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SPECIAL GEOMETRY IN HYPERMULTIPLETS

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ABSTRACT

We give a detailed analysis of pairs of vector and hypermultiplet theories with $N = 2$ supersymmetry in four spacetime dimensions that are related by the (classical) mirror map. The symplectic reparametrizations of the special Kähler space associated with the vector multiplets induce corresponding transformations on the hypermultiplets. We construct the $\mathrm{Sp}(1)\times\mathrm{Sp}(n)$ one-forms in terms of which the hypermultiplet couplings are encoded and exhibit their behaviour under symplectic reparametrizations. Both vector and hypermultiplet theories allow vectorial central charges in the supersymmetry algebra associated with integrals over the Kähler and hyper-Kähler forms, respectively. We show how these charges and the holomorphic BPS mass are related by the mirror map.

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1 Introduction

In four spacetime dimensions with $N = 2$ supersymmetry, there exist two inequivalent matter supermultiplets. One is the vector multiplet, which comprises states of helicity ± 1, $\pm\frac{1}{2}$ and 0, the other is the hypermultiplet with only states of helicity $\pm\frac{1}{2}$ and 0. These supermultiplets appear in effective low-energy field theories for type-II string compactifications on a Calabi-Yau three-fold or for $N = 2$ heterotic string compactifications on $K3 \times T_2$. When considering superstring compactifications on a pair of mirror Calabi-Yau spaces [1] the interesting phenomenon arises that the vector multiplets and the hypermultiplets in the four-dimensional effective action are interchanged. The same interchange is effected when compactifying type-IIA and type-IIB supergravity on the same Calabi-Yau manifold. This implies that, at least in string perturbation theory, the special Kähler and the quaternionic moduli spaces parametrized by the scalars of the vector multiplets and the hypermultiplets, are interchanged. In [2] this interchange was studied in detail, at the level of both supergravity and string theory, by reducing to three dimensions, where differences in helicity content of the multiplets no longer play a role. In that case the target space factorizes into two quaternionic spaces, corresponding to the two sets of inequivalent supermultiplets [3]. The map from special Kähler to quaternionic manifolds was called the **c** map, because of its similarity to Calabi's construction for hyper-Kähler metrics on cotangent bundles of a Kähler manifold [4]. Through string duality [5], there exist further relations between the vector multiplet and hypermultiplet sectors.

In the rigid supersymmetry limit, a quaternionic manifold reduces to a hyper-Kähler manifold. In this paper we study the hyper-Kähler manifolds that are in the image of the **c** map, but we expect that our results can be generalized rather straightforwardly to quaternionic manifolds in the context of an appropriate superconformal multiplet calculus. In this way we cover quite a large class of hyper-Kähler manifolds. While the form of the metric in 'special coordinates' has been known for quite some time [2, 6], we study the behaviour of various geometric quantities, such as the $\mathrm{Sp}(1)\times\mathrm{Sp}(n)$ one-forms in terms of which the hyper-Kähler manifold is defined [7], under the diffeomorphisms induced by the symplectic reparametrizations in the underlying special Kähler space. Our hope is that, eventually, this will help us to constrain the perturbative string corrections for the hypermultiplets in type-II string compactifications, guided by what we know from the vector multiplet side. In other words, we intend to explore the consequences of special geometry for the corresponding systems of hypermultiplets.

In practical applications hyper-Kähler and quaternionic manifolds are difficult to deal with because they are usually encoded in terms of a $(4n)$-dimensional metric, whose equivalence classes are provided by general diffeomorphisms. This in contradistinction to the special Kähler manifolds, which are conveniently encoded in terms of a holomorphic function [8] and whose equivalence classes are described by a more restricted group of reparametrizations, due to supersymmetry and gauge invariance. These reparametrizations are associated with symplectic matrices, which act on the (anti-)selfdual components of the field strenghts. Fixed points of these transformations correspond to invariances of the equations of motion. Such duality invariances have a long history in extended supergravity theories [9, 10]. As it turns out the description in terms of holomorphic functions and the symplectic reparametrizations are essential ingredients in the definition of 'special geometry' [11, 12]. They were important tools in the study of Calabi-Yau manifolds [13], nonperturbative phenomena in supersymmetric gauge theories [14], as well as in studies of low-energy effective actions for vector multiplets arising from $N = 2$ supersymmetric string compactifications and tests of string duality [15].

In the past, important information on quaternionic manifolds was obtained via the **c** map. For instance, it was argued in [2, 16] that the **c** map is closely related to the method employed in [17] for the classification of normal quaternionic spaces (i.e., quaternionic spaces that admit a solvable transitive group of isometries). In [18] a general analysis was presented of the isometry structure of the quaternionic spaces in the image of the **c** map. With the exception of the quaternionic projective spaces, all normal quaternionic spaces are contained in the image of the **c** map. Part of the solvable algebra of isometries coincides with the duality invariances of the underlying special Kähler spaces. The symplectic reparametrizations of the special Kähler space now correspond to a subclass of the diffeomorphisms of the quaternionic space whose effect can be fully incorporated in the holomorphic function on which the quaternionic or hyper-Kähler metric depends.

In view of these applications we will exhibit the effect of the symplectic reparametrizations on various quantities that play a role on the hyper-Kähler side. As alluded to above, among those are the $\mathrm{Sp}(1)\times\mathrm{Sp}(n)$ one-forms, which we determine explicitly from the Kähler side. In order to do this we have to cast the results of [7] on the general hypermultiplet Lagrangians in a different form. Under the symplectic reparametrizations the hyper-Kähler forms transform covariantly. This means that the duality invariances on the Kähler side take the form of triholomorphic isometries of the hyper-Kähler manifold.

We also consider the possible central charges that may be generated as surface terms in the anticommutator of the supersymmetry charges. It turns out that the vector multiplets generate the scalar and pseudoscalar charges associated with the holomorphic BPS mass and a vectorial central charge expressed in terms of the integral over the pull-back of the Kähler form. The hypermultiplets on the other hand only exhibit vectorial charges expressed as integrals over the pull-back of the hyper-Kähler forms. In three dimensions the central charges associated with these two multiplets can be related via the classical mirror map.

We should add that our work has no direct implication for recent studies of nonperturbative supersymmetric gauge dynamics in three dimensions [19]. In those studies one deals with effective actions based on a nonabelian gauge theory, where one of the three spatial dimensions is compactified to a circle of finite or zero length. However, in our work we start from a generic four-dimensional abelian gauge theory (which may be related to the effective action of some underlying supersymmetric gauge theory) without paying attention to its possible dynamic origin. Upon compactification of one dimension to a circle, this theory does not fully capture the dynamics of the underlying theory associated with the circle compactification. So within this setting we have to content ourselves with exploiting the relation between two classes of four-dimensional supersymmetric theories, based on vector multiplets and hypermultiplets, respectively.

This paper is organized as follows. In section 2 we introduce the general action for vector multiplets and the corresponding symplectic reparametrizations. In section 3 we discuss the reduction of these theories to three dimensions and elucidate certain features relevant for finite compactification radius. Furthermore the geometry of the resulting hyper-Kähler target space is studied, including a classification of its isometries. In section 4 we present a short derivation of the general supersymmetric action and transformation rules for hypermultiplets, in a slightly different setting than in [7]. In section 5 we discuss the emergence of an extra SU(2) symmetry group, contained in the automorphism group of the supersymmetry algebra, when descending to three dimensions. We give a detailed treatment of the classical mirror map and use it to determine explicit expressions for the one-forms in terms of which the hyper-Kähler manifold is defined, as well as other quantities of interest. These one-forms transform covariantly under the symplectic reparametrizations induced by the underlying Kähler geometry. We close with a discussion of the various central charges that may emerge in theories with vector multiplets and hypermultiplets and we exhibit their relation under the mirror map.

2 Vector multiplets

We start by a discussion of the features that are relevant for this paper of rigid $N = 2$ supersymmetric systems in four dimensions consisting of vector multiplets. In particular we emphasize the symplectic reparametrizations of systems of abelian vector multiplets. As is well known [8], the general supersymmetric Lagrangian for $N = 2$ vector multiplets is encoded in terms of a holomorphic function $F(X)$. Here the arguments X^I $(I = 1, 2, \ldots, n)$ denote the complex scalar fields of the n vector multiplets. In the context of this paper we restrict ourselves to abelian vector multiplets. The corresponding Lagrangian can be read off from [20] and, after elimination of auxiliary fields and Fierz reordering, is equal to

$$\begin{aligned}
4\pi \mathcal{L} \;=\;& i\Big(\partial_\mu F_I\, \partial^\mu \bar{X}^I - \partial_\mu \bar{F}_I\, \partial^\mu X^I\Big) \\
&+\tfrac{1}{4} i\Big(F_{IJ} F^{-I}_{\mu\nu} F^{-J\,\mu\nu} - \bar{F}_{IJ} F^{+I}_{\mu\nu} F^{+J\,\mu\nu}\Big) \\
&-\tfrac{1}{4} N_{IJ}\Big(\bar{\Omega}^{iI} \partial\!\!\!/ \Omega_i^J + \bar{\Omega}_i^I \partial\!\!\!/ \Omega^{iJ}\Big) - \tfrac{1}{4} i\Big(\bar{\Omega}_i^I \partial\!\!\!/ F_{IJ} \Omega^{iJ} - \bar{\Omega}^{iI} \partial\!\!\!/ \bar{F}_{IJ} \Omega_i^J\Big) \\
&-\tfrac{1}{8} i\Big(F_{IJK} \bar{\Omega}_i^I\, \sigma^{\mu\nu} F^{-J}_{\mu\nu}\, \Omega_j^K\, \varepsilon^{ij} - \bar{F}_{IJK} \bar{\Omega}^{iI}\, \sigma^{\mu\nu} F^{+J}_{\mu\nu}\, \Omega^{jK}\, \varepsilon_{ij}\Big) \\
&+\tfrac{1}{96} i\Big(F_{IJKL} + i N^{MN} (2 F_{MIK} F_{JLN} - \tfrac{1}{2} F_{MIJ} F_{KLN})\Big)\, \bar{\Omega}_i^I \sigma_{\mu\nu} \Omega_j^J \varepsilon^{ij}\, \bar{\Omega}_k^K \sigma^{\mu\nu} \Omega_l^L \varepsilon^{kl} \\
&-\tfrac{1}{96} i\Big(\bar{F}_{IJKL} - i N^{MN} (2 \bar{F}_{MIK} \bar{F}_{JLN} - \tfrac{1}{2} \bar{F}_{MIJ} \bar{F}_{KLN})\Big)\, \bar{\Omega}^{iI} \sigma_{\mu\nu} \Omega^{jJ} \varepsilon_{ij}\, \bar{\Omega}^{kK} \sigma^{\mu\nu} \Omega^{lL} \varepsilon_{kl} \\
&-\tfrac{1}{16} N^{MN} F_{MIJ} \bar{F}_{KLN}\, \bar{\Omega}^{iK} \Omega^{jL}\, \bar{\Omega}_i^I \Omega_j^J \,,
\end{aligned} \tag{2.1}$$

where we use the notation

$$N_{IJ} = -i F_{IJ} + i \bar{F}_{IJ} \,, \qquad N^{IJ} \equiv [N^{-1}]^{IJ} \,, \tag{2.2}$$

with $F_{I_1 \cdots I_k}$ denoting the k-th derivative of F. The fermion fields Ω carry a chiral SU(2) index, $i, j, \ldots = 1, 2$. The spinors with lower SU(2) index, Ω_i^I, are of positive chirality, i.e. $\gamma^5 \Omega_i^I = \Omega_i^I$; the spinors with upper SU(2) index, Ω^{iI}, are of negative chirality. The tensors $F^{\pm}_{\mu\nu}$ are the (anti-)selfdual components of the gauge fields. In the free theory the holomorphic function $F(X)$ is quadratic, and its second derivatives determine the coupling constants g_{IJ} and generalized theta angles θ_{IJ} according to

$$F_{IJ} = \frac{\theta_{IJ}}{2\pi} + i \frac{4\pi}{g^2_{IJ}} \,. \tag{2.3}$$

The nonlinear sigma model contained in (2.1) exhibits an interesting geometry. The complex scalars X^I parametrize an n-dimensional target space with metric $g_{I\bar{J}} = N_{IJ}$. This is a Kähler space: its metric equals $g_{I\bar{J}} = \partial^2 K(X, \bar{X}) / \partial X^I\, \partial \bar{X}^J$, with Kähler potential

$$K(X, \bar{X}) = i X^I\, \bar{F}_I(\bar{X}) - i \bar{X}^I\, F_I(X) \,. \tag{2.4}$$

The resulting geometry is known as *special geometry.* Nonvanishing components of the Levi-Civitá connection and the curvature tensor are given by

$$\Gamma^I_{JK} \;\equiv\; g^{I\bar{L}} \partial_J g_{K\bar{L}} = -i N^{IL} F_{JKL} \,,$$

$$R^I{}_{JK}{}^L \;\equiv\; g^{L\bar L}\,\partial_{\bar L}\Gamma^I_{JK} = -N^{IP}N^{LQ}N^{MN}\,\bar F_{PQM}\,F_{NJK}\,. \tag{2.5}$$

It is possible to choose different coordinates and view the X^I as holomorphic 'sections' $X^I(z)$ [21]. As it is straightforward to cast our results in such a coordinate-independent form, we keep writing them in terms of the X^I, which are sometimes called *special* coordinates[1].

We also record the supersymmetry transformation rules for the vector multiplet components (after elimination of the auxiliary fields),

$$\begin{aligned}
\delta X^I &= \bar\epsilon^i\Omega^I_i\,,\\
\delta A^I_\mu &= \varepsilon^{ij}\,\bar\epsilon_i\gamma_\mu\Omega^I_j + \varepsilon_{ij}\,\bar\epsilon^i\gamma_\mu\Omega^{jI}\,,\\
\delta\Omega^I_i + \Gamma^I_{JK}\,\delta X^J\,\Omega^K_i &= 2\partial\!\!\!/ X^I\epsilon_i - i\varepsilon_{ij}\sigma^{\mu\nu}\epsilon^j\,N^{IJ}\mathcal{G}^-_{\mu\nu J} + \tfrac12 iN^{IJ}\bar F_{JKL}\bar\Omega^{kK}\Omega^{lL}\,\varepsilon_{ik}\varepsilon_{jl}\,\epsilon^j\,,
\end{aligned} \tag{2.6}$$

where Γ denotes the Kähler connection and $\mathcal{G}^-_{\mu\nu I}$ is an anti-selfdual tensor defined by

$$\mathcal{G}^-_{\mu\nu I} = iN_{IJ}F^{-J}_{\mu\nu} - \tfrac14 F_{IJK}\bar\Omega^J_i\sigma_{\mu\nu}\Omega^K_j\,\varepsilon^{ij}\,. \tag{2.7}$$

The significance of the tensor (2.7) and of the particular form of the spinor transformation in (2.6), will be discussed shortly. The supersymmetry transformation parameters, ϵ^i and ϵ_i, are of positive and negative chirality, respectively. They transform as doublets under the chiral $\mathrm{SU}(2)_\mathrm{R}$ group, which belongs to the automorphism group of the supersymmetry algebra. Observe that both the Lagrangian (2.1) and transformation rules (2.6) are consistent with respect to $\mathrm{SU}(2)_\mathrm{R}$, but not, in general, with respect to the $\mathrm{U}(1)_\mathrm{R}$ subgroup of the automorphism group.

It is possible for two different functions $F(X)$ to describe the same theory. The equivalence is provided by symplectic reparametrizations associated with the group $\mathrm{Sp}(2n;\mathbf{Z})$. The discrete nature of this group is tied to nonperturbative effects, as the lattice of electric and magnetic charges should be left invariant. At the perturbative level, the group is $\mathrm{Sp}(2n;\mathbf{R})$. The equivalence follows from rotating the Bianchi identities, $\partial^\mu(F^+ - F^-)^I_{\mu\nu} = 0$, and the field equations for the vector fields, $\partial^\mu(G^+ - G^-)_{\mu\nu I} = 0$, by means of a real symplectic $(2n)$-by-$(2n)$ matrix. The tensors $G^\pm_{\mu\nu I}$ are obtained from the Lagrangian (2.1), and read

$$G^-_{\mu\nu I} = F_{IJ}\,F^{-J}_{\mu\nu} - \tfrac14 F_{IJK}\bar\Omega^J_i\sigma_{\mu\nu}\Omega^K_j\,\varepsilon^{ij}\,, \tag{2.8}$$

[1] Special coordinates are singled out by supersymmetry. They are the lowest component of an N=2 chiral reduced multiplet. Strictly speaking the term 'special geometry' was proposed for systems of vector multiplets with *local* supersymmetry [11]. To make a distinction one occasionally uses the term 'rigid' special geometry.

while $G^+_{\mu\nu I}$ is related to $G^-_{\mu\nu I}$ by complex conjugation. The symplectic rotation between equations of motion and Bianchi identities is induced by

$$\begin{pmatrix} F^{\pm I}_{\mu\nu} \\ G^{\pm}_{\mu\nu I} \end{pmatrix} \longrightarrow \begin{pmatrix} U & Z \\ W & V \end{pmatrix} \begin{pmatrix} F^{\pm I}_{\mu\nu} \\ G^{\pm}_{\mu\nu I} \end{pmatrix}, \qquad (2.9)$$

where $U^I_{\ J}$, $V_I^{\ J}$, W_{IJ} and Z^{IJ} are constant real $n \times n$ submatrices, subject to certain constraints such that the total matrix is an element of $\mathrm{Sp}(2n;\mathbf{R})$. Consistency with (2.8) requires the symmetric tensor F_{IJ} to change as $F_{IJ} \to (V_I{}^K F_{KL} + W_{IL})\,[(U+ZF)^{-1}]^L{}_J$. This is achieved by the transformations of the scalar fields, implied by

$$\begin{pmatrix} X^I \\ F_I \end{pmatrix} \longrightarrow \begin{pmatrix} \tilde X^I \\ \tilde F_I \end{pmatrix} = \begin{pmatrix} U & Z \\ W & V \end{pmatrix} \begin{pmatrix} X^I \\ F_I \end{pmatrix}. \qquad (2.10)$$

In this transformation we include a transformation of F_I. Because the transformation is symplectic, one can show that the new quantities $\tilde F_I$ can be written as the derivatives of a new function $\tilde F(\tilde X)$. The new equations of motion after performing (2.9) and (2.10) then follow straightforwardly from the Lagrangian based on $\tilde F$ (provided we perform suitable redefinitions of the spinor fields, which we will specify shortly).

It is possible to integrate (2.10) and determine the new function $\tilde F$,

$$\begin{aligned} \tilde F(\tilde X) \;&=\; F(X) - \tfrac12 X^I F_I(X) \qquad (2.11) \\ &\quad + \tfrac12 (U^{\mathrm T} W)_{IJ} X^I X^J + \tfrac12 (U^{\mathrm T} V + W^{\mathrm T} Z)_I{}^J X^I F_J + \tfrac12 (Z^{\mathrm T} V)^{IJ} F_I F_J\,, \end{aligned}$$

up to a constant and terms linear in the $\tilde X^I$ (which give no contribution to the Lagrangian (2.1)). In the coupling to supergravity, where the function must be homogeneous of second degree, such terms are excluded.[2] Obviously $F(X)$ does not transform as a function. Such quantities turn out to be rare. Examples are the holomorphic function $F(X) - \frac12 X^I F_I(X)$ and the Kähler potential (2.4). For a discussion of this, we refer to [22, 23]. In practical situations the expression (2.11) is not always useful, as it requires substituting $\tilde X^I$ in terms of X^I, or vice versa. When F remains the same, $\tilde F(\tilde X) = F(\tilde X)$, the theory is *invariant* under the corresponding transformations. These invariances are often called duality invariances and they have been studied extensively in the context of extended supergravity theories [9, 10]. The space of inequivalent couplings of n abelian vector supermultiplets is equal to the space of holomorphic functions of n variables, divided by the $\mathrm{Sp}(2n;\mathbf{R})$ group. This group does not act freely on the space

[2] The terms linear in $\tilde X$ in (2.11) are associated with constant translations in $\tilde F_I$ in addition to the symplectic rotation shown in (2.10). Likewise one may introduce constant shifts in $\tilde X^I$. Henceforth we ignore these shifts. Constant contributions to $F(X)$ are always irrelevant. We note also that terms quadratic in the X^I with real coefficients correspond to total divergences in the action.

of these functions. There are fixed points whenever the equations of motion exhibit duality symmetries. It is not easy to find solutions of $\tilde F(\tilde X) = F(\tilde X)$, unless one considers infinitesimal transformations. In that case the condition reads [10]

$$C_{IJ}\,X^I X^J - 2B^I{}_J\,X^J F_I + D^{IJ}\,F_I F_J = 0\,, \tag{2.12}$$

where the constant matrices $B^I{}_J$, C_{IJ} and D^{IJ} parametrize the infinitesimal form of the $\mathrm{Sp}(2n;\mathbf{R})$ matrix, according to $U \approx \mathbf{1} + B$, $V \approx \mathbf{1} - B^{\mathrm{T}}$, $W \approx C$ and $Z \approx -D$. For finite transformations, a more convenient method is to verify that the substitution $X^I \to \tilde X^I$ into the derivatives $F_I(X)$ correctly induces the symplectic transformations on the 'periods' (X^I, F_J).

It is convenient to employ quantities that transform as tensors under symplectic reparametrization. Before considering some such tensors let us introduce the following notation,

$$\begin{aligned}
\frac{\partial \tilde X^I}{\partial X^J} &\equiv \mathcal{S}^I{}_J(X) = U^I{}_J + Z^{IK}\,F_{KJ}(X)\,,\\
\mathcal{Z}^{IJ}(X) &\equiv [\mathcal{S}^{-1}(X)]^I{}_K\,Z^{KJ}\,.
\end{aligned} \tag{2.13}$$

The holomorphic quantity $\mathcal{Z}^{IJ}$ is symmetric in I and J, because $Z\,U^{\mathrm{T}}$ is a symmetric matrix as a consequence of the fact that U and Z are submatrices of the symplectic matrix indicated in (2.9).

After these definitions we note the following transformation rules,

$$\begin{aligned}
\tilde F_{IJ} &= (V_I{}^K F_{KL} + W_{IL})\,[\mathcal{S}^{-1}]^L{}_J\,,\\
\tilde N_{IJ} &= N_{KL}\,[\bar{\mathcal{S}}^{-1}]^K{}_I\,[\mathcal{S}^{-1}]^L{}_J\,,\\
\tilde N^{IJ} &= N^{KL}\,\bar{\mathcal{S}}^I{}_K\,\mathcal{S}^J{}_L\,,\\
\tilde F_{IJK} &= F_{MNP}\,[\mathcal{S}^{-1}]^M{}_I\,[\mathcal{S}^{-1}]^N{}_J\,[\mathcal{S}^{-1}]^P{}_K\,,\\
\tilde\Omega^I_i &= \mathcal{S}^I{}_J\,\Omega^J_i\,, \qquad \tilde\Omega^{iI} = \bar{\mathcal{S}}^I{}_J\,\Omega^{iJ}\,.
\end{aligned} \tag{2.14}$$

The first three quantities do not remain manifestly symmetric in I, J, but this symmetry is preserved owing to the symplectic nature of the transformation. The Kähler connection transforms as a mixed tensor but also acts as a connection for symplectic reparametrizations, as follows from

$$\begin{aligned}
\tilde\Gamma^I_{JK} &= \bar{\mathcal{S}}^I{}_L\,\Gamma^L_{MN}\,[\mathcal{S}^{-1}]^M{}_J\,[\mathcal{S}^{-1}]^N{}_K\\
&= -\partial_M\mathcal{S}^I{}_N\,[\mathcal{S}^{-1}]^M{}_J\,[\mathcal{S}^{-1}]^N{}_K + \mathcal{S}^I{}_L\,\Gamma^L_{MN}\,[\mathcal{S}^{-1}]^M{}_J\,[\mathcal{S}^{-1}]^N{}_K\,.
\end{aligned} \tag{2.15}$$

From the field strengths $F^{\pm I}$ and $G_I^{\pm}$ we can construct tensors that transform as symplectic vectors. An example is the tensor $\mathcal{G}$ that we defined in (2.7), which follows from[3]

$$\mathcal{G}^-_{\mu\nu I} = G^-_{\mu\nu I} - \bar{F}_{IJ} F^{-J}_{\mu\nu} \,, \tag{2.16}$$

upon substitution of (2.8). This particular combination transforms under symplectic reparametrizations as

$$\tilde{\mathcal{G}}^-_{\mu\nu I} = \mathcal{G}^-_{\mu\nu I} \, [\bar{\mathcal{S}}^{-1}]^J{}_I \,. \tag{2.17}$$

With this result one can verify that the spinor transformation rule in (2.6) is manifestly covariant under symplectic reparametrizations. The same is true for the supersymmetry variation of the scalar field, but not for the variation of the vector field. This is not surprising, because the symplectic reparametrizations are not defined for the gauge fields. The reader may also verify that the Lagrangian (2.1) is invariant under symplectic reparametrizations, but only up to terms proportional to the equations of motion of the vector fields.

3 Reduction to three spacetime dimensions

In this section we reduce the general Lagrangian (2.1) for (abelian) vector multiplets to three spacetime dimensions. This is done by compactifying one of the spatial dimensions (say, the one parametrized by x^3) on a circle with radius R and suppressing all the modes that depend nontrivially on x^3. The four-dimensional gauge fields then decompose into three-dimensional gauge fields A^I_μ and additional scalar fields $A^I \equiv A^I_3$. If we impose the Bianchi identity in three dimensions through addition of a Lagrange multiplier term proportional to $B_I \epsilon^{\mu\nu\rho} \partial_\mu F^I_{\nu\rho}$ and integrate out the field strength, the degrees of freedom of the four-dimensional gauge field are captured in the two scalars A^I and B_I.

Before turning to more explicit results we deal with the consequences of the dimensional reduction for the fermions, which, in four spacetime dimensions, are four-component Majorana spinors. When reducing to three spacetime dimensions, every spinor decomposes into two two-component spinors. In order to discuss this systematically one decomposes the Clifford algebra of the gamma matrices in four dimensions into two mutually *commuting* Clifford algebras: one is the algebra generated by the gamma matrices appropriate to three dimensions and the second one is the algebra generated by γ^3. This is accomplished by defining

$$\gamma^\mu = \gamma^\mu_{(4)} \tilde{\gamma} \,, \quad (\mu = 0, 1, 2) \tag{3.1}$$

[3]Replacing $\bar{F}_{IJ}$ by F_{IJ} in (2.16) leads to a purely fermionic expression, which is also symplectically covariant.

where

$$\tilde{\gamma} \equiv -i\gamma^3\gamma^5\,, \tag{3.2}$$

so that $\gamma^1\gamma^2\gamma^0$ is proportional to the identity matrix. This implies that the Clifford algebra generated by these three-dimensional gamma matrices acts on the two-component spinors in equivalent representations. The gamma matrices γ^3 and γ^5 coincide with their higher-dimensional expressions: $\gamma^3 = \gamma^3_{(4)}$ and $\gamma^5 = \gamma^5_{(4)}$. The matrices $\tilde{\gamma}$, γ^3 and γ^5 *commute* with the three γ^μ. An observation that will be relevant later on, is that $\hat{\sigma}^1 \equiv \gamma^3$, $\hat{\sigma}^2 \equiv \gamma^5$ and $\hat{\sigma}^3 \equiv \tilde{\gamma}$ form an $su(2)$ algebra: $\hat{\sigma}^1\,\hat{\sigma}^2 = i\hat{\sigma}^3$.

Because the Dirac conjugate of a spinor involves the matrix γ^0, it will acquire an extra factor $\tilde{\gamma}$ as compared to the four-dimensional definition. Correspondingly we absorb a factor $\tilde{\gamma}$ into the three-dimensional charge-conjugation matrix C. With this definition we have the following identities

$$C\,\gamma^\mu C^{-1} = -\gamma^{\mu\mathrm{T}}\,,\quad C\,\gamma^3 C^{-1} = \gamma^{3\mathrm{T}}\,,\quad C\,\tilde{\gamma} C^{-1} = \tilde{\gamma}^{\mathrm{T}}\,,\quad C\,\gamma^5 C^{-1} = -\gamma^{5\mathrm{T}}\,. \tag{3.3}$$

It is possible to choose C such that it commutes with γ^3, $\tilde{\gamma}$ and γ^5.

The material of this section is organized in three subsections. First we discuss the actual reduction leading to a supersymmetric nonlinear sigma model in three dimensions. Then we elucidate the geometrical aspects of the target space. Finally, in a last subsection, we discuss the isometry structure of the target space.

3.1 The reduction

Now we turn to the Lagrangian of the compactified theory. After converting the three-dimensional gauge field into a scalar field, the terms in the Lagrangian (2.1) that contain the field strengths, are replaced by

$$\begin{aligned}
&-\tfrac{1}{4}i\Big(\bar{F}_{IJ}\,W^I_\mu W^{J\mu} - F_{IJ}\,\bar{W}^I_\mu \bar{W}^{\mu J}\Big)\\
&-\tfrac{1}{8}i\Big(F_{IJK}\bar{\Omega}^I_i\,\bar{W}^J_\mu\gamma^\mu\gamma_3\Omega^K_j\,\varepsilon^{ij} - \bar{F}_{IJK}\bar{\Omega}^{iI}\,W^J_\mu\gamma^\mu\gamma_3\Omega^{jK}\varepsilon_{ij}\Big)\,,
\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}
W^I_\mu \;&=\; 2iN^{IJ}(\partial_\mu B_J - F_{JK}\,\partial_\mu A^K)\\
&\quad +\tfrac{1}{4}iN^{IJ}\Big(\bar{F}_{JKL}\,\bar{\Omega}^{iK}\gamma_\mu\gamma_3\Omega^{jL}\varepsilon_{ij} + F_{JKL}\,\bar{\Omega}^K_i\gamma_\mu\gamma_3\Omega^L_j\varepsilon^{ij}\Big)\,.
\end{aligned} \tag{3.5}$$

Substituting this into (3.4) and combining with the other terms of the Lagrangian (2.1) yields

$$4\pi\,\mathcal{L} \;=\; i\Big(\partial_\mu F_I\,\partial^\mu\bar{X}^I - \partial_\mu\bar{F}_I\,\partial^\mu X^I\Big) - N^{IJ}(\partial_\mu B_I - F_{IK}\partial_\mu A^K)(\partial^\mu B_J - \bar{F}_{JM}\partial^\mu A^M)$$

$$
\begin{aligned}
&- \tfrac{1}{4} N_{IJ}\Big(\bar\Omega^{iI}\partial\!\!\!/\Omega^J_i + \bar\Omega^I_i\partial\!\!\!/\Omega^{iJ}\Big) \\
&- \tfrac{1}{4} iF_{IJK}\Big(\bar\Omega^I_i\partial\!\!\!/X^J\Omega^{iK} - i\bar\Omega^I_i N^{JL}(\partial\!\!\!/B_L - \bar F_{LM}\partial\!\!\!/A^M)\gamma_3\Omega^K_j\varepsilon^{ij}\Big) \\
&+ \tfrac{1}{4} i\bar F_{IJK}\Big(\bar\Omega^{iI}\partial\!\!\!/\bar X^J\Omega^K_i + i\bar\Omega^{iI} N^{JL}(\partial\!\!\!/B_L - F_{LM}\partial\!\!\!/A^M)\gamma_3\Omega^{jK}\varepsilon_{ij}\Big) \qquad (3.6)\\
&+ \tfrac{1}{96} i\Big(F_{IJKL} + 3iN^{MN}F_{M(IJ}F_{KL)N}\Big)\bar\Omega^I_i\,\gamma_3\gamma_\mu\Omega^J_j\varepsilon^{ij}\bar\Omega^K_k\gamma_3\gamma^\mu\Omega^L_l\varepsilon^{kl} \\
&- \tfrac{1}{96} i\Big(\bar F_{IJKL} - 3iN^{MN}\bar F_{M(IK}\bar F_{JL)N}\Big)\bar\Omega^{Ii}\gamma_3\gamma_\mu\Omega^{Jj}\varepsilon_{ij}\,\bar\Omega^{Kk}\gamma_3\gamma^\mu\Omega^{Ll}\varepsilon_{kl} \\
&- \tfrac{1}{48} N^{MN}F_{MIJ}\bar F_{KLN}\,\Big(2\bar\Omega^I_i\gamma_\mu\Omega^{iK}\,\bar\Omega^J_j\gamma^\mu\Omega^{jL} + \bar\Omega^I_i\gamma_\mu\gamma_3\Omega^J_j\varepsilon^{ij}\,\bar\Omega^{kK}\gamma_\mu\gamma_3\Omega^{lL}\varepsilon_{kl}\Big)\,,
\end{aligned}
$$

where we have suppressed a factor $2\pi R$ corresponding to the integration over the compactified coordinate x^3. Observe that the Lagrangian remains manifestly invariant under $\mathrm{SU}(2)_\mathrm{R}$. Note also that we keep the fermion fields in their original four-dimensional form, i.e. they are doublets of $\frac{1}{2}(1\pm\gamma^5)$ projections of four-dimensional Majorana spinors. Only the definition of the Dirac conjugate has been changed in accord with the rules obtained above.

The above Lagrangian is invariant under the following supersymmetry transformations,

$$
\begin{aligned}
\delta X^I &= -i\bar\epsilon^i\gamma_3\Omega^I_i\,, \\
\delta A^I &= i\varepsilon^{ij}\bar\epsilon_i\Omega^I_j - i\varepsilon_{ij}\bar\epsilon^i\Omega^{jI}\,, \\
\delta B_I &= iF_{IJ}\varepsilon^{ij}\bar\epsilon_i\Omega^J_j - i\bar F_{IJ}\varepsilon_{ij}\bar\epsilon^i\Omega^{jJ}\,, \\
\delta\Omega^I_i &= 2i\partial\!\!\!/X^I\gamma_3\epsilon_i + 2N^{IJ}(\partial\!\!\!/B_J - \bar F_{JK}\partial\!\!\!/A^K)\varepsilon_{ij}\epsilon^j \\
&\quad +iN^{IJ}\delta F_{JK}\,\Omega^K_i - N^{IJ}\bar F_{JKL}N^{KM}(\delta B_M - F_{MN}\delta A^N)\varepsilon_{ij}\gamma_3\Omega^{Lj}\,, \\
\delta\Omega^{Ii} &= -2i\partial\!\!\!/\bar X^I\gamma_3\epsilon^i + 2N^{IJ}(\partial\!\!\!/B_J - F_{JK}\partial\!\!\!/A^K)\varepsilon^{ij}\epsilon_j \\
&\quad -iN^{IJ}\delta\bar F_{JK}\,\Omega^{Ki} - N^{IJ}F_{JKL}N^{KM}(\delta B_M - \bar F_{MN}\delta A^N)\varepsilon^{ij}\gamma_3\Omega^L_j\,. \qquad (3.7)
\end{aligned}
$$

Under symplectic reparametrizations (A,B) transform as a symplectic pair, just as the field strengths (cf. 2.9). From (A^I, B_I) we can construct a complex scalar

$$
Y_I = B_I - F_{IJ}A^J\,, \qquad (3.8)
$$

which transforms as a (co)vector under symplectic reparametrizations,

$$
\tilde Y_I = Y_J\,[\mathcal{S}^{-1}]^J{}_I\,. \qquad (3.9)
$$

The supersymmetry transformation rule for Y_I equals

$$
\delta Y_I - \Gamma^K_{JI}\,\delta X^J\,Y_K = -N_{IJ}\varepsilon_{ij}\bar\epsilon^i\Omega^{jJ} + iF_{IJK}\,N^{JL}\,\bar Y_L\,\bar\epsilon^i\gamma_3\Omega^K_i\,, \qquad (3.10)
$$

where the left-hand side takes the form of a symplectically covariant variation, while the right-hand side is explicitly symplectically covariant. Observe that X^I and Y_I all transform holomorphically, i.e., their supersymmetry variations are proportional to $\bar{\epsilon}^i$ and not to $\bar{\epsilon}_i$. This observation will be relevant in subsection 3.3. All supersymmetry variations take a symplectically covariant form, as follows from using the transformation properties given in section 2.

After the dualization of the vector to scalar fields, the symplectic reparametrizations can be applied to the equations of motion or directly to the Lagrangian. These reparametrizations express the fact that the theory retains its form under certain diffeomorphisms, provided that we simultaneously change the function $F(X)$. As with general diffeomorphisms, this is not an invariance statement, but it characterizes the equivalence classes of the theory as encoded in functions $F(X)$. Henceforth we will use the term 'symplectically invariant' to indicate that quantities retain their form under the combined effect of a certain diffeomorphism and a change of the function $F(X)$. The Lagrangian (3.6) is symplectically invariant. In particular we note that the four-fermion terms are proportional to either the special Kähler curvature or to the symmetric tensor

$$C_{IJKL} = F_{IJKL} + 3iN^{MN}\, F_{M(IJ}\, F_{KL)N}\,. \tag{3.11}$$

Both tensors are symplectically covariant. The latter tensor vanishes for a symmetric Kähler space (defined by the condition that the curvature tensor is covariantly constant). As mentioned above, the above supersymmetry transformations are covariant under symplectic reparametrizations.

If the function $F(X)$ describes an effective four-dimensional gauge theory, based on charged fields which have been integrated out, then the θ-angles are defined up to shifts by 2π (at the nonperturbative level). Consequently the quantity F_{IJ} is only defined up to an additive integer-valued matrix.[4] From this observation it follows that, after compactifying on a circle, we must identify B_I with B_I plus an integer times A^I. Furthermore the fields A^I are only defined up to an integer times $1/R$, as a consequence of four-dimensional gauge transformations with non-trivial winding around the compactified direction.[5] Therefore, consistency requires that also B_I is defined up to an integer times $1/R$. At the perturbative level the corresponding invariance is realized by continuous transformations as can be seen from (3.6), which is invariant under $F_{IJ} \to F_{IJ} + c_{IJ}$ and

[4]In principle the integers are multiplied by a certain constant determined by the embedding of the corresponding U(1) group into the nonabelian gauge group of the underlying field theory. This constant is set to unity.

[5]For simplicity, we have set the I-th elementary charge equal to unity.

$B_I \to B_I + c_{IJ}A^J$, where the constants c_{IJ} constitute an arbitrary real symmetric tensor. These transformations correspond to the continuous Peccei-Quinn symmetries and are consistent with the transformations induced by the symplectic reparametrizations of the underlying vector-multiplet theory. Note that these transformations do not presuppose invariance under continuous shifts of the fields A_I, which, at finite R, do not represent a symmetry at the perturbative level. It is here that our approach fails to capture the dynamical effects associated with the compactification, just because we take $F(X)$ from a four-dimensional setting. This has no direct bearing on the fact that the target space parametrized by the (A^I, B_I) fields constitutes a torus T^{2n}, whose periodicity lattice is in fact directly related to the lattice of dyonic charges. Globally the full space is a fibre bundle over a special Kähler manifold with fibre T^{2n}. In the limit $R \to 0$, the torus decompactifies to $\mathbf{R}^{2n}$.

Let us discuss some properties of the torus at a given point X in the special-Kähler moduli space. First we determine the volume of T^{2n}, which turns out to be independent of X. To see this one integrates the square root of the determinant of the (A^I, B_I) metric given in (3.6) over the torus. Including the factor 4π from the left-hand side of (3.6) and the factor $2\pi R$ from the integration over the compactified coordinate x^3, we find

$$V(T^{2n}) = (4R)^{-n}\,. \tag{3.12}$$

Secondly, consider the invariant lengths of cycles $\gamma(X) : t \mapsto (X^I; A^I(t), B_I(t))$, which depend on the point X in the special-Kähler moduli space. For the cycles γ_{A^I} and γ_{B^I} in the A^I and B^I directions, these lengths are equal to

$$\ell_{A^I}(X) = \frac{1}{R}\sqrt{(FN^{-1}\bar{F})_{II}}\,, \qquad \ell_{B_I}(X) = \frac{1}{R}\sqrt{(N^{-1})^{II}}\,. \tag{3.13}$$

When X^I approaches a point where the Kähler metric becomes singular, one of the cycles $(\gamma_{A^I}, \gamma_{B_I})$ shrinks to zero while the other one grows to infinite length.

At this point it is tempting to identify the torus at X with the Jacobian variety of an auxiliary Riemann surface $\mathcal{M}_X$ that underlies the four-dimensional nonperturbative dynamics of a gauge theory in the Coulomb phase [14]. Its effective action takes the form of (2.1) and the abelian vector multiplets are then associated with the Cartan subalgebra of the underlying gauge group. Singularities in the effective action associated with the emergence of massless states correspond to a pinching of the auxiliary Riemann surface $\mathcal{M}_X$ which in turn leads to a degeneration of its Jacobian variety.[6] According to

[6]In a full three-dimensional treatment, it is possible that nonperturbative effects associated with monopoles wrapping around the circle smooth out some of these singularities; see the first reference of [19].

these arguments one may conclude that the complex scalars Y_I take their values in the (rescaled) Jacobian,

$$J(\mathcal{M}_X) = \mathbf{C}^n/L_X, \qquad L_X = \Big\{ \frac{1}{R}\Big(m_I - F_{IJ}(X)\,n^J\Big)\Big|\, m_I, n^I \in \mathbf{Z}\Big\}, \tag{3.14}$$

where we identify the second derivative of $F(X)$ with the intersection matrix τ of the Riemann surface $\mathcal{M}_X$. The target space parametrized by all the scalar fields thus coincides with the holomorphic Sp($2n;\mathbf{Z}$) bundle of Jacobian varieties over the moduli space of auxiliary Riemann surfaces, with metric as given in (3.6) and transitions functions prescribed by the monodromies of the moduli space.

3.2 Geometric features and symplectic transformations

A number of geometric features of the target space associated with the metric defined in the Lagrangian (3.6) deserves further attention. Note that, as a three-dimensional model, we are dealing with four independent supersymmetries. Therefore the target space must be a hyper-Kähler manifold [24], which, in the case at hand, is completely determined by the holomorphic function $F(X)$. Some of the properties of the special Kähler space are inherited by the ensuing hyper-Kähler space. In particular, when the Kähler space is symmetric or homogeneous, then the hyper-Kähler space is also symmetric or homogeneous, respectively. The material of this subsection covers some of the results presented in [2] and the relation with the work of [6]. Furthermore we discuss the behaviour under symplectic reparametrizations of the special hyper-Kähler manifold.

The bosonic Lagrangian follows from (3.6). It can be rewritten as

$$\begin{aligned} 4\pi\,\mathcal{L} \;=\;& -N_{IJ}\Big(\partial_\mu X^I\,\partial^\mu\bar X^I + \tfrac14\partial_\mu A^I\,\partial^\mu A^J\Big) \\ & -N^{IJ}\Big(\partial_\mu B_I - \tfrac12(F+\bar F)_{IK}\partial_\mu A^K\Big)\Big(\partial^\mu B_J - \tfrac12(F+\bar F)_{JM}\partial^\mu A^M\Big)\,. \end{aligned} \tag{3.15}$$

When the coordinates A and B are frozen to constant values, we have a special Kähler space parametrized by the coordinates X^I. Alternatively, freezing the special Kähler coordinates yields the torus T^{2n}. To describe the resulting $(4n)$-dimensional hyper-Kähler space, one must specify the metric and three covariantly constant complex structures, from which three closed two-forms can be defined.

The metrics (3.15) form a subclass of hyper-Kähler metrics constructed by [6] using the Legendre-transform method. The latter are characterized by the presence of at least n abelian isometries, which are triholomorphic so that they leave the metric as well as the closed two-forms invariant. In (3.15), they correspond to constant shifts in A^I and B_I.

Hyper-Kähler metrics with at least n triholomorphic abelian isometries can be written in the general form [25]

$$\mathrm{d}s^2 = U_{IJ}(x)\,\mathrm{d}\vec{x}^I\cdot\mathrm{d}\vec{x}^J + (U^{-1}(x))^{IJ}\Big(\mathrm{d}\varphi_I + \vec{W}_{IK}(x)\cdot\mathrm{d}\vec{x}^K\Big)\Big(\mathrm{d}\varphi_J + \vec{W}_{JL}(x)\cdot\mathrm{d}\vec{x}^L\Big)\ . \quad (3.16)$$

Here, the coordinates are split according to $\{\vec{x}^I, \varphi_I\}$. The n vectors $\vec{x}^I$ comprise $3n$ components $\vec{x}^{I\Lambda}$, where $\Lambda = 1,2,3$; the remaining coordinates φ_I are subject to the shift isometries. The tensors U_{IJ} and $\vec{W}_{IJ}$ are independent of φ_I and satisfy the hyper-Kähler equations

$$\partial_J^\Lambda W_{KI}^\Sigma - \partial_K^\Sigma W_{JI}^\Lambda = \varepsilon^{\Lambda\Sigma\Pi}\partial_J^\Pi U_{KI}\ , \quad (3.17)$$

where $\partial_I^\Lambda = \partial/\partial x^{\Lambda I}$. From this it follows that $\partial_I^\Lambda U_{JK} = \partial_J^\Lambda U_{IK}$. The three hyper-Kähler two-forms, given in [6], can be rewritten as (see e.g. [26])

$$\omega^\Lambda = (\mathrm{d}\varphi_I + \vec{W}_{IJ}\cdot\mathrm{d}\vec{x}^J)\wedge\mathrm{d}x^{\Lambda I} + U_{IJ}\,\varepsilon^{\Lambda\Sigma\Pi}\mathrm{d}x^{\Sigma I}\wedge\mathrm{d}x^{\Pi J}\ . \quad (3.18)$$

Clearly, they are invariant under constant shifts of φ_I, so that these isometries are indeed triholomorphic. In the case of (3.15), we have coordinates $\vec{x}^I = (\mathrm{Re}X^I, \mathrm{Im}X^I, -\frac{1}{2}A^I)$, $\varphi_I = B_I$ and

$$U_{IJ} = N_{IJ}\,, \qquad \vec{W}_{IJ} = (0,0,F_{IJ}+\bar{F}_{IJ})\ . \quad (3.19)$$

For this solution both U and $\vec{W}$ are determined by a single holomorphic function F, independent of A^I. It can be shown that F is proportional to the holomorphic function that appears in the contour-integral representation (cf. [6]) of the solution (3.16). Note also that $\vec{W}_{IJ}$ is symmetric. Other examples of hyper-Kähler metrics of the type (3.16) are Taub-NUT and the asymptotic metric on the moduli space of N SU(2) BPS monopoles. These metrics appear in the effective actions of three-dimensional $N=4$ SU(N) gauge theories [19]. They are not in the class (3.19).

To write down the hyper-Kähler forms and discuss symplectic transformations, it is convenient to use the complex coordinates Y_I (cf. (3.8)). In terms of the fields X^I and Y_I the bosonic Lagrangian reads

$$\begin{aligned} 4\pi\,\mathcal{L} &= -N_{IJ}\,\partial_\mu X^I\partial^\mu\bar{X}^J \qquad (3.20)\\ &\quad -N^{IJ}\Big(\partial_\mu Y_I + iN^{KL}(Y-\bar{Y})_L\partial_\mu F_{IK}\Big)\Big(\partial^\mu\bar{Y}_J + iN^{MN}(Y-\bar{Y})_N\partial^\mu\bar{F}_{JM}\Big)\,. \end{aligned}$$

At this point we note the identity

$$\partial_\mu Y_I + iN^{JK}(Y-\bar{Y})_K\,\partial_\mu F_{IJ} = (\partial_\mu Y_I - \Gamma^K_{IJ}\,\partial_\mu X^J\,Y_K) - iF_{IJK}\,\partial_\mu X^J\,N^{KL}\bar{Y}_L\,, \quad (3.21)$$

where the first term is just the Kähler covariant derivative of special geometry with the connection given in (2.5); the second term is separately covariant with respect to symplectic reparametrizations, as can be easily verified from (2.14). Therefore the above Lagrangian is invariant under the symplectic reparametrizations, as we claimed already in the previous subsection.

The combined (X, Y) space is a hyper-Kähler space with Kähler potential [2]

$$K(X, Y, \bar{X}, \bar{Y}) = iX^I \bar{F}_I(\bar{X}) - i\bar{X}^I F_I(X) - \tfrac{1}{2}(Y - \bar{Y})_I N^{IJ}(X, \bar{X}) (Y - \bar{Y})_J \,. \quad (3.22)$$

Under symplectic reparametrizations K changes by a Kähler transformation,

$$\tilde{K}(\tilde{X}, \tilde{Y}, \tilde{\bar{X}}, \tilde{\bar{Y}}) = K(X, Y, \bar{X}, \bar{Y}) + \tfrac{1}{2}i\mathcal{Z}^{IJ}(X)\, Y_I Y_J - \tfrac{1}{2}i\, \bar{\mathcal{Z}}^{IJ}(\bar{X})\, \bar{Y}_I \bar{Y}_J \,, \quad (3.23)$$

where $\tilde{K}$ is evaluated on the basis of the new function $\tilde{F}$ and $\mathcal{Z}(X)$ is the symmetric holomorphic tensor defined in (2.13). This does not imply that the Kähler metric takes the form of a symplectically covariant tensor, because the coordinates Y_I, unlike the special Kähler coordinates X^I, do not transform as coordinates but as symplectic vectors.[7] To see this, one first computes the metric from the derivatives of the Kähler potential. In the coordinates $z^a = (X^I, Y_J)$ we find

$$g_{a\bar{b}} = \begin{pmatrix} (N + P\, N^{-1} \bar{P})_{IK} & (P\, N^{-1})_I{}^L \\ \\ (N^{-1}\bar{P})^J{}_K & (N^{-1})^{JL} \end{pmatrix} , \quad (3.24)$$

where we have defined the symmetric tensor $P_{IJ} = iF_{IJK}N^{KL}(Y - \bar{Y})_L$. Under a symplectic reparametrization, P_{IJ} transforms as

$$P_{IJ} \to \Big[P_{KL} + F_{KLM}\, \mathcal{Z}^{MN} Y_N\Big]\, [\mathcal{S}^{-1}]^K{}_I\, [\mathcal{S}^{-1}]^L{}_J \,, \quad (3.25)$$

which implies that the metric is not symplectically covariant.

The inverse metric satisfies the relation

$$\Omega_{ac}\, g^{c\bar{d}}\, \Omega_{\bar{d}\bar{b}} = -g_{a\bar{b}} \,, \quad (3.26)$$

where Ω_{ab} is a covariantly constant antisymmetric tensor,

$$\Omega_{ab} = \begin{pmatrix} 0 & \delta_I{}^L \\ -\delta^J{}_K & 0 \end{pmatrix} . \quad (3.27)$$

The covariant constancy follows from (3.26). As a result Ω commutes with the holonomy group. Complex structures are then defined by

$$J^3 = \begin{pmatrix} -i\delta^a{}_b & 0 \\ 0 & i\delta^{\bar{a}}{}_{\bar{b}} \end{pmatrix} , \qquad J^\alpha = \begin{pmatrix} 0 & \alpha\Omega_a{}^{\bar{b}} \\ \bar{\alpha}\Omega_{\bar{a}}{}^b & 0 \end{pmatrix} , \quad (3.28)$$

[7] A similar situation is present in Calabi's construction of hyper-Kähler spaces on cotangent bundles with coordinates (X^I, Y_I) [4]. The corresponding Kähler potential is $K = i(X^I \bar{F}_I - \bar{X}^I F_I) - Y_I N^{IJ} \bar{Y}_J$, and is invariant under symplectic transformations. An essential difference, however, is that Calabi's metric does not possess the same (triholomorphic) isometries as the metric described above.

with α a phase factor and $\Omega_a{}^{\bar b} = \Omega_{ac} g^{c\bar b}$. Choosing $\alpha = 1, -i$ corresponding to, respectively, J^2, J^1, the matrices satisfy

$$J^\Lambda\, J^\Sigma = -\mathbf{1}\, \delta^{\Lambda\Sigma} - \varepsilon^{\Lambda\Sigma\Pi}\, J^\Pi\,. \tag{3.29}$$

Finally, the hyper-Kähler forms can be computed from (3.18) or, equivalently, from the complex structures. One finds

$$\begin{aligned} \omega^3 &= -iK_{X\bar X}\, \mathrm{d}X \wedge \mathrm{d}\bar X - iK_{X\bar Y}\, \mathrm{d}X \wedge \mathrm{d}\bar Y - iK_{Y\bar X}\, \mathrm{d}Y \wedge \mathrm{d}\bar X - iK_{Y\bar Y}\, \mathrm{d}Y \wedge \mathrm{d}\bar Y\,, \\ \omega^+ &= \mathrm{d}X^I \wedge \mathrm{d}Y_I\,, \qquad \omega^- = \mathrm{d}\bar X^I \wedge \mathrm{d}\bar Y_I\,. \end{aligned} \tag{3.30}$$

Observe that $\omega^\pm$ is purely (anti-)holomorphic, as already mentioned in [4, 6]. This will be important when we discuss the central charges in section 5. These two forms are closed so that locally they can be written as an exterior derivative of the following one-forms,

$$\begin{aligned} A^3 &= \tfrac12 iK_X\, \mathrm{d}X + \tfrac12 iK_Y\, \mathrm{d}Y - \tfrac12 iK_{\bar X}\, \mathrm{d}\bar X - \tfrac12 iK_{\bar Y}\, \mathrm{d}\bar Y\,, \\ A^+ &= \tfrac12 X^I\, \mathrm{d}Y_I - \tfrac12 Y_I\, \mathrm{d}X^I\,, \qquad A^- = \tfrac12 \bar X^I\, \mathrm{d}\bar Y_I - \tfrac12 \bar Y_I\, \mathrm{d}\bar X^I\,. \end{aligned} \tag{3.31}$$

Under symplectic reparametrizations, these one-forms are invariant up to an exact form. For A^3 this follows from (3.23) and for $A^\pm$ this can be seen from noting that the second term is manifestly symplectically invariant, whereas the first term equals the second up to an exact form. Therefore the corresponding hyper-Kähler two-forms are symplectically invariant. For $\omega^\pm$ this can also be seen directly by observing that replacing the one-forms $\mathrm{d}Y$ by the symplectically covariant forms $\mathrm{d}Y_I + iN^{JK} Y_K \mathrm{d}F_{IJ}$, does not change $\omega^\pm$. Note, however, that the corresponding tensors J^Λ are *not* symplectically covariant, just as the metric was not a covariant tensor. This is again related to the fact that the symplectic reparametrizations act differently on the X^I and Y_I.

3.3 Isometries

As explained in detail in [18] the isometry group of a special Kähler manifold extends in a characteristic way when performing the **c** map. The additional isometries are called *extra* symmetries when their origin can be understood directly from the four-dimensional gauge transformations, or *hidden* symmetries when their existence is not generic and depends on special properties of the manifold. In [18] this was discussed for special quaternionic manifolds (i.e., in the case of local supersymmetry). In this subsection, we give a similar discussion for the special hyper-Kähler manifolds. Here the *extra* symmetries follow directly from the gauge symmetry in four dimensions and correspond to constant shifts in A^I and in B_I, as we discussed previously. In the complex basis the extra isometries take the form

$$\delta Y_I = \beta_I - F_{IJ}\alpha^J \ , \tag{3.32}$$

with real parameters. They leave the metric invariant and also the closed two-forms ω^Λ of (3.30). By definition, such isometries are called triholomorphic.

Apart from these, there can be isometries corresponding to duality symmetries of the original four-dimensional action of the vector multiplets. These isometries are associated with the symplectic reparametrizations, leaving the function F unchanged. Because the two-forms ω^Λ are symplectically invariant, these isometries are also triholomorphic. There can be additional isometries of the special Kähler manifold that do not leave the full action invariant [18, 27]. Those isometries do not take the form of symplectic reparametrizations and will in principle not correspond to isometries of the hyper-Kähler manifold.

Just as for the special quaternionic manifolds, we find that *hidden* symmetries for the hyper-Kähler manifolds are subject to certain nontrivial conditions. But unlike the quaternionic case, the conditions seem impossible to satisfy unless one makes a rather simple choice for the function $F(X)$. Before proceeding to derive the conditions for general isometries, we make the following observation. Obviously the commutator of an infinitesimal isometry and a supersymmetry variation defines a fermionic symmetry. However, we know that the fields X^I and Y_I transform only under supersymmetries with positive-chirality parameters. Unless the isometries are holomorphic, we will thus generate new supersymmetries of the wrong chirality. These can not be accomodated by the standard supersymmetry algebra and the theory can only be invariant under them if it contains noninteracting sectors, i.e., if the model is reducible and the target space is a local product space (this argument is identical to the one used in [28] for two-dimensional sigma models with torsion). So without loss of generality, we may assume that the isometries are holomorphic.

With this in mind we first study the variations of the action under an arbitrary infinitesimal isometry that are quadratic in the derivatives of the fields A^I and B_I. This leads to the result that the variation of F_{IJ} must take the form

$$\delta F_{IJ} = N_{IK}\, N_{JL}\, \frac{\partial^2 f}{\partial \bar Y_K \partial \bar Y_L}\,, \tag{3.33}$$

where f is some real function of $Y, \bar Y, X, \bar X$. Furthermore the transformation rule for Y_I can be written as

$$\delta Y_I = -iN_{IJ}\, \frac{\partial}{\partial \bar Y_J}\Big[2f + (Y_K - \bar Y_K)\frac{\partial f}{\partial \bar Y_K}\Big] + iN_{IJ}\Lambda^{JK}\bar Y_K\,, \tag{3.34}$$

where the quantity $\Lambda^{IJ}(X,\bar X)$ is independent of Y and $\bar Y$ and antisymmetric in I,J so that it cannot be incorporated into the first term for δY_I. From the fact that the right-hand side of (3.33) must be independent of $\bar Y$, it follows that the function f depends at most quadratically on $\bar Y$. (Obviously the same conclusion can be drawn for the Y-dependence.) Therefore the first term in (3.34) is $\bar Y$-independent. Because δY itself must also be independent of $\bar Y$, it follows that $\Lambda^{IJ}=0$.

Subsequently, consider the mixed variations in the Lagrangian, proportional to a derivative of A or B and $\bar X$. This leads to conditions for the derivatives of δX^I with respect to A^I and B_I, which can be integrated. Specifically, we find two restrictions,

$$\delta X^I \pm iN^{IJ}\frac{\partial f}{\partial \bar X^J}\Big|_{A,B} = \tfrac12 P^I_{\pm}\,, \tag{3.35}$$

where P^I_+ depends on $X,\bar X,\bar Y$ and P^I_- depends on $X,\bar X,Y$. The holomorphy of δX^I implies that $(P_+ + P_-)^I$ depends only on X and Y. Therefore it follows that $N^{IJ}\partial f/\partial \bar X^J|_{A,B}$ must be independent of $\bar Y$ and $\bar X$, up to terms that depend exclusively on $X,\bar X$. The holomorphy in Y restricts f to the following form,

$$\begin{aligned} f(X,\bar X,Y,\bar Y) &= [N(Y-\bar Y)]^I[N(Y-\bar Y)]^J O_{IJ}(X,Y) \\ &\quad + i[N(Y-\bar Y)]^I\Lambda_I(X,Y) + \tilde f(X,\bar X,Y)\,. \end{aligned} \tag{3.36}$$

The holomorphic functions O_{IJ} and Λ_I can now be expanded in powers of Y. Note that the first one is at most quadratic and the second one at most cubic in Y. Also the nonholomorphic function $\tilde f$ can be expanded in Y, up to fourth order.

The reality of f yields a large number of restrictions. For instance, the Y-expansion coefficients of $\tilde f$ and Λ are related,

$$[\Lambda_I(X) - \bar\Lambda_I(\bar X)]^{J_1\cdots J_n} = inN_{IJ}\,\tilde f^{JJ_1\cdots J_n}(X,\bar X)\,, \tag{3.37}$$

where the $\tilde f^{IJK\cdots}$ must be real. On the other hand, holomorphy in X restricts the $\tilde f^{IJK\cdots}$ to the form

$$\tilde f^{KL\cdots}(X,\bar X) = (\bar X^I F_{IJ} - \bar F_J)\, g^{J,KL\cdots}(X) + h^{KL\cdots}(X)\,. \tag{3.38}$$

Combining these constraints seems to lead inevitably to the conclusion, at least for non-trivial functions $F(X)$, that the $\tilde f^{IJ\cdots}$ must be constant. In that case we may rewrite (3.36) in terms of A and B, and observe that there is no $\bar X$-dependence anymore. However, the function f must be real, so that we conclude that it can be written as the sum of a function of A and B and a function of X and $\bar X$. The latter function can be ignored. The independence of X and $\bar X$ now implies that (3.36) is a real polynomial in A and B that is at most of order two. The terms linear in A and B characterize the shift symmetries (3.32), whereas the quadratic terms correspond to the isometries embedded in the symplectic reparametrizations of the special Kähler space. The latter can be verified by showing that (3.33) and (3.34) take the form of an infinitesimal symplectic reparametrization as follows from the first equation of (2.14) and (3.9), respectively.

4 Hypermultiplets

Hyper-Kähler spaces serve as target spaces for hypermultiplets. One of our goals is to understand the relation between special Kähler and special hyper-Kähler at the level of the full actions for vector multiplets and hypermultiplets, including the fermions. Before doing this we briefly review the derivation of the Lagrangian for hypermultiplets in four spacetime dimensions. Our analysis, which is self-contained, is closely related to the one presented in [7]. However, our results are cast in a somewhat different form in order to facilitate the comparison with the models that emerge from the vector multiplets under the action of the **c** map. Furthermore, we find that a certain restriction found in [7] is unnecessary and in fact too restrictive.

We assume $4n$ real scalars ϕ^A and $2n$ positive-chirality spinors $\zeta^{\bar\alpha}$ and $2n$ negative-chirality spinors ζ^α, which are related by conjugation (so that we have $2n$ Majorana spinors). Therefore, under complex conjugation indices are converted according to $\alpha \leftrightarrow \bar\alpha$, while, just as before, SU(2) indices $i, j, \ldots$ are raised and lowered. The supersymmetry transformations are parametrized in terms of certain ϕ-dependent quantities γ^A and V_A as

$$
\begin{aligned}
\delta\phi^A &= 2\Big(\gamma^A_{i\bar\alpha}\,\bar\epsilon^i\zeta^{\bar\alpha} + \bar\gamma^{Ai}_{\alpha}\,\bar\epsilon_i\zeta^\alpha\Big)\,,\\
\delta\zeta^{\bar\alpha} &= \bar V^{i\bar\alpha}_A\,\partial\!\!\!/\phi^A\epsilon_i - \delta\phi^A\,\bar\Gamma_A{}^{\bar\alpha}{}_{\bar\beta}\,\zeta^{\bar\beta}\,,\\
\delta\zeta^\alpha &= V^\alpha_{A\,i}\,\partial\!\!\!/\phi^A\epsilon^i - \delta\phi^A\,\Gamma_A{}^\alpha{}_\beta\,\zeta^\beta\,.
\end{aligned}
\qquad (4.1)
$$

The definition of Γ and $\bar\Gamma$ will be discussed shortly. As it turns out, with the proper definition, the above ansatz comprises the full supersymmetry transformation laws. Observe that the variations are consistent with a U(1) chiral invariance under which the scalars remain invariant, which we will denote by $\mathrm{U}(1)_{\mathrm{R}}$ to indicate that it is a subgroup of the automorphism group of the supersymmetry algebra. However, for generic γ^A and V_A, the $\mathrm{SU}(2)_{\mathrm{R}}$ part of the automorphism group cannot be realized consistently. In the above, we only used that ζ^α and $\zeta^{\bar\alpha}$ are related by complex conjugation. Our notation is similar but not identical to the one used in [20].

A first condition on the quantities γ^A and V_A follows from the closure of the supersymmetry transformations (4.1) on the scalars. This yields the Clifford-like condition

$$\gamma^A_{i\bar\alpha}\,\bar V^{j\bar\alpha}_B + \bar\gamma^{A\,j}_{\alpha}\,V^{\alpha}_{B\,i} = \delta^j_i\,\delta^A_B\,. \tag{4.2}$$

Subsequently let us turn to the action, which we parametrize as

$$4\pi\,\mathcal{L} = -\tfrac{1}{2}g_{AB}\,\partial_\mu\phi^A\partial^\mu\phi^B - G_{\bar\alpha\beta}\Big(\bar\zeta^{\bar\alpha}\not{D}\zeta^\beta + \bar\zeta^\beta\not{D}\zeta^{\bar\alpha}\Big) + \mathcal{L}(\zeta^4)\,, \tag{4.3}$$

where $G_{\bar\alpha\beta}$ is a hermitean metric[8], and we use the covariant derivatives

$$D_\mu\zeta^\alpha = \partial_\mu\zeta^\alpha + \partial_\mu\phi^A\,\Gamma_A{}^\alpha{}_\beta\,\zeta^\beta\,,\quad D_\mu\zeta^{\bar\alpha} = \partial_\mu\zeta^{\bar\alpha} + \partial_\mu\phi^A\,\bar\Gamma_A{}^{\bar\alpha}{}_{\bar\beta}\,\zeta^{\bar\beta}\,. \tag{4.4}$$

The Noether term thus takes the following form,

$$4\pi\,\mathcal{L}_{\rm N} = \Big[\bar\Gamma_A{}^{\bar\gamma}{}_{\bar\alpha}\,G_{\bar\gamma\beta} - G_{\bar\alpha\gamma}\,\Gamma_A{}^\gamma{}_\beta\Big]\,\bar\zeta^{\bar\alpha}\not\partial\phi^A\zeta^\beta\,. \tag{4.5}$$

Observe that only a linear combination of the two connections appears in the action.

Considering various terms of the supersymmetry variation of the action (4.3) leads to further conditions. Cancellation of the variations proportional to $\partial^2\phi^A$ implies

$$g_{AB}\,\gamma^B_{i\bar\alpha} = G_{\bar\alpha\beta}\,V^\beta_{A\,i}\,,\qquad g_{AB}\,\bar\gamma^{B\,i}_\alpha = G_{\bar\beta\alpha}\,\bar V^{i\bar\beta}_A\,. \tag{4.6}$$

Then variations proportional to $\partial_\mu\phi^B\,\partial_\nu\phi^C$ require

$$\begin{aligned} 2G_{\bar\beta\alpha}\,D_B V^\alpha_{A\,i} + D_B G_{\bar\beta\alpha}\,V^\alpha_{A\,i} &= 0\,,\\ 2G_{\bar\beta\alpha}\,D_B \bar V^{i\bar\beta}_A + D_B G_{\bar\beta\alpha}\,\bar V^{i\bar\beta}_A &= 0\,. \end{aligned} \tag{4.7}$$

Note that the first covariant derivative in (4.7) contains also the Christoffel symbol $\{A;BC\}$. Now redefine the connections according to

$$\begin{aligned} G_{\bar\beta\gamma}\Gamma_A{}^\gamma{}_\alpha + \tfrac{1}{2}D_A G_{\bar\beta\alpha} &\;\to\; G_{\bar\beta\gamma}\hat\Gamma_A{}^\gamma{}_\alpha\,,\\ G_{\bar\gamma\alpha}\bar\Gamma_A{}^{\bar\gamma}{}_{\bar\beta} + \tfrac{1}{2}D_A G_{\bar\beta\alpha} &\;\to\; G_{\bar\gamma\alpha}\hat{\bar\Gamma}_A{}^{\bar\gamma}{}_{\bar\beta}\,. \end{aligned} \tag{4.8}$$

Taking the difference, one sees that this modification does not modify the Noether term. Furthermore, one can verify that the γ^A tensors are covariantly constant with respect to the connection $\hat\Gamma$, and so is the metric $G_{\bar\alpha\beta}$. Thus we replace the connections everywhere by the new connection and drop the caret. These are then the connections that appear

[8] A possible antihermitean part can be absorbed into the Noether term, modulo a total derivative. In principle, it is possble to absorb G into the definition of the fermion fields, but we refrain from doing so for reasons that will become clear in due course.

in the variations of the spinor fields in (4.1) and, as it turns out, no additional terms quadratic in the spinor fields are required in these transformation rules.

According to the above results we define four real, antisymmetric covariantly constant tensors,

$$J^{\Lambda}_{AB} = i\gamma_{Ai\bar\alpha}\,\bar V_B^{j\bar\alpha}\,(\sigma^\Lambda)^i{}_j\,, \qquad (\Lambda=1,2,3) \tag{4.9}$$

and

$$C_{AB} = i(\gamma_{Ai\bar\alpha}\,\bar V_B^{i\bar\alpha} - g_{AB})\,. \tag{4.10}$$

It follows that C must vanish, so that γ and $\bar V$ are each others inverse,

$$\bar V_A^{i\bar\alpha}\,\gamma^A_{j\bar\beta} = \delta^i_j\,\delta^{\bar\alpha}_{\bar\beta}\,. \tag{4.11}$$

The precise analysis leading to this result is somewhat subtle, and is based on an extension of the arguments used in [24]. It makes use of the fact that the five covariantly constant two-rank tensors, the metric, the J^Λ and C, and products thereof, must commute with the curvature tensor and therefore with the holonomy group. The latter can act reducibly, so that the target space factorizes and the model decomposes into the sum of several independent models. If the holonomy group acts irreducibly, then according to Schur's lemma, the algebra generated by the above tensors must be a division algebra. This implies a degeneracy between the tensors (4.9) and (4.10). Combining this fact with the Clifford property leads to (4.11).

From (4.11) it then follows directly that the J^Λ are complex structures, satisfying

$$J^\Lambda\,J^\Sigma = -\mathbf{1}\delta^{\Lambda\Sigma} - \varepsilon^{\Lambda\Sigma\Pi}J^\Pi\,, \tag{4.12}$$

reflecting the well-known result that the target space must be hyper-Kähler.

Furthermore we note the existence of covariantly constant antisymmetric tensors,

$$\Omega_{\bar\alpha\bar\beta} = \tfrac12\varepsilon^{ij}\,g_{AB}\,\gamma^A_{i\bar\alpha}\,\gamma^B_{j\bar\beta}\,, \quad \bar\Omega^{\bar\alpha\bar\beta} = \tfrac12\varepsilon_{ij}\,g^{AB}\,\bar V_A^{i\bar\alpha}\,\bar V_B^{j\bar\beta}\,, \tag{4.13}$$

satisfying

$$\varepsilon_{ij}\,\Omega_{\bar\alpha\bar\beta}\,\bar V_A^{j\bar\beta} = g_{AB}\,\gamma^B_{i\bar\alpha}\,. \tag{4.14}$$

According to (4.6) and (4.14) the γ and V tensors are linearly related and pseudo-real. Therefore the tensor Ω is also pseudo-real and it satisfies

$$\Omega_{\bar\alpha\bar\gamma}\,\bar\Omega^{\bar\gamma\bar\beta} = -\delta^{\bar\beta}_{\bar\alpha}\,. \tag{4.15}$$

The existence of covariantly constant tensors implies a variety of integrability conditions for the curvature tensors. From the constancy of $G_{\bar\alpha\beta}$ and $\Omega_{\bar\alpha\bar\beta}$ we obtain,

$$R_{AB}{}^{\bar\beta}{}_{\bar\alpha} = -G_{\bar\alpha\gamma}\,G^{\delta\bar\beta}\,R_{AB}{}^\gamma{}_\delta\,, \qquad R_{AB}{}^{\bar\gamma}{}_{[\bar\alpha}\,\Omega_{\bar\beta]\bar\gamma} = 0\,. \tag{4.16}$$

These conditions imply that ${R_{AB}}^{\alpha}{}_{\beta}$ takes values in $sp(n) \cong usp(2n;\mathbf{C})$ so that the holonomy group acts symplectically on the fermions.

Furthermore, constancy of the γ tensor implies

$$R_{ABD}{}^{C}\,\gamma^{D}_{i\bar\alpha} - R_{AB}{}^{\bar\gamma}{}_{\bar\alpha}\,\gamma^{C}_{i\bar\gamma} = 0\,. \tag{4.17}$$

From this result one proves that Riemann curvature and the Sp(n) curvature are related,

$$\begin{aligned} R_{AB}{}^{\bar\beta}{}_{\bar\alpha} &= \tfrac{1}{2} R_{ABE}{}^{C}\,\gamma^{E}_{i\bar\alpha}\,\bar V^{i\bar\beta}_{C}\,,\\ R_{ABD}{}^{C} &= R_{AB}{}^{\bar\beta}{}_{\bar\alpha}\,\gamma^{C}_{i\bar\beta}\,\bar V^{i\bar\alpha}_{D}\,. \end{aligned} \tag{4.18}$$

Using the pair-exchange property of the Riemann tensor and contracting with $\gamma^{C}\,\bar\gamma^{D}$ one derives

$$R_{AB}{}^{\bar\beta}{}_{\bar\alpha} = \tfrac{1}{2} W_{\bar\alpha\epsilon\bar\gamma\delta}\,\bar V^{i\bar\gamma}_{A}\,V^{\delta}_{Bi}\,G^{\epsilon\bar\beta}\,, \tag{4.19}$$

where

$$W_{\bar\alpha\beta\bar\gamma\delta} = R_{AB}{}^{\bar\epsilon}{}_{\bar\gamma}\,\gamma^{A}_{i\bar\alpha}\,\bar\gamma^{iB}_{\beta}\,G_{\bar\epsilon\delta} = \tfrac{1}{2} R_{ABCD}\,\gamma^{A}_{i\bar\alpha}\,\bar\gamma^{iB}_{\beta}\,\gamma^{C}_{j\bar\gamma}\,\bar\gamma^{jD}_{\delta}\,. \tag{4.20}$$

The tensor W can be written as $W_{\alpha\beta\gamma\delta}$ by contracting with the metric G and the antisymmetric tensor Ω. It then follows that $W_{\alpha\beta\gamma\delta}$ is symmetric in symmetric index pairs $(\alpha\beta)$ and $(\gamma\delta)$. Using the Bianchi identity for Riemann curvature, which implies $g_{D[A}R_{BC]}{}^{\bar\beta}{}_{\bar\alpha}\,\gamma^{D}_{i\bar\beta} = 0$, one shows that it is in fact symmetric in all four indices.

Hence all the curvatures are expressed in terms of the fully symmetric tensor $W_{\alpha\beta\gamma\delta}$. From this result many other identities for the curvatures can be derived. In particular we note the identity

$$R_{AB}{}^{\bar\gamma}{}_{[\bar\alpha}\,\gamma^{B}_{\bar\beta]i} = 0\,, \tag{4.21}$$

which plays a crucial role in proving the supersymmetry of the action. For that, one needs to include a four-fermion interaction into the Lagrangian, equal to

$$4\pi\,\mathcal{L}(\zeta^4) = -\tfrac{1}{4} W_{\bar\alpha\beta\bar\gamma\delta}\,\bar\zeta^{\bar\alpha}\gamma_\mu\zeta^{\beta}\,\bar\zeta^{\bar\gamma}\gamma^{\mu}\zeta^{\delta}\,. \tag{4.22}$$

All the above results are closely related to the ones derived long ago in [7]. One feature that is different is the presence of a fermionic metric, which, as we will demonstrate in the next section, is important in exhibiting the effect of symplectic reparametrizations for models in the image of the **c** map. Another feature concerns the condition imposed in [7] that $\gamma^{B}_{i\bar\alpha}\,\bar\gamma^{C\,i}_{\beta} + \gamma^{C}_{i\bar\alpha}\,\bar\gamma^{B\,i}_{\beta}$ be proportional to the product of g^{BC} and $G_{\bar\alpha\beta}$ and inversely proportional to the number of hypermultiplets n. We found no need for this condition. In fact, it is in contradiction with the case of free fields, where no $1/n$ terms can arise.

5 Applying the mirror map

From the material of the previous sections we will explicitly extract the basic quantities of the hyper-Kähler space that emerges from the four-dimensional $N=2$ vector multiplets under the action of the **c** map. Before doing so, it is important that we first discuss the extension of the chiral $\mathrm{SU}(2)_\mathrm{R}\times\mathrm{U}(1)_\mathrm{R}$ automorphism group of the supersymmetry algebra in four spacetime dimensions to SO(4). Of course, it is well known that the automorphism group in three dimensions contains SO(4), but we are interested in the way this extension is realized, namely by promoting the $\mathrm{U}(1)_\mathrm{R}$ group to SU(2). With the aforementioned $\mathrm{SU}(2)_\mathrm{R}$ one thus obtains the group $(\mathrm{SU}(2)\times\mathrm{SU}(2))/\mathbf{Z}_2\cong\mathrm{SO}(4)$.

In section 3 we already made reference to the fact that the independent combinations of four-dimensional gamma matrices that commute with the three-dimensional ones, constitute an $su(2)$ algebra. Therefore spinors ϵ^i in a four-dimensional spacetime, which transform under a chiral $\mathrm{SU}(2)_\mathrm{R}\times\mathrm{U}(1)_\mathrm{R}$ group, can in principle transform under a bigger group after descending to three dimensions. However, we are not interested in any such extension, but only in those that constitute a subgroup of the automorphism group of the supersymmetry algebra in three spacetime dimensions.

To understand the fate of the $su(2)$ let us momentarily consider $N=1$ supersymmetry in four spacetime dimensions. The four-dimensional automorphism group contains a chiral U(1). According to the above arguments this group can be extended to SU(2) in the reduction to three spacetime dimensions; its generators are just proportional to the three hermitean matrices $\hat\sigma^1,\hat\sigma^2,\hat\sigma^3$ that were defined in section 3. This SU(2) group is consistent with the supersymmetry algebra, but it cannot be realized on Majorana spinors. The Majorana constraint requires the phases appropriate to the group SL(2), which, in turn, is not consistent with the supersymmetry algebra. So, unless one doubles the spinors, the automorphism group U(1) remains unextended when descending to three spacetime dimensions.

Starting from $N=2$ in four dimensions, on the other hand, naturally incorporates such a doubling of spinors. The spinor doublets then transform under chiral $\mathrm{SU}(2)_\mathrm{R}\times\mathrm{U}(1)_\mathrm{R}$ and the extension of the U(1) group to SU(2) is automatic. Starting with a (nonchiral) Majorana doublet ϵ^i (which comprises eight real independent components), the $\mathrm{SU}(2)_\mathrm{R}$ transformations act according to

$$\epsilon^i\to\Big[U^i{}_j\,\Big(\frac{\mathbf{1}+\hat\sigma^2}{2}\Big)+\overline{U^i{}_j}\,\Big(\frac{\mathbf{1}-\hat\sigma^2}{2}\Big)\Big]\epsilon^j\,. \tag{5.1}$$

In three spacetime dimensions the group $\mathrm{U}(1)_\mathrm{R}$ is extended to SU(2), with matrices $\hat U$

that are asssociated with the generators $\hat\sigma^a$. They act according to

$$\epsilon^i \to \Big[\hat U\Big(\frac{\mathbf{1}+\sigma^2}{2}\Big)^i{}_j + \bar{\hat U}\Big(\frac{\mathbf{1}-\sigma^2}{2}\Big)^i{}_j\Big]\epsilon^j\,, \tag{5.2}$$

where $(\sigma^2)^i{}_j$ equals the skew-symmetric imaginary σ-matrix. This extra SU(2) commutes with $\mathrm{SU(2)_R}$ by virtue of the fact that they both have a skew-symmetric invariant tensor, $(\sigma^2)^i{}_j$ and $\hat\sigma^2 = \gamma^5$, satisfying $\bar U = \sigma^2 U\,\sigma^2$ and likewise for $\hat U$ and $\hat\sigma^2$. It is convenient to write the above transformations in infinitesimal form employing chiral spinor components. Defining $\hat U \approx \mathbf{1} + \frac12 i\hat\alpha_a\,\hat\sigma^a$, one obtains

$$\begin{aligned} \delta\epsilon^i &= \tfrac12 i\hat\alpha_2\,\epsilon^i + \tfrac12\varepsilon^{ij}(\hat\alpha_1 + i\hat\alpha_3)\gamma^3\epsilon_j\,,\\ \delta\epsilon_i &= -\tfrac12 i\hat\alpha_2\,\epsilon_i + \tfrac12\varepsilon_{ij}(\hat\alpha_1 - i\hat\alpha_3)\gamma^3\epsilon^j\,. \end{aligned} \tag{5.3}$$

The above results show that a proper basis for the extra SU(2) transformations is obtained by choosing

$$\begin{aligned} \epsilon^+ &= \tfrac12\sqrt2\gamma^3(\epsilon_1 - i\epsilon_2)\,, \qquad & \epsilon^- &= \tfrac12\sqrt2(\epsilon^1 - i\epsilon^2)\,,\\ \epsilon_+ &= \tfrac12\sqrt2\gamma^3(\epsilon^1 + i\epsilon^2)\,, \qquad & \epsilon_- &= \tfrac12\sqrt2(\epsilon_1 + i\epsilon_2)\,. \end{aligned} \tag{5.4}$$

These spinors are eigenstates under both σ^2 and $\hat\sigma^2$ and transform under phase transformations under both $\mathrm{U(1)_R}$ as the SO(2) subgroup of $\mathrm{SU(2)_R}$. Upper- and lower-index spinors are related by conjugation.

Now let us consider the reduction to three dimensions of the actions presented in sections 2 and 4 for vector multiplets and hypermultiplets. As pointed out previously, the vector multiplet Lagrangian and supersymmetry transformations are manifestly covariant with respect to the $\mathrm{SU(2)_R}$ group, but not to the group $\mathrm{U(1)_R}$ (at least, not in the general case). Consequently, when descending to three dimensions, the symmetry group is not enhanced and we are left with the $\mathrm{SU(2)_R}$ transformations and the symplectic reparametrizations. On the other hand, the hypermultiplet Lagrangian and supersymmetry transformations are generically only covariant with respect to the group $\mathrm{U(1)_R}$ and when descending to three dimensions, this group is enhanced to a full SU(2) group, with elements $\hat U$. However, consistency requires that this extra SU(2) group commutes with the holonomy group and therefore its action incorporates the antisymmetric tensor $\Omega_{\bar\alpha\bar\beta}$ constructed in the previous section. Infinitesimally the SU(2) transformations act on the hypermultiplet fermions according to

$$\begin{aligned} \delta\zeta^\alpha &= \tfrac12 i\hat\alpha_2\,\zeta^\alpha - \tfrac12 G^{\alpha\bar\gamma}\Omega_{\bar\gamma\bar\beta}\,(\hat\alpha_1 + i\hat\alpha_3)\gamma^3\zeta^{\bar\beta}\,,\\ \delta\zeta^{\bar\alpha} &= -\tfrac12 i\hat\alpha_2\,\zeta^{\bar\alpha} - \tfrac12\bar\Omega^{\bar\alpha\bar\gamma}G_{\bar\gamma\beta}\,(\hat\alpha_1 - i\hat\alpha_3)\gamma^3\zeta^\beta\,. \end{aligned} \tag{5.5}$$

In other words, when systems based on both vector multiplets and hypermultiplets are reduced to a three-dimensional spacetime, the target space factorizes into two hyper-Kähler manifolds which will both possess an independent SU(2) invariance group, corresponding to different factors of the SO(4) automorphism group of the supersymmetry algebra. This reflects the general situation in $N = 4$ supersymmetric sigma models in three dimensions, even when coupled to supergravity. In the latter case the sigma model target space factorizes into two quaternionic spaces, whose Sp(1) holonomy groups consitute the two different factors of the SO(4) group [3]. This situation is typical for the case of $N = 4$ supersymmetry.

The above observations are essential to reconcile the fermionic supersymmetry transformations (3.7) with those of the hypermultiplet (4.1), after dimensional reduction. The SO(2) subgroup of $\mathrm{SU(2)_R}$ will play the role of $\mathrm{U(1)_R}$ after applying the mirror map and returning to four spacetime dimensions. Consequently, we must identify the fields ζ^α and $\zeta^{\bar\alpha}$ with combinations of the vector multiplet spinor fields, Ω_i^I and Ω^{iI}, that transform as eigenspinors under the SO(2) group with the proper phase transformations. For the spinor parameters, this means that we must convert to the previously introduced spinor parameters $\epsilon^\pm$ and $\epsilon_\pm$ (cf. 5.4). These requirements motivate us to make the following identification,

$$
\begin{aligned}
\zeta^\alpha &= \Big(-\tfrac{1}{2}\sqrt{2}\gamma^3(\Omega_1^I - i\Omega_2^I), \tfrac{1}{2}\sqrt{2}(\Omega^{1I} - i\Omega^{2I})\Big)\,,\\
\zeta^{\bar\alpha} &= \Big(-\tfrac{1}{2}\sqrt{2}\gamma^3(\Omega^{1I} + i\Omega^{2I}), \tfrac{1}{2}\sqrt{2}(\Omega_1^I + i\Omega_2^I)\Big)\,,
\end{aligned}
\tag{5.6}
$$

where the relation between ζ^α and $\zeta^{\bar\alpha}$ proceeds via Dirac conjugation and the Majorana condition.

Let us first comment on the various factors in (5.6). As explained above, the identification is such that the ζ^α transform under the SO(2) subgroup of $\mathrm{SU(2)_R}$ with a uniform phase. The $\zeta^{\bar\alpha}$ then transform with the opposite phase. The relative factors γ^3 follow from the requirement that the fermions on the right-hand side, whose supersymmetry transformations follow from (3.7), will take a form similar to the transformations of the hypermultiplet fermions, as given in (4.1), when descending to three dimensions. Both the overall and relative factors of γ^3 are required to match the chirality of both sides of the equations. The phase factors adopted for the various components in (5.6), are somewhat arbitrary. They can be changed a posteriori by performing certain redefinitions. The same comment applies to the phase factors adopted in the definitions of the spinors (5.4).

In three dimensions, (5.6) and (5.4) represent simply a different basis for the spinors that play a role in the vector multiplet. However, from the point of view of the four-dimensional Lorentz group, this choice of basis has nontrivial implications. When assuming that the newly defined spinor fields transform in the conventional way under the four-dimensional Lorentz transformations, one implicitly exchanges the $\mathrm{SU}(2)_{\mathrm{R}}$ and the extra SU(2) group that contains $\mathrm{U}(1)_{\mathrm{R}}$. More precisely, taking the vector multiplet to three dimensions, the four-dimensional gamma matrices are related to the three-dimensional ones, properly combined with the SU(2) generators denoted by $\hat\sigma^a$. Returning to four dimensions in the same way as before, but on the basis of the newly defined spinors, implies that the four-dimensional gamma matrices are now formed from the three-dimensional gamma matrices combined with the $\mathrm{SU}(2)_{\mathrm{R}}$ generators σ^a. Thus the mere switch in the spinor basis suffices to correctly implement the mirror map.

The fermion basis (5.6) shows an obvious decomposition of the index α according to $\alpha = (I, r)$ with the index r taking values $r = 1, 2$; a similar decomposition holds for $\bar\alpha$. This decomposition will be used below. For instance, the $\mathrm{Sp}(1)\times\mathrm{Sp}(n)$ one-forms, can be written as $V_{Ai}^{\alpha}\,\mathrm{d}\phi^A = (V_A^I\,\mathrm{d}\phi^A)^r{}_i$. Using (5.6) we can now identify these one-forms as well as the $\mathrm{Sp}(n)$ connections for a hypermultiplet theory that originates from a four-dimensional vector multiplet theory by comparing the fermion supersymmetry transformations on vector and hypermultiplet sides. We thus find (strictly speaking the indices i now run over $+,-$),

$$V_{Ai}^{\alpha}\,\mathrm{d}\phi^A = \Big(V_A^I\,\mathrm{d}\phi^A\Big)^r{}_i = 2\begin{pmatrix} \mathrm{d}X^I & N^{IK}\bar{\mathcal{W}}_K \\ -N^{IK}\mathcal{W}_K & \mathrm{d}\bar X^I \end{pmatrix}, \tag{5.7}$$

where $\mathcal{W}_I = \mathrm{d}B_I - F_{IJ}\mathrm{d}A^J$ and $\alpha = (I, r)$, and

$$\Gamma_A{}^{\alpha}{}_{\beta}\,\mathrm{d}\phi^A = \Big(\Gamma_A\,\mathrm{d}\phi^A\Big)^{I\,r}{}_{J\,s} = \begin{pmatrix} -iN^{IK}\,\mathrm{d}F_{KJ} & -iN^{IK}\bar F_{KJL}N^{LM}\mathcal{W}_M \\ -iN^{IK}F_{KJL}N^{LM}\bar{\mathcal{W}}_M & iN^{IK}\,\mathrm{d}\bar F_{KJ} \end{pmatrix}, \tag{5.8}$$

with $\alpha = (I, r)$ and $\beta = (J, s)$. Observe that the above quantities all take their values in the quaternions.

From the transformation rules and/or the action we can now determine all the relevant quantities in the hypermultiplet sector, such as the metric, the complex structures and the antisymmetric tensor Ω. They are all consistent with the general results for hypermultiplets, derived in the previous section. Let us first give the expressions for the fermionic metric $G_{\bar\alpha\beta}$ and the antisymmetric tensor $\Omega_{\bar\alpha\bar\beta}$,

$$G_{\bar\alpha\beta} = \tfrac14 N_{IJ}\,\delta_{rs}\,, \qquad \Omega_{\bar\alpha\bar\beta} = \tfrac14 N_{IJ}\,\varepsilon_{rs}\,. \tag{5.9}$$

The one-forms γ^A take the form,

$$\gamma_{i\bar{\alpha}\,A}\,\mathrm{d}\phi^A = \Big(\gamma_{AI}\,\mathrm{d}\phi^A\Big)_{ri} = \tfrac{1}{2}\begin{pmatrix} N_{IK}\mathrm{d}X^K & \bar{\mathcal{W}}_I \\ -\mathcal{W}_I & N_{IK}\mathrm{d}\bar{X}^K \end{pmatrix}\,, \tag{5.10}$$

where $\bar{\alpha} = (r, I)$. Furthermore, we present the fermionic Lagrangian that follows from (3.6) and (5.6), which exhibits most of the geometric quantities, such as the tensor W defined in (4.20),

$$\begin{aligned} 4\pi\,\mathcal{L}_{\rm ferm} \;=\;& -\tfrac{1}{4}N_{IJ}\Big(\bar{\zeta}^{\overline{I1}}\partial\!\!\!/\zeta^{J1} + \bar{\zeta}^{\overline{I2}}\partial\!\!\!/\zeta^{J2} + \text{h.c.}\Big) \\ & +\tfrac{1}{4}iF_{IJK}\Big(\bar{\zeta}^{\overline{I1}}\partial\!\!\!/X^J\zeta^{K1} - \bar{\zeta}^{\overline{I2}}\partial\!\!\!/X^J\zeta^{K2} + \\ & \qquad 2N^{JL}\,\bar{\zeta}^{\overline{I2}}(\partial\!\!\!/B_L - F_{LM}\partial\!\!\!/A^M)\zeta^{K1}\Big) + \text{h.c.} \\ & -\tfrac{1}{24}i\Big(F_{IJKL} + 3iN^{MN}F_{M(IJ}F_{KL)N}\Big)\,\bar{\zeta}^{\overline{I2}}\gamma_\mu\zeta^{J1}\,\bar{\zeta}^{\overline{K2}}\gamma^\mu\zeta^{L1} + \text{h.c.} \\ & -\tfrac{1}{24}N^{MN}F_{MIJ}\bar{F}_{NKL}\Big(\bar{\zeta}^{\overline{K1}}\gamma_\mu\zeta^{I1} - \bar{\zeta}^{\overline{I2}}\gamma_\mu\zeta^{K2}\Big) \\ & \qquad \times\Big(\bar{\zeta}^{\overline{L1}}\gamma^\mu\zeta^{J1} - \bar{\zeta}^{\overline{J2}}\gamma^\mu\zeta^{L2}\Big) \\ & -\tfrac{1}{12}N^{MN}F_{MIJ}\bar{F}_{NKL}\,\bar{\zeta}^{\overline{I1}}\gamma_\mu\zeta^{J2}\,\bar{\zeta}^{\overline{K1}}\gamma^\mu\zeta^{L2}\,. \end{aligned} \tag{5.11}$$

The tensor W defined in (4.20), is thus expressed in terms of the tensor C_{IJKL}, defined in (4.10), and the curvature tensor of the special Kähler space given in (2.5). Both these tensors, and therefore the tensor W, are covariant with respect to the symplectic reparametrizations of the underlying special Kähler manifold. The tensor W fully encodes the curvature tensor of the special hyper-Kähler manifold. We refrain from giving explicit formulae, but wish to point out that these expressions allow for a coordinate-independent characterization of the special hyper-Kähler manifolds. We have also verified that the tensor W becomes fully symmetric when written in purely (anti)holomorphic indices, employing the result for the tensors $G_{\bar{\alpha}\beta}$ and $\Omega_{\bar{\alpha}\bar{\beta}}$ given above.

It is clear from their index structure that the one-forms (5.7) transform covariantly under the symplectic reparametrizations of the underlying vector multiplet by multiplication from the left with matrices

$$S^{Ir}{}_{Js} = \begin{pmatrix} \mathcal{S}^I{}_J & 0 \\ 0 & \bar{\mathcal{S}}^I{}_J \end{pmatrix}\,, \tag{5.12}$$

while the one-forms (5.10) transform from the left with $[\bar{S}^{-1}]^J{}_I$. In general these transformations are not contained in the holonomy group $\mathrm{Sp}(n)$.

The above thus constitutes the full construction of a hypermultiplet model in four space-time dimensions associated with a specific theory based on vector multiplets. The detour through three dimensions only serves as a means to arrive at these results. Unlike the corresponding theory of vector multiplets, the hypermultiplet theory does not exhibit an $\mathrm{SU}(2)_{\mathrm{R}}$ invariance, at least not in the generic case. Only a manifest $\mathrm{U}(1)_{\mathrm{R}}$ invariance remains. All the isometries of the vector-multiplet target space that represent invariances of the full set of equations of motion, remain present as isometries of the hypermultiplet target space. The symplectic reparametrizations of the vector multiplets induce corresponding transformations on the hyper-Kähler side. In this way we deal with a large class of hyper-Kähler spaces. They can be expressed in terms of certain restrictions on the curvature tensor.

We should stress that the general hypermultiplet action is encoded in the one-forms V_i^α, but one has to provide one extra ingredient, such as the fermionic metric $G_{\bar\alpha\beta}$, or the antisymmetric tensor $\Omega_{\bar\alpha\bar\beta}$. The expressions given above for these quantities concern the special hyper-Kähler spaces and are given in special coordinates. As already alluded to earlier, it is straightforward to write them in a coordinate-independent way. In the case of local supersymmetry, the one-forms will become Sp(1) sections subject to an appropriate projective condition.

As a last application of the mirror map we turn to the possible central charges that can emerge in the supersymmetry algebra for a theory based on vector multiplets or hypermultiplets. As the symplectic reparametrizations can be performed in a supergravity background [10], the algebra and therefore the expressions for the central charges should be invariant under these reparametrizations. Likewise, the charges should be consistent with the underlying Kähler or hyper-Kähler geometry. We will determine the central charges by evaluating the possible surface terms on the right-hand side of the anticommutator of two supercharges. To determine this anticommutator we use canonical quantization. This approach is the same as the one followed in [29] for the elementary super-Yang-Mills system. Here we apply it for an arbitrary function $F(X)$ and arbitrary hyper-Kähler metrics.

Let us first present the supercurrent for the vector multiplet and hypermultiplet theories,

$$\begin{aligned} J_{\mu\, i} &= \frac{1}{8\pi}\Big\{N_{IJ}\,\partial\!\!\!/\bar X^I\gamma_\mu\Omega_i^J + \tfrac{1}{2}i\varepsilon_{ij}\,\mathcal{G}^-_{\rho\sigma\, I}\,\sigma^{\rho\sigma}\gamma_\mu\Omega^{jI} + \tfrac{1}{12}i\bar F_{IJK}\,\gamma_\mu\Omega^{kI}\,\bar\Omega^{lJ}\Omega^{jK}\,\varepsilon_{ij}\varepsilon_{kl}\Big\}\,, \\ J_{\mu\, i} &= \frac{1}{4\pi}\,g_{AB}\,\gamma^A_{i\bar\alpha}\,\partial\!\!\!/\phi^B\gamma_\mu\zeta^{\bar\alpha}\;. \end{aligned} \tag{5.13}$$

where $\mathcal{G}^-$ was defined in (2.7). The other chirality components follow by complex conjugation. Observe that the first one is invariant under symplectic reparametrizations. Obviously the second expression for the hypermultiplet current is invariant under the hyper-Kähler holonomy group. The reader may be surprised that the vector-multiplet current contains terms cubic in the fermion fields, whereas the hypermultiplet current is linear in the fermion fields. Still one can verify, by performing the duality transformation in the presence of the gravitino field coupling to the supercurrent, that the expressions for the two currents become compatible upon reduction to three dimensions.

To determine the central charges one needs only the Dirac brackets for the fermions, as the bosonic brackets lead to terms at least quadratic in the fermion fields, which represent supersymmetric completions of bosonic terms that are already present in the algebra. In this way we find the following commutators for the vector multiplet,

$$\begin{aligned} \{Q_i, \bar{Q}^j\} &= i\hbar\, \frac{1-\gamma^5}{2}\, \delta_i{}^j \left\{\gamma_\mu\, P^\mu + \gamma_a\, Z^a\right\}, \\ \{Q_i, \bar{Q}_j\} &= -i\hbar\, (1-\gamma^5)\, \varepsilon_{ij} \left\{\bar{X}^I\, q_{\mathrm{e}I} - \bar{F}_I\, q_{\mathrm{m}}^I\right\}, \end{aligned} \tag{5.14}$$

where the vector central charge, Z^a, is defined by $(a,b,c$ denote spatial indices),

$$Z^a = \frac{i}{8\pi}\varepsilon^{abc} \int \mathrm{d}^3x\; N_{IJ}\, \partial_b X^I\, \partial_c \bar{X}^J\,, \tag{5.15}$$

which is an integral over the Kähler two-form; the second anticommutator yields the anti-holomorphic BPS mass expressed in terms of the values of $\bar{X}^I$ and $\bar{F}_I$ taken at spatial infinity (to obtain this result we used the field equations for the vector fields) and the electric and magnetic charges.[9] Obviously the central charges are invariant under symplectic reparametrizations, as predicted above. For the case of a quadratic function F our result for the second commutator coincides with that in [29]. The Kähler form contribution was presented in [30].

For the hypermultiplets we find a similar result for the anticommutators,

$$\begin{aligned} \{Q_i, \bar{Q}^j\} &= i\hbar\, \frac{1-\gamma^5}{2} \left\{\delta_i{}^j\, \gamma_\mu\, P^\mu + (\sigma^\Lambda)_i{}^j\, \gamma_a\, Z^{\Lambda\, a}\right\}, \\ \{Q_i, \bar{Q}_j\} &= 0\,, \end{aligned} \tag{5.16}$$

where we now have three vector central charges defined by

$$Z^{\Lambda\, a} = -\frac{1}{16\pi}\varepsilon^{abc} \int \mathrm{d}^3x\; J^\Lambda_{AB}\, \partial_b\phi^A \partial_c\, \phi^B\,. \tag{5.17}$$

[9]The charges $q_{\mathrm{e}I}$ and q_{m}^I are related to electric and magnetic charges and are defined in terms of flux integrals over closed spatial surfaces that surround the charged objects (quantized on a lattice with elementary area equal to $2\hbar$),

$$2\pi\, q_{\mathrm{m}}^I = \oint_{\partial V} (F^+ + F^-)^I\,, \qquad 2\pi\, q_{\mathrm{e}I} = \oint_{\partial V} (G^+ + G^-)_I\,.$$

This definition shows that the charges $(q_{\mathrm{m}}^I, q_{\mathrm{e}I})$ transform under symplectic reparametrizations precisely as the field strengths (F^I, G_I).

The J^Λ are the three complex structures of the hyper-Kähler space defined in (4.9).

There is a clear systematics in the above results. Note that the central charges for the vector multiplet are singlets under $\mathrm{SU}(2)_\mathrm{R}$, whereas those for the hypermultiplets transform as a triplet under this group. In addition to the BPS mass, we find certain integrals over the pull back of the Kähler form (for the vector multiplet) and the hyper-Kähler forms (for the hypermultiplet). Naively, all these integrals vanish, as we can write (locally in the target space) these two-forms as the exterior derivative of corresponding one-forms. This then allows us to write the integrands as total derivatives in the base space, which can be dropped subject to certain reasonable assumptions on the asymptotic values of the scalar fields. Hence the question whether these charges are actually realized depends on the kind of boundary conditions that one wishes to impose. For instance, in $3+1$ dimensions, if one imposes boundary conditions at spatial infinity such that the fields converge in all directions to the same value, with the derivatives vanishing sufficiently fast so as to ensure finite energy, then the central charges associated with the two-forms will vanish. In $2+1$ dimensions, the situation is different. In that case the central charges are expressed as integrals of the (hyper-)Kähler two-forms over the image of ϕ. Topologically this image is S^2, so that the central charges are enumerated by the second homotopy group of the target-space manifold. Obviously the central charges set a BPS bound in the usual fashion.

From the perspective of this paper it is of interest to see how the central charges of the vector multiplet sector and the hypermultiplet sector are related by mirror symmetry. When suppressing the dependence on the compactified coordinate x^3 the central charges Z^3 and $Z^{\Lambda 3}$ can be finite. It is then straightforward to write down the supersymmetry algebra corresponding to (5.14) in three dimensions. One subtlety is that the momentum in the third direction is also a surface integral, which should be added to the central charge associated with the Kähler form. As it turns out, the resulting two-form corresponds then precisely with the Kähler form ω^3 defined in (3.28) for the hyper-Kähler space.

In order to apply the mirror map, we write the charges in an alternative basis in correspondence with the new basis (5.4) for the supersymmetry parameters,

$$\begin{aligned} Q^+ &= \tfrac{1}{2}\sqrt{2}\gamma^3(Q_1 - iQ_2)\,, \qquad & Q^- &= \tfrac{1}{2}\sqrt{2}(Q^1 - iQ^2)\,, \\ Q_+ &= \tfrac{1}{2}\sqrt{2}\gamma^3(Q^1 + iQ^2)\,, \qquad & Q_- &= \tfrac{1}{2}\sqrt{2}(Q_1 + iQ_2)\,. \end{aligned} \tag{5.18}$$

With these definitions, the three-dimensional version of (5.14) reads

$$\{Q_\pm, \bar{Q}^\pm\} \;=\; i\hbar\frac{1-\gamma^5}{2}\left\{\gamma_\mu P^\mu \mp iZ'\right\},$$

$$
\begin{aligned}
\{Q_+,\bar Q^-\} &= -2i\hbar\frac{1-\gamma^5}{2}\left\{X^I\,q_{\mathrm{e}I}-F_I\,q^I_{\mathrm{m}}\right\},\\
\{Q_-,\bar Q^+\} &= 2i\hbar\frac{1-\gamma^5}{2}\left\{\bar X^I\,q_{\mathrm{e}I}-\bar F_I\,q^I_{\mathrm{m}}\right\},
\end{aligned}
\tag{5.19}
$$

where Z' is now defined in terms of the hyper-Kähler two-form ω^3. This result coincides with the algebra relevant to the hypermultiplets upon reduction to three spacetime dimensions, which reads,

$$
\begin{aligned}
\{Q_i,\bar Q^j\} &= i\hbar\,\frac{1-\gamma^5}{2}\left\{\delta_i{}^j\,\gamma_\mu\,P^\mu+i(\sigma^\Lambda)_i{}^j\;Z^{\Lambda\,3}\right\},\\
\{Q_i,\bar Q_j\} &= 0\,.
\end{aligned}
\tag{5.20}
$$

This demonstrates that the supersymmetry algebra remains consistent with the mirror map in the presence of the central charge configurations. A gratifying feature of this result is that the holomorphic BPS mass of the vector multiplets is mapped to the holomorphic hyper-Kähler two-forms, $\omega^\pm$, defined in (3.28).

Although the above results do not capture the full dynamics of the four-dimensional gauge theories in a circle compactification, they are consistent with the results derived in the context of three-dimensional gauge dynamics [19]. There the two sets of central charges are associated with explicit mass terms and Fayet-Iliopoulos terms, which are interchanged under the quantum mirror symmetry. The relation of the central charges with integrals of the hyper-Kähler two-forms also arose in that context.

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