counterterms. Finally we show how to implement background charges into the BV language in order to cancel the anomaly with the appropriate counterterms. Again we use the theorem to simplify the calculations, which agree with previous results.

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1 Introduction

The Batalin–Vilkovisky (BV) [1] method uses the nice mathematical structure of Poisson–like brackets (antibrackets), canonical transformations, ... which are the attractive features of the Hamiltonian language. But one can use the Lagrangian formalism and in this way keep the advantages of a covariant formalism. Antifields, Φ^*_A, are the canonically conjugate variables of the ordinary fields Φ^A (which include already the ghosts). It is well–known that the BV method is useful for constructing a gauge–fixed action for a broad range of gauge theories. It can be applied for all gauge theories which are known today. This includes the cases that the algebra of gauge transformations closes only modulo field equations ('open algebras'), that there are structure functions rather than constants ('soft algebras'), that the transformations may not be independent ('reducible algebras'), ... In this language gauge fixing is just a canonical transformation in the space of fields and antifields [2, 3, 4, 5]. On the other hand, the formalism also has been proven to be useful to study one–loop anomalies [3, 5, 6]. In these quantum aspects, it is close to the formalism introduced by Zinn–Justin where sources of the BRST–transformation were introduced [7]. These sources are the antifields of the BV formalism. Also from a geometrical point of view the antifields have a natural origin, as was explained in [8]. More recently, the geometry of the BV formalism was discussed in [9]. However, including these antifields seems unattractable and very laborious. In this article we will show how one can simplify the calculations, not including the full load of antifields in intermediate steps, and still obtain the terms depending on antifields at the end. We will make use of the ideas of the Koszul–Tate [10] differential introduced in [11] in this subject. An important theorem, which we will prove below, will be our main tool. This theorem considers the existence of a local function $F(\Phi, \Phi^*)$ satisfying a consistency condition. It shows that this function is determined by the terms of non–negative ghost numbers (which in some basis, the 'classical basis', is $F(\Phi, 0)$). Further it shows in how far the remaining terms are fixed by the consistency condition. In the proofs we follow the ideas of [11]. However, we present new proofs which show the locality and covariance of the expressions.

We will apply our methods to the calculation of the one–loop anomalies in chiral W_3–gravity. These have already been obtained in [12, 13, 14, 15, 16]. The procedure which we follow is based on a Pauli–Villars (PV) regularization [17, 18, 19]. The latter can be done at the level of the action (the path integral), and this guarantees that one obtains a consistent anomaly. (We will also prove this directly from the final expression for the one–loop anomaly in appendix A). Performing the anomaly calculations for the W_3–model including the antifields would lead to unmanageable expressions. But the theorem mentioned before, allows us to work with simple expressions, and at the end to find the complete result. This calculation will demonstrate how the formalism can be used in more complicated situations than those discussed previously [18, 3, 19, 20, 21, 22].

We will start this paper with a short review of the BV formalism. Antifields, the antibracket and the classical extended action are the ingredients to explain gauge fixing by a canonical transformation. Further also the one–loop quantum theory is discussed (section 2). We repeat the essential formulae to obtain anomalies, and explain why antifields have to be included. Inclusion of background fields becomes unnecessary. Section 3 is more formal. In section 3.1 we define some important technical ingredients like locality, stationary surface and properness of the extended action. In section 3.2 we introduce the

Koszul–Tate differential and in section 3.3 we prove its acyclicity, respecting the locality and covariance of the theory. Then we come to our main theorem, (section 3.4), where we prove the equivalence between the antibracket cohomology and the weak BRST cohomology. This theorem can be applied for the analysis of anomalies. Indeed, in section 4 we calculate the one–loop anomaly for chiral W_3 gravity using the simplifications. This calculation is a clear example of the general method. Finally, in the section 5, we introduce background charges in order to cancel the anomaly. We show how this modifies the usual expansion of the quantum master equation, but it still fits in the general framework. The theorem on the antibracket cohomology can again be used to simplify the analysis. In the concluding section we will also comment on the higher loop anomalies.

2 The essential ingredients of the BV–formalism

We will give here a summary of the way in which we use the BV–formalism. Our methods are now simpler and more transparant than the original ones of Batalin and Vilkovisky. A short review has been given in [5], which will be supplemented in [23]. A complete review will be given in [6].

2.1 The classical formalism

We denote by $\{\Phi^A\}$ the complete set of fields. It will includes the ghosts for all the gauge symmetries, and possibly auxiliary fields introduced for gauge fixing. We will see below that sometimes we do not need more than the classical fields and the ghosts. Then one doubles the space of field variables by introducing antifields Φ^*_A, which play the role of canonical conjugate variables with respect to a Poisson–like structure. This is defined by means of 'antibrackets', whose canonical structure is

$$(\Phi^A,\, \Phi^*_B) = \delta^A{}_B \ ; \qquad (\Phi^A, \Phi^B) = (\Phi^*_A, \Phi^*_B) = 0 \ . \tag{2.1}$$

The antifields Φ^*_A have opposite statistics than their canonically conjugate field Φ^A. In general the antibracket of $F(\Phi^A, \Phi^*_A)$ and $G(\Phi^A, \Phi^*_A)$ is defined by

$$(F, G) = F\overleftarrow{\partial}_A \cdot \overrightarrow{\partial}^A G - F\overleftarrow{\partial}^A \cdot \overrightarrow{\partial}_A G \ , \tag{2.2}$$

using the notations

$$\partial_A = \frac{\partial}{\partial \Phi^A} \ ; \qquad \partial^A = \frac{\partial}{\partial \Phi^*_A}, \tag{2.3}$$

and $\overleftarrow{\partial}$ and $\overrightarrow{\partial}$ stand for right and left derivatives acting on the object before, resp. behind, the symbol ∂. The separating symbol $\cdot$ is often useful to indicate up to where the derivatives act, if they are not enclosed in brackets. Note that this antibracket is a fermionic operation, in the sense that the statistics of the antibracket (F, G) is opposite to that of FG. The antibracket also satisfies some graded Jacobi–relations:

$$(F, (G, H)) + (-)^{FG+F+G}(G, (F, H)) = ((F, G), H). \tag{2.4}$$

We assign **ghost numbers** to fields and antifields. These are integers such that

$$gh(\Phi^*) + gh(\Phi) = -1 \ , \tag{2.5}$$

and therefore the antibracket (2.2) raises the ghost number by 1.

We will often perform canonical transformations in this space of fields and antifields [24]. These are the transformations such that the new basis again satisfies (2.1). We will also always respect the ghost numbers. It is clear that interchanging the name field and antifield of a canonical conjugate pair ($\phi' = \phi^*$ and $\phi'^* = -\phi$) is such a transformation. The new antifield has the ghost number of the old field. From (2.5) we see then that there is always a basis in which all fields have positive or zero ghost numbers, and the antifields have negative ghost numbers. We will often use that basis. It is the natural one from the point of view of the classical theory, and therefore we will denote it as the 'classical basis'. We will see below that it is not the most convenient from the point of view of the path integral.

One defines an 'extended action', $S(\Phi^A, \Phi^*_A)$, of ghost number zero, whose antifield independent part $S(\Phi^A, 0)$ is at this point the classical action, and which satisfies the *master equation*

$$(S, S) = 0 \; . \tag{2.6}$$

This equation contains the statements of gauge invariances of the classical action, their algebra, closure, Jacobi identities, ... In the enlarged field space, gauge fixing is obtained by a canonical transformation (see example below). This transformation is chosen such that the new antifield–independent part of S has no gauge invariances. In this new basis some antifields will have positive or zero ghost numbers, so it is not any more of the type mentioned above. Choosing different canonical transformations satisfying the above requirement amounts to different gauge choices.

A simple example is 2–dimensional chiral gravity. The classical action is

$$S_0 = \int d^2x \left[-\tfrac{1}{2}\partial X^\mu \cdot \bar\partial X^\mu + \tfrac{1}{2} h \, \partial X^\mu \cdot \partial X^\mu \right] \; . \tag{2.7}$$

In the extended action appears a ghost c related to the reparametrization invariance. The fields are then $\Phi^A = \{X^\mu, h, c\}$. The extended action is

$$\begin{aligned} S = S_0 + \int d^2x \Big[& X^*_\mu \, c\partial X^\mu + h^* \left(\bar\partial - h\partial + (\partial h) \right) c \\ & -c^* \, c \, \partial c \Big] \; . \end{aligned} \tag{2.8}$$

Added to the classical action, one finds here the antifields multiplied with the transformation rules of the classical fields in their BRST form. One may then check with the above definitions that the vanishing of $(S, S)|_{\Phi^*=0}$ expresses the gauge invariance. The second line contains in the same way the BRST transformation of the ghost. It is clear that this is determined by the previous line and (2.6) (BRST invariance). This is a trivial example of a principle which we want to stress in this article. Often antifield dependent terms are already determined by requirements as (2.6), such that one does not have to specify them explicitly, and one knows that a solution exist without having to make tedious checks of the higher ghost terms[1]. In order that these terms are uniquely defined (up to canonical transformations) we will need a requirement of 'properness', which in simple words means that ghosts are introduced for *all* the gauge invariances, with the appropriate terms in the extended action. An exact definition will be given in section 3.1.

[1]Some may remember the checks of quintic ghost terms in the BRST transformations of supergravity actions.

In this example gauge fixing is obtained by the canonical transformation where h and h^* are replaced by b and b^*:

$$b = h^* \; ; \qquad b^* = -h \; . \tag{2.9}$$

One checks then that the part of S depending only on the new 'fields', i.e. X^μ, b and c, which is $\left[-\frac{1}{2}\partial X^\mu \cdot \bar{\partial} X^\mu + b\bar{\partial}c\right]$, has no gauge invariances. b, which has its origin as antifield of the classical field h, is now considered as a field (the antighost) of negative ghost number. In more complicated situations one may first need to include 'non–minimal' fields, i.e. other fields than classical or ghost fields, to be able to perform a canonical transformation such that the action in the new fields is 'gauge–fixed'. When quantizing, we will work in this basis, to which we will refer as the 'gauge–fixed basis'. It is such that the extended action takes the form

$$S(\Phi, \Phi^*) = S_{gauge-fixed}(\Phi) + \text{ antifield–dependent terms} \; . \tag{2.10}$$

'Gauge fixed' means that in the new definitions of fields the matrix of second derivatives w.r.t. fields, S_{AB}, is non–singular when setting the field equations equal to zero. For more details on this procedure we refer to [1, 4, 5, 6].

2.2 The one–loop theory

This gauge–fixed action is still a classical action in the sense that no loops ($\hbar$ corrections) are included. The full quantum (extended) action $W(\Phi, \Phi^*)$ can be expanded in powers of $\hbar$:

$$W = S + \hbar M_1 + \hbar^2 M_2 + \ldots \; . \tag{2.11}$$

In the previous subsection we discussed the form of S, but there is no reason to assume that this would have no $\hbar$ dependent corrections, which are the M_i. For a local field theory we want also integrals of local functions for these M_i. Note that as we are discussing the extended action, this includes at once the quantum corrections to the transformation laws. The expansion (2.11) is the usual one, but we will see in section 5 that terms of order $\sqrt{\hbar}$ can appear.

This quantum action appears in the path integral

$$Z(J, \Phi^*) = \int \mathcal{D}\Phi \exp\left(\frac{i}{\hbar} W(\Phi, \Phi^*) + J(\Phi)\right) \; , \tag{2.12}$$

where we have introduced sources $J(\Phi)$, and in this subsection we use the gauge–fixed basis. The gauge fixing which we discussed, can be seen as the procedure to select out of the $2N$ variables, Φ^A and Φ^*_A, the N variables over which one integrates in this integral (the 'Lagrangian submanifold'). To define the path integral properly one has to discuss regularization, which can be seen as a way to define the measure. In gauge theories, in general one can not find a regularization which respects all the gauge symmetries. Then it is possible that symmetries of the classical theory are not preserved in the quantum theory. Anomalies are the expression of this non–invariance. If there are no anomalies, then the quantum theory does not depend on the gauge fixing, i.e. on the particular choice of variables Φ^A used for the integral in (2.12) (as long as S_{AB} is non–singular). That does not hold when there are anomalies. In that case the quantum theory will have a different content than the classical theory. One can obtain in this way induced theories, which from our point of view are theories where antifields become propagating fields (this point will be explained in [23]). In [3] it was shown how to use the BV framework to investigate the anomaly structure. For a theory to be free of anomalies, W has to be a solution of the master equation

$$(W,W) = 2i\hbar\Delta W \ , \tag{2.13}$$

where

$$\Delta = (-)^A \vec{\partial}_A \vec{\partial}^A \, . \tag{2.14}$$

In powers of $\hbar$ the first two equations (zero–loop and one–loop) are

$$\begin{aligned} &(S,S) = 0 \\ &i\mathcal{A} \equiv i\Delta S - (M_1, S) = 0 \ . \end{aligned} \tag{2.15}$$

The first one is the (classical) master equation discussed before. The second one is an equation for M_1. In a local field theory we will moreover demand that M_1 is an integral of a local function (exact definitions are given in section 3.1). If there does not exist such an M_1 then $\mathcal{A}$ is called the anomaly. It is clearly not uniquely defined, as M_1 is arbitrary. It satisfies

$$(\mathcal{A}, S) = 0 \ , \tag{2.16}$$

which is a reformulation of the Wess–Zumino consistency conditions [25]. But as mentioned, we need a regularization procedure. In the expressions above, we notice this because on any integral of a local function, the Δ operator is proportional to $\delta(0)$.

To do so within the context of path integrals, one can not use dimensional regularization as there is no good definition of the action in arbitrary (non–integer) dimension. The method of [18, 3, 19, 5, 6] is based on Pauli–Villars regularization [17]. This can be done in the path integral and in this way one obtains consistent anomalies. We will use this method in section 4, and will now briefly repeat the essential features.

The new ingredient is a mass matrix which is introduced for the PV fields: T_{AB}, such that

$$S^{PV}_M = -\tfrac{1}{2} M^2 \Phi^A_{PV} T_{AB} \Phi^B_{PV} \ , \tag{2.17}$$

where Φ^A_{PV} are the PV partners of all the fields. This matrix should be invertible, at least in the set of propagating fields. Its inverse serves as index–raising metric in the space of fields. The regulator, $\mathcal{R}$, is defined from the (non–singular) matrix of second derivatives of the extended action

$$\mathcal{R}^A{}_B = (T^{-1})^{AC} \, S_{CB} \ ; \qquad S_{AB} = \vec{\partial}_A S \overleftarrow{\partial}_B \ . \tag{2.18}$$

Note that in general S_{AB} contains antifields. The anomaly is then obtained as the non–invariance of the mass term (2.17) after integrating over the PV fields. This leads to

$$\Delta S = \lim_{M^2 \to \infty} Tr \left[J \frac{1}{1 - \mathcal{R}/M^2} \right] \ , \tag{2.19}$$

where J is given in terms of the transformation matrix K, the derivative of the extended action w.r.t. an antifield and a field, as

$$J^A{}_B = K^A{}_B + \tfrac{1}{2} (T^{-1})^{AC} \, (T_{CB}, S) \, (-)^B \ ; \qquad K^A{}_B = S^A{}_B \ . \tag{2.20}$$

Note that in general S_{AB} contains antifields (even for closed algebras). Also the other matrices may be antifield dependent, and thus ΔS also contains antifields. After some further steps, we may write this regularized value of ΔS also as

$$\Delta S = \lim_{M^2\to\infty} Tr\left[J\ \exp(\mathcal{R}/M^2)\right]\ , \tag{2.21}$$

The regularized value thus depends on the choice of the matrix T. Different T's correspond to different regularization schemes. In our example we will use matrices T which have neither fields nor antifields, and thus $J = K$. Sometimes it is useful to take a field–dependent mass matrix [19]. In appendix A we explicitly check that this expression for ΔS satisfies the consistency condition

$$(S, \Delta S) = 0\ . \tag{2.22}$$

Alternative formulations of the expression for the anomaly are given in [26] (see also appendix A). They show that there is always an M_1 such that (2.15) is satisfied. However, this expression is in general non–local. Anomalies appear if no local expression M_1 can be found satisfying that equation.

To evaluate the traces, one can put the operator between a basis of plane waves. So

$$\begin{aligned} Tr\left[K\exp\frac{\mathcal{R}}{M^2}\right] &= \int d^dx \int d^dy \delta(x-y)K(x)e^{\mathcal{R}_x/M^2}\delta(x-y) \\ &= \int d^dx \int \frac{d^dk}{(2\pi)^d} e^{-ikx}Ke^{\mathcal{R}_x/M^2}e^{ikx} \end{aligned} \tag{2.23}$$

Then one pulls the e^{ikx} to the left, replacing derivatives by $\partial + ikx$, and one takes the trace. However, this whole procedure is included in the results of the heat kernel method [27]. The heat kernel is the expression

$$e^{t\mathcal{R}_x}\delta(x-y) = G(x,y;t;\Phi)\ , \tag{2.24}$$

which we thus need for $t = 1/M^2$. This has been considered for general second order differential operators $\mathcal{R}$

$$\mathcal{R}_x[\Phi] = \frac{1}{\sqrt{g}}\left(\partial_\alpha \mathbb{1} + \mathcal{Y}_\alpha\right)\sqrt{g}\, g^{\alpha\beta}\left(\partial_\beta \mathbb{1} + \mathcal{Y}_\beta\right) + E\ , \tag{2.25}$$

where $\Phi = \{g^{\alpha\beta}, \mathcal{Y}_\alpha, E\}$. The latter two can be matrices in an internal space. The restriction here is that the part which contains second derivatives is proportional to the unit matrix in internal space. (Note that in the application for W_3, which will be considered in section 4, we will have a more general regulator, but we will be able to reduce it to this case by some expansion). Further in principle g should be a positive definite matrix. For Minkowski space, we have to perform first a Wick rotation, which introduces a factor $-i$ in the final expression, and thus we have to use

$$Tr\left[K\exp\frac{\mathcal{R}}{M^2}\right] = -i\int d^dx \int d^dy \delta(x-y)K(x)G\left(x,y;\frac{1}{M^2}\right)\ . \tag{2.26}$$

The ‘early time’ expansion of this heat kernel is as follows

$$G(x,y;t;\Phi) = \frac{\sqrt{g(y)}\,\Delta^{1/2}(x,y)}{(4\pi t)^{d/2}} e^{-\sigma(x,y)/2t} \sum_{n=0} a_n(x,y,\Phi) t^n \ , \tag{2.27}$$

where $g = |det\ g_{\alpha\beta}|$, and $\sigma(x,y)$ is the ‘world function’, which is discussed at length in [28]. It is half the square of the geodesic distance between x and y

$$\sigma(x,y) = \frac{1}{2} g_{\alpha\beta}(y-x)^\alpha (y-x)^\beta + \mathcal{O}(x-y)^3 \ . \tag{2.28}$$

Further, $\Delta(x,y)$ is defined from the ‘Van Vleck–Morette determinant’ (for more information on this determinant, see [29])

$$\mathcal{D}(x,y) = \left| det \left(-\frac{\partial^2 \sigma}{\partial x^\alpha \partial y^\beta} \right) \right| \tag{2.29}$$

as

$$\mathcal{D}(x,y) = \sqrt{g(x)}\,\sqrt{g(y)}\,\Delta(x,y) \ . \tag{2.30}$$

At coincident points it is 1, and its first derivative is zero.

The ‘Seeley–DeWitt’ coefficients $a_n(x,y,\Phi)$ have been obtained using various methods for the most important cases. In two dimensions we only need a_0 and a_1. For most applications we only need their value and first and second derivatives at coincident points. Let us still note that this expansion also appears in quantum mechanics, and vice versa results of quantum mechanics can be used to prove parts of the heat kernel expansion [30].

If the calculations are done in a specific gauge, one can not see at the end whether one has obtained gauge–independent results. Therefore one often includes parameters in the choice of gauge, or background fields. The question then remains whether this choice was ‘general enough’ to be able to trigger all possible anomalies. E.g. for the above example the gauge $h = 0$ would not show the anomaly, which is of the form $\int d^2x\, c\, \partial^3 h$. Instead, a gauge choice $h = H$, where the latter is a background field, is sufficient. For Yang–Mills gauge theories on the other hand, it is difficult to see how much generalization of the gauge $\partial^\mu A_\mu = 0$ is necessary to obtain all possible anomalies. In the BV formalism, no background fields are necessary, but one keeps the antifield–dependent terms through all calculations. The anomalies are reflected at the end by dependence on these antifields. In our example, we would obtain an anomaly $\int d^2x\, c\, \partial^3 b^*$. In this formalism it is then also clear how anomalies change by going to other gauges. The relation is given by canonical transformations.

3 Antifield dependence by Koszul–Tate acyclicity

Inclusion of antifields in all calculations can make them very difficult and tedious. In this section we will show that in various expressions, in particular for the anomaly, the knowledge of the part which does not depend on fields of negative ghost number, determines the full expression up to certain counterterms. Indeed, the knowledge that the anomaly satisfies the consistency condition (2.22) gives extra information. We will here investigate what we can derive from it. In general the problem which we have to solve, is to determine a quantity $F(\Phi^A, \Phi^*_A)$ restricted by

$$(F, S) = 0 \ , \tag{3.1}$$

when we know F^0, which is the part of F where all fields with negative ghost number are omitted. In this section we will use the 'classical basis', where all the fields at negative ghost number are considered to be the antifields, so $F^0 = F(\Phi^A, 0)$. We will see that the problem only has a solution if F^0 satisfies a certain condition. In that case all the other terms of F can be determined up to terms (G, S) using a certain expansion which we will define below. We thus solve a cohomology problem of the nilpotent operator $\mathcal{S}F = (F, S)$, knowing the antifield–independent part of F. In section 4 we will illustrate how this simplifies the calculation of the one–loop anomaly for W_3–gravity.

The essential ingredient in the proofs is the Koszul–Tate (KT) differential, working on the space of antifields. This operator and its properties have been studied in detail in [11]. It has been used first of all to prove the existence and uniqueness of a solution to the master equation.

Most of the statements in this section have been derived already in [11]. The difference in the derivation is that in these articles it was assumed that in gauge theories, the field equations can be split in dependent and independent ones. E.g. in Maxwell theory, the field equations $y_\mu \equiv \partial^\nu F_{\mu\nu} = 0$ are four equations, which are dependent: $\partial^\mu y_\mu = 0$. It is indeed true that these can be split in three independent ones, and one dependent on the others. However one can not do this splitting in a local (and/or covariant) way. There is no local expression of one of the constraints in terms of the others. If that were possible, then we could just drop the corresponding field and the symmetry. In our proofs we will always avoid such splittings. We will only work with a set of functions, to be denoted by the set of 'local functions' (see subsection 3.1), which does not allow such a split. Therefore all the quantities which we construct will be local functions (and possibly covariant under a rigid symmetry group). We are then able to prove the statements about the cohomology for these local functions.

There are of course certain weak conditions on the theories under considerations. E.g. the field equations should allow at least one solution. Further conditions were referred to in [31, 11] as defining 'regular theories'. We will extend the set of theories for which the theorems are applicable. The previous definition of 'regular theories' excluded already the simple 2–dimensional theories which we use as examples here. We will introduce the notion of evanescent functions to be added to the functions on the stationary surface.

It is clear that we first need a set of definitions (subsection 3.1). We define the 'stationary surface', using 'classical' field equations, and specify the meaning of the 'local functions' which we consider. Also the 'properness condition' will be discussed. Then we will be able to define our expansion and the KT differential in the second subsection. The main property of the KT differential, its acyclicity, will be the subject of subsection 3.3, where we give the essential ideas, while the technical proofs are in appendix B. Similarly, the ideas of the antibracket cohomology are in subsection 3.4, while the corresponding proofs are in appendix C.

We will work here in general under the assumption that there is an extended action $S(\Phi, \Phi^*)$, which satisfies the master equation (2.6). However, with slight modifications the strategy of these proofs can be used also to prove first the KT–acyclicity before we have the extended action. Then this provides the proof of the existence and uniqueness of the extended action as in [11]. Our improvements thus apply also to these statements, and the proofs will be presented in that way in [6].

3.1 The technical ingredients.

As mentioned, we use the 'classical basis' where antifields have negative ghost numbers, and fields have zero or positive ghost numbers. For further use, we now give special names to the fields of each ghost number. The fields of ghost number zero are denoted by ϕ^i. Those of ghost number 1 are denoted by c^a, and those of ghost number $k+1$ are written as c^{a_k}. (In this way the index i can also be denoted as a_{-1} and for c^a we can also write c^{a_0}).

$$\{\Phi^A\} \equiv \{\phi^i,\, c^a,\, c^{a_1}, \ldots\}\ . \tag{3.2}$$

When one starts from a classical action, one directly obtains the above structure, where ϕ^i are the classical fields, c^a are the ghosts, and the others are 'ghosts for ghosts'. In fact, the structure below with the KT differential is used in the proof that starting from a classical action one can always construct solutions of the master equation with zero ghost number [31, 11, 6].

The antifields thus have negative ghost number and we define the **antifield number** (afn) as

$$afn(\Phi^*_A) = -gh(\Phi^*_A) > 0\ ; \qquad afn(\Phi^A) = 0\ . \tag{3.3}$$

With the above designation of names to the different fields we thus have

$$afn(\phi^*_i) = 1\ ; \qquad afn(c^*_{a_k}) = k+2\ . \tag{3.4}$$

Then every expression can be expanded in terms with definite antifield number. E.g. for the extended action, we can define

$$S = \sum_{k=0} S^k\ ; \qquad S^k = S^k\left(c^*_{a_{k-2}}, c^*_{a_{k-3}}, \ldots, \phi^i, \ldots, c^{a_{k-1}}\right)\ , \tag{3.5}$$

where the range of fields and antifields which can occur in each term follows from the above definitions of antifield and ghost numbers, and the requirement $gh(S) = 0$.

When constructing a theory, one should specify the set of functions of ϕ^i, denoted by $\mathcal{F}^0$, which is considered. An important aspect of the quantization procedure which we present here is that we consider **'local functions'**. This means that they depend on ϕ^i and a finite number of their derivatives at one space–time point (no integrals). In general we need more restrictions, which depend on the theory. E.g. one should specify whether a square root of the field is in the set $\mathcal{F}^0$. This set should contain at least the fields themselves, and other functions which appear in the action and transformation rules. For some applications one may also consider non–local functions (see e.g. [20, 22]). In each case it will be important to define exactly what is meant by the set of functions $\mathcal{F}^0$. For convenience we will call this set 'the local functions'.

S^0 depends only on ϕ^i. This is 'the classical action'. From this action we obtain the classical field equations

$$y_i \equiv S^0 \overleftarrow{\partial}_i \approx 0 \ . \tag{3.6}$$

The field configurations which satisfy these field equations form the **stationary surface**. We define the symbol ≈ 0 (weakly zero) by

$$F \approx 0 \iff F = y_i G^i \ , \tag{3.7}$$

where G^i are some local and regular functions $\in \mathcal{F}^0$.

In some cases one can define $\mathcal{F}^0$ such that any function in this set which vanishes on the stationary surface is proportional to the field equations, and thus ≈ 0. Then the theory is called *regular*. This is not always the case. In our example of 2–dimensional chiral gravity, (2.7), the field equations are

$$\begin{aligned} y_\mu &= -\bar{\partial}\partial X^\mu + \partial\left(h(\partial X^\mu)\right) \approx 0 \\ y_h &= -\tfrac{1}{2}\partial X^\mu \cdot \partial X^\mu \approx 0 \ . \end{aligned} \tag{3.8}$$

Now ∂X^μ vanishes on the stationary surface, but it is not proportional to a field equation. Therefore we say that ∂X^μ is not weakly zero. Functions vanishing on the stationary surface, but not weakly zero, can be called 'evanescent functions'.

Within the set of functions $\mathcal{F}^0$ the relation $\approx$, defines equivalence classes. These are the functions on the stationary surface and the classes of evanescent functions. Further on, we will use the terminology 'functions on the stationary surface' in the sense that the evanescent functions are included. We do so because the definition (3.7) will be important further. To characterize the equivalence classes, we select from each one a representative function. The set of representative functions will be denoted by $\mathcal{F}^0_s \subset \mathcal{F}^0$. In the example, this set can be defined in the following way. We use the first field equation to write any function of $\bar{\partial}\partial X$ as $\partial\left(h(\partial X^\mu)\right)$, so that the functions $\mathcal{F}^0_s$ depend on X, $\bar{\partial}^n X$, $\partial^n X$, but not on $\bar{\partial}^m\partial^n X$ for $m, n > 0$. The second field equation further restricts this set of functions. It removes functions proportional to $\partial X^\mu \cdot \partial X^\mu$.

Consider now any function $F \in \mathcal{F}^0$. Then we denote the representative of the class of functions which equals F on the stationary surface by $F_0 \in \mathcal{F}^0_s$. We have thus $F = F_0 + y_i G^i$. The G^i may be not unique. This is the case if there are gauge invariances:

$$y_i R^i{}_a = 0 \ . \tag{3.9}$$

Note that we use a, the index of the ghosts, for denoting all the relations of this kind (we will give below a more exact definition). We will see that we need a ghost corresponding to each such relation. Anyway, we can build an expansion

$$F = F_0 + y_i F_1^i + \tfrac{1}{2} y_i y_j F_2^{ji} + \dots \ . \tag{3.10}$$

The coefficient functions $F_0, F_1^i, F_2^{ij}, \dots$ should belong to the restricted set of the functions $\mathcal{F}^0_s$. This expansion has of course graded symmetric coefficients. It defines a derivative

$$\frac{\overrightarrow{\partial}}{\partial y_i} F \equiv F_1^i + y_j F_2^{ji} + \dots \ , \tag{3.11}$$

which is not uniquely defined if there are gauge invariances, (3.9). Then the derivative of F w.r.t. y_i is only defined up to terms $R^i{}_a\epsilon^a$, where ϵ^a is an arbitrary local function. We demand there is no other indefiniteness in the definition of the derivative $\partial F/\partial y_i$. This is equivalent to the statement that our set of coefficient functions $R^i{}_a$ be complete in the following sense: if for any set of local functions $T^i(x) \in \mathcal{F}^0$

$$y_i T^i(x) = 0 \implies T^i(x) = R^i{}_a \mu^a(x) + y_j v^{ji}(x) \tag{3.12}$$

where $\mu^a(x)$ and $v^{ij}(x)$ are local functions, the latter being graded antisymmetric

$$v^{ij}(x) = (-)^{ij+1} v^{ji}(x) \ . \tag{3.13}$$

To prove this equivalence, one constructs μ^a perturbatively in powers of y_i.

The fields ϕ^i are only a subset of the total set of fields Φ^A and Φ^*_A, which we will collectively denote by z^α. The set of functions of ϕ^i denoted by $\mathcal{F}^0$ can be extended to functions of z^α. We will define the set $\mathcal{F}$ as polynomials in the fields (and antifields) of ghost number different from zero, with coefficients in $\mathcal{F}^0$. We have then the following structure

$$\mathcal{F} \supset \mathcal{F}^0 \supset \mathcal{F}^0_s \ . \tag{3.14}$$

Further we can define mappings

$$\begin{aligned} P \ &: \ \mathcal{F} \rightarrow \mathcal{F}^0 \\ &\quad F(z) \mapsto P(F)(\phi^i) = F\left(\Phi^*_A = 0, \phi^i, c^{a_k} = 0\right) \\ p \ &: \ \mathcal{F}^0 \rightarrow \mathcal{F}^0_s \\ &\quad f(\phi^i) \mapsto f_0(\phi^i) \in \mathcal{F}^0_s \text{ and } f_0 \approx f \ . \end{aligned} \tag{3.15}$$

For general local functions $F \in \mathcal{F}$, the symbol $\approx$ will be used in the following sense. These functions are defined as an expansion in the fields of non–zero ghost number with coefficients in $\mathcal{F}^0$. Now one applies in each coefficient the relation $\approx$ in $\mathcal{F}^0$. In other words, one uses the field equations of $S^0(\phi)$ and leaves the fields of non–zero ghost number as they are. So (3.7) still applies.

The master equation $(S,S) = 0$ implies relations typical for a general gauge theory. In the collective notation of fields and antifields, the master equation takes the form

$$S\overleftarrow{\partial}_\alpha \cdot \omega^{\alpha\beta} \cdot \overrightarrow{\partial}_\beta S = 0 \ \text{ with } \ \omega^{\alpha\beta} = (z^\alpha, z^\beta) \ . \tag{3.16}$$

We also introduce the Hessian

$$Z_{\alpha\beta} \equiv P\left(\overrightarrow{\partial}_\alpha S \overleftarrow{\partial}_\beta\right) \ . \tag{3.17}$$

Because S has zero ghost number, $Z_{\alpha\beta}$ is only non–zero if $gh(z^\alpha) + gh(z^\beta) = 0$. This implies that its non–zero elements are $Z_{ij} = S^0_{ij} = \overrightarrow{\partial}_i S^0 \overleftarrow{\partial}_j$, $Z^i{}_a$, $Z^a{}_{a_1}$, ..., $Z^{a_k}{}_{a_{k+1}}$. Note that upper indices of Z appear here because derivatives are taken w.r.t. antifields Φ^*_A. Of course also elements as $Z_{a_{k+1}}{}^{a_k}$ are non–zero, being the supertransposed of the above expressions.

From the ghost number requirements we can determine that S is of the form

$$S = S^0 + \phi_i^* Z^i{}_a(\phi) c^a + \sum_{k=0} c^*_{a_k} Z^{a_k}{}_{a_{k+1}} c^{a_{k+1}} + \ldots \ , \tag{3.18}$$

where ... stands for terms cubic or higher order in fields of non–zero ghost number.

Considering the master equation at antifield number zero, we obtain

$$0 = \tfrac{1}{2} \left. (S,S) \right|_{\Phi^*=0} = S^0 \overleftarrow{\partial}_i \cdot \overrightarrow{\partial}^i S^1 = y_i Z^i{}_a c^a \ . \tag{3.19}$$

These relations are the same as (3.9). The transformation matrices $R^i{}_a$ can thus be taken to be $Z^i{}_a$.

Taking two derivatives of (3.16), and applying the projection P, we get

$$Z_{\alpha\gamma} \omega^{\gamma\delta} Z_{\delta\beta} = (-)^{\alpha+1} y_i \ P \left(\overrightarrow{\partial}^i \overrightarrow{\partial}_\alpha S \overleftarrow{\partial}_\delta \right) \ . \tag{3.20}$$

This says that the Hessian $Z_{\alpha\beta}$ is weakly nilpotent. Explicitly, we obtain (apart from the derivative of (3.9))

$$R^i{}_a Z^a{}_{a_1} = 2 y_j f^{ji}{}_{a_1} \approx 0 \tag{3.21}$$

$$Z^{a_k}{}_{a_{k+1}} Z^{a_{k+1}}{}_{a_{k+2}} \approx 0 \ , \tag{3.22}$$

where

$$f^{ji}{}_{a_1} = \tfrac{1}{2} (-)^i \ P \left(\overrightarrow{\partial}^j \overrightarrow{\partial}^i S \overleftarrow{\partial}_{a_1} \right) \ . \tag{3.23}$$

In the first relation the r.h.s. is written explicitly because it exhibits a graded antisymmetry in $[ij]$.

A requirement on the extended action which was already briefly mentioned in the introduction is the **properness condition**. It concerns the rank of the Hessian. As the latter is weakly nilpotent its maximal (weak) rank is half its dimension. The properness condition is now the requirement that this matrix is of maximal rank, which means that for any (local) function $v^\alpha(z)$

$$Z_{\alpha\beta} v^\beta \approx 0 \Longrightarrow v^\beta \approx \omega^{\beta\gamma} Z_{\gamma\delta} w^\delta \ , \tag{3.24}$$

for a local function w^δ.

The properness conditions (3.24) can now be written explicitly as

$$\begin{aligned} S^0_{ij} v^j \approx 0 &\quad\Longrightarrow\quad v^j \approx R^j{}_a w^a \\ R^i{}_a v^a \approx 0 &\quad\Longrightarrow\quad v^a \approx Z^a{}_{a_1} w^{a_1} \\ Z^{a_k}{}_{a_{k+1}} v^{a_{k+1}} \approx 0 &\quad\Longrightarrow\quad v_{a_{k+1}} \approx Z^{a_{k+1}}{}_{a_{k+2}} w^{a_{k+2}} \\ &\text{and} \\ v_i R^i{}_a \approx 0 &\quad\Longrightarrow\quad v_i \approx w^j S^0_{ji} \\ v_a Z^a{}_{a_1} \approx 0 &\quad\Longrightarrow\quad v_a \approx w_i R^i{}_a \\ v_{a_k} Z^{a_k}{}_{a_{k+1}} \approx 0 &\quad\Longrightarrow\quad v_{a_k} \approx w_{a_{k-1}} Z^{a_{k-1}}{}_{a_k} \ . \end{aligned} \tag{3.25}$$

The second group of equations follows from the first group, using that the right and left ranks of matrices are equal. The first one implies (3.12) if there are no non–trivial symmetries which vanish at stationary surface. By the latter we mean that there would be relations

$$y_i y_j T^{ji} = 0 \;\; \text{where } y_j T^{ji} \neq R^i{}_a \epsilon^a \tag{3.26}$$

and T^{ij} is graded symmetric. If such non–trivial symmetries would exist, then (3.12) is an extra requirement with T^i replaced by $y_j T^{ji}$.

3.2 Antifield expansion and the Koszul–Tate differential

An expansion according to antifield number is often very useful. E.g. to construct the extended action, one can always solve the master equation expanding in this way. To show the existence and uniqueness of such perturbative solutions, the essential ingredient is the 'Koszul–Tate' (KT) differential. It acts on antifields, is a nilpotent operation and has an acyclicity property. This means that its cohomology will consist only of the functions on the stationary surface.

We start to define an expansion according to antifield number for the antibracket operation:

$$\begin{aligned}
(F,G) &= \sum_{k=1}(F,G)_k \\
(F,G)_1 &= \sum_i (F\overleftarrow{\partial}_i \cdot \overrightarrow{\partial}^i G - F\overleftarrow{\partial}^i \cdot \overrightarrow{\partial}_i G) \\
(F,G)_{k+2} &= \sum_{a_k} (F\overleftarrow{\partial}_{a_k} \cdot \overrightarrow{\partial}^{a_k} G - F\overleftarrow{\partial}^{a_k} \cdot \overrightarrow{\partial}_{a_k} G) \; .
\end{aligned} \tag{3.27}$$

Note that the last sum is taken with the fixed value of k, but over all values of a_k. The subindex is chosen such that

$$afn\left((F,G)_k\right) = afn(F) + afn(G) - k \; . \tag{3.28}$$

We will often have to consider functions $F(\Phi, \Phi^*)$ satisfying

$$\mathcal{S}F \equiv (F,S) = \sum_{k,\ell,m} (F^k, S^\ell)_m = 0 \; , \tag{3.29}$$

where we expanded as well F, S as the antibracket in antifield numbers. Let us denote the ghost number of F by f. If f is negative, then all terms of F should contain antifields, and the sum over k starts at $k=-f$. Otherwise this sum, as that over ℓ, starts at $k,\ell = 0$ (while the sum over m starts at $m=1$). Concerning the ghosts it is useful to introduce a 'pureghost' number, which is equal to the ghost number for fields, and zero for antifields. We thus have

$$\begin{aligned}
&puregh(c^{a_k}) = k+1 \; ; \qquad puregh(\phi^*_{a_k}) = 0 \\
&gh \; = \; puregh \; - \; afn \; .
\end{aligned} \tag{3.30}$$

For the ranges of k,ℓ and m in the sum (3.29) we can make restrictions using that the pureghost number of F^k is $f+k$, and in the term $(\cdot,\cdot)_m$ there are derivatives w.r.t.

antifields of antifield number m and w.r.t. fields of pureghost number $m-1$. So we have respectively if F^k is derived w.r.t. a field and S^ℓ w.r.t. an antifield, or in opposite order

$$\begin{array}{lcl} m-1 \leq f+k\ ; & & m \leq \ell\ , \\ & \text{or} & \\ m \leq k\ ; & & m-1 \leq \ell\ . \end{array} \tag{3.31}$$

We collect the terms with equal value of the antifield number of $\mathcal{S}F$, so $k+\ell-m=n$. According to the above inequalities, there is then just one case where $\ell-m$ is negative, so where the antifield number of $\mathcal{S}F$ is lower than that of F. We split this term from the others and obtain

$$(\mathcal{S}F)^n = (-)^F \delta_{KT} F^{n+1} + D^n F(S^1, \dots S^{\tilde{n}}, F^0, \dots, F^n) \tag{3.32}$$

$$\delta_{KT} F = \sum_{k=0} (S^k, F)_{k+1} = \sum_{k=-1} S^{k+1} \overleftarrow{\partial}_{a_k} \cdot \overrightarrow{\partial}^{a_k} F\ , \tag{3.33}$$

$$D^n F \equiv \sum_{k=0}^{n} \sum_{m=1}^{\tilde{k}} (F^k, S^{n-k+m})_m\ , \tag{3.34}$$

where $\tilde{k}=k$ if $f<0$ and $\tilde{k}=k+f+1$ for $f \geq 0$. The second line defines the Koszul–Tate differential. We have chosen it to act from the left in accordance with previous references. Note the following important facts about it. First it is a fermionic operation, which means that $\delta_{KT}F$ has opposite statistics as F. It lowers the antifield number by one, and it acts only non–trivially on antifields. Using (3.18) we find

$$\begin{aligned} \delta_{KT}\phi_i^* &= y_i \\ \delta_{KT} c_a^* &= \phi_i^* R^i{}_a \\ \delta_{KT} c_{a_k}^* &= c_{a_{k-1}}^* Z^{a_{k-1}}{}_{a_k} + M_{a_k}(\phi; \phi_i^*, c_a^*, \dots, c_{a_{k-2}}^*) \text{ with } k \geq 1\ , \end{aligned} \tag{3.35}$$

where $M_{a_k}(\phi; \phi_i^*, c_a^*, \dots, c_{a_{k-2}}^*)$ is determined by the parts not explicitly written in (3.18).

The important properties of this operation are its nilpotency and acyclicity. These are the remnants of respectively the master equation and the properness condition of the extended action. First, the nilpotency is easy to show when we already know that $(S,S)=0$ [2]. In fact on functions of a definite antifield number we have

$$\begin{aligned} \mathcal{S}F^k &= (-)^F \delta_{KT} F^k + \sum_{n \geq k} D^n F^k \\ \delta_{KT} F^k &= (S, F^k)^{k-1}\ , \end{aligned} \tag{3.36}$$

where the superscript means that we restrict to the terms of antifield number $k-1$ (and thus omit the terms of higher antifield number). Then

$$\delta_{KT}\delta_{KT} F^k = \Big(S, (S, F^k)^{k-1}\Big)^{k-2} = \Big(S, (S, F^k)\Big)^{k-2} = 0\ . \tag{3.37}$$

Indeed, we can omit the superscript of the first antibracket, as the other terms have after the two antibracket operations an antifield number which is higher than $k-2$. Then we use the Jacobi–identities (2.4) and the master equation. Therefore we can define a cohomology.

[2]The KT differential is introduced e.g. in [11] in a different way to prove that there is a solution to the master equation. Then the proof of the nilpotency is more involved.

3.3 Acyclicity of the KT differential.

A more difficult, but essential, property of the KT differential is its acyclicity. As the operation only acts on antifields, we now have to consider the function space $\mathcal{F}^* \equiv \{F(\phi^i, \Phi^*_A)\} \subset \mathcal{F}$. The acyclicity statement is that the cohomology of the KT differential on this space contains only the functions on the stationary surface $\mathcal{F}^0_s$.

$$\delta_{KT} F(\phi, \Phi^*) = 0 \;\Rightarrow\; F = \delta_{KT} H + f(\phi) \tag{3.38}$$

where $f \in \mathcal{F}^0_s$.

The proof of this statement goes perturbatively in the level of antifields which are included in the set. We define $\mathcal{F}^*_k = \{F(\phi^i, \phi^*_i, \ldots, c^*_{a_{k-2}})\}$ as the functions which depend at most on antifields with antifield number k, such that

$$\mathcal{F} \supset \mathcal{F}^* \supset \ldots \supset \mathcal{F}^*_k \supset \ldots \supset \mathcal{F}^*_1 \supset \mathcal{F}^*_0 \equiv \mathcal{F}^0 \supset \mathcal{F}^0_s \; . \tag{3.39}$$

In $\mathcal{F}^*_0 = \mathcal{F}^0$ we have that the functions which should not be in the cohomology according to (3.38) are the functions which are proportional to a field equation y_i. This is $\delta_{KT}\phi^*_i$. So we need functions in $\mathcal{F}^*_1$ to remove these. Lemma B.3 in the appendix B proves the acyclicity in $\mathcal{F}^*_1$ apart from terms proportional to $\phi^*_i R^i{}_a = \delta_{KT} c^*_a$. Then one proves a similar lemma in $\mathcal{F}^*_2$, where now only terms proportional to $\delta_{KT} c^*_{a_1}$ are left in the cohomology (apart from the functions on the stationary surface). Continuing in this way we can perturbatively construct the function H in (3.38). The essential ingredient for all these proofs is of course the properness condition. It is this condition which is translated to the acyclicity of the KT differential.

The proofs are in appendix B because they are rather long and tedious. Although they are related to the proofs of [11] they are fundamentally different, as was explained in the beginning of section 3.

In this appendix B we first show the relation between the KT differential in different sets of coordinates, and prove that if acyclicity is true in one set of coordinates, then it is also true in a canonically related set. Of course, these canonical transformations do not include the interchange of fields and antifields, as canonical transformations preserve the ghost number and we always have to reserve the word 'antifields' here for those fields of negative ghost number. Then the proofs are given for the first levels, such that the generalization is obvious. It shows that perturbatively in the functions containing antifields of higher antifield numbers, the KT differential contains no non–trivial cycles apart from the (antifield independent) functions at ghost number zero in $\mathcal{F}^0_s$ which are the functions on the stationary surface.

This acyclicity statement applies for local (x–dependent) functions. It does not apply in general for integrals. Indeed, consider the following example

$$\begin{aligned} S &= \int d^d x \, \tfrac{1}{2} \partial_\alpha X^\mu \cdot \partial^\alpha X^\mu \; , \quad \text{with } \mu = 1,2 \;\; \alpha = 1, ..., d \\ F &= \int d^d x \left(X^*_1 X^2 - X^*_2 X^1 \right) \;\rightarrow\; \delta_{KT} F = 0 \; . \end{aligned} \tag{3.40}$$

Nevertheless, F can not be written as $\delta_{KT} G$. This violation of acyclicity for integrals is due to rigid symmetries, such that (3.12) does not apply for integrals. However, for F a local integral (integral of a local function) of antifield number 0 (and thus obviously

$\delta_{KT}F = 0$), which vanishes on the stationary surface, we have by definition $F = \int y_i F^i$ and $F = \delta_{KT} \int (\phi_i^* F^i)$.

The acyclicity was proven here for functions independent of ghosts (elements of $\mathcal{F}^*$). If one considers functions $F(\Phi^*, \phi, c) \in \mathcal{F}$ depending on ghosts, then the acyclicity can be used when we first expand in c. Therefore the modified acyclicity statement is then

$$\begin{aligned} \delta_{KT}F(\Phi^*, \phi, c) = 0 \quad &\Rightarrow \quad F = \delta_{KT}H + f(\phi, c) \\ \text{and if } f(\phi, c) \approx 0 \quad &\Rightarrow \quad f = \delta_{KT}G(\Phi^*, \phi, c) \ , \end{aligned} \tag{3.41}$$

where, as mentioned before, $\approx$ stands for using the field equations of $S^0(\phi)$ for ϕ, while the ghosts c remain unchanged. If F is an integral, where the integrand contains ghosts, then we apply the acyclicity to the coefficient functions of the ghosts which are local functions. If $gh(F) > 0$ (or even if just $puregh(F) > 0$) then each term can be treated in this way and the statement (3.41) holds even when F is an integral of a local quantity. Also if $gh(F) = 0$ then any term has either a ghost, or it just depends on ϕ^i in which case the acyclicity statement also applies for integrals.

3.4 Antibracket cohomology

In this section we present our main theorem, mentioned in the introduction. We consider the problem that we have to find a certain function $F(\Phi, \Phi^*)$, which is known to satisfy $(F, S) = 0$. We want to know F modulo a part $F = (G, S)$. We thus investigate here the antibracket cohomology.

First we have to consider which functions are invariant under the $\mathcal{S}$ operation: see (3.32), and (3.36). Especially the last expression of $\mathcal{S}$ is useful, as it shows that the equation at lowest antifield number of $\mathcal{S}F^k = 0$ is $\delta_{KT}F^k = 0$ $(k \geq 1)$. This is the clue for solving $\mathcal{S}F = 0$ perturbatively in antifield numbers.

First, in theorem C.1 it is proven that there is no cohomology for local functions of negative ghost number. For non–negative ghost numbers the situation is more complicated. The equation $\mathcal{S}F = 0$ at zero antifield number is by (3.32)

$$- (-)^F \delta_{KT} F^1 = D^0 F^0 = \sum_{m=1}^{f+1} (F^0, S^m)_m \ . \tag{3.42}$$

D^0 is a fermionic right derivative operator, which acts on fields only, and is given by

$$D^0 F^0 = (F^0, S)\Big|_{\Phi^*=0} \ . \tag{3.43}$$

For antifield number 0, the KT differential is acyclic on functions which vanish on the stationary surface. Therefore F^1 exists if $D^0F^0 \approx 0$, and in lemma C.2 it is proven that then also the full F can be constructed perturbatively in antifield number such that $\mathcal{S}F$=0.

This operator D^0 raises the pureghost number by 1. It is nilpotent on the classical stationary surface: $D^0D^0F^0 \approx 0$. One can define a (weak) cohomology of this operator (lemma C.1) on functions of fields only, and this is graded by the pureghost number p. The main result is that this cohomology is equivalent to the (strong) cohomology of $\mathcal{S}$ for functions of ghost number p (theorem C.2). This is the theorem which determines the antibracket cohomology. We can state it as follows

Theorem 3.1 *: Any local function F of negative ghost number which satisfies $\mathcal{S}F = 0$ can be written as $F = \mathcal{S}G$.*
For functions of non–negative ghost number, the following statement holds. For a local function or a local integral $F^0(\Phi)$ (not containing antifields)

$$D^0F^0 \approx 0 \;\Leftrightarrow\; \exists F(\Phi, \Phi^*) \;\, : \mathcal{S}F = 0 \tag{3.44}$$

where $F(\Phi, 0) = F^0$. Further

$$F^0 \approx D^0G^0 \;\Leftrightarrow\; \exists G(\Phi, \Phi^*) \;\, : F = \mathcal{S}G \;, \tag{3.45}$$

where again $G^0 = G(\Phi, 0)$, and F is a function determined by (3.44). The ghost numbers of F and G are equal to the pureghost numbers of F^0 and G^0.

The inclusion of local integrals for the second part of the theorem follows from the fact that for non–negative ghost numbers we could at the end of subsection 3.3 include these in the acyclicity statement, and this was the only ingredient of the proof.

At ghost number zero the antibracket cohomology gives the functions on the stationary surface, where two such functions which differ by gauge transformations are identified . Indeed, the KT cohomology reduced the functions to those on the stationary surface. D^0 acts within the stationary surface, and its cohomology reduces these functions to the gauge invariant ones, as for (pure) ghost number 0,

$$D^0F^0 = F_0\overleftarrow{\partial}_i \cdot R^i{}_a c^a \;, \tag{3.46}$$

gives the gauge transformation of F^0. Here we clearly see how the antibracket and BRST formalisms are connected. D^0 is in fact the BRST operator.

At ghost number 1, we can apply the theorem for the analysis of anomalies. We have seen in section 2 that anomalies $\mathcal{A}$ are local integrals of ghost number 1. They satisfy the Wess–Zumino consistency relations [25] in the form[3] $\mathcal{S}\mathcal{A} = 0$, while anomalies can be absorbed in local counterterms if $\mathcal{A} = \mathcal{S}M$. The anomalies are thus in fact elements of the cohomology of $\mathcal{S}$ at ghost number 1 in the set of local integrals. We have found here that in the classical basis, these anomalies are completely determined by their part $\mathcal{A}^0$ which is independent of antifields and just contains 1 ghost of ghost number 1. On this part there is the consistency condition $D^0\mathcal{A}^0 \approx 0$. If this equality is strong, then $\mathcal{A}$ does not need antifield–dependent terms for its consistency. If it is weak, then these are necessary, but we know that they exist. If for an anomaly $\mathcal{A}^0 \approx D^0M^0$, then we know that it can be cancelled by a local counterterm. Consequently the anomalies (as elements of the cohomology) are determined by their part without antifields, and with ghost number one, and can thus be written as

$$\mathcal{A} = \mathcal{A}_a(\phi)c^a + \ldots \;, \tag{3.47}$$

where the written part determines the .... Therefore we can thus split the anomalies in parts corresponding to the different symmetries represented by the index a. Indeed, people usually talk about anomalies in a certain symmetry (although this can still have different forms according to the particular representant of the cohomological element which one considers), and we show here that this terminology can always be maintained for the general gauge theories which the BV formalism can describe.

[3]We consider here only 1–loop effects.

4 Consistent Anomalies in W_3 gravity

4.1 The classical theory and the extended action

The classical action for chiral W_3 gravity is [32][4, 5]

$$S^0 = -\tfrac{1}{2}(\partial X^\mu)(\bar\partial X^\mu) + \tfrac{1}{2}h(\partial X^\mu)(\partial X^\mu) + \tfrac{1}{3}d_{\mu\nu\rho}B(\partial X^\mu)(\partial X^\nu)(\partial X^\rho) \qquad (4.1)$$

where $d_{\mu\nu\rho}$ is a symmetric tensor satisfying the nonlinear identity

$$d_{\mu(\nu\rho}d_{\sigma)\tau\mu} = \kappa\delta_{(\nu\sigma}\delta_{\rho)\tau} \qquad (4.2)$$

for some arbitrary, but fixed, parameter κ. The general solution of this equation was found in [33][6]. The model contains n scalar fields $X^\mu, \mu = 1, ..., n$ and two gauge fields h and B. The field equations are

$$\begin{aligned} y_\mu &= \partial\bar\partial X^\mu - \partial(h(\partial X^\mu)) - d_{\mu\nu\rho}\partial(B(\partial X^\nu)(\partial X^\rho)) \\ y_h &= \tfrac{1}{2}(\partial X^\mu)(\partial X^\mu) \\ y_B &= \tfrac{1}{3}d_{\mu\nu\rho}(\partial X^\mu)(\partial X^\nu)(\partial X^\rho)\,. \end{aligned} \qquad (4.3)$$

As one can check, by using (4.2), these field equations are not independent, but satisfy

$$y_i R^i{}_a = 0 \quad i = \{\mu, h, B\} \quad a = 1, 2\,. \qquad (4.4)$$

It expresses that the action (4.1) has two gauge invariances $\delta\phi^i = R^i{}_a\epsilon^a$, corresponding to the local reparameterization and W_3 symmetry. Therefore, we introduce the ghosts c and u, and $R^i{}_a$ is given by

$$\begin{aligned} S^1 = \phi^*_i R^i{}_a c^a = \; & X^*_\mu\left[(\partial X^\mu)c + d_{\mu\nu\rho}(\partial X^\nu)(\partial X^\rho)u\right] \\ & + h^*\left[(\nabla^{-1}c) + \tfrac{\kappa}{2}(\partial X^\mu)(\partial X^\mu)(D^{-2}u)\right] \\ & + B^*\left[(D^{-1}c) + (\nabla^{-2}u)\right] \end{aligned} \qquad (4.5)$$

[4]We will use the notations $\partial = \partial_+$ and $\bar\partial = \partial_-$, where $x^\pm = \rho(x^1 \pm x^0)$, and we leave the factor ρ undetermined.

[5]We omit the $\int d^2x$ in all the actions below. Similarly for matrices as S_{AB} there is a factor $\delta(x-y)$ if the DeWitt index A contains the point x, and B the point y. Derivatives which will still act to the right in these matrices (all those not enclosed in brackets) act on these delta functions. We will not write the latter explicitly.

[6]The general solutions are related to specific realizations of real Clifford algebras $\mathcal{C}(D,0)$ of positive signature. We have then $n = 1 + D + r$, where r is the dimension of the Clifford algebra realization. We obtain solutions of (4.2) for the following values : ($D = 0, r = 0$), ($D = 1, r$ arbitrary), ($D = 2, r = 2$), ($D = 3, r = 4$) (these are the $SU(3)$ d–symbols), ($D = 5, r = 8$), ($D = 9, r = 16$). The latter four are the so–called 'magical cases'. Then the Clifford algebra representation is irreducible, and the d–symbols are traceless as it is the case for ($D = 1, r = 0$). All the solutions can be given in the following way. μ takes the values $1, a$ or i, where a runs over D values and i over r values. The non–zero coefficients are (for $\kappa = 1$) $d_{111} = 1$, $d_{1ab} = -\delta_{ab}$, $d_{1ij} = \frac{1}{2}\delta_{ij}$, and $d_{aij} = \frac{\sqrt{3}}{2}(\gamma_a)_{ij}$. For $D = 1$ the gamma matrix is $(\gamma_a)_{ij} = \delta_{ij}$ (reducible). In that case the form of the solution can be simplified by a rotation between the index 1, and a, which takes only one value, see below: (5.8). All representations of all real Clifford algebras $\mathcal{C}(D,0)$ appear as solutions of a generalization of (4.2) and classify the homogeneous special Kähler and quaternionic spaces [34]

and

$$\nabla^j = \bar{\partial} - h\partial - j(\partial h)$$
$$D^j = -2B\partial - j(\partial B)\,. \qquad (4.6)$$

The number j is the spin (the number of unwritten + indices). These spins are given in table 6 together with other properties of derivative operators and fields.

	gh	j	$dim - j$		gh	j	$dim - j$
X	0	0	0	X^*	-1	0	2
B	0	-3	2	$B^* = v$	-1	3	0
h	0	-2	2	$h^* = b$	-1	2	0
c	1	-1	0	c^*	-2	1	2
u	1	-2	0	u^*	-2	2	2
∂	0	1	0				
$\bar{\partial}$, ∇	0	-1	2				
D	0	-2	2				

Table 1: We give here the properties of fields and derivative operators. gh is the ghost number and j is the spin. Further one can assign an 'engineering' dimension to the fields and derivatives, such that the Lagrangian has dimension 2. We defined $dim(\Phi^*) = 2 - dim(\Phi)$ (the number 2 is arbitrary, this freedom is related to redefinitions proportional to the ghost number). When we subtract the spin from this dimension, we find that only a few fields are dimensionfull. We mentioned already the new names of fields in the gauge fixed theory

.

The matrix $R^i{}_a$ is of maximal rank, so there are no further zero modes, i.e. $Z^{a_k}{}_{a_{k+1}} = 0, k \geq 0$ and we have no further ghosts. The calculation of S^2 amounts to calculate the structure functions T and nonclosure functions E of the gauge algebra:

$$S^2 = (-)^b \tfrac{1}{2} c^*_a T^a{}_{bc} c^c c^b + (-)^{i+a} \tfrac{1}{4} \phi^*_i \phi^*_j E^{ji}{}_{ab} c^b c^a\ . \qquad (4.7)$$

After calculating (S^1, S^1) we see that we can choose the structure functions $T^a{}_{bc}$ either such that this term in (S^2, S^1) cancels all terms with X^*_μ, or such that all terms with B^* are cancelled. So one can choose which of the two antifields appear in S^2. In other words, in the first case the algebra is chosen to close off–shell on X^μ and to close only on–shell on B, while this is reversed in the second case. In fact one can give the following expression for S^2 which depends on a free parameter α

$$\begin{aligned} S^2 \;=\;& c^*\left[(\partial c)c + \kappa(1-\alpha)(\partial X^\mu)(\partial X^\mu)(\partial u)u\right] + u^*\left[2(\partial c)u - c(\partial u)\right] \\ & -2\kappa\alpha h^*(D^3 B^* + \nabla^2 h^*)(\partial u)u - 2\kappa(\alpha+1)X^*_\mu h^*(\partial u)u(\partial X^\mu)\ . \end{aligned} \qquad (4.8)$$

$\alpha = -1$ corresponds to the first choice discussed above, while $\alpha = 0$ corresponds to the second choice, and this form has been written in [12, 14]. Remark also the choice $\alpha = 1$ which gives the simplest structure functions. The relation between these actions is given by the canonical transformation: starting from the action with $\alpha = 0$ the transformation with generating function

$$f = 2\kappa\alpha h'^*(\partial u)uc'^* \,. \tag{4.9}$$

gives in the primed coordinates the action with this arbitrary parameter α. This (fermionic) generating function $f(\Phi, \Phi'^*)$ determines a canonical transformation by [24, 3, 6]

$$\Phi'^A = \Phi^A + \frac{\partial}{\partial \Phi'^*_A} f(\Phi, \Phi'^*) \; ; \qquad \Phi^*_A = \Phi'^*_A + \frac{\partial}{\partial \Phi^A} f(\Phi, \Phi'^*) \; . \tag{4.10}$$

Note that this type of freedom is always possible. A canonical transformation with $f = c'^*_a \phi'^*_i T^{ia}{}_{bc} c^c c^b (-)^{a+b}$ adds a term with a field equation in the structure functions and changes the non–closure functions. In general this can also change the higher antifield terms of the extended action. In the case at hand however, the master equation is already satisfied for all α without the need of a term S^3. This is equivalent to the statement that the Jacobi identities

$$(-)^b \left(T^a{}_{bc} c^c c^b \right) \overleftarrow{\partial}_i \cdot R^i{}_d \, c^d + (-)^{b+d} T^a{}_{bc} \, c^c \, T^b{}_{de} \, c^e \, c^d \approx 0 \tag{4.11}$$

are strongly satisfied (with = rather than $\approx$). The canonical transformation (4.9) does not generate terms in S of antifield number 3 or higher. So the full extended action is

$$S = S^0 + S^1 + S^2 \; , \tag{4.12}$$

where the three terms are given in (4.1), (4.5) and (4.8) respectively.

Now, in order to obtain a gauge fixed action, we perform the canonical transformation from $\{h, h^*, B, B^*\}$ (fields and antifields of the classical basis) to new fields and antifields $\{b, b^*, v, v^*\}$ (the gauge–fixed basis):

$$h = -b^*; \;\; h^* = b \;\; \text{and} \;\; B = -v^*; \;\; B^* = v \,. \tag{4.13}$$

One may check now that the new antifield independent action, which depends thus on $\{X^\mu, b, c, v, u\}$, has no gauge invariances. These are thus the fields that appear in loops.

4.2 Calculation of the one–loop anomaly without antifields

Now, we will calculate the one–loop anomaly following the prescription summarized in section 2.2. We need first the (invertible) matrix of second derivatives w.r.t. the fields of the gauge–fixed basis. The theorems proven in section 3.4 will imply that we will only need in these second derivatives the terms without fields of negative ghost numbers and at most linear in c and u, the fields of ghost number 1. We will therefore in the entries still use the names of fields and antifields as in the classical basis. In the following matrices we first write the entries corresponding to the bosons X^μ, and then order the fermions according to ghost number and spin: $\Phi'^A = \{X^\mu, v = B^*, b = h^*, c, u\}$ we have $S'_{AB} = \tilde{S}'_{AB}$+ terms of antifield number non–zero + terms of pureghost number > 1.

$$\tilde{S}'_{AB} \;\; = \;\; \begin{pmatrix} S_{\mu\nu} & \underset{\sim}{q}_\mu \\ -q^T_\nu & \nabla \end{pmatrix}$$

$$S_{\mu\nu} \;\; = \;\; \delta_{\mu\nu} \nabla^1 \partial + d_{\mu\nu\rho} D^2 (\partial X^\rho) \partial$$

$$
\begin{aligned}
q_\mu &= (\,0 \quad \kappa\partial(D^{-2}u)(\partial X^\mu) \quad 0 \quad 0\,) \\
\tilde{\nabla} &= \begin{pmatrix} 0 & 0 & D^{-1} & \nabla^{-2} \\ 0 & 0 & \nabla^{-1} & \kappa y_h D^{-2} \\ D^3 & \nabla^2 & 0 & 0 \\ \nabla^3 & \kappa D^4 y_h & 0 & 0 \end{pmatrix} ,
\end{aligned} \tag{4.14}
$$

where y_h is given by (4.3). A lot of zeros follow already from the ghost number requirements.

To obtain the regulator we have, according to (2.18), to choose a mass matrix T and multiply its inverse with $\tilde{S}'$. Looking at the table 6 for the spins of the fields, we notice that some fermionic fields (e.g. c) do not have a partner of opposite spin, which we need for a mass term, if we want to preserve these rigid symmetries (the PV partners of fields have the same properties as the corresponding fields). Further, the regulator will not regularize (2.21) because the fermion sector of (4.14) is only linear in the derivatives. These two problems can be solved by first introducing extra PV fields (this procedure was already used in [3]). They have no interaction in the massless sector and do not transform under any gauge transformation. Inspecting the ghost numbers and spins of the fermions in $\{\Phi'^A\}$, we find that we need extra PV fields with ghost numbers and spin as in table 4.2. Then we have to choose the matrix T. It determines the mass matrix in

	gh	j	$dim - j$
$\bar{u}$	1	-3	1
$\bar{c}$	1	-2	1
$\bar{b}$	-1	1	1
$\bar{v}$	-1	2	1

Table 2: The extra non–interacting and gauge invariant PV fields. The ghost numbers and spins are chosen in order to be able to construct mass terms for c, u, b and v. The names are chosen such that $gh(\bar{x}) = gh(x)$ and $j(\bar{x}) = j(x) - 1$. The dimensions follow from the kinetic terms, although this would still allow less symmetric choices.

the PV action as in (2.17). We can choose it as

$$
T = \begin{pmatrix} \delta_{\mu\nu} & 0 & 0 \\ 0 & 0 & \frac{1}{M}\mathbb{1} \\ 0 & -\frac{1}{M}\mathbb{1} & 0 \end{pmatrix} , \tag{4.15}
$$

where the second entry refers to the fermions from above, and the third line to the four new fermions. The latter are thus ordered as in the table. The kinetic part of their action can be chosen such that the enlarged matrix S'_{AB} is

$$
S'_{AB} = \begin{pmatrix} S_{\mu\nu} & q_\mu & 0 \\ -q^T_\nu & \tilde{\nabla} & 0 \\ 0 & 0 & -\tilde{\partial} \end{pmatrix} ; \qquad \tilde{\partial} \equiv \begin{pmatrix} 0 & 0 & 0 & \partial \\ 0 & 0 & \partial & 0 \\ 0 & \partial & 0 & 0 \\ \partial & 0 & 0 & 0 \end{pmatrix} . \tag{4.16}
$$

To put everything together, we find as regulator

$$\mathcal{R} = \mathcal{R}_0 + M\mathcal{R}_1$$
$$\mathcal{R}_0 = \begin{pmatrix} S_{\mu\nu} & q_\mu & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \; ; \qquad \mathcal{R}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \tilde{\partial} \\ -q^T_\nu & \tilde{\nabla} & 0 \end{pmatrix} \; , \tag{4.17}$$

and the transformation matrix K defined in (2.20) is, dropping again terms of pureghost number 2 or antifield number 1,

$$K = \begin{pmatrix} K^\mu{}_\nu & K^\mu{}_F & 0 \\ K^F{}_\nu & K^F{}_F & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned}
K^\mu{}_\nu &= \delta^\mu_\nu c\partial + 2ud_{\mu\nu\rho}(\partial X^\rho)\partial \\
K^\mu{}_F &= \begin{pmatrix} 0 & 0 & (\partial X^\mu) & d_{\mu\rho\sigma}(\partial X^\rho)(\partial X^\sigma) \end{pmatrix} \\
K^F{}_\nu &= \begin{pmatrix} -d_{\nu\rho\sigma}(\partial X^\rho)(\partial X^\sigma)\partial \\ -(\partial X_\nu)\partial \\ 0 \\ 0 \end{pmatrix} \\
K^F{}_F &= \begin{pmatrix} -(c\partial)_3 & -2\kappa[y_h(u\partial)_2 + u(\partial y_h)] & 0 & 0 \\ -2(u\partial)_{\frac{3}{2}} & -(c\partial)_2 & 0 & 0 \\ 0 & 0 & -(c\partial)_{-1} & -2\kappa(1-\alpha)y_h(u\partial)_{-1} \\ 0 & 0 & -2(u\partial)_{-\frac{1}{2}} & -(c\partial)_{-2} \end{pmatrix} ,
\end{aligned} \tag{4.18}$$

where we used the shorthand

$$(c\partial)_x = c\partial + x(\partial c) \; . \tag{4.19}$$

Note that the second line of (4.8) can not play a role for these expressions up to the order in which we have to calculate.

The expression of $\mathcal{R}$ is so far still linear in derivatives for the fermionic sector. This we solve as in [18] by multiplying in (2.19) the numerator and the denominator by $(1+\mathcal{R}_1/M)$. We obtain

$$(\Delta S)^0 = Tr\left[K(1+\frac{1}{M}\mathcal{R}_1)\frac{1}{\left(1-\frac{1}{M^2}\mathcal{R}_0 - \frac{1}{M}\mathcal{R}_1\right)\left(1+\frac{1}{M}\mathcal{R}_1\right)}\right] \; , \tag{4.20}$$

which leads to the regulator

$$\mathcal{R}_0 + \mathcal{R}_1^2 + \frac{1}{M}\mathcal{R}_0\mathcal{R}_1 = \begin{pmatrix} S_{\mu\nu} & q_\mu & \frac{1}{M}q_\mu\tilde{\partial} \\ -\tilde{\partial}q^T_\nu & \tilde{\partial}\tilde{\nabla} & 0 \\ 0 & 0 & \tilde{\nabla}\tilde{\partial} \end{pmatrix} \; . \tag{4.21}$$

On the other hand

$$K\mathcal{R}_1 = \begin{pmatrix} 0 & 0 & K^\mu{}_F\tilde{\partial} \\ 0 & 0 & K^F{}_F\tilde{\partial} \\ 0 & 0 & 0 \end{pmatrix} \; . \tag{4.22}$$

This can not contribute to the trace because grouping the first two entries together, this regulator is also upper triangular. Therefore we can omit the term $K\mathcal{R}_1$, and then only the first two rows and columns of the regulator can play a role. This eliminates also the $(1/M)$ terms in the regulator. So we effectively have to calculate

$$(\Delta S)^0 = Tr\left[\begin{pmatrix} K^\mu{}_\nu & K^\mu{}_F \\ K^F{}_\nu & K^F{}_F \end{pmatrix} \exp\frac{1}{M^2}\tilde{\mathcal{R}}\right] \; ; \qquad \mathcal{R} = \begin{pmatrix} S_{\mu\nu} & q_\mu \\ -\partial q^T_\nu & \partial\tilde{\nabla} \end{pmatrix} \; . \tag{4.23}$$

As explained in section 2.2 such calculations can be done in the heat kernel method [27], was it not that our regulator contains terms with second derivatives which are not proportional to the unit matrix in internal space. We can anticipate on the form of the anomaly which we have to obtain. The anomaly ΔS should be an integral of a local quantity of spin 0 and dimension 2. From the $dim - j$ column of table 6, we then see that the anomaly can be split in a part without B, and a linear part in B, which can not contain h:

$$(\Delta S)^0 = \Delta^h S + \Delta^B S \; . \tag{4.24}$$

So we split $\tilde{\mathcal{R}} = \mathcal{R}^h + \mathcal{R}_1$

$$\begin{aligned} \mathcal{R}^h &= diag\left(\delta_{\mu\nu}\partial\nabla^0, \partial\nabla^3, \partial\nabla^2, \partial\nabla^{-1}, \partial\nabla^{-2}\right) \\ \mathcal{R}^B &= \mathbb{1}\Box + \mathcal{R}_1 \\ \mathcal{R}_1 &= \begin{pmatrix} d_{\mu\nu\rho}D^2(\partial X^\rho)\partial & 0 & \kappa\partial(D^{-2}u)(\partial X^\mu) & 0 & 0 \\ 0 & 0 & \kappa\partial D^4 y_h & 0 & 0 \\ \kappa\partial(\partial X^\mu)(D^{-2}u)\partial & 0 & 0 & 0 & \kappa\partial y_h D^{-2} \\ 0 & 0 & 0 & \partial D^{-1} & 0 \end{pmatrix} \; . \end{aligned} \tag{4.25}$$

In the last expression $\Box$ is a flat $\partial\bar{\partial}$.

First for the calculation at $B = 0$ we need for each entry the expression of

$$\mathcal{A}_j = -i\int dx\, dy\, \delta(x-y)(c\partial)_j G(x, y; \tfrac{1}{M^2}, \Phi^{(j)}) \; , \tag{4.26}$$

with

$$\Phi^{(j)} = \left\{g^{\alpha\beta} = \begin{pmatrix} -h & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \mathcal{Y}_+ = 0, \mathcal{Y}_- = -j(\partial h), E = -\tfrac{1}{2}j(\partial^2 h)\right\} \; . \tag{4.27}$$

Using the Seeley–DeWitt coefficients[7]

$$\begin{aligned} a_0| = 1 \; ; &\qquad \nabla_\alpha a_0(x,y)| = 0 \\ a_1| = E - \tfrac{1}{6}R = \frac{1-3j}{6}(\partial^2 h) \; ; &\qquad \nabla_\alpha a_1(x,y)| = \tfrac{1}{2}\nabla_\alpha(E - \tfrac{1}{6}R) + \tfrac{1}{6}\nabla^\beta W_{\alpha\beta} \; , \end{aligned} \tag{4.28}$$

where $|$ stands for the value at coincident points $x = y$ (after taking the derivatives). The derivatives ∇_α are covariant w.r.t. gravitation. We will need only ∇_+ for which the connection is zero:

$$\partial a_1(x,y)| = \frac{1-3j}{12}(\partial^3 h) - \frac{j}{12}(\partial^3 h) \; . \tag{4.29}$$

This leads to

$$\mathcal{A}_j = \frac{-i}{24\pi}\int dx\, (6j^2 - 6j + 1)c\,\partial^3 h \; . \tag{4.30}$$

[7]The conventions are $R^\alpha{}_{\beta\gamma\delta} = \partial_\gamma\Gamma^\alpha_{\beta\delta} - ...$ and $R = R^\gamma{}_{\alpha\beta\gamma}g^{\alpha\beta}$. Further $W_{\alpha\beta} = \partial_\alpha\mathcal{Y}_\beta - \partial_\beta\mathcal{Y}_\alpha + [\mathcal{Y}_\alpha, \mathcal{Y}_\beta]$.

For the overall normalization of this anomaly, we used $\sqrt{g} = 2$, in accordance with the form of $g_{\alpha\beta}$ which follows from (4.27). In this way, we thus use the coordinates $x^\alpha = \{x^+, x^-\}$, and the integration measure dx in the above integral is then $dx^+\, dx^-$. If one uses $dx = dx^0\, dx^1$ then the scalars as R and E do not change, but $\sqrt{g} = \rho^{-2}$, where ρ is the parameter in the definitions in footnote 4. Therefore the overall factor $1/(24\pi)$ in the above formula, gets replaced by $1/(48\pi\rho^2)$. One can either interpret dx as $dx^+\, dx^-$ or as $dx^0\, dx^1$ with $\rho = 1/\sqrt{2}$. The overall normalization in fact depends just on the transformation law: if S^1 contains $X^* c\partial X$ then the anomaly for one scalar is

$$\mathcal{A}_0 = \frac{-i}{48\pi} \int dx^0\, dx^1\, c\, \partial(\partial_0 + \partial_1)^2 h \ , \tag{4.31}$$

independent of the definition of ∂. In all further expressions for anomalies we again omit $\int dx$ with the normalization as explained above.

Denoting the ghost combination in the transformation of the bosons as

$$\tilde{c}^\mu_\nu = c\delta^\mu_\nu + 2ud_{\mu\nu\rho}(\partial X^\rho) \ , \tag{4.32}$$

we obtain

$$\Delta^h S = \Delta^h_{XX} S + \Delta^h_{FF} S = \tfrac{1}{24\pi}(\tilde{c}^\mu_\mu - 100\, c)\partial^3 h \ , \tag{4.33}$$

where Δ_{XX} is the contribution from the matter entries in the matrices, and Δ_{FF} comes from the fermions and gives the factor -100.

For $\Delta^B S$ we have to evaluate expressions as[8]

$$\begin{aligned} \mathcal{A}_B &= tr\left[K e^{t(\Box + \mathcal{R}_1)}\right] = \frac{1}{t}\frac{d}{d\lambda}\, tr\left[e^{t(\Box + \mathcal{R}_1 + \lambda K)}\right]\Big|_{\lambda=0} \\ &= \frac{d}{d\lambda}\, tr\left[\mathcal{R}_1 e^{t(\Box + \lambda K)}\right]\Big|_{\lambda=0} \ , \end{aligned} \tag{4.34}$$

where the last step could be done because $\mathcal{R}_1$ is linear in B, and we know that the result should be linear in B. We have for K and $\mathcal{R}_1$ the general forms

$$K = k_0 + k_1\partial \ ; \qquad R = r_0 + r_1\partial + r_2\partial^2 \ . \tag{4.35}$$

Then we have to evaluate the heat kernel with

$$\Phi = \left\{ g^{\alpha\beta} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \mathcal{Y}_+ = 0, \mathcal{Y}_- = \lambda k_1, E = \lambda(k_0 - \tfrac{1}{2}\partial k_1) \right\} \ . \tag{4.36}$$

As the metric is flat, there is no non–trivial contribution from $\Delta^{1/2}$ and $\sigma(x,y)$ in (2.27). The only new coefficient which we need are the second derivatives at coincident points. For a flat metric, and being interested only in linear terms in E and $\mathcal{Y}$, the coefficients are

$$\nabla_\alpha\nabla_\beta a_0| = \tfrac{1}{2}W_{\alpha\beta} \ ; \qquad \nabla_\alpha\nabla_\beta a_1| = \tfrac{1}{3}\partial_\alpha\partial_\beta E + \tfrac{1}{6}\partial^\gamma\partial_{(\alpha}W_{\beta)\gamma} + \mathcal{O}(\lambda^2) \ , \tag{4.37}$$

[8] The calculation was first done without using this trick. Then the exponential has to be evaluated using Campbell–Baker–Hausdorff expansions. The manipulations described here simplify the calculations considerably and were found in a discussion with Walter Troost.

where the symmetrization $(\alpha,\beta)=\frac{1}{2}(\alpha\beta+\beta\alpha)$. This gives

$$2\pi i\mathcal{A}_B = k_0\left(r_0-\tfrac{1}{2}\partial r_1+\tfrac{1}{3}\partial^2 r_2\right)+(\partial k_1)\left(-\tfrac{1}{2}r_0+\tfrac{1}{6}\partial r_1-\tfrac{1}{12}\partial^2 r_2\right)\ , \tag{4.38}$$

which can be used to obtain

$$\begin{aligned}
\Delta^B S &= \Delta^B_{XX}S+\Delta^B_{XF}S+\Delta^B_{FF}S\\
i\Delta^B_{XX}S &= \frac{1}{12\pi}\tilde{c}^\mu_\nu\partial^3 d_{\mu\nu\rho}B(\partial X^\rho)\\
i\Delta^B_{XF}S &= -\frac{\kappa}{2\pi}(u\partial B-B\partial u)(\partial^3X^\mu)(\partial X^\mu)\\
i\Delta^B_{FF}S &= \frac{\kappa}{6\pi}y_h\Big\{(-5+3\alpha)u(\partial^3 B)+5(\partial^3 u)B\\
&\qquad +(12-5\alpha)(\partial u)(\partial^2 B)-12(\partial^2 u)(\partial B)\Big\}\ .
\end{aligned}\tag{4.39}$$

We have thus obtained the anomaly at antifield number 0. It consists of three parts. The first can be written in the form

$$i(\Delta S)^0_X = i\Delta^h_{XX}S+i\Delta^B_{XX}S=\tfrac{1}{24\pi}\tilde{c}_{\mu\nu}\partial^3\tilde{h}_{\mu\nu}\ , \tag{4.40}$$

where $\tilde{c}_{\mu\nu}$ was given in (4.32), and

$$\tilde{h}_{\mu\nu}=h\delta_{\mu\nu}+2d_{\mu\nu\rho}B\partial X^\rho\ . \tag{4.41}$$

It is the total contribution from the matter loops, and agrees with [12, 13]. The other two parts originate in loops with fermions. They are

$$\begin{aligned}
i(\Delta S)^0_F &= i\Delta_{XF}S+i\Delta^h_{FF}S=-\frac{100}{24\pi}\,c\,\partial^3 h-\frac{\kappa}{2\pi}(u\partial B-B\partial u)(\partial^3X^\mu)(\partial X^\mu)\\
(\Delta S)^0_W &= \Delta^B_{FF}S\approx 0\ .
\end{aligned}\tag{4.42}$$

The upper index 0 indicates that we have so far only the terms of antifield number 0. These results agree with [14, 15, 16].

4.3 Consistency and antifield terms

From the general arguments in section 2.2 and appendix A we know that the anomaly is consistent. At antifield number zero this implies that $D^0(\Delta S)^0\approx 0$. We may check this now, and at the same time obtain the anomaly at antifield number 1. This will e.g. include the contributions of h^*, which according to (4.13) is the antighost. Note that this subsection is just a check, and allows to compare with results of other authors, but in principle the anomalies are determined and one may go directly to section 5.

It will turn out that the three parts mentioned above, $(\Delta S)^0_X$, $(\Delta S)^0_F$ and $(\Delta S)^0_W$, are separately invariant under D^0

$$D^0(\Delta S)^0_X\approx 0\qquad D^0(\Delta S)^0_F\approx 0\ , \tag{4.43}$$

while this is obvious for $(\Delta S)^0_W\approx 0$. To check this, one first obtains that

$$\begin{aligned}
D^0\tilde{c}_{\mu\nu} &= -\tilde{c}_{\rho(\mu}\partial\tilde{c}_{\nu)\rho}+2\kappa u(\partial u)\left[2(\partial X^\mu)(\partial X^\nu)+(\alpha+1)y_h\delta_{\mu\nu}\right]\\
D^0\tilde{h}_{\mu\nu} &= \left(\delta_{\rho(\mu}\bar\partial-\tilde{h}_{\rho(\mu}\partial+(\partial\tilde{h}_{\rho(\mu)})\right)\tilde{c}_{\nu)\rho}\\
&\quad +2\kappa\left[2(\partial X^\mu)(\partial X^\nu)+\delta_{\mu\nu}y_h\right](B\partial u-u\partial B)-2d_{\mu\nu\rho}y_\rho u\ .
\end{aligned}\tag{4.44}$$

According to the theorem 3.1, this implies that the consistent anomaly can be split in

$$\Delta S = (\Delta S)_X + (\Delta S)_F + (\Delta S)_W \ , \tag{4.45}$$

where each term separately is invariant under $\mathcal{S}$, and starts with the expressions in (4.40) and (4.42). Theorem 3.1 implies that the full expressions are obtainable from the consistency requirement.

Indeed, from (4.43) one can use (3.42) to determine $(\Delta S)^1_X, (\Delta S)^1_F$ and $(\Delta S)^1_W$: they are obtained by replacing the field equations y_h, y_B and y_X in the variation under D^0 by h^*, B^* and X^*. For the first one we obtain:

$$\begin{aligned} i(\Delta S)^1_X &= \frac{1}{24\pi}\tilde{c}_{\mu\nu}\partial^3[-2\kappa\delta_{\mu\nu}h^*(B\partial u - u\partial B) + 2X^*_\rho d_{\mu\nu\rho}u] \\ &+ \frac{1}{24\pi}2\kappa u(\partial u)(1+\alpha)h^*\delta_{\mu\nu}(\partial^3\tilde{h}_{\mu\nu}) \\ &+ \frac{1}{24\pi}h^*[-8\kappa(\partial^3 c)(B\partial u - u\partial B) + 8\kappa u(\partial u)(\partial^3 h)] \end{aligned} \tag{4.46}$$

The first two terms can be absorbed in $(\Delta S)^0_X$ ((4.40)) by adding to $\tilde{c}_{\mu\nu}$ and $\tilde{h}_{\mu\nu}$:

$$\begin{aligned} \tilde{c}^{(1)}_{\mu\nu} &= 2\kappa(1+\alpha)h^*u(\partial u)\delta_{\mu\nu} \\ \tilde{h}^{(1)}_{\mu\nu} &= -2\kappa\delta_{\mu\nu}h^*(B\partial u - u\partial B) + 2d_{\mu\nu\rho}X^*_\rho u \ , \end{aligned} \tag{4.47}$$

This part originated in the matter–matter entries of the transformation matrix K and the regulator $S_{\mu\nu}$ including all antifields. If we consider these entries completely, we get as anomaly

$$i(\Delta S)_m = \tfrac{1}{24\pi}c_{\mu\nu}\partial^3 h_{\mu\nu} \tag{4.48}$$

where

$$\begin{aligned} c_{\mu\nu} &= \tilde{c}_{\mu\nu} + \tilde{c}^{(1)}_{\mu\nu} \\ h_{\mu\nu} &= \tilde{h}_{\mu\nu} + \tilde{h}^{(1)}_{\mu\nu} + 2\kappa(1-\alpha)c^*(\partial u)u\delta_{\mu\nu} \ . \end{aligned} \tag{4.49}$$

Note therefore that computing the matter anomaly by using just these entries would not give rise to the last term of (4.46):

$$i(\Delta S)_X - i(\Delta S)_m = \frac{\kappa}{3\pi}h^*[-(\partial^3 c)(B\partial u - u\partial B) + u(\partial u)(\partial^3 h)] + \text{ terms of } afn \geq 2\,. \tag{4.50}$$

As $(S, (\Delta S)_X) = 0$, it follows that $(\Delta S)_m$ is not a consistent anomaly ! Indeed, the proof of consistency given in appendix A requires that we trace over all the fields in the theory. One may check that the violation of the consistency condition for $(\Delta S)_m$ agrees with (A.19). We will see that for $\alpha = 0$ this extra term will be cancelled when adding the fermion contributions.

For the other parts of the anomaly we obtain at antifield number 1:

$$\begin{aligned} \frac{6\pi i}{\kappa}(\Delta S)^1_W &= h^*\Big\{\Big(-49(\partial^3 c) + 9(\partial^2 c)\partial\Big)(u\partial B - B\partial u) \\ &\qquad +\alpha\Big[u(\partial^2 c)\partial^2 B + 10(\partial u)B(\partial^3 c) + 15(\partial u)(\partial B)(\partial^2 c) \end{aligned}$$

$$
\begin{aligned}
&\quad -15u(\partial^3 c)\partial B - 6u(\partial^4 c)B\Big] \\
&\qquad +49(\partial^3 h)(\partial u)u - 9(\partial^2 h)(\partial^2 u)u + 3\alpha u(\partial^3\nabla u) - 5\alpha(\partial u)(\partial^2\nabla u)\Big\} \\
&+B^*\left\{9(\partial^3 u)(u\partial B - B\partial u) + (-9+\alpha)(\partial u)u\partial^3 B + 10\alpha(\partial^2 u)u\partial^2 B\right\} \\
&+X^*_\mu(\partial X^\mu)\left\{5u\partial^3 u + 12(\partial^2 u)\partial u\right\}
\end{aligned}
$$

$$
\begin{aligned}
\frac{6\pi i}{\kappa}(\Delta S)^1_F \;=\;& -50h^*\left\{(1-\alpha)(\partial u)u\partial^3 h + (\partial^3 c)(B\partial u - u\partial B)\right\} \\
& -3h^*\left\{\left((\partial^3 c) + 3(\partial^2 c)\partial\right)(u\partial B - B\partial u) + u(\partial u)\partial^3 h + 3u(\partial^2 u)\partial^2 h\right\} \\
& -3B^*\left\{-3(u\partial B - B\partial u)\partial^3 u + 3u(\partial u)\partial^3 B\right\} \\
& -3X^*_\mu\left\{-2u(\partial u)\partial^3 X^\mu - (\partial X^\mu)(\partial(u\partial^2 u)) - 2u(\partial^2 u)\partial^2 X^\mu\right\}
\end{aligned} \tag{4.51}
$$

Remarkable simplifications occur for the full anomaly:

$$
\begin{aligned}
i(\Delta S)^0 + i(\Delta S)^1 \;=\;& i(\Delta S)^0_m + i(\Delta S)^1_m + i(\Delta S)^0_F + i(\Delta S)^0_W \\
& + \frac{\kappa}{6\pi}X^*_\mu\left\{6\partial(u(\partial u)\partial^2 X^\mu) + 9(\partial^2 u)(\partial u)\partial X^\mu + 8u(\partial^3 u)\partial X^\mu\right\} \\
& + \frac{\alpha\kappa}{6\pi}h^*\Big\{u(\partial^2 c)\partial^2 B + 10(\partial u)B(\partial^3 c) + 15(\partial u)(\partial B)(\partial^2 c) \\
&\qquad -15u(\partial^3 c)\partial B - 6u(\partial^4 c)B \\
&\qquad +50(\partial u)u\partial^3 h + 3u\partial^3\nabla u + 5(\partial^2\nabla u)\partial u\Big\} \\
& + \frac{\alpha\kappa}{6\pi}B^*\left\{(\partial u)u\partial^3 B + 10(\partial^2 u)u\partial^2 B\right\}\ ,
\end{aligned} \tag{4.52}
$$

where $(\Delta S)^0_m$ and $(\Delta S)^1_m$ are the terms of antifield number 0 and 1 in (4.48). Note especially the simplification when using the parametrization with $\alpha = 0$. The terms with the 'antighost' h^* are included in the 'matter anomaly' $(\Delta S)_m$, (4.48), and B^* disappears completely.

Let us recapitulate what we have determined. First remark that the regularization depends on an arbitrary matrix T, and this implies that, not specifying T, ΔS is only determined up to (G, S), where G is a local integral [3]. We have chosen a regularization (a specific matrix T). This determines the value of ΔS. However, we have calculated only the part of ΔS at antifield number 0 (including the weakly vanishing terms). If we would have calculated up to field equations (which would in principle be sufficient to establish whether the theory has anomalies), then we would have determined ΔS up to (G, S), where G has only terms with antifield number 1 or higher. In our calculation of section 4.2, we determined also the weakly vanishing terms. Therefore the value of ΔS has been fixed up to the above arbitrariness with terms G of antifield number 2 or higher. Indeed, looking at (3.42) one can always shift $(\Delta S)^1 \to (\Delta S)^1 + \delta_{KT}G^2$, for some arbitrary function G^2 of antifield number two. This arbitrariness can also be seen from the fact that changing T at a certain antifield number, would change the counterterms starting at the same antifield number. Our calculation above did not depend on possible new terms of T at antifield number 2 or higher (changing T at antifield number 1, would change the weakly vanishing terms of the anomaly). Of course, the remaining arbitrariness is not important when we want to investigate whether a theory has anomalies.

To obtain the complete form of ΔS up to (S, G) one can continue the calculations of this subsection to determine the terms of antifield number 2.

5 Background charges in W_3 gravity

It is by now well–known that the anomalies can be cancelled in chiral W_3 gravity by including background charges [35]. We will see in this section how this can be implemented in the BV language.

To cancel the anomalies by local counterterms, we first note that $(\Delta S)^0_W \approx 0$, and theorem 3.1 thus implies that there is a local counterterm, which starts with (we take in this section $\alpha = 0$, but of course these steps can also be done in parametrizations with $\alpha \neq 0$, as this involves only a canonical transformation).

$$M_{W1}{}^1 = -\frac{\kappa}{\pi} B \left[\frac{5}{6} u (\partial^3 h^*) + \frac{9}{2} (\partial u)(\partial^2 h^*) + \frac{17}{2} (\partial^2 u)(\partial h^*) + \frac{17}{3} (\partial^3 u) h^* \right] , \qquad (5.1)$$

which is the right hand side of the last expression in (4.39), with y_h replaced by h^*, where we added a total derivative for later convenience. The other terms in (4.45) can not be countered by a local integral.

Background charges are terms with $\sqrt{\hbar}$ in the (extended) action. We thus have

$$W = S + \sqrt{\hbar}\, M_{1/2} + \hbar M_1 + \dots \ . \qquad (5.2)$$

Therefore the expansion (2.15) of the master equation (2.13) is now changed to

$$\begin{aligned} (S, M_{1/2}) &= 0 & (5.3) \\ (S, M_1) &= i\Delta S - \frac{1}{2}(M_{1/2}, M_{1/2})\,. & (5.4) \end{aligned}$$

Relevant terms $M_{1/2}$ are those which are in the antibracket cohomology. Indeed, if $M_{1/2}$ which solves (5.4) has a part (S, G), then we find also a solution by omitting that part of $M_{1/2}$, and adding to M_1 a term $\frac{1}{2}\left(M_{1/2}, G\right)$. These terms $M_{1/2}$ are thus again determined by their part at antifield number zero. In chiral W_3 gravity one considers

$$M_{1/2}{}^0 = (2\pi)^{-1/2} \left[a_\mu h (\partial^2 X^\mu) + e_{\mu\nu} B (\partial X^\mu)(\partial^2 X^\nu) \right] , \qquad (5.5)$$

where the numbers a_μ and $e_{\mu\nu}$ are the background charges, and the numerical factor is for normalization in accordance with previous literature. We first consider (5.3). Using our theorem, $D^0 M_{1/2}{}^0$ should be weakly zero in order to find a solution. This gives the following conditions on the background charges:

$$\begin{aligned} e_{(\mu\nu)} - d_{\mu\nu\rho} a_\rho &= 0 \\ d_{\mu\nu\rho}(e_{\rho\sigma} - e_{\sigma\rho}) + 2 e_{(\mu}{}^\rho d_{\nu)\sigma\rho} &= b_\sigma \delta_{\mu\nu} \ , \end{aligned} \qquad (5.6)$$

where b_σ is determined to be $2\kappa a_\sigma$ by the consistency of the symmetric part $(\mu\nu\sigma)$ of the last equation with the first one and (4.2). If these conditions are satisfied, we know that we can construct the complete $M_{1/2}$ perturbatively in antifield number. The solution is

$$\begin{aligned}
M_{1/2} &= {M_{1/2}}^0 + {M_{1/2}}^1 + {M_{1/2}}^2 \\
{M_{1/2}}^1 &= (2\pi)^{-1/2}\Big[-a_\mu X^*_\mu(\partial c) + e_{\mu\nu}X^*_\mu u(\partial^2 X^\nu) + e_{\nu\mu}(\partial X^*_\mu)u(\partial X^\nu) \\
&\qquad + \kappa a_\mu h^*(D^{-2}u)(\partial^2 X^\mu)\Big] \\
{M_{1/2}}^2 &= (2\pi)^{-1/2} 2\kappa a_\mu \Big[(\partial X^*_\mu)h^* - c^*(\partial^2 X^\mu)\Big] u(\partial u) \ .
\end{aligned} \tag{5.7}$$

We calculated here the terms of antifield number 2 (and checked that there are no higher ones), but note that these are not necessary for the analysis below.

In the set of solutions of (4.2) as explained in footnote 6, we have shown that there is only a solution for these equations in the case $D = 1$ and r arbitary [9]. There is thus exactly one solution for each value of n, the range of the index μ. For these models, the solution of (4.2) can be simply written as

$$d_{111} = -\sqrt{\kappa}\ ; \qquad d_{1ij} = \sqrt{\kappa}\,\delta_{ij}\ , \tag{5.8}$$

where $i = 2, \ldots, n$. The solution to (5.6) is

$$\begin{aligned}
&e_{00} = -\sqrt{\kappa}\,a_0\ ; \qquad e_{ij} = \sqrt{\kappa}\,a_0\delta_{ij} \\
&e_{0i} = 2\sqrt{\kappa}\,a_i\ ; \qquad e_{i0} = 0\ ,
\end{aligned} \tag{5.9}$$

where a_μ is arbitrary.

The final relation for absence of anomalies up to one loop is that we have to find an M_1 such that the last equation of (5.4) is satisfied. Again we only have to verify this at zero antifield number, and up to field equations of S^0, due to theorem 3.1. We calculate $Q \equiv i\Delta S - \frac{1}{2}(M_{1/2}, M_{1/2})$ at antifield number zero. It contains terms proportional to $c\partial^3 h$, which can not be removed by a local counterterm. So in order to have no anomaly, the multiplicative factor of this term has to vanish. This implies the relation

$$c_{mat} \equiv n - 12 a_\mu a_\mu = 100 \tag{5.10}$$

For the other terms in Q, one needs a counterterm

$${M_{B1}}^0 = \frac{1}{6\pi} e_{\mu\nu} a_\mu B \partial^3 X\ , \tag{5.11}$$

and one imposes the relations

$$\begin{aligned}
&2e_{\mu\nu}a_\mu - 6a_\mu e_{\nu\mu} + d_{\nu\mu\mu} = 0 \\
&e_{\mu\rho}e_{\rho\nu} = \kappa Y \delta_{\mu\nu} \\
&-e_{\mu\rho}e_{\nu\rho} + \tfrac{1}{3}d_{\mu\rho\sigma}d_{\nu\rho\sigma} + \tfrac{2}{3}d_{\mu\nu\rho}e_{\sigma\rho}a_\sigma = \kappa\delta_{\mu\nu}Z \\
&-4\kappa a_\mu a_\nu + e_{\rho\mu}e_{\rho\nu} + 2a_\rho e_{\rho\sigma}d_{\sigma\mu\nu} = (3Z + 4Y - 2)\kappa\delta_{\mu\nu}\ ,
\end{aligned} \tag{5.12}$$

[9] In [35] this solution was found, and no solution was found for the other cases, but this was also not excluded. Now we have confirmed that the other cases, with the 4 ‘magical solutions’, do not allow a solution.

where Y and Z are arbitrary numbers, to be determined by consistency requirements. This set of equations on the background charges are exactly the same as in [35]. The solutions (5.8) and (5.9) give now

$$\begin{aligned} {a_0}^2 &= -\frac{49}{8}\ ; \qquad {a_i}^2 = \frac{-53+2n}{24} \\ Y &= {a_0}^2\ ; \qquad Z = \frac{1}{3}\left(2-{a_0}^2\right) = \frac{87}{8}\ . \end{aligned} \tag{5.13}$$

Then we obtain that

$$\begin{aligned} &2\pi\left((M^0_{1/2}, M^1_{1/2}) - i\Delta S + (M^0_{B1}, S^1)\right) = -\tfrac{1}{3}e_{\nu\mu}a_\nu(\partial^2 u)y_\mu \\ &+\kappa B\left(2Z(\partial^3 u)y_h + 3Z(\partial^2 u)\partial y_h + (3Z-2+2Y)(\partial u)\partial^2 y_h + (Z+Y-1)u\partial^3 y_h\right)\ . \end{aligned} \tag{5.14}$$

This determines then the counterterm M_{B1} at antifield number 1. Inserting the values for Y and Z gives

$$\begin{aligned} {M_{B1}}^1 &= \frac{\kappa}{16\pi}B\left[30u(\partial^3 h^*) + 147(\partial u)(\partial^2 h^*) + 261(\partial^2 u)(\partial h^*) + 174(\partial^3 u)h^*\right] \\ &\quad + \frac{1}{6\pi}X^*_\mu e_{\nu\mu}a_\nu\partial^2 u\ . \end{aligned} \tag{5.15}$$

Combining this term with (5.1), we get for the total counterterm at antifield number one

$$\begin{aligned} {M_1}^1 &= \frac{25\kappa}{48\pi}B\left[2u(\partial^3 h^*) + 9(\partial u)(\partial^2 h^*) + 15(\partial^2 u)(\partial h^*) + 10(\partial^3 u)h^*\right] \\ &\quad + \frac{1}{6\pi}X^*_\mu e_{\nu\mu}a_\nu\partial^2 u\ . \end{aligned} \tag{5.16}$$

The form of this counterterm is also obtained in [16, 14, 36]. As mentioned before, this does not only determine the quantum corrections to the action, but also the corrections to the BRST transformations, by looking to the linear terms in antifields in the gauge–fixed basis.

The value of κ has been irrelevant here. In fact, one can remove κ rescaling $d_{\mu\nu\rho}$, $e_{\mu\nu}$, B^* and u^* with $\sqrt{\kappa}$ and B and u by $(1/\sqrt{\kappa})$. For the usual normalizations in operator product expansions of the W–algebra one takes

$$\kappa = \frac{1}{6Z} = \frac{8}{22+5c_{mat}} = \frac{4}{261}\ . \tag{5.17}$$

6 Conclusions

By applying the BV formalism and the regularization procedure initiated in [18, 19, 3] to chiral W_3 gravity, we have shown how it can be applied to more difficult theories. Gauge fixing could be performed by the most simple canonical transformation from the classical extended action. We have seen how the theorem on antibracket cohomology can be effectively used in calculations. Background charges have been introduced as terms of order $\sqrt{\hbar}$. They lead to a new expansion of the quantum master equation: (5.3), (5.4). This method allows an easy verification of the cancellation of anomalies.

We performed only the one–loop calculations. The higher loops should be regulated using higher derivative regularization (still combined with Pauli–Villars) [37]. This is again a regularization within the path integral. Recently it has been successfully used in Wess–Zumino–Witten models to obtain the anomalies at all loops [22].

It would be interesting to consider also W_3 gravity as an example for higher loop anomalies. Usually in gauge theories one may use covariant derivatives to built the terms with higher derivatives for the regularization. Then these terms are invariant under the gauge transformations, and therefore they produce no anomalies. The only anomalies which occur, originate in the Pauli–Villars mass terms, which are genuine one–loop anomalies. This situation is different in W gravity, because in these theories there are no covariant derivatives (with a finite number of terms). Therefore one expects here new anomalies from higher loops, which indeed are known to appear [38].

Acknowledgments.

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Appendix

A Consistency of the regulated anomaly

The PV regularization should give a consistent anomaly, as all manipulations can be done at the level of the path integral. Here we want to check this explicitly from the final expression for the anomaly after integrating out the PV fields. Then we will consider the expression which is obtained after integrating out only parts of the fields. We will see that in that case the resulting expression does not satisfy this consistency equation.

The expression for the anomaly depends on an invertible matrix T_{AB}:

$$\Delta S = str\left[J\frac{1}{\mathbb{1}-\mathcal{R}/M^2}\right] , \tag{A.1}$$

where

$$\begin{aligned} K^A{}_B &= \overrightarrow{\partial}^A S\overleftarrow{\partial}_B \ ; \qquad & \underline{S}_{AB} &= \overrightarrow{\partial}_A S\overleftarrow{\partial}_B \\ J &= K+\tfrac{1}{2}T^{-1}(\mathcal{S}T) \ ; \qquad & \mathcal{R} &= T^{-1}\underline{S} \ , \end{aligned} \tag{A.2}$$

and for matrices $M^A{}_B$ or M_{AB} we define the nilpotent operation

$$(\mathcal{S}M)_{AB} = (M_{AB},S)\,(-)^B \ . \tag{A.3}$$

Matrices can be of bosonic or fermionic type. The Grassmann parity $(-)^M$ of a matrix is the statistic of $M^A{}_B(-)^{A+B}$. So of the above matrices, J, K and $(\mathcal{S}T)$ are fermionic, the other are bosonic. The definition of supertraces and supertransposes depend on the position of the indices.

$$\begin{aligned} \left(R^T\right)_{BA} &= (-)^{A+B+AB+R(A+B)}R_{AB} \ ; \qquad & \left(T^T\right)^{BA} &= (-)^{AB+T(A+B)}T_{AB} \\ \left(K^T\right)_B{}^A &= (-)^{B(A+1)+K(A+B)}K^A{}_B \ ; \qquad & \left(L^T\right)^A{}_B &= (-)^{B(A+1)+L(A+B)}L_B{}^A \\ str\ K &= (-)^{A(K+1)}K^A{}_A \ ; \qquad & str\ L &= (-)^{A(L+1)}L_A{}^A \ . \end{aligned} \tag{A.4}$$

This leads to the rules

$$\begin{aligned} &\left(M^T\right)^T = M \ ; \qquad (MN)^T = (-)^{MN}N^TM^T \ ; \qquad \left(M^T\right)^{-1} = (-)^M\left(M^{-1}\right)^T \\ &str\ (MN) = (-)^{MN}str\ (NM) \ ; \qquad str\ (M^T) = str\ (M) \\ &\mathcal{S}(M\,N) = M(\mathcal{S}N)+(-)^N(\mathcal{S}M)N \ ; \qquad \mathcal{S}\,(str\,M) = str\ (\mathcal{S}M)\,. \end{aligned} \tag{A.5}$$

The second derivatives of the master equation lead to

$$\begin{aligned} \mathcal{S}\underline{S} &= -K^T\underline{S}-\underline{S}\,K \\ \mathcal{S}K &= \overline{S}\,\underline{S}-K\,K \ , \end{aligned} \tag{A.6}$$

where,

$$\overline{S} = \overrightarrow{\partial}^A S\overleftarrow{\partial}^B \ ; \qquad (K^T)_A{}^B = -\overrightarrow{\partial}_A S\overleftarrow{\partial}^B \ , \tag{A.7}$$

the first being a graded antisymmetric matrix, and the second is in accordence with (A.2) and the previous rules of supertransposes. We rewrite the expression of the anomaly as

$$\Delta S = M^2 \; str \left[T\, J\, P^{-1} \right] \;, \tag{A.8}$$

where

$$P = M^2\, T - \underline{S} \;. \tag{A.9}$$

We also easily derive the following properties

$$\begin{aligned} \mathcal{S}(T\,J) &= \mathcal{S}(T\,K) = T\overline{S}\underline{S} - TKK - (\mathcal{S}T)K \\ \mathcal{S}P &= M^2(TJ + J^T T) - (PK + K^T P) \;. \end{aligned} \tag{A.10}$$

This leads to

$$\begin{aligned} \mathcal{S}\Delta S &= M^2 \; str \Big[\Big(T\overline{S}\underline{S} - TKK - (\mathcal{S}T)K \Big) \, P^{-1} \\ &\qquad -TJP^{-1} \left(M^2(TJ + J^T T) - (PK + K^T P) \right) P^{-1} \Big] \end{aligned} \tag{A.11}$$

In the first term of the first line we write $\underline{S} = M^2 T - P$. The trace of both these terms is zero due to (A.5) and the (a)-symmetry properties of the matrices given above. For the same reason the second term of the second line vanishes. The first term of the second line is a square of a fermionic matrix which vanishes under the trace. The remaining terms again combine into matrices which are traceless by using their symmetry and by (A.5). This means we have proven that PV-regularization guarantees consistent one–loop anomalies,

$$\mathcal{S}\Delta S = 0 \;. \tag{A.12}$$

In [26] a formula has been given for the non–local counterterm for any ΔS defined by (A.1):

$$\Delta S = -\frac{1}{2}\mathcal{S}\left(str \ln \frac{\mathcal{R}}{M^2 - \mathcal{R}} \right) \;. \tag{A.13}$$

This can be proven also from the above formulas.

Consider the part of the anomaly originating from the path integral over some subset of fields, e.g. the matter fields in W_3. The question arises whether this gives already a consistent anomaly. To regulate this anomaly we only have to introduce PV–partners for this subset of fields. In the space of all fields, and taking the basis with first the fields which are integrated, we write the T–matrix as

$$T = \begin{pmatrix} \tilde{T} & 0 \\ 0 & 0 \end{pmatrix} \;. \tag{A.14}$$

Because only this subsector is integrated, we have to project out the other sectors (mixed and external) in the full matrix of second derivatives $\underline{S}$ before inverting it to define a propagator. This can be done by defining the projection operator Π as

$$\Pi = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix} \;. \tag{A.15}$$

The inverse propagator is then $Z = (\Pi P \Pi)$. Further we understand by inverses, as e.g. T^{-1}, the inverse in the subspace. So we have

$$TT^{-1} = T^{-1}T = \Pi \ . \tag{A.16}$$

The matter anomaly takes the form

$$(\Delta S)_m = M^2 \ str[TJZ^{-1}] \ , \tag{A.17}$$

and (A.10) remains valid. In the variation of this anomaly, we now often encounter

$$P\,Z^{-1} = \Pi + Y \qquad \text{where} \qquad Y = -(1-\Pi)\underline{S}Z^{-1} \ . \tag{A.18}$$

Using again the properties of supertraces and transposes, we obtain

$$\mathcal{S}(\Delta S)_m = M^2 \ str\left[-T\overline{S}Y - TK(\mathbb{1}-\Pi)KZ^{-1} - Y^TKZ^{-1}(TJ + J^TT)\right] \ . \tag{A.19}$$

This thus shows that one does in general not obtain a consistent anomaly when integrating in the path integral only over part of the fields. One can see that for ordinary chiral gravity the structure of the extended action implies that each term of (A.19) vanishes. For chiral W_3 gravity however, some terms remain. A similar result for Ward identities was obtained in [39].

B Lemmas and proofs concerning the KT acyclicity.

Lemma B.1 *: After a canonical transformation, the new KT differential is expressed in terms of the old one as follows*

$$\delta'_{KT}F(\Phi'^*, \Phi') = \delta_{KT}\Phi'^*_A|_{c'=0}\,\overrightarrow{\partial}'^A F \ . \tag{B.1}$$

The primed derivatives are taken with all other primed fields constant, and also the notation $c' = 0$ means that the considered expression is first written in terms of primed variables, and then all 'ghosts' are put equal to zero. The word 'ghosts' stands for all fields with positive definite ghost number.

First, we can use the following definition of the KT differential

$$\delta_{KT}F(\Phi^*, \Phi) = \left(S\overleftarrow{\partial}_A\right)_{c=0}\overrightarrow{\partial}^A F \ . \tag{B.2}$$

One may check that this is equivalent with (3.33). Now we obtain for the primed KT differential

$$\begin{aligned}\delta'_{KT}F(\Phi'^*, \Phi') &= \left(S\overleftarrow{\partial}'_A\right)_{c'=0}\overrightarrow{\partial}'^A F \\ &= \left(S\overleftarrow{\partial}_B \cdot \Phi^B\overleftarrow{\partial}'_A\Big|_{\Phi'^*} + S\overleftarrow{\partial}^B \cdot \Phi^*_B\overleftarrow{\partial}'_A\Big|_{\Phi'}\right)_{c'=0}\overrightarrow{\partial}'^A F \ . \end{aligned} \tag{B.3}$$

In the last expression the second term in the brackets can be omitted because $\partial^B S$ has positive ghost number, and will thus vanish when all fields of positive ghost number in the primed basis are put equal to zero. (Canonical transformations conserve the ghost number). For the first term we use eq.(A.30) in [3]

$$\Phi^B \overleftarrow{\partial}'_A \Big|_{\Phi'^*} = \left(M^{-1}\right)^B{}_A = \overrightarrow{\partial}^B \Phi'^*_A \Big|_{\Phi} \; . \tag{B.4}$$

Therefore this term is by definition equal to $\delta_{KT}\Phi'^*_A$, which gives the result (B.1) ■

The previous lemma implies that for any function $F(z')$

$$\delta'_{KT}F(z') = \left(\delta_{KT}F(z'(z))\right)_{c'=0} + \; c'\text{–dependent terms} \; . \tag{B.5}$$

The only non–trivial step in this new formulation of the lemma is that $\delta_{KT}\Phi'^A$ gives only c'–dependent terms.

Theorem B.1 : *If the KT differential is acyclic in one set of coordinates, then it is also acyclic after a canonical transformation.*

Suppose the KT differential is acyclic in primed coordinates. Then consider $F(z_0)$ such that $\delta_{KT}F(z_0) = 0$, where we use the notation $z = \{\Phi^*, \phi, c\}$ and z_0 is the same set with $c = 0$. Now define

$$F'(z'_0) \equiv F(z_0(z'_0)) \tag{B.6}$$

where $z(z')$ is the canonical transformation. This function is thus just rewriting F in primed coordinates and then putting $c' = 0$, thus

$$F(z_0(z')) = F'(z'_0) + c'\text{–dependent terms.} \tag{B.7}$$

From our starting point we thus have

$$0 = \delta_{KT}F'(z'_0) + \delta_{KT}c'\text{–dependent terms} \; , \tag{B.8}$$

and the last term is still proportional to c' because of ghost number conservation of the canonical transformation, and $\delta_{KT}c = 0$.

Now we use (B.5) for F'. As F' does not depend on c', also $\delta'_{KT}F'$ does not depend on c', and therefore the first term gives the complete result. Eq. (B.8) with the remark following it, then implies that $\delta'_{KT}F' = 0$.

The assumption of acyclicity of δ'_{KT} implies that

$$F'(z'_0) = \delta'_{KT}H'(z'_0) + h'(\phi') \; . \tag{B.9}$$

We now follow the same method as above with primed and unprimed fields interchanged. We define

$$\begin{aligned} H(z_0) &\equiv H'(z'_0(z_0)) \\ H'(z'_0(z)) &= H(z_0) + \; c\text{–dependent terms} \\ \delta'_{KT}H'(z'_0(z)) &= \delta'_{KT}H(z_0) + \; c\text{–dependent terms} \; . \end{aligned} \tag{B.10}$$

Now we use the previous lemma in the reverse order

$$\begin{aligned} \delta_{KT}H(z_0) &= \left(\delta'_{KT}H(z_0)\right)_{c=0} \\ &= \left(\delta'_{KT}H'(z'_0(z)\right)_{c=0} \\ &= \left(F'(z'_0) - h'(\phi')\right)_{c=0} \end{aligned} \tag{B.11}$$

and then with (B.7) this implies (at $c = 0$, the transformation $\phi'(z)$ is restricted to $\phi'(\phi)$)

$$F(z_0) = \delta_{KT} H(z_0) + h'(\phi'(\phi)) \ . \tag{B.12}$$

If the definition of the functions $\mathcal{F}^0_s$ has been taken in both variables such that the subpart of the canonical transformation (the fields at ghost number zero) $\phi'(\phi)$ leaves this set of functions invariant, this finishes the proof. Otherwise, one uses lemma B.3 to write the difference as another δ_{KT}–exact expression. This proves the acyclicity in unprimed coordinates. ■

Lemma B.2 : *One can perform a canonical transformation such that* $f^{ij}{}_{a_1}$ *defined by (3.23) does not have parts proportional to* $R^i{}_a$ *or* $R^j{}_a$.

The exact definition of 'proportional' is explained in lemma B.4. The extended action is

$$\begin{aligned} S &= S^0(\phi) + \phi^*_i R^i{}_a c^a \\ &\quad + \left(c^*_a Z^a{}_{a_1} + (-)^i \phi^*_i \phi^*_j f^{ji}{}_{a_1} \right) c^{a_1} + \ldots \end{aligned} \tag{B.13}$$

(where ... are terms of antifield number 3 or more, or quadratic in ghosts). We consider now the case that

$$(-)^i \phi^*_i \phi^*_j f^{ji}{}_{a_1} = (-)^a y_a y_b g^{ba}{}_{a_1} + (-)^i \phi^*_i y_a g^{ai}{}_{a_1} + (-)^i \phi^*_i \phi^*_j g^{ji}{}_{a_1} \ . \tag{B.14}$$

where $y_a = \phi^*_i R^i{}_a$. We perform a canonical transformation using the generating function (the transformation being defined by (4.10))

$$f = (-)^a c'^*_a \phi'^*_i \left(g^{ja}{}_{a_1} + R^j{}_b g^{ba}{}_{a_1} \right) c^{a_1} \ . \tag{B.15}$$

One may check that the action in terms of the primed fields is again as in (B.13), with f replaced by g. ■

First we will prove acyclicity in $\mathcal{F}^*_1$:

Lemma B.3 : *The KT differential is acyclic on functions of* ϕ^i *and* ϕ^*_i*, modulo terms proportional to* $\phi^*_i R^i{}_a$.

First define an operator d which is

$$dF = \phi^*_i \frac{\overrightarrow{\partial}}{\partial y_i} F + \phi^*_i R^i{}_a \epsilon^a \ . \tag{B.16}$$

The derivative w.r.t. y_i was defined in (3.11), where we noticed that its value is undetermined, corresponding to an arbitrary function ϵ^a in the above equation. In $\mathcal{F}^*_1$, we have

$$\delta_{KT} F = y_i \frac{\overrightarrow{\partial}}{\partial \phi^*_i} F \ . \tag{B.17}$$

This implies that

$$(\delta_{KT}d + d\delta_{KT})\, F = y_i \frac{\vec{\partial}}{\partial y_i} F + \phi_i^* \frac{\partial}{\partial \phi_i^*} F + \phi_i^* R^i{}_a \epsilon'^a \ . \tag{B.18}$$

Now we expand any function as well according to (3.10) as in its number of antifields. This defines

$$F = \sum_{m,n=0}^{\infty} F^{m,n} \ , \tag{B.19}$$

where m counts the number of antifields and n the number of field equations in each term. We have now also

$$(\delta_{KT}d + d\delta_{KT})\, F^{m,n} = (m+n) F^{m,n} + \phi_i^* R^i{}_a \epsilon'^a . \tag{B.20}$$

As δ_{KT} does not change the value of $m+n$, any function which vanishes under δ_{KT}, also has $\delta_{KT} F_N = 0$, where F_N has all terms with $m+n=N$. Therefore with the previous formula, F_N for $N>0$ is the δ_{KT} of something, modulo $\phi_i^* R^i{}_a \epsilon^a$. The functions with $m+n=0$ are those of $\mathcal{F}_s^0$. We thus obtain for $F \in \mathcal{F}_1^*$

$$\delta_{KT} F = 0 \ \Rightarrow \ F = \delta_{KT} H + \phi_i^* R^i{}_a \epsilon^a + f(\phi) \ \text{ with } f(\phi) \in \mathcal{F}_s^0 \ . \tag{B.21}$$

■

Now we will go one step further. The relations (3.21) with (3.23) will be important.

Lemma B.4 *: The KT differential is also acyclic on $\mathcal{F}_2^*$ (antifields up to c_a^*), modulo terms proportional to*

$$\delta_{KT} c^*_{a_1} = c_a^* Z^a{}_{a_1} + (-)^i \phi_i^* \phi_j^* f^{ji}{}_{a_1} \ . \tag{B.22}$$

We define $y_a \equiv \phi_i^* R^i{}_a$. We will expand functions $F(\phi^i, \phi_i^*, c_a^*)$ in powers of c_a^* and y_a. The first expansion is obvious. The expansion in y_a is similar to the one in y_i explained before. We have to choose a representant between the antifields ϕ_i^* to define when we have terms proportional y_a. E.g. for the 2–dimensional gravity model of section 2, the second term in (2.8) is yc (where here $y \equiv y_a$). We choose here to look first at the terms with h^* and write all terms proportional to $\bar{\partial} h^*$ as

$$\bar{\partial} h^* = -y + X_\mu^* \partial X^\mu + \partial(h^* h) + h^*(\partial h) \ . \tag{B.23}$$

The relation $y_i(F^i - G^i) = 0$ defines again an equivalence relation between functions $F^i(\phi)$ with a free index of type i. The properness condition (3.12) gives the possibilities for the difference $F^i - G^i$. We further reduce these functions by identifying those which are equal on the stationary surface for all i. The above procedure defines for each equivalence class one representative function. We denote the set of these representative functions as $\mathcal{F}_s^1$. We thus obtain

$$y_i(F^i - G^i) = 0 \ \text{ or } F^i \approx G^i \text{ where } F^i, G^i \in \mathcal{F}_s^1 \ \Longrightarrow \ F^i = G^i \ . \tag{B.24}$$

We can also generalize this to functions with more free indices of type i. Then the above requirement should hold for each such free index. (If there are no free i indices then this is just the reduction to $\mathcal{F}_s^0$).

Functions $F(\phi^i, \phi^*_i)$ are now expanded as follows

$$F(\phi^i, \phi^*_i) = \sum_{m,n,p=0} (y_a)^m \, (y_i)^n \, (\phi^*_i)^p \, F_{m,n,p}(\phi) \ , \tag{B.25}$$

where $F_{m,n,p} \in \mathcal{F}^1_s$, and it has m indices of type a and $n+p$ indices of type i, which are not written. This restriction on the coefficients is possible because if the expansion would have a coefficient $F_{m,n,p}$ which does not belong to $\mathcal{F}^1_s$, then $F_{m,n,p} - F^s_{m,n,p}$, where $F^s_{m,n,p} \in \mathcal{F}^1_s$, is either proportional to a field equation (and can thus be absorbed in terms of higher value of n), or proportional to $R^i{}_a$. In the latter case, if i is one of the middle indices, the difference can be omitted in (B.25). If i belongs to the third set of indices this difference can be written using an extra y_a factor.

The remaining indefiniteness of this expansion is that $y_a F^a_{m,n,p}$ could be rewritten as $\phi^*_i G^i$, for a certain G^i of the form $G^i = \sum_{n'=0} (y_j)^{n'} G^i_{m-1,n+n',p+1}$, where the coefficients belong to $\mathcal{F}^1_s$. However, as δ_{KT} on the first expression gives zero, $y_i G^i = 0$. Therefore, as we suppose that G^i is not proportional to $R^i{}_a$, it follows from (3.12) that $G^i = y_j G^{ji}$ (antisymmetric in $[ij]$). This ambiguity thus only occurs if $R^i{}_a F^a \approx 0$, which implies, due to the second line of (3.25), that $F^a \approx Z^a{}_{a_1} \epsilon^{a_1}$. The ambiguous expression is thus $y_a Z^a{}_{a_1}$.

For the functions of ϕ^i, ϕ^*_i and c^*_a, we will use an expansion

$$F(\phi^i, \phi^*_i, c^*_a) = \sum_{m,n=0} (c^*_a)^m \, (y_a)^n \, F_{m,n}(\phi, \phi^*) \ , \tag{B.26}$$

where $F_{m,n}$ has $m+n$ extra indices of type a which are not explicitly written here, and compared with (B.25), we have performed here the sum over n and p. The previous expansion (B.25) was just necessary to define this one in a proper way. As explained above, this expansion is not unique: by (3.21) we have

$$\phi^*_i R^i{}_a Z^a{}_{a_1} - 2\phi^*_i y_j f^{ji}{}_{a_1} = 0 \ . \tag{B.27}$$

The first term appears at level $n = 1$ when we expand as above. The second term is of order zero. Indeed, if f^{ij} would have terms proportional to $R^i{}_a$, we can remove them by canonical transformations (lemma B.2). We have shown that if the KT differential is acyclic in one set of variables, then it is also true for the canonically transformed variables (theorem B.1). The equation (B.27) gives an indefiniteness in the expression for the derivative w.r.t. y_a.

We define an operator d (unrelated to the one of the previous lemma)

$$dF = c^*_a \frac{\overrightarrow{\partial}}{\partial y_a} F + c^*_a Z^a{}_{a_1} \epsilon^{a_1} \ . \tag{B.28}$$

As explained, the derivative in the first term is not unique, and the form of this indefiniteness can be seen by applying d to (B.27).

On the other hand, the KT differential for these functions is

$$\delta_{KT} = y_i \frac{\overrightarrow{\partial}}{\partial \phi^*_i} + y_a \frac{\overrightarrow{\partial}}{\partial c^*_a} \ . \tag{B.29}$$

We will consider the terms of (B.26) in decreasing order of $n+m$. So we define

$$F_{(p)}=\sum_{m=0}^{p}(c_a^*)^m(y_a)^{p-m}F_{m,p-m} \tag{B.30}$$

which leads to

$$(\delta_{KT}d+d\delta_{KT})F_{(p)}=pF_{(p)}+c_a^*Z^a{}_{a_1}\epsilon^{a_1}+G_{(p-1)}\ , \tag{B.31}$$

where ϵ^{a_1} and $G_{(p-1)}$ are undetermined. The latter, of order at most $p-1$ in the expansion variables y_a and c_a^*, is produced by

$$\delta_{KT}\left(c_a^*Z^a{}_{a_1}\right)=2\phi_i^*y_jf^{ji}{}_{a_1}\ . \tag{B.32}$$

For future use, it is more appropriate to have the indefiniteness proportional to a KT–invariant term. This can be done by changing the terms of order $p-1$ with a term $(-)^i\phi_i^*\phi_j^*f^{ji}{}_{a_1}\epsilon^{a_1}$. We then have

$$\begin{aligned}(\delta_{KT}d+d\delta_{KT})F_{(p)}&=pF_{(p)}+y_{a_1}\epsilon^{a_1}+G_{(p-1)}\\ y_{a_1}&\equiv c_a^*Z^a{}_{a_1}+(-)^i\phi_i^*\phi_j^*f^{ji}{}_{a_1}\end{aligned} \tag{B.33}$$

Consider now a function $F\in\mathcal{F}_2^*$ with $\delta_{KT}F=0$. Suppose that in the expansion of the previous lines, the highest level is $(p)=(N)$. Then, as the operator δ_{KT} and d do not raise the level (p), $\delta_{KT}F_{(N)}$ is of lower level, and $d\delta_{KT}F_{(N)}$ as well, except for the indeterminacy mentioned before, which is taken into account in (B.33). Therefore this formula implies

$$F=\delta_{KT}H+y_{a_1}\epsilon^{a_1}+G_{(N-1)}\ , \tag{B.34}$$

where the undetermined quantities ϵ^{a_1} and $G_{(N-1)}$ are not the same as those above. The latter has terms of level $p\le N-1$. Applying δ_{KT} to this expression shows that $\delta_{KT}G_{(N-1)}$ is proportional to y_{a_1}. Then $d\delta_{KT}G_{(N-1)}$ gives zero modulo $c_a^*Z^a{}_{a_1}$. So we may repeat the previous step, obtaining finally (B.34) with $G_{(N-1)}$ replaced by $G_{(0)}$ of order 0, which is a function of ϕ and ϕ^* only, having no terms proportional to $\phi_i^*R^i{}_a$. The KT operation on these functions can not produce c_a^*, and as $\delta_{KT}G_{(0)}$ could only be proportional to y_{a_1}, we have $\delta_{KT}G_{(0)}=0$, and we can apply the previous lemma, to obtain

$$F=\delta_{KT}H+y_{a_1}\epsilon^{a_1}+f(\phi)\ , \tag{B.35}$$

where $f(\phi)\in\mathcal{F}_s^0$. ■

Lemma B.5 *: The KT differential is also acyclic on $\mathcal{F}_3^*$ (antifields up to $c_{a_1}^*$), modulo terms proportional to*

$$\delta_{KT}c_{a_2}^*=c_{a_1}^*Z^{a_1}{}_{a_2}+M_{a_2}(\phi^i,\phi_i^*,c_a^*)\equiv y_{a_2}\ . \tag{B.36}$$

The proof of this lemma is very similar to that of the previous lemma. However, some steps are a bit different here because some indices i of the previous lemma remain (the zero modes are weak relations) while some are replaced by indices a. We therefore present again the first part of the proof (where differences occur), and at the end we can refer to the previous proof.

We will expand functions $F(\phi^i,\phi^*_i,c^*_a,c^*_{a_1})$ in powers of $c^*_{a_1}$ and y_{a_1}. The first expansion is obvious. The expansion in y_{a_1} needs more care. We have to choose a representant between the antifields c^*_a to define when we have terms proportional y_{a_1}.

The relation $y_a(F^a-G^a)\approx 0$ defines again an equivalence relation between functions $F^a(\phi,\phi^*_i)$ with a free index of type a. The properness condition is the second line of (3.25):

$$\text{If } F^a\sim G^a \text{ then } F^a-G^a = Z^a{}_{a_1}\epsilon^{a_1}+y_iK^{ia}\ , \tag{B.37}$$

where ϵ and K are undetermined. The above procedure defines for each equivalence class one representative function. We denote the set of these representative functions as $\mathcal{F}^2_s$. We thus obtain

$$y_a(F^a-G^a)\approx 0 \ \text{ and } F^a,G^a\in\mathcal{F}^2_s \implies F^a=G^a\ , \tag{B.38}$$

which again can be generalized to functions with more free indices of type a.

Functions $F(\phi^i,\phi^*_i,c^*_a)$ are now first expanded as follows

$$F(\phi^i,\phi^*_i,c^*_a)=\sum_{m,n,p=0}(y_{a_1})^m\,(y_i)^n\,(c^*_a)^p\,F_{m,n,p}(\phi,\phi^*)\ , \tag{B.39}$$

where $F_{m,n,p}\in\mathcal{F}^2_s$, and it has m indices of type a_1, n indices of type i, and p indices of type a, which are not written. This restriction on the coefficients is possible because if the expansion would have a coefficient $F_{m,n,p}$ which does not belong to $\mathcal{F}^2_s$, then it is in the equivalence class of some function $F^s_{m,n,p}\in\mathcal{F}^2_s$, and by (B.37), $F_{m,n,p}-F^s_{m,n,p}$ is either proportional to a field equation (and can thus be absorbed in terms of higher value of n), or proportional to $Z^a{}_{a_1}$. In the latter case, this difference can be written using an extra y_{a_1} factor.

The remaining indefiniteness of this expansion is that $y_{a_1}F^{a_1}_{m,n,p}$ could be rewritten as $c^*_aG^a$, for a certain G^a of the form $G^a=\sum_{n'=0}(y_j)^{n'}G^i_{m-1,n+n',p+1}$, where the coefficients belong to $\mathcal{F}^2_s$. However, taking the derivative w.r.t. c^*_a, we obtain $Z^a{}_{a_1}F^{a_1}_{m,n,p}=G^a$. In the equivalence relation which we defined, the l.h.s. is equivalent to zero, and the r.h.s. is equivalent to the $n'=0$ term in the expansion of G^a. Therefore, G^a is proportional to y_i. This ambiguity thus only occurs if $Z^a{}_{a_1}F^{a_1}\approx 0$, which implies, due to the properness conditions, that $F^{a_1}\approx Z^{a_1}{}_{a_2}\epsilon^{a_2}$. The ambiguous expression is thus $y_{a_1}Z^{a_1}{}_{a_2}$.

We again take the sum over n and p and write

$$F(\phi^i,\phi^*_i,c^*_a,c^*_{a_1})=\sum_{m,n=0}\left(c^*_{a_1}\right)^m(y_{a_1})^n\,F_{m,n}(\phi,\phi^*,c^*)\ . \tag{B.40}$$

The non–uniqueness of this expansion is due to

$$Z^a{}_{a_1}Z^{a_1}{}_{a_2}-y_if^{ia}{}_{a_2}=0\ , \tag{B.41}$$

for some functions $f^{ia}{}_{a_2}(\phi)$. Therefore

$$y_{a_1}Z^{a_1}{}_{a_2}-(-)^i\phi^*_i\phi^*_jf^{ji}_{a_1}Z^{a_1}{}_{a_2}-c^*_ay_if^{ia}{}_{a_2}=0\ . \tag{B.42}$$

The first term appears at level $n = 1$ when we expand as above. The other terms are of level zero. Indeed, if f^{ia} would have terms proportional to $Z^a{}_{a_1}$, we can remove them by a similar canonical transformations as in lemma B.2.

We can then define a new operator d, and from here on there is no difference between this proof and the previous one, replacing indices a there with a_1, ... ■

The proofs for higher levels are similar to the last one.

C Proofs of the antibracket cohomology.

We first make a remark. It is clear from (3.32) that if $\mathcal{S}F = 0$, then $\delta_{KT} D^n F = 0$. But later we will need the latter equation when we only know $(\mathcal{S}F)^i = 0$ for $i < n$. This is proven as follows. Define $F_n \equiv \sum_{k=0}^{n} F^k$, so omitting from F the terms F^i with $i > n$. Now the nilpotency property $\mathcal{S}\left(\mathcal{S}F_n\right) = 0$, gives at antifield level $n - 1$

$$(-)^{F+1} \delta_{KT} \left(\mathcal{S}F_n\right)^n + D^{n-1} \mathcal{S}F_n = 0 \ . \tag{C.1}$$

The last term depends at most on $(\mathcal{S}F_n)^{n-1}$. As $(\mathcal{S}F)^i$ for $i < n$ does not depend on F_i for $i > n$ (see (3.32)), we can replace here $(\mathcal{S}F_n)^i$ by $(\mathcal{S}F)^i$. So the last term vanishes if $(\mathcal{S}F)^i = 0$ for $i < n$. On the other hand, (3.32) gives $(\mathcal{S}F_n)^n = D^n F$. So we obtain

$$\text{If} \quad (\mathcal{S}F)^i = 0 \text{ for } i < n \ \Rightarrow \delta_{KT} D^n F = 0 \ . \tag{C.2}$$

With these results we can obtain the theorems about the cohomology. The first one is

Theorem C.1 : *The antibracket cohomology on local functions of negative ghost number is empty.*

If F has ghost number $f < 0$ then its lowest antifield level is $-f \geq 1$. We start from $\mathcal{S}F = 0$ at antifield number $-f - 1$. From (3.32) at $n = -f - 1$, we find $\delta_{KT} F^{-f} = 0$. Therefore $F^{-f} = (-)^G \delta_{KT} G^{-f+1}$ for some local G^{-f+1}. Then we also have

$$\left(\mathcal{S}G^{-f+1}\right)^{-f} = (-)^G \delta_{KT} G^{-f+1} = F^{-f} \ , \tag{C.3}$$

and $\mathcal{S}G^{-f+1}$ has no terms of lower antifield number. Now consider

$$F' \equiv F - \mathcal{S}G^{-f+1} \ . \tag{C.4}$$

It satisfies $\mathcal{S}F' = 0$ and starts at antifield number $-f + 1$. So we can repeat the previous steps, constructing a G^{-f+2} such that $F - \mathcal{S}G^{-f+1} - \mathcal{S}G^{-f+2}$ starts at antifield number $-f + 2$. Continuing in this way we obtain perturbatively a function G such that any local KT–closed function F of negative antifield number is equal to $\mathcal{S}G$. ■

Lemma C.1 : *D^0 is a weak nilpotent operator, which preserves the weak relations: if $M \approx 0$ then $D^0 M \approx 0$. Therefore we can define a weak cohomology $H^p(D^0)$, on functions of fields with pureghost number p, by the 'weak relations'*

$$D^0 M \approx 0, \qquad M \sim M' \approx M + D^0 N \ . \tag{C.5}$$

The nilpotency follows from

$$
\begin{aligned}
D^0D^0F^0 &= \left((F^0,S)\Big|_{\Phi^*=0},S\right)\Big|_{\Phi^*=0} \\
&= \left((F^0,S),S\right)\Big|_{\Phi^*=0} + (F^0,S)\overleftarrow{\partial}^A\Big|_{\Phi^*=0}\cdot\overrightarrow{\partial}_A S\Big|_{\Phi^*=0} \approx 0\ .
\end{aligned} \tag{C.6}
$$

Indeed the last factor is only non–zero for the classical action, and this A index is thus automatically restricted to i.

The second statement follows from the fact that D^0 acts on classical fields as gauge transformations, and that field equations transform to field equations under gauge transformations. Indeed

$$D^0\phi^i = R^i{}_a c^a\ , \tag{C.7}$$

and

$$M\approx 0 \ \Rightarrow\ M = M^i(\overrightarrow{\partial}_i S^0)\ \Rightarrow\ D^0M \approx M^i(\overrightarrow{\partial}_i S^0\overleftarrow{\partial}_j)R^j{}_a c^a \approx M^i\overrightarrow{\partial}_i\left((S^0\overleftarrow{\partial}_j)R^j{}_a c^a\right) = 0\ . \tag{C.8}$$

■

We will show that this D^0 cohomology is equivalent to the antibracket cohomology. We have first the following lemma

Lemma C.2 : *Consider a local function F^0 of antifield number 0 and of non–negative ghost number. If it satisfies $D^0F^0\approx 0$, a function F exists satisfying $\mathcal{S}F = 0$ which has F^0 as its part of antifield number 0. Any 2 solutions F and F' of these requirements differ by $\mathcal{S}G$ for a function G with terms of antifield number 2 and higher.*

If F^0 satisfies $D^0F^0\approx 0$, then (3.42) allows a solution for F^1 due to the acyclicity of the KT differential. This implies $(\mathcal{S}F)^0 = 0$ for $F = F^0 + F^1$+terms with antifield number ≥ 2. Then (C.2) implies that $\delta_{KT}D^1F = 0$. The acyclicity property implies that there exist a function F^2 such that $D^1F = (-)^{F+1}\delta_{KT}F^2$, and thus with (3.32), we have $(\mathcal{S}F)^1 = 0$. This procedure can be iterated to find F completely.

If F^0 is given, the vanishing of (3.32) for $n = 0$ (or (3.42)) determines F^1 up to $\delta_{KT}G^2$ for an arbitrary G^2. With (3.32) we obtain

$$F'^1 - F^1 = \delta_{KT}G^2 = \left((-)^G\mathcal{S}G^2\right)^1\ . \tag{C.9}$$

Then $F' - F - (-)^G\mathcal{S}G^2$ is closed under $\mathcal{S}$ and has its first term at antifield number 2. Therefore (3.32) on this function at $n = 1$ implies that $\left(F' - F - (-)^G\mathcal{S}G^2\right)^2$ is KT closed and the procedure can be continued to construct G completely to obtain

$$\text{If } F'^0 = F^0 \ \text{ and } \ \mathcal{S}F = \mathcal{S}F' = 0\ \Rightarrow\ F' = F + \mathcal{S}G \ \text{ with } afn(G)\geq 2. \tag{C.10}$$

■

The previous lemma makes the following theorem easy.

Theorem C.2 : *The antibracket cohomology at non–negative ghost number is equal to the (weak) cohomology of D^0.*

First let us make clear what we have to prove.

1. For any $F^0(\phi, c)$ with $D^0F^0 \approx 0$ there is a $F(\Phi^*, \Phi)$ such that $\mathcal{S}F = 0$.
2. If $F'^0 - F^0 \approx D^0G^0$ then $F' - F = \mathcal{S}G$ for a local function G, where F and F' are the functions constructed in the first step from F^0 and F'^0.
3. If $\mathcal{S}F = 0$, then one can find an F^0 with $D^0F^0 \approx 0$, and this relation is the inverse of the one in point 1.
4. If $\mathcal{S}F = \mathcal{S}F' = 0$ and $F - F' = \mathcal{S}G$, then $F^0 - F'^0 \approx D^0G^0$ for F^0 and F'^0 defined in the previous steps.

Lemma C.2 made already the construction of F mentioned in point 1, and the inverse (point 3) of this is just its restriction to antifield number 0. The equation $D^0F^0 \approx 0$ follows then from (3.42) using that δ_{KT} on terms of antifield number 1 produces field equations. For point 2, we have that $F'^0 - F^0 - D^0G^0$ is weakly vanishing and thus equal to $\delta_{KT}G^1$ for some local G^1. Therefore

$$F'^0 = F^0 + D^0G^0 + (-)^G\delta_{KT}G^1 = F^0 + \left(\mathcal{S}(G^0 + G^1)\right)^0 \ . \qquad \text{(C.11)}$$

As F' and $F + \mathcal{S}(G^0 + G^1)$ are both closed under $\mathcal{S}$ and equal at antifield number 0, it follows from the second part of the lemma that they differ by $\mathcal{S}$ on some function starting at antifield number 2, which proves point 2. Finally for point 4, the assumption at antifield number 0 reads

$$F^0 - F'^0 = (-)^G\delta_{KT}G^1 + D^0G^0 \approx D^0G^0 \ . \qquad \text{(C.12)}$$

■

References

[1] I.A. Batalin and G.A. Vilkovisky, Phys. Lett. **B102** (1981) 27; Phys. Rev. **D28** (1983) 2567 (E:**D30** (1984) 508).

[2] W. Siegel, Int. J. Mod. Phys. **A3** (1988) 2707.

[3] W. Troost, P. van Nieuwenhuizen and A. Van Proeyen, Nucl. Phys. **B333** (1990) 727.

[4] E.A. Bergshoeff, R. Kallosh and A. Van Proeyen, Class. Quantum Grav. **9** (1992) 321.

[5] A. Van Proeyen, in Proc. of the Conference *Strings & Symmetries 1991*, Stony Brook, May 20–25, 1991, eds. N. Berkovits et al., (World Sc. Publ. Co., Singapore, 1992), pp. 388.

[6] W. Troost and A. Van Proeyen, *An introduction to Batalin–Vilkovisky Lagrangian quantization*, Leuven Notes in Math. Theor. Phys., in preparation.

[7] J. Zinn–Justin, in *Trends in Elementary Particle Theory*, Lecture notes in Physics **37**, Int. Summer Inst. on Theor. Phys., Bonn 1974, eds. H. Rollnik and K. Dietz, Springer, Berlin, 1975; Nucl. Phys. **B246** (1984) 246.

[8] E. Witten, Mod. Phys. Lett. **A5** (1990) 487.

[9] A. Schwarz, UC Davis preprint, hepth 9205088;
O.M. Khudaverdian and A.P. Nersessian, UGVA–93/03–807;
S. Aoyama and S. Vandoren, preprint–KUL–TF–93/15, hepth 9305087.

[10] J.L. Koszul, Bull. Soc. Math. France **78** (1950) 5;
J. Tate, Illinois J. Math. **1** (1957) 14.

[11] J. Fisch, M. Henneaux, J. Stasheff and C. Teitelboim, Commun. Math. Phys. **120** (1989) 379;
J. Fisch and M. Henneaux, Commun. Math. Phys. **128** (1990) 627;
J.M.L. Fisch, *On the Batalin–Vilkovisky antibracket–antifield BRST formalism and its applications*, Ph.D. thesis, ULB TH2/90–01;
M. Henneaux, Nucl. Phys. **B** (Proc. Suppl.) **18A** (1990) 47;
M. Henneaux and C. Teitelboim, *Quantization of gauge systems*, Princeton University Press, Princeton 1992.

[12] C.M. Hull, Phys. Lett. **B265** (1991) 347.

[13] K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, Nucl. Phys. **B364** (1991) 584;

[14] K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, preprint ITP–SB–91–13, to appear in the proceedings of the Januari 1991 Miami workshop on Quantum Field Theory, Statistical Mechanics, Quantum Groups and Topology.

[15] C.M. Hull, Int. J. Mod. Phys. **A8** (1993) 2419.

[16] C.N. Pope, L.J. Romans and K.S. Stelle, Phys. Lett. **B268** (1991) 167;

[17] W. Pauli and F. Villars, Rev. Mod. Phys. **21** (1949) 434.

[18] A. Diaz, W. Troost, P. van Nieuwenhuizen and A. Van Proeyen, Int. J. Mod. Phys. **A4** (1989) 3959.

[19] M. Hatsuda, W. Troost, P. van Nieuwenhuizen and A. Van Proeyen, Nucl. Phys. **B335** (1990) 166.

[20] F. De Jonghe, R. Siebelink and W. Troost, Phys. Lett. **B288** (1992) 47.

[21] F. De Jonghe, R. Siebelink and W. Troost, Phys. Lett. **B306** (1993) 295;
F. De Jonghe, R. Siebelink, W. Troost, S. Vandoren, P. van Nieuwenhuizen and A. Van Proeyen, Phys. Lett. **B289** (1992) 354.

[22] A. Sevrin, R. Siebelink and W. Troost, preprint KUL-TF-93/25.

[23] W. Troost and A. Van Proeyen, *Regularization and the BV formalism*, to be published in the Proc. of the Conference *Strings 1993*, Berkeley, May 24–29, 1993.

[24] I.A. Batalin and G.A. Vilkovisky, Nucl. Phys. **B234** (1984) 106.

[25] J. Wess and B. Zumino, Phys. Lett. **B37** (1971) 95;
W.A. Bardeen and B. Zumino, Nucl. Phys. **B244** (1984) 421.

[26] J. Gomis and J. París, preprint UB–ECM–PF 93/14, UTTG–16–93, in preparation.

[27] J. Schwinger, Phys. Rev. **82** (1951) 664;
B. DeWitt, *Dynamical Theory of Groups and Fields* (Gordon and Breach) 1965; Phys. Rep. **19** (1975) 295;
R. Seeley, Am. Math. Soc. Proc. Symp. Pure Math. **10** (1967) 288; CIME (1968) 167;
P.B. Gilkey, J. Diff. Geo. **10** (1975) 601; *Invariance Theory, the Heat Equation and the Atiyah–Singer Index Theorem* (Publish or Perish, Inc.) Math. Lect. series no. 11, 1984;
A.O. Barvinsky and G.A. Vilkovisky, Phys. Rep. **119** (1985) 1.

[28] J.L. Synge, *Relativity: the general theory*, North Holland Amsterdam, 1960.

[29] M. Visser, Phys. Rev. **D47** (1993) 2395.

[30] F. Bastianelli and P. van Nieuwenhuizen, Nucl. Phys. **B389** (1993) 53.

[31] I.A. Batalin and G.A. Vilkovisky, J. Math. Phys. **26** (1985) 172.

[32] C.M. Hull, Phys. Lett. **B240** (1990) 110.

[33] M. Günaydin, G. Sierra and P.K. Townsend, Phys. Lett. **B133** (1983) 72; Nucl. Phys. **B242** (1984) 244, **B253** (1985) 573;
E. Cremmer, C. Kounnas, A. Van Proeyen, J.P. Derendinger, S. Ferrara, B. de Wit and L. Girardello, Nucl. Phys. **B250** (1985) 385.

[34] B. de Wit and A. Van Proeyen, Commun. Math. Phys. **149** (1992) 307.

[35] L.J. Romans, Nucl. Phys. **B352** (1990) 829.

[36] E. Bergshoeff, A. Sevrin and X. Shen, Phys. Lett. **B296** (1992) 95.

[37] L. D. Faddeev and A. A. Slavnov, *Gauge Fields: Introduction to Quantum Theory*, Frontiers in Physics series 50, The Benjamin/Cummings Publishing Company, Reading, Massachussetts (1980).

[38] Y. Matsuo, in proceedings of the meeting *Geometry and Physics*, Lake Tahoe, July 1989.

[39] F. Bastianelli, Phys. Lett. **B263** (1991) 411.