

CONFORMAL BLOCKS REVISITED

EDUARD LOOIJENGA

ABSTRACT. We give a simple coordinate free description of the WZW connection and derive its main properties.

This paper is based on a seminar talk whose original goal was to define the (projective) Wess-Zumino-Witten connection and to show its flatness. My reason for writing this up was to demonstrate the simplicity of its construction, a fact not so apparent from the literature known to me, and to recover in a selfcontained (and hopefully, more transparent) manner all the main results of the fundamental paper by Tsuchiya-Ueno-Yamada [10]. Because of its consistent coordinate free approach, the present discussion is mostly algebraic in character, which (as so often) is not only conceptually more satisfying, but also tends to simplify the arguments. (Another such approach insofar it involves the WZW connection, can be found in a paper by Y. Tsuchimoto [8].)

In sections 1 and 2 basically redoes parts of the book of Kac-Raina [4] in a more algebraic setting, replacing for instance the ring of complex Laurent polynomials by a complete local field containing \mathbb{Q} (or rather a direct sum of these), which is then also allowed to ‘depend on parameters’. What may be new is the extension 2.4 of the Sugawara representation to a relative situation involving a Leibniz rule in the horizontal direction. I regard this construction as the origin of WZW-connection and its projective flatness. Section 3 defines the connection in question. We use the occasion for a painless derivation of related results such as coherence and what is called the propagation of vacua. We treat the genus zero case in some detail and show how the WZW-connection is related to the Knizhnik-Zamolodchikov connection. Finally, in Section 4 we derive the basic results associated to a double point degeneration (such as local freeness, factorization and monodromy).

The reader be warned that the bibliography is merely a record of papers consulted, rather than anything else.

1. CANONICAL CONSTRUCTION OF THE VIRASORO ALGEBRA

In this section we fix a \mathbb{Q} -algebra R and a R -algebra \mathcal{O} isomorphic to the formal power series ring $R[[t]]$. In other words, \mathcal{O} comes with a principal ideal \mathfrak{m} so that \mathcal{O} is complete for the \mathfrak{m} -adic topology and $\mathcal{O}/\mathfrak{m}^j$ is for $j = 1, 2, \dots$ a free R -module of rank j . A generator t of \mathfrak{m} will then identify \mathcal{O} with $R[[t]]$. We denote by L the localization of \mathcal{O} obtained by inverting a generator of \mathfrak{m} . For $N \in \mathbb{Z}$, $\mathfrak{m}^N \subset L$ has the obvious meaning as a \mathcal{O} -submodule of L . The \mathfrak{m} -adic topology on L is the topology that has the collection of cosets $\{f + \mathfrak{m}^N\}_{f \in L, N \in \mathbb{Z}}$ as a basis of open subsets. We sometimes write $F^N L$ for \mathfrak{m}^N . We further denote by θ the L -module of continuous R -derivations from L into L and by ω the L -dual of θ . These L -modules come with filtrations (making them principal filtered L -modules): $F_N \theta$ consists of the derivations that take \mathfrak{m} to \mathfrak{m}^{N+1} and $F^N \omega$ consists of the

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L -homomorphisms $\theta \rightarrow L$ that take $F^0\theta$ to \mathfrak{m}^N . So in terms of the generator t above, $L = R((t))$, $\theta = R((t))\frac{d}{dt}$, $F^N\theta = R[[t]]t^{N+1}\frac{d}{dt}$, $\omega = R((t))dt$ and $F^N\omega = R[[t]]t^{N-1}dt$.

The residue map $\text{Res} : \omega \rightarrow R$ which assigns to an element of $R((t))dt$ the coefficient of $t^{-1}dt$ is independent of the choice of t . The R -bilinear map

$$r : L \times \omega \rightarrow R, \quad (f, \alpha) \mapsto \text{Res}(f\alpha)$$

is a topologically perfect pairing of filtered R -modules: we have $r(t^k, t^{-l-1}dt) = \delta_{k,l}$ so that any R -linear $\phi : L \rightarrow R$ which is continuous (i.e., ϕ zero on \mathfrak{m}^N for some N) is definable by an element of ω (namely by $\sum_{k < N} \phi(t^k)t^{-k-1}dt$) and likewise for a R -linear continuous map $\omega \rightarrow R$.

A trivial Lie algebra. If we regard L^\times as an algebraic group over R (or rather as a group object in a category of ind schemes over R), then its Lie algebra, denoted here by \mathfrak{l} , is L , regarded as a R -module with trivial Lie bracket. It comes with a decreasing filtration $F^\bullet\mathfrak{l}$ (as a Lie algebra) defined by the valuation. The universal enveloping algebra $U\mathfrak{l}$ is clearly $\text{Sym}_R^\bullet(\mathfrak{l})$. The ideal $U_+\mathfrak{l} \subset U\mathfrak{l}$ generated by \mathfrak{l} is also a right \mathcal{O} -module (since \mathfrak{l} is). We complete it \mathfrak{m} -adically: given an integer $N \geq 0$, then an R -basis of $U_+\mathfrak{l}/(U\mathfrak{l} \circ F^N\mathfrak{l})$ is the collection $t^{k_1} \circ \dots \circ t^{k_r}$ with $k_1 \leq k_2 \leq \dots \leq k_r < N$. So elements of the completion

$$U_+\mathfrak{l} \rightarrow \overline{U_+\mathfrak{l}} := \varprojlim_N U_+\mathfrak{l}/U\mathfrak{l} \circ F^N\mathfrak{l}$$

are series of the form $\sum_{i=1}^\infty r_i t^{k_{i,1}} \circ \dots \circ t^{k_{i,r_i}}$ with $r_i \in R$, $k_{1,i} \leq k_{2,i} \leq \dots \leq k_{i,r_i}$, $\{k_{i,1}\}_i$ bounded from below and $\lim_{i \rightarrow \infty} k_{i,r_i} = \infty$. We put $\overline{U\mathfrak{l}} := R \oplus \overline{U_+\mathfrak{l}}$, which could of course have been defined directly as

$$U\mathfrak{l} \rightarrow \overline{U\mathfrak{l}} := \varprojlim_N U\mathfrak{l}/U\mathfrak{l} \circ F^N\mathfrak{l}.$$

We will refer to this construction as the *\mathfrak{m} -adic completion on the right*. (In the present case, this is no different from \mathfrak{m} -adic completion on the left, because \mathfrak{l} is commutative.)

Any $D \in \theta$ defines an R -linear map $\omega \rightarrow L$ which is selfadjoint relative our topological pairing: $r(\langle D, \alpha \rangle, \beta) = r(\alpha, \langle D, \beta \rangle)$. We use that pairing to identify D with an element of the closure of $\text{Sym}^2\mathfrak{l}$ in $\overline{U\mathfrak{l}}$. Let $C(D)$ be half this element, so that in terms of the above topological basis:

$$C(D) = \frac{1}{2} \sum_{l \in \mathbb{Z}} \langle D, t^{-l-1}dt \rangle \circ t^l.$$

In particular we have for $D = D_k = t^{k+1}\frac{d}{dt}$

$$C(D_k) = \frac{1}{2} \sum_{i+j=k} : t^i \circ t^j : \quad .$$

We here adhered to the normal ordering convention (the factor with the highest index comes last and hence acts first), to make the righthand side look like an element of $\overline{U\mathfrak{l}}$, although there is no need for this as $t^i \circ t^j = t^j \circ t^i$. Observe that the map $C : \theta \rightarrow \overline{U\mathfrak{l}}$ is continuous.

Oscillator algebra. The residue map defines a central extension of \mathfrak{l} , the *oscillator algebra* $\hat{\mathfrak{l}}$, which as a R -module is simply $\mathfrak{l} \oplus \hbar R$ and has Lie bracket

$$[f + \hbar r, g + \hbar s] := \hbar \text{Res}(g df).$$

So $[t^k, t^{-l}] = \hbar k \delta_{k,l}$ and $R\hbar$ is central. We filter $\hat{\mathfrak{l}}$ by letting $F^N\hat{\mathfrak{l}}$ be $F^N\mathfrak{l}$ for $N > 0$ and $F^N\mathfrak{l} + R\hbar$ for $N \leq 0$. We complete $U\hat{\mathfrak{l}}$ \mathfrak{m} -adically on the right:

$$U\hat{\mathfrak{l}} \rightarrow \overline{U\hat{\mathfrak{l}}} := \varprojlim_N U\hat{\mathfrak{l}}/U\hat{\mathfrak{l}} \circ F^N\mathfrak{l}.$$

Since \hbar is in the center of \mathfrak{l} , this is a $R[\hbar]$ -algebra (for a similar reason it is even a $R[e, \hbar]$ -algebra if $e = t^0$ denotes the unit element of L viewed as an element of \mathfrak{l}). As a $R[\hbar]$ -algebra it is a quotient of the tensor algebra of \mathfrak{l} (over R) tensored with $R[\hbar]$, $\otimes_R \mathfrak{l} \otimes_R R[\hbar]$, by the two-sided ideal generated by the elements $f \otimes g - g \otimes f - \hbar \text{Res}(gdf)$. As a $R[\hbar]$ -module it has for topological basis the collection $t^{k_1} \circ \dots \circ t^{k_r}$ with $r \geq 0$, $k_1 \leq k_2 \leq \dots \leq k_r$. Since $\hat{\mathfrak{l}}$ is not abelian, the left and right \mathfrak{m} -adic topologies differ. For instance, $\sum_{k \geq 1} t^k \circ t^{-k}$ does not converge in $\overline{U\hat{\mathfrak{l}}}$, whereas $\sum_{k \geq 1} t^{-k} \circ t^k$ does. Notice that the obvious surjection $\pi : U\hat{\mathfrak{l}} \rightarrow U\mathfrak{l}$ is simply the reduction modulo \hbar of $U\hat{\mathfrak{l}}$ and likewise for their completions. Observe that the filtrations of \mathfrak{l} and $\hat{\mathfrak{l}}$ determine decreasing filtrations of the their (completed) universal enveloping algebras. For instance, $F^N U\hat{\mathfrak{l}} = \sum_{r \geq 0} \sum_{n_1 + \dots + n_r \geq N} F^{n_1} \hat{\mathfrak{l}} \circ \dots \circ F^{n_r} \hat{\mathfrak{l}}$.

Denote by \mathfrak{l}_2 the image of $\mathfrak{l}^{\otimes 2} \subset \hat{\mathfrak{l}}^{\otimes 2} \rightarrow U\hat{\mathfrak{l}}$. Under the reduction modulo \hbar , \mathfrak{l}_2 maps onto $\text{Sym}^2(\mathfrak{l}) \subset U\mathfrak{l}$ with kernel $R\hbar$. Its closure $\bar{\mathfrak{l}}_2$ in $\overline{U\hat{\mathfrak{l}}}$ maps onto the closure of $\text{Sym}^2(\mathfrak{l})$ in $\overline{U\mathfrak{l}}$ with the same kernel. We denote by $\hat{\theta}$ the set of pairs $(D, u) \in \theta \times \bar{\mathfrak{l}}_2$ for which $C(D)$ is the mod \hbar reduction of u so that we have an exact sequence

$$0 \rightarrow \hbar R \rightarrow \hat{\theta} \rightarrow \theta \rightarrow 0$$

of R -modules and a natural R -homomorphism $\hat{C} : \hat{\theta} \rightarrow \overline{U\hat{\mathfrak{l}}}$. The generator t defines a (noncanonical) section of $\hat{\theta} \rightarrow \theta$:

$$D \in \theta \mapsto \hat{D} := (D, \frac{1}{2} \sum_{j \in \mathbb{Z}} \langle D, t^{-j-1} dt \rangle \circ t^j) \in \theta \times \hat{\mathfrak{l}}.$$

Lemma 1.1. *We have*

- (i) $[\hat{C}(\hat{D}), f] = -\hbar D(f)$ as an identity in $\overline{U\hat{\mathfrak{l}}}$ (where $f \in \mathfrak{l} \subset \hat{\mathfrak{l}}$) and
- (ii) $[\hat{C}(\hat{D}_k), \hat{C}(\hat{D}_l)] = -\hbar(l-k)\hat{C}(\hat{D}_{k+l}) + \hbar^2 \frac{1}{12}(k^3 - k)\delta_{k+l,0}$.

Proof. For the first statement we compute $[\hat{C}(\hat{D}_k), t^l]$. The only terms in the expansion of $\sum_{i+j=k} t^i \circ t^j$ that can contribute are for the form $[t^{k+l} \circ t^{-l}, t^l]$ or $[t^{-l} \circ t^{k+l}, t^l]$, (depending on whether $k+2l \leq 0$ or $k+2l \geq 0$) and with coefficient $\frac{1}{2}$ if $k+2l = 0$ and 1 otherwise. In all cases the result is $\hbar l t^{k+l} = \hbar D_k(t^l)$.

Formula (i) implies that

$$\begin{aligned} [\hat{C}(\hat{D}_k), \hat{C}(\hat{D}_l)] &= \lim_{N \rightarrow \infty} \sum_{|i| \leq N} \frac{1}{2} (D_k(t^i) \circ t^{l-i} + t^i \circ D_k(t^{l-i})) \\ &= -\hbar \lim_{N \rightarrow \infty} \sum_{|i| \leq N} (i t^{k+i} \circ t^{l-i} + t^i \circ (l-i) t^{k+l-i}) \end{aligned}$$

This is up to a reordering equal to $-\hbar(l-k)\hat{C}(\hat{D}_{k+l})$. The terms which do not commute and are in the wrong order are those for which $0 < k+i = -(l-i)$ (with coefficient i) and for which $0 < i = -(k+l-i)$ (with coefficient $(l-i)$). This accounts for the extra term $\hbar^2 \frac{1}{12}(k^3 - k)\delta_{k+l,0}$. \square

This lemma suggests we rescale \hat{C} as

$$T := -\frac{1}{\hbar} \hat{C} : \hat{\theta} \rightarrow \overline{U\hat{\mathfrak{l}}}\left[\frac{1}{\hbar}\right]$$

and write c_0 for $(0, -\hbar) \in \hat{\theta}$ since then

- (i) T is injective and maps $\hat{\theta}$ onto a Lie subalgebra of $\overline{U\hat{\mathfrak{l}}}\left[\frac{1}{\hbar}\right]$ with $c_0 \in \hat{\theta} \mapsto 1$,

(ii) if we transfer the Lie bracket to $\hat{\theta}$, we find that

$$[\hat{D}_k, \hat{D}_l] = (l - k)\hat{D}_{k+l} + \frac{k^3 - k}{12}\delta_{k+l,0}c_0,$$

(iii) and $\text{ad}_{T(\hat{D})}$ leaves \mathfrak{l} invariant (as a subspace of $\overline{U\hat{\mathfrak{l}}}$) and acts on that subspace by derivation with respect to D ,

Thus we get a central extension of θ with a one-dimensional center canonically isomorphic to R (c_0 corresponds to $1 \in R$), the *Virasoro algebra* (of the R -algebra L).

Remark 1.2. An alternative coordinate free definition of the Virasoro algebra, based on the algebra of pseudodifferential operators on L , can be found in [2].

Fock representation. It is clear that $F^{\mathfrak{l}} = F^1\hat{\mathfrak{l}}$ is an abelian subalgebra of $\hat{\mathfrak{l}}$. We put

$$\mathbb{F} := (U\hat{\mathfrak{l}}/U\hat{\mathfrak{l}} \circ F^1\mathfrak{l})[\frac{1}{\hbar}].$$

This is a representation of \mathfrak{l} over R ; at the same time it is a $R[e, \hbar, \hbar^{-1}]$ -module and as such it is free with basis the collection $t^{-k_r} \circ \dots \circ t^{-k_1} v_0$, where v_0 denotes the image of 1, $r \geq 0$ and $1 \leq k_1 \leq k_2 \leq \dots \leq k_r$ (for $r = 0$, read v_0). This shows that we may identify \mathbb{F} with $(\overline{U\hat{\mathfrak{l}}}/\overline{U\hat{\mathfrak{l}}} \circ F^1\mathfrak{l})[\frac{1}{\hbar}]$, which makes it a representation for $\hat{\theta}$ over R . This is the *Fock representation* of $\hat{\theta}$. The $R[e, \hbar, \hbar^{-1}]$ -subalgebra of $U\hat{\mathfrak{l}}$ generated by the elements t^{-k} , $k = 1, 2, \dots$ is a polynomial algebra (in infinitely many variables) which projects isomorphically onto \mathbb{F} . But this subalgebra depends on t ; \mathbb{F} itself does not seem to be an R -algebra in a canonical way. On the other hand, it is naturally filtered with $F^N\mathbb{F} = F^N\overline{U\hat{\mathfrak{l}}}(v_0)$.

It follows from Lemma 1.1 that when $D \in \theta$, then

$$\begin{aligned} T(\hat{D})t^{-k_r} \circ \dots \circ t^{-k_1} v_0 &= \\ &= \left(\sum_{i=1}^r X_{\alpha_i} t^{-k_r} \circ \dots \circ D(t^{-k_i}) \circ \dots \circ t^{-k_1} \right) v_0 + t^{-k_r} \circ \dots \circ t^{-k_1} T_{\mathfrak{g}}(\hat{D})v_0. \end{aligned}$$

For $D \in F^0\theta$ we have $T_{\mathfrak{g}}(\hat{D})v_0 = 0$ and so $F^0\theta$ acts on \mathbb{F} by coefficientwise derivation.

2. THE SUGAWARA CONSTRUCTION

In this section, we fix a base field k of characteristic zero and a simple Lie algebra \mathfrak{g} over k of finite dimension. We retain the data and the notation of Section 1, except that we now assume R to be a k -algebra.

Loop algebras. The space of symmetric invariant bilinear forms $\mathfrak{g} \times \mathfrak{g} \rightarrow k$ is of dimension one and has a canonical generator (\mid) (namely the form which takes the value 2 on the small coroots). We choose an orthonormal basis $\{X_{\kappa}\}_{\kappa}$ of \mathfrak{g} relative to this form. The dual of this line is the space \mathfrak{g} -invariants in $\text{Sym}^2 \mathfrak{g}$; we shall denote this line by \mathfrak{c} . The form (\mid) singles out a generator of \mathfrak{c} , namely the Casimir element $c = \sum_{\kappa} X_{\kappa} \otimes X_{\kappa}$. It is well-known and easy to prove that \mathfrak{c} maps to the center of $U\mathfrak{g}$. In particular, c acts in any finite dimensional irreducible representation of \mathfrak{g} by a scalar. In the case of the adjoint representation we denote half this scalar by \hbar (for this happens to be the *dual Coxeter number* of \mathfrak{g}) so that

$$\sum_{\kappa} [X_{\kappa}, [X_{\kappa}, Y]] = 2\hbar Y \quad \text{for all } Y \in \mathfrak{g}.$$

Let $L\mathfrak{g}$ stand for $\mathfrak{g} \otimes_k L$, but considered as a filtered R -Lie algebra (so we restrict the scalars to R): $F^N L\mathfrak{g} = \mathfrak{g} \otimes_k \mathfrak{m}^N$. An argument similar as for r shows that the pairing

$$r_{\mathfrak{g}} : (\mathfrak{g} \otimes_k L) \otimes (\mathfrak{g} \otimes_k \omega) \rightarrow R, \quad (Xf, Y\kappa) \mapsto (X|Y) \text{Res}(f\kappa)$$

is topologically perfect; the basis dual to $(X_{\kappa} t^l)_{\kappa, l}$ is $(X_{\kappa} t^{-l-1} dt)_{\kappa, l}$.

For an integer $N \geq 0$, the quotient $U_+ L\mathfrak{g}/UL\mathfrak{g} \circ F^N L\mathfrak{g}$ is a free R -module with generators $X_{\kappa_1} t^{k_1} \circ \dots \circ X_{\kappa_r} t^{k_r}$, $k_1 \leq \dots \leq k_r < N$. We complete $UL\mathfrak{g}$ \mathfrak{m} -adically on the right:

$$\overline{UL\mathfrak{g}} := \varprojlim_N UL\mathfrak{g}/UL\mathfrak{g} \circ F^N L\mathfrak{g}.$$

A central extension $\widehat{L\mathfrak{g}}$ of $L\mathfrak{g}$ by \mathfrak{c} is defined by endowing $L\mathfrak{g} \oplus \mathfrak{c}$ with the Lie bracket

$$[Xf + cr, Yg + cs] := [X, Y]fg + c \text{Res}(gdf)(X|Y).$$

We filter $\widehat{L\mathfrak{g}}$ by letting for $N > 0$, $F^N \widehat{L\mathfrak{g}} = F^N L\mathfrak{g}$ and for $N \leq 0$, $F^N \widehat{L\mathfrak{g}} = F^N L\mathfrak{g} + \mathfrak{c}$. Then $U\widehat{L\mathfrak{g}}$ is a filtered $R[c]$ -algebra whose reduction modulo c is $UL\mathfrak{g}$. Since the residue is zero on \mathcal{O} , the inclusion of $F^0 L\mathfrak{g}$ in $\widehat{L\mathfrak{g}}$ is a homomorphism of Lie algebras. The \mathfrak{m} -adic completion on the right

$$\overline{U\widehat{L\mathfrak{g}}} := \varprojlim_N U\widehat{L\mathfrak{g}}/(U\widehat{L\mathfrak{g}} \circ F^N L\mathfrak{g})$$

is still a $R[c]$ -algebra and the obvious surjection $\overline{U\widehat{L\mathfrak{g}}} \rightarrow \overline{UL\mathfrak{g}}$ is the reduction modulo c . These (completed) enveloping algebras inherit a decreasing filtration from L .

Sugawara representation. If c is regarded as an element of $\mathfrak{g} \otimes_k \mathfrak{g}$, then tensoring with it defines the R -linear map

$$\mathfrak{l} \otimes_R \mathfrak{l} \rightarrow L\mathfrak{g} \otimes_R L\mathfrak{g}, \quad f \otimes g \mapsto \sum_{\kappa} X_{\kappa} f \otimes X_{\kappa} g,$$

which, composed with $L\mathfrak{g} \otimes_R L\mathfrak{g} \subset \widehat{L\mathfrak{g}} \otimes_R \widehat{L\mathfrak{g}} \rightarrow U\widehat{L\mathfrak{g}}$, yields $\gamma : \mathfrak{l} \otimes_R \mathfrak{l} \rightarrow U\widehat{L\mathfrak{g}}$. Since $\gamma(f \otimes g - g \otimes f) = \sum_{\kappa} [X_{\kappa} f, X_{\kappa} g] = c \dim \mathfrak{g} \text{Res}(gdf)$, γ drops and extends naturally to an R -module homomorphism $\hat{\gamma} : \mathfrak{l}_2 \rightarrow U\widehat{L\mathfrak{g}}$ which sends \hbar to $c \dim \mathfrak{g}$. It extends continuously to a map from the closure $\bar{\mathfrak{l}}_2$ of \mathfrak{l}_2 in $\overline{U\widehat{L\mathfrak{g}}}$ to $\overline{U\widehat{L\mathfrak{g}}}$. As $\bar{\mathfrak{l}}_2$ contains the image of $C : \hat{\theta} \rightarrow \overline{U\widehat{L\mathfrak{g}}}$, we get a natural R -homomorphism

$$\hat{C}_{\mathfrak{g}} := \hat{\gamma} \hat{C} : \hat{\theta} \rightarrow \overline{U\widehat{L\mathfrak{g}}}.$$

We may also describe $\hat{C}_{\mathfrak{g}}$ in the spirit of Section 1: given $D \in \theta$, then the R -linear map

$$1 \otimes D : \mathfrak{g} \otimes_k \omega \rightarrow \mathfrak{g} \otimes_k L$$

is continuous and selfadjoint relative to $r_{\mathfrak{g}}$ and the perfect pairing $r_{\mathfrak{g}}$ allows us to identify it with an element of $\overline{UL\mathfrak{g}}$; this element is our $\hat{C}_{\mathfrak{g}}(D)$. The choice of t yields the lift

$$\hat{C}_{\mathfrak{g}}(\hat{D}) = \frac{1}{2} \sum_{\kappa} \sum_l : X_{\kappa} \langle D, t^{-l-1} dt \rangle \circ X_{\kappa} t^l : \in \overline{U\widehat{L\mathfrak{g}}}$$

(so that in particular, $\hat{C}_{\mathfrak{g}}(\hat{D}_k) = \frac{1}{2} \sum_{\kappa, l} : X_{\kappa} t^{k-l} \circ X_{\kappa} t^l :$). This formula can be used to define $\hat{C}_{\mathfrak{g}}$, but this approach does not exhibit its naturality.

Lemma 2.1. *We have*

- (i) $[C_{\mathfrak{g}}(\hat{D}), Xf] = -(c + \check{h})XD(f)$, for every $D \in \theta$, $X \in \mathfrak{g}$ and $f \in L$ and
- (ii) $[C_{\mathfrak{g}}(\hat{D}_k), C_{\mathfrak{g}}(\hat{D}_l)] = (c + \check{h})(k - l)C_{\mathfrak{g}}(\hat{D}_{k+l}) + c(c + \check{h})\delta_{k+l,0} \frac{1}{12} \dim \mathfrak{g}(k^3 - k)$.

For the proof (which is a bit tricky, but not very deep), we refer to Lecture 10 of [4] (our $C_{\mathfrak{g}}(\hat{D}_k)$ is their T_k).

Corollary 2.2 (Sugawara representation). *The R -linear map*

$$T_{\mathfrak{g}} := \frac{-1}{c+\hbar} \hat{C}_{\mathfrak{g}} : \hat{\theta} \rightarrow \overline{U\widehat{L\mathfrak{g}}}\left[\frac{1}{c+\hbar}\right]$$

which sends the central $c_0 \in \hat{\theta}$ to $c(c+\hbar)^{-1} \dim \mathfrak{g}$ is a homomorphism of R -Lie algebras. Moreover, if $D \in \theta$, then $\text{ad}_{T_{\mathfrak{g}}(\hat{D})}$ leaves $L\mathfrak{g}$ invariant (as a subspace of $\overline{U\widehat{L\mathfrak{g}}}$) and acts on that subspace by derivation with respect to D .

Fock type representation for \mathfrak{g} . Consider the $U\widehat{L\mathfrak{g}}$ -module

$$\mathbb{F}(L\mathfrak{g}) := (U\widehat{L\mathfrak{g}}/UF^1L\mathfrak{g})\left[\frac{1}{c+\hbar}\right]$$

If $v_0 \in \mathbb{F}(L\mathfrak{g})$ denotes the image of 1, then as a $R\left[\frac{1}{c+\hbar}\right]$ -module it has for basis is the collection $\{X_{\kappa_r} t^{-k_r} \circ \dots \circ X_{\kappa_1} t^{-k_1}(v_0) : r \geq 0, 0 \leq k_1 \leq k_2 \leq \dots \leq k_r\}$. It is easy to see that we can identify $\mathbb{F}(L\mathfrak{g})$ with $\overline{U\widehat{L\mathfrak{g}}}/\overline{UF^1L\mathfrak{g}}\left[\frac{1}{c+\hbar}\right]$, so that $\hat{\theta}$ acts on $\mathbb{F}(L\mathfrak{g})$. It follows from Corollary 2.2 that when $D \in \theta$, then

$$\begin{aligned} T_{\mathfrak{g}}(\hat{D})X_{\kappa_r} t^{-k_r} \circ \dots \circ X_{\kappa_1} t^{-k_1}(v_0) &= \\ &= \left(\sum_{i=1}^r X_{\kappa_r} t^{-k_r} \circ \dots \circ X_{\kappa_i} D(t^{-k_i}) \circ \dots \circ X_{\kappa_1} t^{-k_1} \right) v_0 + \\ &\quad + X_{\kappa_r} t^{-k_r} \circ \dots \circ X_{\kappa_1} t^{-k_1} T_{\mathfrak{g}}(\hat{D})v_0. \end{aligned}$$

Thus $\hat{\theta}$ is faithfully represented as a Lie algebra of $R\left[\frac{1}{c+\hbar}\right]$ -linear endomorphisms of $\mathbb{F}(L\mathfrak{g})$. If $D \in F^0\theta$, then clearly $T_{\mathfrak{g}}(\hat{D})v_0 = 0$ and hence:

Lemma 2.3. *The Lie subalgebra $F^0\theta$ of $\hat{\theta}$ acts on $\mathbb{F}(L\mathfrak{g})$ by coefficientwise derivation.*

This lemma has an interesting corollary. In order to state it, consider the module of k -derivations $R \rightarrow R$ (denoted here simply by θ_R instead of the more correct $\theta_{R/k}$) and the module of k -derivations of L which preserve $R \subset L$ (denoted by $\theta_{L,R}$). Since $L \cong R((t))$ as an R -algebra, every k -derivation $R \rightarrow R$ extends to one from L to L . So $\theta_{L,R}/\theta$ can be identified with θ_R .

Corollary 2.4. *The central extension $\hat{\theta}$ of θ by Rc_0 naturally extends to a central extension of R -Lie algebras $\hat{\theta}_{L,R}$ of $\theta_{L,R}$ by Rc_0 (so with $\hat{\theta}_{L,R}/\hat{\theta} = \theta_{L,R}/\theta \cong \theta_R$) and the Sugawara representation $T_{\mathfrak{g}}$ of $\hat{\theta}$ on $\mathbb{F}(L\mathfrak{g})$ extends to $\hat{\theta}_{L,R}$ in such a manner that it preserves any $U\widehat{L\mathfrak{g}}$ -submodule of $\mathbb{F}(L\mathfrak{g})$ and for every $D \in \theta_{L,R}$, the following relations hold in $\text{End}(\mathbb{F}(L\mathfrak{g}))$:*

- (i) $[T_{\mathfrak{g}}(\hat{D}), Xf] = X(Df)$ for $X \in \mathfrak{g}$, $f \in L$, and
- (ii) (Leibniz rule) $[T_{\mathfrak{g}}(\hat{D}), r] = Dr$ for $r \in R$.

These assertions also hold for the oscillator representation T on \mathbb{F} .

Proof. The generator t can be used to define a section of $\theta_{L,R} \rightarrow \theta_R$: the set of elements of $\theta_{L,R}$ which kill t is a k -Lie subalgebra of $\theta_{L,R}$ which projects isomorphically onto θ_R . Now if $D \in \theta_{L,R}$, write $D = D_{\text{vert}} + D_{\text{hor}}$ with $D_{\text{vert}} \in \theta$ and $D_{\text{hor}}(t) = 0$ and define a $k\left[\frac{1}{c+\hbar}\right]$ -linear operator \hat{D} in $\mathbb{F}(L\mathfrak{g})$ as the sum of $T_{\mathfrak{g}}(\hat{D}_{\text{vert}})$ and coefficientwise derivation

by D_{hor} . This map clearly satisfies the two properties and preserves any $U\widehat{L\mathfrak{g}}$ -submodule of $\mathbb{F}(L\mathfrak{g})$. As to its dependence on t : another choice yields a decomposition of the form $D = (D_{\text{hor}} + D_0) + (D_{\text{vert}} - D_0)$ with $D_0 \in F^0\theta$ and in view of Lemma 2.3 \hat{D}_0 acts in $\mathbb{F}(L\mathfrak{g})$ by coefficientwise derivation.

The same argument works for the oscillator representation. \square

Semilocal case. This refers to the situation where we allow the R -algebra L to be a finite direct sum of R -algebras isomorphic to $R((t))$: $L = \bigoplus_{i \in I} L_i$, where I is a nonempty finite index set and L_i as before. We use the obvious convention, for instance, \mathcal{O} , \mathfrak{m} , ω , \mathfrak{l} are now the direct sums over I (as filtered objects) of the items suggested by the notation. If $r : L \times \omega \rightarrow R$ denotes the sum of the residue pairings of the summands, then r is still topologically perfect. In this setting, the oscillator algebra $\hat{\mathfrak{l}}$ is of course not the direct sum of the $\hat{\mathfrak{l}}_i$, but rather the quotient of $\bigoplus_i \hat{\mathfrak{l}}_i$ that identifies the central generators $c_{0,i}$ of the summands with a single c_0 . We thus get a Virasoro extension $\hat{\theta}$ of θ by c_0R and a (faithful) oscillator representation of $\hat{\theta}$ in $\overline{U\hat{\mathfrak{l}}}$. The decreasing filtrations are the obvious ones. In likewise manner we define $\widehat{L\mathfrak{g}}$ (a central extension of $\bigoplus_{i \in I} L\mathfrak{g}_i$) and construct $\hat{\theta}_{L,R}$ and the associated Sugawara representation: Corollaries 2.2 and 2.4 continue to hold.

3. THE WZW CONNECTION

From now on we assume that our base field k is algebraically closed and R is a noetherian k -algebra. We place ourselves in the semi-local case. In what follows, we often allow \mathfrak{g} to be replaced by k , viewed as an abelian Lie algebra, where the substitutions are the obvious ones; for instance, $\widehat{L\mathfrak{g}}$, $T_{\mathfrak{g}}$ and $\mathbb{F}(L\mathfrak{g})$ become $\hat{\mathfrak{l}}$, T and \mathbb{F} .

Abstract conformal blocks. Let A be a R -subalgebra of L and let $\theta_{A/R}$ have the usual meaning as the Lie algebra of R -derivations $A \rightarrow A$. We assume that:

- (A₁) as a R -algebra, A is flat and of finite type,
- (A₂) $A \cap \mathcal{O} = R$ and the R -module $L/(A + \mathcal{O})$ is locally free of finite rank,
- (A₃) the annihilator $\text{ann}(A)$ of A in ω contains the image of dA and the resulting map $\text{Hom}_A(\text{ann}(A), A) \rightarrow \theta_{A/R}$ is an isomorphism.

We denote $\theta_{A,R}$ the Lie algebra of k -derivations $A \rightarrow A$ which preserve R . So the quotient $\theta_R^A := \theta_{A,R}/\theta_{A/R}$ consists of the k -derivations $R \rightarrow R$ that extend to one of A and hence is a submodule of θ_R . It is clear that $\theta_{A,R} \subset \theta_{L,R}$. We denote by $\hat{\theta}_{A,R}$ the preimage of $\theta_{A,R}$ in $\hat{\theta}_{L,R}$ and by $\hat{\theta}_R^A$ the quotient $\hat{\theta}_{A,R}/\theta_{A/R}$. These are central extensions of $\theta_{A,R}$ resp. θ_R^A by $\mathfrak{c} \otimes_k R = cR$.

We put

$$A\mathfrak{g} := \mathfrak{g} \otimes_k A$$

and view it as a Lie subalgebra of $L\mathfrak{g}$. Since the residue vanishes on $\text{ann}(A)$, the inclusion $A\mathfrak{g} \subset \widehat{L\mathfrak{g}}$ is a Lie algebra homomorphism. We define the *universal covacuum space* as the space of $A\mathfrak{g}$ -covariants in $\mathbb{F}(L\mathfrak{g})$, $\mathbb{F}(L\mathfrak{g})_{A\mathfrak{g}} := \mathbb{F}(L\mathfrak{g})/A\mathfrak{g}\mathbb{F}(L\mathfrak{g})$.

Proposition 3.1. *For $D \in \theta_{A/R}$, $T_{\mathfrak{g}}(\hat{D})$ lies in the closure of $A\mathfrak{g} \circ \widehat{L\mathfrak{g}}$ in $\overline{U\widehat{L\mathfrak{g}}}$.*

The Sugawara representation of the Lie algebra $\hat{\theta}_{A,R}$ on $\mathbb{F}(L\mathfrak{g})$ preserves the space of $A\mathfrak{g}$ -covariants in $\mathbb{F}(L\mathfrak{g})$, $\mathbb{F}(L\mathfrak{g})_{A\mathfrak{g}} := \mathbb{F}(L\mathfrak{g})/A\mathfrak{g}\mathbb{F}(L\mathfrak{g})$, and acts on $\mathbb{F}(L\mathfrak{g})_{A\mathfrak{g}}$ via the central extension $\hat{\theta}_R^A$ of θ_R^A (with the central c_0 acting as multiplication by $(c + \hbar)^{-1}c \dim \mathfrak{g}$).

These assertions also hold if we replace \mathfrak{g} by the abelian Lie algebra k (in particular, $T(\hat{D})$ lies in the closure of $\mathfrak{a} \circ \hat{\mathfrak{l}}$ in $\overline{U\hat{\mathfrak{l}}}$).

Proof. We only do this for \mathfrak{g} . Since D maps $\text{ann}(A)$ to $A \subset L$, $1 \otimes D$ maps the submodule $\mathfrak{g} \otimes \text{ann}(A)$ of $\mathfrak{g} \otimes \omega$ to the submodule $\mathfrak{g} \otimes A = A\mathfrak{g}$ of $\mathfrak{g} \otimes L = L\mathfrak{g}$. It is clear that $\mathfrak{g} \otimes \text{ann}(A)$ and $A\mathfrak{g}$ are each others annihilator relative to the pairing $r_{\mathfrak{g}}$. This implies that $\widehat{C}(\widehat{D})$ lies in the closure of the image of $A\mathfrak{g} \otimes_k L\mathfrak{g} + L\mathfrak{g} \otimes_k A\mathfrak{g}$ in $\widehat{UL\mathfrak{g}}$. It follows that $\widehat{C}(\widehat{D})$ has the form $cr + \sum_{\kappa} \sum_{n \geq 1} X_{\kappa} f_{\kappa, n} \circ X_{\kappa} g_{\kappa, n}$ with $r \in R$, one of $f_{\kappa, n}, g_{\kappa, n} \in L$ being in A and the order of $f_{\kappa, n}$ smaller than that of $g_{\kappa, n}$ for almost all κ, n . Since the elements of A have order ≤ 0 and $X_{\kappa} f_{\kappa, n} \circ X_{\kappa} g_{\kappa, n} \equiv X_{\kappa} g_{\kappa, n} \circ X_{\kappa} g_{\kappa, n} \pmod{cR}$, we can assume that all $f_{\kappa, n}$ lie in A and so the first assertion follows.

If $D \in \theta_{A, R}$, then for $X \in \mathfrak{g}$ and $f \in A$, we have $[D, Xf] = X(Df)$, which is an element of $A\mathfrak{g}$ (since $Df \in A$). This shows that $T_{\mathfrak{g}}(\widehat{D})$ preserves $A\mathfrak{g}\mathbb{F}(L\mathfrak{g})$. If $D \in \theta_{A/R}$, then it follows from the proven part that $T_{\mathfrak{g}}(\widehat{D})$ maps $\mathbb{F}(L\mathfrak{g})$ to $A\mathfrak{g}\mathbb{F}(L\mathfrak{g})$ and hence induces the zero map in $\mathbb{F}(L\mathfrak{g})_{A\mathfrak{g}}$. So $\widehat{\theta}_{A, R}$ acts on $\mathbb{F}(L\mathfrak{g})_{A\mathfrak{g}}$ via $\widehat{\theta}_R^A$. \square

More relevant than $\mathbb{F}(L\mathfrak{g})_{A\mathfrak{g}}$ will be certain finite dimensional quotients thereof obtained as follows. Let ℓ be a positive integer and let $V : i \mapsto V_i$ be a map which assigns to every $i \in I$ a finite dimensional irreducible representation V_i of \mathfrak{g} . Make $\otimes_{i \in I} V_i$ a k -representation of $F^0 L\mathfrak{g}$ by letting c act as multiplication by ℓ and $\mathfrak{g} \otimes_k \mathcal{O}_i$ on the i th factor via its projection onto \mathfrak{g} . If we induce this up to $\widehat{L\mathfrak{g}}$ we get a representation $\widetilde{\mathbb{H}}_{\ell}(V)$ of $\widehat{L\mathfrak{g}}$ which clearly is a quotient of $\mathbb{F}(L\mathfrak{g})$. Its irreducible quotient is denoted by $\mathbb{H}_{\ell}(V)$. The following is known (see the book of Kac [3]). First of all, $\mathbb{H}_{\ell}(V)$ is nonzero precisely when each \mathfrak{g} -representation V_i is of level $\leq \ell$, i.e., has the property that for every nilpotent $X \in \mathfrak{g}$, $X^{\ell+1}$ yields the zero map in V_i . Assume this is the case. Then $\mathbb{H}_{\ell}(V)$ is integrable as a $\widehat{L\mathfrak{g}}$ -module: if $Y \in \mathfrak{g}$ is nilpotent and $f \in L$, then Yf acts locally nilpotently in $\mathbb{H}_{\ell}(V)$.

The set of isomorphism classes of finite dimensional irreducible \mathfrak{g} -representations of level $\leq \ell$ is finite. It is clear that this set, which we denote by P_{ℓ} , is invariant under dualization (and more generally, under all outer automorphisms of \mathfrak{g}).

Adopting the physicists terminology, we might call the R -module of R -linear forms on $\mathbb{H}_{\ell}(V)_{A\mathfrak{g}}$ the *conformal block* attached to A . The following proposition says that it is of finite rank and describes in essence the WZW-connection.

Proposition 3.2 (Finiteness). *The space $\mathbb{H}_{\ell}(V)$ is finitely generated as a $UA\mathfrak{g}$ -module (so that $\mathbb{H}_{\ell}(V)_{A\mathfrak{g}}$ is finitely generated R -module). The Lie algebra $\widehat{\theta}_R^A$ acts on $\mathbb{H}_{\ell}(V)_{A\mathfrak{g}}$ via the Sugawara representation with the central c_0 acting as multiplication by $\frac{c}{c+\hbar} \dim \mathfrak{g}$.*

Proof. Pick a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, let $\Delta \subset \mathfrak{h}^*$ be its root system and $\mathfrak{g} = \mathfrak{h} \oplus_{\alpha \in \Delta} \mathfrak{g}_{\alpha}$ the associated decomposition. Recall that \mathfrak{g}_{α} has a nilpotent generator and $[\mathfrak{g}_{\alpha}, \mathfrak{g}_{\beta}]$ is contained in $\mathfrak{g}_{\alpha+\beta}$ when $\alpha + \beta \in \Delta$ and is contained in \mathfrak{h} otherwise. Recall also that \mathfrak{g} is spanned by $\oplus_{\alpha \in \Delta} \mathfrak{g}_{\alpha}$ and that \mathfrak{h} normalizes each \mathfrak{g}_{α} . Choose a generator t_i of \mathfrak{m}_i . Using a simple (PBW-type) argument we see that $\mathbb{H}_{\ell}(V)$ is spanned over R by the subspaces

$$(\mathfrak{g}_{\alpha_k} q_k)^{\circ r_k} \circ \cdots \circ (\mathfrak{g}_{\alpha_1} q_1)^{\circ r_1} V$$

with $k \geq 0$, each q_{ρ} a negative power of some t_i and such that $r_1 \geq \cdots \geq r_k$. Choose a finite set Q of powers of t_i 's that spans an R -supplement of $\mathcal{O} + A$ in L and let M be the R -span of the above subspaces for which all q_{ρ} lie in Q . It is clear that then M generates $\mathbb{H}_{\ell}(V)$ as a $UA\mathfrak{g}$ -module. Since each line $\mathfrak{g}_{\alpha} q$ acts locally nilpotently in $\mathbb{H}_{\ell}(V)$, there exists a positive integer N such that $(\mathfrak{g}_{\alpha} q)^{\circ N} V = 0$ for all $\alpha \in \Delta, q \in Q$. So the only generating lines that contribute to M have $r_1 < N$ and are hence finite in number. This shows that M is a finitely generated R -module. The rest follows from 3.1. \square

Remark 3.3. We expect the R -module $\mathbb{H}_\ell(V)_{A\mathfrak{g}}$ is also flat and that this is a consequence from a related property for the $UA\mathfrak{g}$ -module $\mathbb{H}_\ell(V)$. Such a result, or rather an algebraic proof of it, might simplify the argument in [10] (see Section 4 for our version) which shows that the sheaf of conformal blocks attached to a degenerating family of pointed curves is locally free.

Propagation principle. The following proposition is a bare version of what is known as *propagation of vacua*, or rather the generalization of this fact that can be found in Beauville [1]. It often allows us to reduce the discussion to the case where I is a singleton.

Proposition 3.4. *Let $J \subsetneq I$ be such that A maps onto $\bigoplus_{j \in J} L_j / \mathcal{O}_j$. Denote by $B \subset A$ the kernel, so that we have a surjective Lie homomorphism $B\mathfrak{g} \rightarrow \mathfrak{g}^J$ via which $B\mathfrak{g}$ acts on $\bigotimes_{j \in J} V_j$. Then the map of $B\mathfrak{g}$ -modules $\mathbb{H}_\ell(V|I - J) \otimes \bigotimes_{j \in J} V_j \rightarrow \mathbb{H}_\ell(V)$ induces an isomorphism on covariants:*

$$\left(\mathbb{H}_\ell(V|I - J) \otimes \bigotimes_{j \in J} V_j \right)_{B\mathfrak{g}} \xrightarrow{\cong} \mathbb{H}_\ell(V)_{A\mathfrak{g}}.$$

Proof. It suffices to do the case when J is a singleton $\{o\}$. The hypotheses clearly imply that $\mathbb{H}_\ell(V|I - \{o\}) \otimes V_o \rightarrow \mathbb{H}_\ell(V)_{A\mathfrak{g}}$ is onto. The kernel is easily shown to be $B\mathfrak{g}(\mathbb{H}_\ell(V|I - \{o\}) \otimes V_o)$. \square

Remark 3.5. This proposition is sometimes used in the opposite direction: if $\mathfrak{m}_o \subset A$ is a principal ideal with the property that for a generator $t \in \mathfrak{m}_o$, the \mathfrak{m}_o -adic completion of A gets identified with $R((t))$, then let \tilde{I} be the disjoint union of I and $\{o\}$, \tilde{V} the extension of V to \tilde{I} which assigns to o the trivial representation and $\tilde{A} := A[[t^{-1}]]$. With $(\tilde{I}, \{o\})$ taking the role of (I, J) , we then find that $\mathbb{H}_\ell(V)_{A\mathfrak{g}} \cong \mathbb{H}_\ell(\tilde{V})_{\tilde{A}\mathfrak{g}}$.

Conformal blocks in families. We transcribe the preceding in more geometric terms. This leads us to sheafify many of the notions we introduced earlier; we shall modify our notation (or its meaning) accordingly. Suppose given a proper and flat morphism between k -varieties $\pi : \mathcal{C} \rightarrow S$ whose base S is smooth and fibers are reduced connected curves which have complete intersection singularities only (we are here not assuming that \mathcal{C} is smooth over k). Since the family is flat, the arithmetic genus of the fibers is locally constant; we simply assume it is globally so and denote this constant genus by g . We also suppose given disjoint sections $S_i \subset \mathcal{C}$, indexed by the finite nonempty set I whose union $\bigcup_{i \in I} S_i$ lies in the smooth part of \mathcal{C} and meets every irreducible component of a fiber. The last condition ensures that if $j : \mathcal{C}^\circ := \mathcal{C} - \bigcup_{i \in I} S_i \subset \mathcal{C}$ is the inclusion, then πj is an affine morphism.

We denote by $(\mathcal{O}_i, \mathfrak{m}_i)$ the formal completion of $\mathcal{O}_{\mathcal{C}}$ along S_i , by \mathcal{L}_i the subsheaf of fractions of \mathcal{O}_i with denominator a local generator of \mathfrak{m}_i and by $\mathcal{O}, \mathfrak{m}$ and \mathcal{L} the corresponding direct sums. But we keep on using $\omega, \theta, \hat{\theta}$ etc. for their sheafified counterparts. So these are now all \mathcal{O}_S -modules and the residue pairing is also one of \mathcal{O}_S -modules: $r : \mathcal{L} \times \omega \rightarrow \mathcal{O}_S$. We write \mathcal{A} for $\pi_* j_* j^* \mathcal{O}_{\mathcal{C}}$ (a sheaf of \mathcal{O}_S -algebras) and often identify this with its image in \mathcal{L} . We denote by $\theta_{\mathcal{A}/S}$ the sheaf of \mathcal{O}_S -derivations $\mathcal{A} \rightarrow \mathcal{A}$ and by $\omega_{\mathcal{A}/S}$ for the direct image on S of the the relative dualizing sheaf of \mathcal{C}°/S , (in the present situation the relative dualizing sheaf of $\pi, \omega_{\mathcal{C}/S}$, is simply the direct image on \mathcal{C} of the sheaf of relative differentials on the open subset of \mathcal{C} where π is smooth). So $\omega_{\mathcal{A}/S}$ is torsion free and hence embeds in ω .

Lemma 3.6. *The properties A_1, A_2 and A_3 hold for the sheaf \mathcal{A} . Precisely,*

(A_1) \mathcal{A} is as a sheaf of \mathcal{O}_S -algebras flat and of finite type,

- (\mathcal{A}_2) $\mathcal{A} \cap \mathcal{O} = \mathcal{O}_S$ and $R^1\pi_*\mathcal{O}_C = \mathcal{L}/(\mathcal{A} + \mathcal{O})$ is locally free of rank g ,
 (\mathcal{A}_3) we have $\theta_{\mathcal{A}/S} = \text{Hom}_{\mathcal{A}}(\omega_{\mathcal{A}/S}, \mathcal{A})$ and $\omega_{\mathcal{A}/S}$ is the annihilator of \mathcal{A} with respect to the residue pairing.

Proof. Property \mathcal{A}_1 is clear. It is also clear that $\mathcal{O}_S = \pi_*\mathcal{O}_C \rightarrow \mathcal{A} \cap \mathcal{O}$ is an isomorphism. The long exact sequence defined by the functor π_* applied to the short exact sequence

$$0 \rightarrow \mathcal{O}_C \rightarrow j_*j^*\mathcal{O}_C \rightarrow \mathcal{L}/\mathcal{O} \rightarrow 0$$

tells us that $R^1\pi_*\mathcal{O}_C = \mathcal{L}/(\mathcal{A} + \mathcal{O})$; in particular, the latter is locally free of rank g . Hence \mathcal{A}_2 holds as well.

In order to verify \mathcal{A}_3 , we note that $\pi_*\omega_{C/S}$ is the \mathcal{O}_S -dual of $R^1\pi_*\mathcal{O}_S$, and hence is locally free of rank g . The first part of \mathcal{A}_3 from the corresponding local property $\theta_{C/S} = \text{Hom}_{\mathcal{O}_C}(\omega_{C/S}, \mathcal{O}_C)$ by apply π_*j^* to either side. This local property is known to hold for families of curves with complete intersection singularities. (A proof under the assumption that \mathcal{C} is smooth—which is does not affect the generality, since π is locally the restriction of that case and both sides are compatible with base change—runs as follows: if $j' : \mathcal{C}' \subset \mathcal{C}$ denotes the locus where π is smooth, then its complement is of codimension ≥ 2 everywhere. Clearly, $\theta_{C/S}$ is the \mathcal{O}_C -dual of $\omega_{C/S}$ on \mathcal{C}' and since both are inert under $j'_*j'^*$, they are equal everywhere.)

The last assertion essentially restates the well-known fact that the polar part of a rational section of $\omega_{C/S}$ must have zero residue sum, but can otherwise be arbitrary. More precisely, the image of $\omega_{\mathcal{A}/S}$ in $\omega/F^1\omega$ is the kernel of the residue map $\omega/F^1\omega \rightarrow \mathcal{O}_S$. The intersection $\omega_{\mathcal{A}/S} \cap F^1\omega$ is $\pi_*\omega_{C/S}$ and is hence locally free of rank g . Since $\text{ann}(F^1\omega) = \mathcal{O}$, it follows that $\text{ann}(\omega_{\mathcal{A}/S}) \cap \mathcal{O}$ and $\mathcal{L}/(\text{ann}(\omega_{\mathcal{A}/S}) + \mathcal{O})$ are locally free of rank 1 and g respectively. Since \mathcal{A} has these properties also and is contained in $\text{ann}(\omega_{\mathcal{A}/S})$, we must have $\mathcal{A} = \text{ann}(\omega_{\mathcal{A}/S})$. \square

For what follows one usually supposes that the fibers are stable I -pointed curves (meaning that every fiber of πj has only ordinary double points as singularities and has finite automorphism group) and is versal (so that in the discriminant D_π of π is a reduced normal crossing divisor), but we shall not make these assumptions yet. Instead, we assume the considerable weaker property that the sections of the sheaf $\theta_S(\log D_\pi)$ of vector fields on S tangent to D_π lift locally on S to vector fields on \mathcal{C} . This is for instance the case if \mathcal{C} is smooth and π is multi-transversal with respect to the (Thom) stratification of $\text{Hom}(TC, \pi^*TS)$ by rank [6]. Notice that the restriction homomorphism $\theta_S(\log D_\pi) \otimes \mathcal{O}_{D_\pi} \rightarrow \theta_{D_\pi}$ is an isomorphism.

Let $\theta_{C,S} \subset \theta_C$ denote the sheaf of derivations which preserve $\pi^*\mathcal{O}_S$. Applying $\pi_*j_*j^*$ to the exact sequence $0 \rightarrow \theta_{C/S} \rightarrow \theta_C \rightarrow \theta_{C,S}/\theta_{C/S} \rightarrow 0$ yields the exact sequence

$$0 \rightarrow \theta_{\mathcal{A}} \rightarrow \theta_{\mathcal{A},S} \rightarrow \theta_S(\log D_\pi) \rightarrow 0.$$

(using our liftability assumption and the fact that πj is affine).

We have defined $\hat{\theta}_{\mathcal{A},S}$ as the preimage of $\theta_{\mathcal{A},S}$ in $\hat{\theta}_{\mathcal{L},S}$ and $\hat{\theta}_S(\log D_\pi)$ as the quotient $\hat{\theta}_{\mathcal{L},S}/\theta_{\mathcal{A}}$. These centrally extend $\theta_{\mathcal{A},S}$ and θ_S by $c_0\mathcal{O}_S$.

Observe that $\mathcal{L}\mathfrak{g} = \mathfrak{g} \otimes_{\mathcal{O}_S} \mathcal{L}$ is now a sheaf of Lie algebras over \mathcal{O}_S . The same applies to $\hat{\mathfrak{g}}$ and so we have a Virasoro extension $\hat{\theta}_S$ of θ_S by $c_0\mathcal{O}_S$. We have also defined $\mathcal{A}\mathfrak{g} = \mathfrak{g} \otimes_k \mathcal{A}$, which is a Lie subsheaf of $\mathcal{L}\mathfrak{g}$ as well as of $\hat{\mathcal{L}}\mathfrak{g}$ and the Fock type $\hat{\mathcal{L}}\mathfrak{g}$ -module $\mathcal{F}(\hat{\mathcal{L}}\mathfrak{g})$. The *universal covacuum module* is the sheaf of $\mathcal{A}\mathfrak{g}$ -covariants in the latter,

$$\mathcal{F}(\hat{\mathcal{L}}\mathfrak{g})_{\mathcal{C}} := \mathcal{F}(\hat{\mathcal{L}}\mathfrak{g})_{\mathcal{A}\mathfrak{g}} := \mathcal{F}(\hat{\mathcal{L}}\mathfrak{g})/\mathcal{A}\mathfrak{g}\mathcal{F}(\hat{\mathcal{L}}\mathfrak{g}).$$

From Proposition 3.1 we get:

Corollary 3.7. *The representation of the Lie algebra $\hat{\theta}_{\mathcal{A},S}$ on $\mathcal{F}(\widehat{\mathcal{L}\mathfrak{g}})$ preserves $\text{Ag}\mathcal{F}(\widehat{\mathcal{L}\mathfrak{g}})$ and acts on $\mathcal{F}(\widehat{\mathcal{L}\mathfrak{g}})_{\mathcal{C}}$ via the central extension $\hat{\theta}_S(\log D_\pi)$ of $\theta_S(\log D_\pi)$ (with the central c_0 acting as multiplication by $(c + \hbar)^{-1}c \dim \mathfrak{g}$). This construction has a base change property along the smooth part D_π° of the discriminant: we may identify $\mathcal{F}(\widehat{\mathcal{L}\mathfrak{g}})_{\mathcal{C}} \otimes \mathcal{O}_{D_\pi^\circ}$ with $\mathcal{F}(\widehat{\mathcal{L}}|_{D_\pi^\circ} \widehat{\mathfrak{g}})_{\mathcal{C}|_{D_\pi^\circ}}$ and the action of $\hat{\theta}_S(\log D_\pi) \otimes \mathcal{O}_{D_\pi^\circ}$ on it factors through $\hat{\theta}_{D_\pi^\circ}$.*

The bundle of integrable representations $\mathcal{H}_\ell(V)$ over S is defined in the expected manner: it is obtained as a quotient of $\mathcal{F}(\widehat{\mathcal{L}\mathfrak{g}})$ in the way $\mathbb{H}_\ell(V)$ is obtained from $\mathbb{F}(\mathcal{L}\mathfrak{g})$. According to Corollary 3.7, $\mathcal{H}_\ell(V)$ comes with a k -linear representation of the Lie algebra $\hat{\theta}_S(\log D_\pi)$ that satisfies the Leibniz rule and on which c_0 acts as multiplication by $\ell(\ell + \hbar)^{-1} \dim \mathfrak{g}$. We write $\mathcal{H}_\ell(V)_{\mathcal{C}}$ for $\mathcal{H}_\ell(V)_{\text{Ag}}$.

Corollary 3.8 (WZW-connection). *The \mathcal{O}_S -module $\mathcal{H}_\ell(V)_{\mathcal{C}}$ is of finite rank; It is also locally free over $S - D_\pi$ and the Lie action of $\hat{\theta}_S(\log D_\pi)$ on $\mathcal{H}_\ell(V)_{\mathcal{C}}$ defines a flat connection on the associated bundle of projective spaces, $\mathbb{P}_S(\mathcal{H}_\ell(V)_{\mathcal{C}}) \rightarrow S$ with a logarithmic singularity along D_π . The same base change property holds along the smooth part D_π° of the discriminant as in Corollary 3.7.*

Proof. The first assertion follows from 3.2. The rest is clear except perhaps the local freeness of $\mathcal{H}_\ell(V)_{\mathcal{C}}$ on $S - D_\pi$. But this follows from the local existence of a connection in the \mathcal{O}_S -module $\mathcal{H}_\ell(V)_{\mathcal{C}}$. \square

Remark 3.9. In case the base field k is \mathbb{C} we could equally well work in the complex-analytic category. The preceding corollary then leads to an interesting topological functor: if Σ is a compact smooth oriented surface, $I \subset \Sigma$ a finite nonempty subset such that for every $i \in I$ is given a \mathfrak{g} -representation V_i , then there is canonically associated to these data a projective space $\mathbb{P}(\Sigma, I, V)$: for every complex structure on Σ compatible with the given smooth structure and orientation we have defined a conformal block as above. The complex structures with these property form a simply connected (even contractible) space $\mathcal{C}(\Sigma)$. To see this: such a structure amounts to giving a conformal structure on Σ , that is, a Riemann metric up to multiplication by the exponential of a real valued function. The space of Riemann metrics is a convex cone in some linear space (hence contractible) and the linear space of real valued functions on Σ acts sufficiently nice on it to ensure that the orbit space remains simply connected. So any two such complex structures can be joined a path whose homotopy class is unique. Corollary 3.8 then implies that the WZW-connection enables us to identify their associated conformal blocks in a unique manner. Perhaps the best way to describe $\mathbb{P}(\Sigma, I, V)$ is as the space of flat sections of the flat projective space bundle of conformal blocks over $\mathcal{C}(\Sigma)$. The naturality implies that the group of orientation preserving diffeomorphisms of the pair Σ which fix I pointwise acts on $\mathbb{P}(\Sigma, I, V)$. Since its identity component is contractible, the action will be through the mapping class group $\Gamma(\Sigma, I)$. It is conjectured that this action leaves invariant a Fubini-Study metric on $\mathbb{P}(\Sigma, I, V)$.

Propagation principle continued. In the preceding subsection we made the assumption throughout that a union \tilde{S} of sections of $\mathcal{C} \rightarrow S$ is given to ensure that $\mathcal{C} - \tilde{S} \rightarrow S$ is affine. However, the propagation principle permits us to abandon that assumption. In fact, this leads us to let V stand for any map which assigns to every closed point p of \mathcal{C} an irreducible \mathfrak{g} -representation V_p of level $\leq \ell$, subject to the condition that its *support*, $\text{Supp}(V)$ (i.e., the set of p for which V_p is not the trivial representation), is a trivial finite cover over S and contained in the locus where $\pi : \mathcal{C} \rightarrow S$ is smooth. Our earlier defined $\mathcal{H}_\ell(V)$ is now better written as $\mathcal{H}_\ell(V|_{\text{Supp}(V)})$. Since $\mathcal{C} - \text{Supp}(V)$ need not be affine over S , this does

not yield the right notion of conformal block. We can find however, at least locally over S , additional sections S'_i of $\mathcal{C} - \text{Supp}(V) \rightarrow S$ so that $\mathcal{C}^\circ := \mathcal{C} - \text{Supp}(V) - \cup_i S'_i$ is affine over S . Then we can form $\mathcal{H}_\ell(V|\mathcal{C} - \mathcal{C}^\circ)$ and Remark 3.5 shows that the resulting conformal block $\mathcal{H}_\ell(V|\mathcal{C} - \mathcal{C}^\circ)_{\mathfrak{g}[\pi_*\mathcal{O}_{\mathcal{C}^\circ}]}$ is independent of the choices made. This suggests that we let $\mathcal{H}_\ell(V)$ stand for the sheaf associated to the presheaf

$$S \supset U \mapsto \varprojlim_{\tilde{S}} \mathcal{H}_\ell(V|_{\tilde{S}}),$$

where \tilde{S} runs over the closed subvarieties of $\pi^{-1}U$ that trivial finite covers of S , contain $\text{Supp}(V)$ and have affine complement in $\pi^{-1}U$ and perhaps also justifies our custom of writing $\mathcal{H}_\ell(V)_{\mathcal{C}}$ for the associated sheaf of conformal blocks. It is clear that in this set-up there is also no need anymore to insist that the fibers of π be connected.

The genus zero case and the KZ-connection. We take $C \cong \mathbb{P}^1$ and let ℓ and $z \in C \mapsto V_z$ be as usual. Choose an affine coordinate z on C such that $\infty \notin \text{Supp}(V)$ and let $z_1, \dots, z_N \in \mathbb{C}$ enumerate the distinct points of $\text{Supp}(V)$. Write V_i for V_{z_i} . The local field attached to ∞ has parameter $t_\infty = z^{-1}$. If \mathbb{H}_ℓ denotes the representation of $\mathfrak{g}(\widehat{(z^{-1})})$ attached to the trivial representation (with generator v_0), then by the propagation principle 3.4 we have $\mathbb{H}_\ell(V)_{\mathcal{C}} = (\mathbb{H}_\ell \otimes V_1 \otimes \dots \otimes V_N)_{\mathfrak{g}[[z]]}$, where $\mathfrak{g}[[z]]$ acts on V_i via its evaluation at z_i . According to [3], the $\mathfrak{g}[[z]]$ -homomorphism $U\mathfrak{g}[[z]] \rightarrow \mathbb{H}_\ell$, $u \mapsto uv_0$ is surjective and its kernel is the left ideal generated by the constants \mathfrak{g} and $(Xz)^{1+\ell}$, where $X \in \mathfrak{g}$ is a regular nilpotent element (as these form a single orbit under the adjoint representation, it does not matter which one we take). This implies that $\mathbb{H}_\ell(V)_{\mathbb{P}^1}$ can be identified with a quotient of the space of \mathfrak{g} -covariants $(V_1 \otimes \dots \otimes V_N)_{\mathfrak{g}}$, namely its biggest quotient on which $(\sum_{i=1}^N z_i X_{(i)})^{1+\ell}$ acts trivially (where $X_{(i)}$ acts on V_i as X and on the other tensor factors V_j , $j \neq i$, as the identity).

Now regard z_1, \dots, z_N as variables. If S denotes the open subset of $(z_1, \dots, z_N) \in \mathbb{C}^N$ of pairwise distinct N -tuples, then we have in an evident manner a family \mathcal{C} over S , $\pi : \mathcal{C} \times S \rightarrow S$, with $N+1$ sections (including the one at infinity) so that we also have defined $\mathcal{C}^\circ \rightarrow S$. This determines a sheaf of conformal blocks $\mathcal{H}_\ell(V)_{\mathcal{C}}$ over S . According to the preceding, we have an exact sequence

$$(V_1 \otimes \dots \otimes V_N)_{\mathfrak{g}} \otimes_k \mathcal{O}_S \xrightarrow{(\sum_i z_i X_{(i)})^{1+\ell}} (V_1 \otimes \dots \otimes V_N)_{\mathfrak{g}} \otimes_k \mathcal{O}_S \rightarrow \mathcal{H}_\ell(V)_{\mathcal{C}} \rightarrow 0.$$

We identify its WZW connection, or rather, a natural lift of that connection to $V_1 \otimes \dots \otimes V_N \otimes_k \mathcal{O}_S$. In order to compute the covariant derivative with respect to the vector field $\partial_i := \frac{\partial}{\partial z_i}$ on S , we follow our recipe and lift it to \mathcal{C} in the obvious way (with zero component along C). We continue to denote that lift by ∂_i and determine its (Sugawara) action on $\mathcal{H}_\ell(V)$. We first observe that ∂_i is tangent to all the sections, except the i th. Near that section we decompose it as $(\frac{\partial}{\partial z} + \partial_i) - \frac{\partial}{\partial z}$, where the first term is tangent to the i -th section and the second term is vertical. The action of the former is easily understood (in terms of an obvious trivialization of $\mathcal{H}_\ell(V)$, it acts as derivation with respect to z_i). The vertical term, $-\frac{\partial}{\partial z}$, acts via the Sugawara representation, that is, it acts on the i th slot as $-\frac{1}{\ell+\hbar} \sum_{\kappa} X_{\kappa}(z-z_i)^{-1} \circ X_{\kappa}$ and as the identity on the others. This action does not induce one in $V_1 \otimes \dots \otimes V_N \otimes_k \mathcal{O}_S$. To make it so, we add to this the action of the element

$$\left(\frac{1}{\ell+\hbar} \sum_{\kappa} X_{\kappa}(z-z_i)^{-1} \right) \circ X_{\kappa}^{(i)} \in \mathfrak{g}[\mathcal{C}^\circ]U\widehat{\mathcal{L}}\mathfrak{g},$$

for this sum then acts in $V_1 \otimes \cdots \otimes V_N \otimes_k \mathcal{O}_S$ as

$$\frac{1}{\ell + \hbar} \sum_{j \neq i} \frac{1}{z_j - z_i} X_{\kappa}^{(j)} X_{\kappa}^{(i)}.$$

Let us regard the Casimir element c as a tensor of $\mathfrak{g} \otimes_k \mathfrak{g}$, and denote by $c^{(i,j)}$ its action in $V_1 \otimes \cdots \otimes V_N$ on the i th and j th factor (since c is symmetric, we have $c^{(i,j)} = c^{(j,i)}$, so that we need not worry about the order here). We conclude that the WZW-connection is induced by the connection on $V_1 \otimes \cdots \otimes V_N \otimes_k \mathcal{O}_S$ defined by the connection form

$$\frac{1}{\ell + \hbar} \sum_{i=1}^N \sum_{j \neq i} \frac{dz_i}{z_j - z_i} c^{(i,j)} = -\frac{1}{\ell + \hbar} \sum_{1 \leq i < j \leq N} \frac{d(z_i - z_j)}{z_i - z_j} c^{(i,j)}$$

This lift of the WZW-connection is known as the Knizhnik-Zamolodchikov connection. It is not difficult to verify that it is still flat (see for instance [5]).

4. DOUBLE POINT DEGENERATION

Factorization. Let $\ell, \pi : \mathcal{C} \rightarrow S$ and V be as in the preceding section so that we have defined the sheaf of conformal blocks $\mathcal{H}_\ell(V)_{\mathcal{C}}$. We assume in addition that we are given a section S_0 along which π has an ordinary double point and a partial normalization $\nu : \tilde{\mathcal{C}} \rightarrow \mathcal{C}$ which only separates the branches. So ν is an isomorphism over the complement of S_0 and S_0 has two disjoint lifts to $\tilde{\mathcal{C}}$, which we shall denote by S_+ and S_- . The *factorization principle* expresses $\mathcal{H}_\ell(V)_{\mathcal{C}}$ in terms of conformal blocks attached to the partial normalization of $\tilde{\mathcal{C}}$.

Recall that P_ℓ denotes the set of isomorphism classes of irreducible \mathfrak{g} -representations of level $\leq \ell$ and is invariant under dualization: if $\lambda \in P_\ell$, then $\lambda^* \in P_\ell$. Let V_λ be a \mathfrak{g} -representation in the equivalence class $\lambda \in P_\ell$ and choose \mathfrak{g} -equivariant dualities $b_\lambda : V_\lambda \otimes V_{\lambda^*} \rightarrow k$, $\lambda \in P_\ell$.

Proposition 4.1. *Let $\tilde{V}_{\lambda, \lambda^*}$ be the representation valued map on $\tilde{\mathcal{C}}$ which is constant equal to V_λ resp. V_{λ^*} on S_+ resp. S_- and on any other closed point of the representation already assigned to its image in $\tilde{\mathcal{C}}$. Then the contractions b_λ define an isomorphism*

$$\bigoplus_{\lambda \in P_\ell} \mathcal{H}_\ell(\tilde{V}_{\lambda, \lambda^*})_{\tilde{\mathcal{C}}} \xrightarrow{\cong} \mathcal{H}_\ell(V)_{\mathcal{C}}.$$

This is almost a formal consequence of:

Lemma 4.2. *Let M be a finite dimensional representation of $\mathfrak{g} \times \mathfrak{g}$ which is of level $\leq \ell$ relative to both factors. If M^δ denotes that same space viewed as \mathfrak{g} -module with respect to the diagonal embedding $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \times \mathfrak{g}$, then the contraction $\bigoplus_{\lambda \in P_\ell} M \otimes (V_\lambda \boxtimes V_{\lambda^*}) \rightarrow M$ (each component is defined by b^λ ; the symbol \boxtimes stands for the exterior tensor product of representations) induces an isomorphism between covariants:*

$$\bigoplus_{\lambda \in P_\ell} (M \otimes (V_\lambda \boxtimes V_{\lambda^*}))_{\mathfrak{g} \times \mathfrak{g}} \xrightarrow{\cong} M_{\mathfrak{g}}^\delta.$$

Proof. Without loss of generality we may assume that M is irreducible, or more precisely, equal to $V_\mu \boxtimes V_{\mu'}$ for some $\mu, \mu' \in P_\ell$. Then $M^\delta = V_\mu \otimes V_{\mu'}$. By Schur's lemma, $M_{\mathfrak{g}}^\delta$ is one-dimensional if $\mu' = \mu^*$ and trivial otherwise. That same lemma applied to $\mathfrak{g} \times \mathfrak{g}$ shows that $(M \otimes (V_\lambda \boxtimes V_{\lambda^*}))_{\mathfrak{g} \times \mathfrak{g}}$ is zero unless $(\mu, \mu') = (\lambda^*, \lambda)$, in which case it is one-dimensional. The lemma follows. \square

Proof of 4.1. The issue is local on S and so we may assume that there exists an affine open-dense subset \mathcal{C}° of \mathcal{C} which contains its singular locus, is disjoint with $\text{Supp}(V)$ and is such that its complement $\tilde{S} := \mathcal{C} - \mathcal{C}^\circ$ is a trivial finite cover over S . Then $\mathcal{H}_\ell(V)_\mathcal{C} = \mathcal{H}_\ell(V|\tilde{S})_{\pi_*\mathcal{O}_{\mathcal{C}^\circ}[\mathfrak{g}]}$. Let $\tilde{\mathcal{C}}^\circ := \nu^{-1}\mathcal{C}^\circ$. Evaluation in S_0 resp. S_+, S_- defines epimorphisms $\pi_*\mathcal{O}_{\mathcal{C}^\circ} \rightarrow \mathcal{O}_S$ resp. $\pi_*\nu_*\mathcal{O}_{\tilde{\mathcal{C}}^\circ} \rightarrow \mathcal{O}_S \oplus \mathcal{O}_S$ whose kernels may be identified by means of ν . If we denote that common kernel by \mathcal{I} and $\mathcal{I}\mathfrak{g}$ has the evident meaning, then the argument used to prove Proposition 3.2 shows that $M := \mathcal{H}_\ell(V|\tilde{S})_{\mathcal{I}\mathfrak{g}}$ is a \mathcal{O}_S -module of finite rank. It underlies a representation of $\mathcal{O}_S\mathfrak{g} \oplus \mathcal{O}_S\mathfrak{g}$ of level $\leq \ell$ relative to both factors and is such that $M_\mathfrak{g}^\delta = \mathcal{H}_\ell(V)_\mathcal{C}$. Now apply Lemma 4.2. \square

Local freeness. We continue with the situation of Proposition 4.1, except that we now assume (for simplicity only, actually) that the base S is $\text{Spec}(k)$: C is a reduced complete curve with complete intersection singularities only which has at $S_0 \in C$ an ordinary double point. Choose generators t_\pm of the maximal ideals of the completed local rings of C at S_\pm . There is then a canonical *smoothing* of C , that is, a way of making C the closed fiber of a flat morphism $\tau : \mathcal{C} \rightarrow \Delta$, with Δ the spectrum of the discrete valuation ring $k[[\tau]]$, such that the generic fiber is smooth: in the product $\tilde{C} \times \Delta$, blow up (S_\pm, o) and let \tilde{C} be the formal neighborhood of the strict transform of $\tilde{C} \times \{o\}$. So at the preimage of (S_\pm, o) on the strict transform of $\tilde{C} \times \{o\}$ we have the formal coordinate chart $(t_\pm, \tau/t_\pm)$. Now let \mathcal{C} be the quotient of \tilde{C} obtained by identifying these formal charts up to order: $(t_+, \tau/t_+) = (\tau/t_-, t_-)$, so that (t_+, t_-) is now a formal chart of \mathcal{C} on which we have $\tau = t_+t_-$ (in either domain τ represents the same regular function). We thus have defined a flat morphism $\tau : \mathcal{C} \rightarrow \Delta$ whose closed fiber may be identified with C . The domain \mathcal{C} is smooth over k and that the generic fiber of τ is smooth over $k((\tau))$. This is our canonical smoothing of C . Let us also notice that \mathcal{C} is at every $p \in C - \{S_0\}$ canonically identified with $(C, p) \times \Delta$ with τ given as the projection on the second factor.

Let I be a finite subset of the smooth part of C such that $C - I$ is affine and V is a \mathfrak{g} -representation valued map on I . We extend V canonically to \mathcal{C} by letting it be zero at S_0 and constant on $\{p\} \times \Delta$ when $p \neq S_0$; we denote that extension by V/Δ . Then we have defined the $k[[\tau]]$ -module $\mathcal{H}_\ell(V/\Delta)$. It is clear that $\mathcal{H}_\ell(V/\Delta)$ is naturally identified with $\mathbb{H}_\ell(V)[[\tau]]$. We assume that the complement of the support of V is affine and we denote by A its k -algebra of regular functions. Then the complement of the support of V/Δ is affine over Δ and if \mathcal{A} denotes the corresponding $k[[\tau]]$ -algebra of regular functions, then $A = \mathcal{A}/\tau\mathcal{A}$. According to Proposition 3.2, the conformal block $\mathcal{H}_\ell(V/\Delta)_\mathcal{C}$ is a finitely generated $k[[\tau]]$ -module; its reduction modulo τ is clearly $\mathbb{H}_\ell(V)_{A\mathfrak{g}}$. If we denote by B the algebra of regular functions on $C - \text{Supp}(V) - S_0 = \tilde{C} - \text{Supp}(V) - (S_+ \cup S_-)$, then Proposition 4.1 identifies the latter with $\bigoplus_{\lambda \in P_\ell} \mathbb{H}_\ell(\tilde{V}_{\lambda, \lambda^*})_{B\mathfrak{g}}$. It is our goal to extend this identification to one of $\mathcal{H}_\ell(V/\Delta)_\mathcal{C}$ with $\bigoplus_{\lambda \in P_\ell} \mathbb{H}_\ell(\tilde{V}_{\lambda, \lambda^*})_{B\mathfrak{g}}[[\tau]]$ (which will imply that $\mathcal{H}_\ell(V/\Delta)_\mathcal{C}$ is a free $k[[\tau]]$ -module).

Put $\mathcal{O}_* := k[[t_+]] \oplus k[[t_-]]$ and $L_* := k((t_+)) \oplus k((t_-))$.

Lemma 4.3. *The rational map $\tilde{C} \times \Delta \leftarrow \tilde{C} \rightarrow \mathcal{C}$ identifies $k[[t_+, t_-]]$ with the subalgebra of $L_*[[\tau]]$ of elements of the form $\sum_{n \geq 0, m \geq 0} a_{m,n} (t_+^{m-n} \tau^n, t_-^{n-m} \tau^m)$. Furthermore, any continuous k -derivation $k[[t_+, t_-]] \rightarrow k[[t_+, t_-]]$ which fixes $\tau = t_+t_-$ defines in L_* an operator of the form*

$$(t_+ \frac{\partial}{\partial t_+}, 0) + \sum_{m, n \geq 0} a_{m,n} (t_+^{m-n+1} \tau^n \frac{\partial}{\partial t_+}, -t_-^{n-m+1} \tau^m \frac{\partial}{\partial t_-}).$$

Proof. Let $f = \sum_{n,m \geq 0} a_{m,n} t_+^m t_-^n$. If we substitute $t_- = \tau/t_+$, this becomes at (S_+, o) : $\sum_{n \geq 0} (\sum_{m \geq 0} a_{m,n} t_+^{m-n}) \tau^n$. So the coefficient of τ^n has polar part $\sum_{m=0}^{n-1} a_{m,n} t_+^{m-n}$ and constant term $a_{n,n}$. Similarly at (S_-, o) , f is written as $\sum_{m \geq 0} (\sum_{n \geq 0} a_{m,n} t_+^{n-m}) \tau^m$ with the coefficient of τ^m having polar part $\sum_{n=0}^{m-1} a_{m,n} t_+^{n-m}$ and constant term $a_{m,m}$. The polar parts and constant terms of these expressions determine all the $a_{n,m}$.

The second assertion is left as an exercise. \square

Given $\lambda \in P_\ell$, then the Casimir element $c = \sum_{\kappa} X_{\kappa} \circ X_{\kappa}$ acts in V_λ as a scalar, a scalar we shall denote by $c(\lambda)$. Observe that $c(\lambda^*) = c(\lambda)$.

Let $\mathbb{H}_\ell^\pm(V_\lambda)$ denote the representation attached to V_λ of the central extension of $\mathfrak{g}((t_\pm))$, so that $\mathbb{H}_\ell^+(V_\lambda) \otimes \mathbb{H}_\ell^-(V_{\lambda^*})$ is a representation of the central extension $\widehat{L_*\mathfrak{g}}$ of $L\mathfrak{g}$.

Lemma 4.4. *There exists a tensor valued Laurent series*

$$\varepsilon^\lambda = \sum_{d=0}^{\infty} \varepsilon_d^\lambda \tau^d \in (\mathbb{H}_\ell^+(V_\lambda) \otimes \mathbb{H}_\ell^-(V_{\lambda^*}))[[\tau]] \quad \text{with } \varepsilon_d^\lambda \in F^{-d}\mathbb{H}_\ell^+(V_\lambda) \otimes F^{-d}\mathbb{H}_\ell^-(V_{\lambda^*})$$

whose constant term $\varepsilon_0^\lambda \in V_\lambda \otimes V_{\lambda^*}$ is the dual of b_λ and which is annihilated by the image of $\mathfrak{g}[[t_+, t_-]]$ in $\widehat{L_*\mathfrak{g}}$. Moreover, for any continuous k -derivation $D : k[[t_+, t_-]] \rightarrow k[[t_+, t_-]]$ which fixes τ , ε^λ is an eigenvector of $T_{\mathfrak{g}}(D)$ with eigenvalue $-\frac{c(\lambda)}{2(\ell+h)}$.

Proof. We first observe that the choice of the coordinates t_+ and t_- defines a grading on all the relevant objects on which we have defined the associated filtration F (e.g., the degree zero summand of $\mathbb{H}_\ell(V_\lambda)$ is V_λ). It is known [3] that the pairing $b_\lambda : V_\lambda \times V_{\lambda^*} \rightarrow k$ extends (in a unique manner) to a perfect pairing

$$b_\lambda : \mathbb{H}_\ell^+(V_\lambda) \times \mathbb{H}_\ell^-(V_{\lambda^*}) \rightarrow k$$

characterized by the property that $b_\lambda(X t_+^k u, u') + b_\lambda(u, X t_-^{-k} u') = 0$ for all $X \in \mathfrak{g}$ and $k \in \mathbb{Z}$. It follows from that the restriction of b_λ to $\mathbb{H}_\ell^+(V_\lambda)_{-d} \times \mathbb{H}_\ell^-(V_{\lambda^*})_{-d'}$ is zero when $d \neq d'$ and is perfect when $d = d'$. So if $\varepsilon_d^\lambda \in \mathbb{H}_\ell^+(V_\lambda)_{-d} \otimes \mathbb{H}_\ell^-(V_{\lambda^*})_{-d}$ denotes the latter's inverse, then we have for all $k \in \mathbb{Z}$, $X \in \mathfrak{g}$

$$(X t_+^k, 0) \varepsilon_{d+k}^\lambda + (0, X t_-^{-k}) \varepsilon_d^\lambda = 0.$$

This amounts to the property that $(X t_+^k, \tau^k X t_-^{-k})$ annihilates $\sum_{d \geq 0} \varepsilon_d^\lambda \tau^d$. Since Lemma 4.3 says that the elements of $\mathfrak{g}[[t_+, t_-]]$ have series expansions in $(X t_+^k, X t_-^{-k}) \in \widehat{L_*\mathfrak{g}}$ with coefficients in $k[[\tau]]$, the first statement of the lemma follows.

The second will be a straightforward computation. Using Lemma 4.3, we write D as an operator in L_* . We then find that we need to verify the following two assertions:

- (i) For every pair $m, n \geq 0$, $\tau^n T_{\mathfrak{g}}(t_+^{m-n+1} \frac{\partial}{\partial t_+}) - \tau^m T_{\mathfrak{g}}(t_-^{n-m+1} \frac{\partial}{\partial t_-})$ kills ε^λ and
- (ii) $T_{\mathfrak{g}}(t_+ \frac{\partial}{\partial t_+})(\varepsilon^\lambda) = -\frac{c(\lambda)}{2(\ell+h)} \varepsilon^\lambda$.

As to (i), if we substitute

$$T_{\mathfrak{g}}(t_+^{m-n+1} \frac{\partial}{\partial t_+}) = -\frac{1}{2(\ell+h)} \sum_{j \in \mathbb{Z}} \sum_{\kappa} : X_{\kappa} t_+^{m-n-j} \circ X_{\kappa} t_+^j :$$

and do likewise for $T_{\mathfrak{g}}(t_-^{n-m+1} \frac{\partial}{\partial t_-})$, this assertion follows easily.

For (ii) we first observe that $T_{\mathfrak{g}}(t_+ \frac{\partial}{\partial t_+})$ preserves the grading of $\mathbb{H}_\ell^+(V_\lambda)$ and acts on $\mathbb{H}_\ell^+(V_\lambda)_0 = V_\lambda$ as $-(2\ell + 2\check{h})^{-1} \sum_{\kappa} X_{\kappa} \circ X_{\kappa}$. This is just multiplication by $-\frac{c(\lambda)}{2(\ell+h)}$. If $u \in \mathbb{H}_\ell^+(V_\lambda)_{-d}$ is written $u = Y_r t_+^{-k_r} \circ \dots \circ Y_1 t_+^{-k_1} \circ v$ with $v \in V_\lambda$, $Y_\rho \in \mathfrak{g}$, $d = k_r + \dots +$

k_1 , then $T_{\mathfrak{g}}(t_+ \frac{\partial}{\partial t_+})(u) = -du + Y_r t_+^{-k_r} \circ \dots \circ Y_1 t_+^{-k_1} \circ T_{\mathfrak{g}}(t_+ \frac{\partial}{\partial t_+})(v) = (-d - \frac{c(\lambda)}{2(\ell+h)})u$. Since $t_+ \frac{\partial}{\partial t_+} \tau^d = d\tau^d$, it follows that $\varepsilon_d^\lambda \tau^d$ is an eigenvector of $T_{\mathfrak{g}}(t_+ \frac{\partial}{\partial t_+})$ with eigenvalue $-\frac{c(\lambda)}{2(\ell+h)}$. \square

Theorem 4.5. *The $k[[\tau]]$ -homomorphism*

$$E = (E_\lambda)_\lambda : \mathcal{H}_\ell(V/\Delta) = \mathbb{H}_\ell(V)[[\tau]] \rightarrow \bigoplus_{\lambda \in P_\ell} \mathbb{H}_\ell(\tilde{V}_{\lambda, \lambda^*})[[\tau]],$$

$$U = \sum_{k \geq 0} u_k \tau^k \mapsto \sum_{\lambda \in P_\ell} U \varepsilon^\lambda := \sum_{\lambda \in P_\ell} \sum_{k, d \geq 0} u_k \otimes \varepsilon_d^\lambda \tau^{k+d}$$

is also a map of $A_{\mathfrak{g}}$ -representations if we let $A_{\mathfrak{g}}$ act on the left hand side via the inclusion $A \subset B[[\tau]]$. The resulting $k[[\tau]]$ -homomorphism of covariants,

$$\mathcal{H}_\ell(V/\Delta)_C \cong \bigoplus_{\lambda \in P_\ell} \mathbb{H}_\ell(\tilde{V}_{\lambda, \lambda^*})_{B_{\mathfrak{g}}}[[\tau]] = \bigoplus_{\lambda \in P_\ell} \mathbb{H}_\ell(\tilde{V}_{\lambda, \lambda^*})_{\tilde{C}}[[\tau]]$$

is an isomorphism. In particular, $\mathcal{H}_\ell(V/\Delta)_C$ is a free $k[[\tau]]$ -module. Moreover, covariant derivation in $\mathcal{H}_\ell(V/\Delta)_C$ by $\tau \frac{d}{d\tau}$ respects each summand $\mathbb{H}_\ell(\tilde{V}_{\lambda, \lambda^*})_{\tilde{C}}[[\tau]]$ and acts there as the first order differential operator $\tau \frac{d}{d\tau} + \frac{c(\lambda)}{2(\ell+h)}$.

Proof. The first statement is immediate from Lemma 4.4. So the map on covariants is defined and is $k[[\tau]]$ -linear. If we reduce modulo τ we get the map

$$u \in \mathbb{H}_\ell(V)_{A_{\mathfrak{g}}} \mapsto \sum_{\lambda \in P_\ell} u \otimes \varepsilon_0^\lambda \in \bigoplus_{\lambda \in P_\ell} \mathbb{H}_\ell(\tilde{V}_{\lambda, \lambda^*})_{B_{\mathfrak{g}}}.$$

We recognize its domain and range as $\mathcal{H}_\ell(V/\Delta)_C$ and $\bigoplus_{\lambda \in P_\ell} \mathbb{H}_\ell(\tilde{V}_{\lambda, \lambda^*})_{\tilde{C}}$ and we observe that the map itself is just the inverse of the isomorphism of Proposition 4.1. Since the range is a free $k[[\tau]]$ -module, this implies that the $k[[\tau]]$ -homomorphism of covariants is an isomorphism.

According to Corollary 2.4 covariant derivation with respect to $\tau \frac{d}{d\tau}$ in $\mathcal{H}_\ell(V/\Delta)_C$ is defined by means of a k -derivation D of \mathcal{A} which lifts $\tau \frac{d}{d\tau}$: if we write $D = \tau \frac{d}{d\tau} + \sum_{n \geq 0} \tau^n D_n$, where D_n is a vector field on the smooth part of C , then the covariant derivative is induced by $T_{\mathfrak{g}}(D) = \tau \frac{d}{d\tau} + \sum_{n \geq 0} \tau^n T_{\mathfrak{g}}(D_n)$ acting on $\mathbb{H}_\ell(V)[[\tau]] = \mathcal{H}_\ell(V/\Delta)_C$. From the last clause of Lemma 4.4 we get that when $U \in \mathbb{H}_\ell(V)[[\tau]]$,

$$\begin{aligned} T_{\mathfrak{g}}(D)E_\lambda(U) &= T_{\mathfrak{g}}(D)(U\varepsilon^\lambda) = \\ &= T_{\mathfrak{g}}(D)(U)\varepsilon^\lambda - \frac{c(\lambda)}{2(\ell+h)}U\varepsilon^\lambda = E_\lambda T_{\mathfrak{g}}(D)(U) - \frac{c(\lambda)}{2(\ell+h)}E_\lambda(U). \end{aligned}$$

Since $T_{\mathfrak{g}}(D)$ acts on $\mathbb{H}_\ell(\tilde{V}_{\lambda, \lambda^*})_{\tilde{C}}[[\tau]]$ simply as derivation by $\tau \frac{d}{d\tau}$, the last clause follows. \square

Corollary 4.6. *The monodromy of the WZW connection acting on $\mathcal{H}_\ell(V/\Delta)_C$ has finite order and acts in the summand $\mathbb{H}_\ell(\tilde{V}_{\lambda, \lambda^*})_{\tilde{C}}$ as multiplication by $\exp(\pi\sqrt{-1}\frac{c(\lambda)}{\ell+h})$.*

Proof. The multivalued flat sections of $\mathcal{H}_\ell(V/\Delta)_C$ decompose under E as a direct sum labeled by P_ℓ . The summand corresponding to P_ℓ is the set of solutions of the differential equation $\tau \frac{d}{d\tau}U + \frac{c(\lambda)}{2(\ell+h)}U = 0$. These are clearly of the form $u\tau^{-c(\lambda)/2(\ell+h)}$ with $u \in \mathbb{H}_\ell(\tilde{V}_{\lambda, \lambda^*})_{\tilde{C}}$. If we let τ run over the unit circle, then we see that the monodromy is as asserted. Finally, we observe that $\frac{c(\lambda)}{\ell+h} \in \mathbb{Q}$. \square

Verlinde formula. Theorem 4.5 and its Corollary 4.6 extend to the case where the base S is an arbitrary k -variety as in Proposition 4.1 and the smoothing is arbitrary. This is based on a versality argument, which shows that our smoothing construction is not so special as it may appear. To be concrete, suppose that we are given a family of smooth curves $\tilde{\mathcal{C}} \rightarrow S$ with pairwise disjoint sections $\{S_i\}_{i \in I} \cup \{S_-, S_+\}$ and a generator t_{\pm} of the ideal defining S_{\pm} in the formal completion along \mathcal{C} . Assume that the complement of the union of these sections is affine over S and that this family is *versal* as a family of pointed curves. Then $\tilde{\mathcal{C}} \rightarrow S$ factors through a family whose fibers have a single double point: $\tilde{\mathcal{C}} \rightarrow \mathcal{C} \rightarrow S$, where $\tilde{\mathcal{C}}$ is obtained by identifying the sections S_+ and S_- . We regard the latter as endowed with the sections $\{S_i\}_{i \in I}$ so that $(\mathcal{C}, \{S_i\}_{i \in I}) \rightarrow S$ is a family of pointed curves. Then the smoothing of \mathcal{C} over $S \times \Delta$ defined by t_{\pm} with its sections $\{S_i \times \Delta\}_{i \in I}$ will be a versal (as a family of pointed curves) so that any deformation of $(\mathcal{C}, \{S_i\}_{i \in I}) \rightarrow S$ is obtained from this one by means of a base change. As any two versal deformations are isomorphic, it follows that 4.5 and 4.6 apply to any versal deformation of $(\mathcal{C}, \{S_i\}_{i \in I}) \rightarrow S$.

The preceding leads to a formula of the dimension of a conformal block. By theorem 4.5, or rather the generalization discussed above, the dimension of a conformal block stays the same under a degeneration. Since every pointed curve degenerates into one with ordinary double points whose normalization consists of curves of genus zero, it suffices to do the computation for such a degenerate curve. But then we may invoke 4.1 to reduce to the case of a smooth rational curve, which can be dealt with using our discussion of the genus zero case. We can even arrange that the support of the representation map meets every component in at most three points. A more refined approach involves the notion of a fusion ring [1].

Remark 4.7. In case $k = \mathbb{C}$, one can work in the complex-analytic category. Then the fact that the singularity is formally regular singular ensures that the flat multivalued sections converge on simply connected sectors based at o (see for instance [7]) so that the monodromy has the expected interpretation. If $C - \{S_0\}$ is smooth, then we have an associated fibration over the punctured unit disk whose monodromy is given by a Dehn twist along a vanishing circle on the general fiber. In terms of the setting of Remark 3.9: an isotopy class α of embedded circles in Σ defines the class of a Dehn twist $D_{\alpha} \in \Gamma(\Sigma, I)$ and the action of D_{α} on $\mathbb{P}(\Sigma, I, V)$ is given by the above formula (hence is of finite order).

REFERENCES

- [1] A. Beauville: *Conformal blocks, fusion rules and the Verlinde formula*, Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry, 75–96, Israel Math. Conf. Proc., 9, Bar-Ilan Univ., Ramat Gan (1996).
- [2] A.A. Beilinson, Yu.I. Manin, V.V. Schechtman: *Sheaves of the Virasoro and Neveu-Schwarz algebras*, in: *K-theory, arithmetic and geometry (Moscow, 1984–1986)*, 52–66, Lecture Notes in Math., 1289, Springer, Berlin (1987).
- [3] V.G. Kac: *Infinite-dimensional Lie algebras*, 3rd ed. Cambridge University Press, Cambridge (1990), xxii+400 pp.
- [4] V.G. Kac, A.K. Raina: *Bombay lectures on highest weight representations of infinite-dimensional Lie algebras*, Advanced Series in Mathematical Physics, 2. World Scientific Publishing Co., Inc., Teaneck, NJ (1987), xii+145 pp.
- [5] T. Kohno: *Integrable connections related to Manin and Schechtman's higher braid groups*, Ill. J. Math. 34 (1990), 476–484.
- [6] E.J.N. Looijenga: *Isolated singular points on complete intersections*, LMS Lecture Note Series, 77. Cambridge University Press, Cambridge (1984), xi+200 pp.
- [7] B. Malgrange: *Sur les points singuliers des équations différentielles*, L'Enseignement Mathématique 20 (1974), 147–176.
- [8] Y. Tsuchimoto: *On the coordinate-free description of the conformal blocks*, J. Math. Kyoto Univ. 33 (1993), no. 1, 29–49.

- [9] Ch. Sorger: *La formule de Verlinde*, Sém. Bourbaki Exp. 794, Astérisque 237 (1996), 87–114.
- [10] A. Tsuchiya, K. Ueno, Y. Yamada: *Conformal field theory on universal family of stable curves with gauge symmetries*. Integrable systems in quantum field theory and statistical mechanics, 459–566, Adv. Stud. Pure Math., 19, Academic Press, Boston, MA (1989).

MATHEMATISCH INSTITUUT, UNIVERSITEIT UTRECHT, P.O. BOX 80.010, NL-3508 TA UTRECHT,
NEDERLAND

E-mail address: looi jeng@math.uu.nl