

Special Geometry and Compactification on a Circle

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Abstract: We discuss some consequences of our previous work on rigid special geometry in hypermultiplets in 4-dimensional Minkowski spacetime for supersymmetric gauge dynamics when one of the spatial dimensions is compactified on a circle.

1 Introduction

Special geometry appears in the context of $N = 2$ supersymmetric gauge theories in 4 spacetime dimensions. Such theories arise as low-energy effective actions of type-II strings compactified on a Calabi-Yau manifold \mathcal{X} . The vector multiplets exhibit special geometry, with the corresponding scalars parametrising a so-called special Kähler manifold. The hypermultiplet moduli space (which is a quaternionic space) of type IIB/A is related to the vector-multiplet special Kähler manifold of type IIA/B by string duality (at least in string perturbation theory). This connection was first studied in [1] in terms of the **c**-map which converts special Kähler into quaternionic manifolds. The **c**-map can be induced by dimensional reduction from 4 to 3 spacetime dimensions. It relates the classical moduli spaces of vector multiplets and hypermultiplets. To go beyond this description, one may consider type-II strings compactified on $\mathcal{X} \times S^1$ and perform a T-duality on the circle [2]. Such a T-duality relates the IIA and IIB strings [3]. When studying the 4-dimensional theories with a compactified coordinate, it is therefore of interest to go beyond a trivial dimensional reduction and retain all the modes associated with the S^1 -compactification.

Motivated by this, we study supersymmetric gauge theories with one coordinate compactified on S^1 . We restrict ourselves to rigid $N = 2$ supersymmetry throughout. In [4] we considered the zero-modes of the vector multiplets in an S^1 -compactification for general abelian supersymmetric actions. After dualising vector multiplets into hypermultiplets, one can thus define a notion of (rigid) special geometry in hypermultiplets. The aim there was to identify and study the special-geometry features for the corresponding hyper-Kähler geometries. In [4] the dynamical effects associated with the S^1 -compactification did not play a role. In this note we discuss the results of [4], also taking into account certain effects caused by the massive modes. It is known that the supersymmetric gauge theories exhibit interesting dynamics when compactified on S^1 . In [5] this was analysed for the gauge group $SU(2)$.

Before turning to the hyper-Kähler manifolds let us devote the remainder of this section to a discussion of vector multiplets and symplectic reparametrisations. Vector multiplets contain bosonic fields X^I and A_μ^I , with $I = 1, \dots, n$ labeling the vector multiplets. The complex scalar fields X^I parametrise a Kähler manifold characterised by a holomorphic function $F(X)$. The Kähler potential is given by

$$K(X, \bar{X}) = -i\bar{X}^I F_I + i\bar{F}_I X^I, \quad (1)$$

where F_I denotes the derivative of F with respect to X^I . The symplectic group $\text{Sp}(2n, \mathbf{R})$ which acts on the (anti-)selfdual components of the field strenghts also acts on the scalar fields and on the function $F(X)$. This can be expressed by introducing ‘sections’ (X^I, F_I) which transform under the symplectic group by multiplication with an $\text{Sp}(2n, \mathbf{R})$ matrix,

$$\begin{pmatrix} X^I \\ F_I \end{pmatrix} \longrightarrow \begin{pmatrix} \tilde{X}^I \\ \tilde{F}_I \end{pmatrix} = \begin{pmatrix} U & Z \\ W & V \end{pmatrix} \begin{pmatrix} X^I \\ F_I \end{pmatrix}. \quad (2)$$

From this, one can find the transformation of F_{IJ} , which is of the same form as the transformation of a period matrix of a genus- n Riemann surface under a redefinition of the homology basis (eventually the symplectic reparametrisations will also be restricted to $\text{Sp}(n, \mathbf{Z})$). Denoting $\mathcal{S}^I_J = \partial \tilde{X}^I / \partial X^J$, one has

$$\tilde{F}_{IJ} = (V_I^K F_{KL} + W_{IL}) [\mathcal{S}^{-1}]^L_J. \quad (3)$$

From these quantities one can construct symplectically covariant objects like, for instance, the Kähler potential. Other well-known examples are [6]

$$C_I = F_I - F_{IJ} X^J = \partial_I G, \quad G = 2F - X^I F_I, \quad F_{IJK}. \quad (4)$$

Symplectically covariant means that $\tilde{C}_I(\tilde{X}) = C_J(X) [\mathcal{S}^{-1}]^J_I$; likewise F_{IJK} transforms as a 3-rank tensor and G transforms as a function, i.e. $\tilde{G}(\tilde{X}) = G(X)$. For a single vector multiplet, using standard notation C_I equals

$$C = a_D - \tau a. \quad (5)$$

where τ denotes the second derivative of $F(a)$. The matrix \mathcal{S} is then equal to $\mathcal{S} = U + Z\tau$, so that, when expressing the functions C , G and the third derivative of F (which can be written as $d\tau/da = -1/(d^2C/d\tau^2)$) in terms of τ , the symplectic reparametrisations take the form of modular transformations acting on modular forms of weights -1 , 0 and -3 , respectively. When the $\text{U}(1)$ theory describes the Wilsonian effective action of an underlying pure $\text{SU}(2)$ gauge theory [7], the function G is *invariant* under the subgroup $\Gamma(2)$ of $\text{Sp}(2, \mathbf{Z})$ that is associated with the monodromies [6]. The function G was studied in [8, 9] and shown to measure the scaling violation of the underlying microscopic theory. The modular form C is the central object in Nahm’s approach [10] to construct the Seiberg-Witten solution. Knowing the poles of C as a function of τ determines it up to a multiplicative factor. The periods then follow from the relation $a = -dC/d\tau$. A generalisation of this approach for higher-rank gauge groups has not been given so far. This would require the study of automorphic forms, of which much less is known. These automorphic forms also appear in the context of $N = 2, d = 4$ heterotic or type-II string compactifications, see e.g. [14].

As was shown in [11] (see also [12]) F_{IJK} satisfies a WDVV-like [13] equation. Denoting matrices $(F_I)_{MN} = F_{IMN}$, the equation takes the form

$$F_I F_K^{-1} F_J = F_J F_K^{-1} F_I. \quad (6)$$

It is obviously consistent with symplectic transformations because F_{IJK} transforms as a tensor. In the case of $\text{SU}(2)$ and $\text{SU}(3)$, the equation is trivial.

Because the supercharges are symplectic invariants, central charges in the supersymmetry algebra are also symplectically invariant objects. An example of this is the well-known BPS mass, which is defined in terms of the electric and magnetic charges that transform under symplectic reparametrisations as a $2n$ -component vector. These charges parametrise a certain lattice and the symplectic group must be restricted to the integer-valued subgroup that leaves this lattice invariant. We return to the central charges in the next section.

2 Special geometry and hyper-Kähler manifolds

In this section, we discuss how symplectic transformations in the hyper-Kähler manifolds are induced by those on the vector-multiplet side. As we will see, a number of new symplectically covariant objects can

be defined in terms of hypermultiplets. Let us first consider the reduction to 3 dimensions, where we only retain the zero-modes on S^1 . We start from a 4-dimensional (effective) theory of abelian vector multiplets based on a function $F(X)$ and a corresponding Kähler potential K . Upon dimensional reduction, one obtains new complex scalars Y_I originating from the gauge fields,

$$Y_I = B_I - F_{IJ}A^J. \quad (7)$$

Here A^I are the gauge fields A_μ^I with the index μ corresponding to the compactified coordinate. The scalar fields B_I come from dualising the 3-dimensional abelian gauge fields. They are the Lagrange multipliers that were introduced to enforce the Bianchi identities. Under symplectic transformations (A^I, B_I) transforms as a $2n$ -component vector (cf. (2)). Consequently Y_I transforms as a co-vector, i.e. $\bar{Y}_I = Y_J[\mathcal{S}^{-1}]^J_I$. The existence of $N = 4$ supersymmetry in 3 dimensions implies that the manifold parametrised by the complex scalars (X^I, Y_I) must be hyper-Kähler. It was shown in [1, 4] that a corresponding Kähler potential is given by

$$K(X, Y, \bar{X}, \bar{Y}) = -i\bar{X}^I F_I + i\bar{F}_I X^I - \frac{1}{2}(Y - \bar{Y})_I N^{IJ}(Y - \bar{Y})_J, \quad (8)$$

which is invariant under symplectic transformations up to Kähler transformations. Here N^{IJ} denotes the inverse of $N_{IJ} = -iF_{IJ} + i\bar{F}_{IJ}$.

Assuming that the radius of the circle is equal to R , one imposes periodic boundary conditions and expands the fields in Fourier modes. The massless zero-mode discussed above is the only one that survives the (naive) $R \rightarrow 0$ limit, as the other modes have masses proportional to $1/R$. When studying the effective actions of an underlying microscopic theory, these massive modes are integrated out in the Wilsonian sense. For the moment we will neglect them. From gauge transformations with non-trivial winding in the compactified direction, it follows that the (zero-mode) fields A^I are defined modulo $1/R$; from the periodicity of the generalised theta angles it follows that also the fields B_I are periodic with the same period (provided one chooses a suitable normalisation of the Lagrange multipliers). The fields A^I and B_I span a torus T^{2n} above each point of the special Kähler moduli space whose volume is equal to $(4R)^{-n}$. At this point one may identify the torus at a given point X in the special Kähler space with the Jacobian variety of an auxiliary Riemann surface \mathcal{M}_X that can be associated with some underlying 4-dimensional dynamics of a nonabelian gauge theory in the Coulomb phase [7], with its period matrix given by $F_{IJ}(X)$. The scalars Y_I take their values in this Jacobian. We now note the existence of the following holomorphic one-forms, which are manifestly covariant under symplectic reparametrisations,

$$\mathcal{W}_I = dB_I - F_{IJ}dA^J. \quad (9)$$

These forms appear in the symplectically covariant $\text{Sp}(1) \times \text{Sp}(n)$ one-forms that characterise the hypermultiplet couplings and supersymmetry transformations [4]. A symplectically invariant holomorphic two-form is

$$\omega = R dX^I \wedge \mathcal{W}_I = R dX^I \wedge dY_I = R \left(dX^I \wedge dB_I - dF_I \wedge dA^I \right). \quad (10)$$

This two-form, its conjugate and the Kähler two-form corresponding to (8) are symplectically invariant and closed. We return to these three hyper-Kähler forms later.

Assuming that the sections (X^I, F_I) can be written in terms of a number of modular parameters u^α , we consider the following symplectically invariant one-forms on the Jacobian variety,

$$\lambda_\alpha = R \frac{\partial X^I}{\partial u^\alpha} \mathcal{W}_I = R \left(\frac{\partial X^I}{\partial u^\alpha} dB_I - \frac{\partial F_I}{\partial u^\alpha} dA^I \right). \quad (11)$$

The integrals of these one-forms along the one-cycles α^I and β_I associated with the coordinates A^I and B_I , yield

$$\partial_\alpha X^I(u) = \oint_{\beta_I} \lambda_\alpha, \quad \partial_\alpha F_I(u) = - \oint_{\alpha^I} \lambda_\alpha. \quad (12)$$

The symplectic transformations on the left-hand side of these equations are now induced by the transformations of the homology cycles that leave the canonical intersection matrix invariant.

The result (12) can also be formulated in terms of the corresponding homology cycles of the underlying Riemann surface. In the special case of $SU(2)$, the Jacobian variety and the Riemann surface can be identified with the same torus with modular parameter τ [5]. The complex coordinate we use on the torus is $Y = B - \tau A$. One can easily compute the period “matrix” from the lengths of the A and B cycles (see [4]). It is independent of the compactification radius,

$$\frac{l_A}{l_B} = |\tau| . \quad (13)$$

One thus expects that there is a relation between the Seiberg-Witten holomorphic one-form λ and $\mathcal{W} = dB - \tau dA$. Following the discussion of [5] (section 3.1), adapted to our notation, one has

$$\lambda = \frac{dx}{y} = R \frac{da}{du} \mathcal{W} = R \left(\frac{da}{du} dB - \frac{da_D}{du} dA \right) , \quad (14)$$

where the torus is parametrised by e.g. $y^2 = (x-1)(x+1)(x-u)$. The holomorphic two-form ω over the hyper-Kähler manifold, defined in (10), equals

$$\omega = R da \wedge dY = \frac{du \wedge dx}{y} . \quad (15)$$

The hyper-Kähler two-forms corresponding to ω , its complex conjugate and the Kähler form, play an important role in the discussion of central charges. From the anticommutator of the supercharges, it follows that both vector multiplets and hypermultiplets may be subject to three central charges in the susy algebra. For vector multiplets one has the Kähler two-form central charge and the (anti-)holomorphic BPS mass. All three are symplectically invariant. For hypermultiplets, the central charges are integrals over the three hyper-Kähler two-forms. As was discussed in [4], the central charges in 3 dimensions are enumerated by the second homotopy group of the target-space manifold. For hyper-Kähler spaces that are in the image of the \mathbf{c} -map, they are symplectically invariant. To realise these central charges explicitly, we take the hyper-Kähler manifold with the torus T^{2n} fibered over the special Kähler space. The two-forms we have to integrate over are closed and thus locally exact. Indeed, for the holomorphic two-form ω one has

$$dX^I \wedge dY_I = d(X^I dB_I - F_I dA^I) . \quad (16)$$

The corresponding central charge can be written as

$$Z = R \int_{C_1} (X^I dB_I - F_I dA^I) , \quad (17)$$

with C_1 a non-vanishing one-cycle in the hyper-Kähler space. A relevant cycle can be chosen on the hyper-torus T^{2n} and decomposed in terms of a canonical homology basis of one-cycles α^I and β_I as

$$C_1 = q_{eI} \alpha^I + q_{mI}^I \beta_I , \quad (18)$$

with integer coefficients q_e and q_m . This leads to

$$Z = X^I q_{eI} - F_I q_{mI}^I . \quad (19)$$

Of course, because of the \mathbf{c} -map, this is precisely the central charge of the vector multiplet model we started with. It illustrates how the torus is related to the lattice of electric and magnetic charges.

3 The effective actions and Kaluza-Klein modes

We now consider some dynamical effects when compactifying the theory on a circle of fixed radius R . For simplicity, we consider the case of pure $SU(2)$ supersymmetric Yang-Mills theory. In the decompactification limit $R \rightarrow \infty$, the classical plus perturbative contributions to the function F that encodes the Wilsonian effective action, is given by

$$F(X) = \frac{1}{2}\tau_0 X^2 + \frac{i}{2\pi} X^2 \ln \frac{X^2}{\Lambda^2} . \quad (20)$$

Here, τ_0 describes the bare coupling and theta parameter according to $\tau_0 = \theta_0/2\pi + 4\pi i/g_0^2$. The Kähler potential and metric are equal to

$$K_{4d} = \frac{8\pi}{g_0^2} X \bar{X} + \frac{2X\bar{X}}{\pi} \left[\ln \frac{X\bar{X}}{\Lambda^2} + 1 \right], \quad N_{X\bar{X}} = \frac{8\pi}{g_0^2} + \frac{2}{\pi} \left[\ln \frac{X\bar{X}}{\Lambda^2} + 3 \right] . \quad (21)$$

The constant in front of the perturbative corrections is equal to the one-loop beta function of $SU(2)$ supersymmetric Yang-Mills theory.

In 4 dimensions, the effective action with at most two derivatives is encoded in a holomorphic function $F(X)$. This follows from requiring that the action depends only on gauge-covariant objects, such as field strengths and covariant derivatives. When one of the dimensions is compactified on S^1 , the corresponding zero modes A^I associated with the phase of the Wilson line around S^1 , is gauge invariant and can appear in the effective action in a less restricted way. The only implication from gauge invariance is that the effective action is invariant under shifts of A^I with a multiple of $1/R$. The presence of the fields A^I causes a change in the holomorphic structure of the various quantities.

To illustrate this more concretely, let us compute the renormalisation of the coupling constant in the S^1 -compactification for $SU(2)$. We first note that the unrenormalised coupling constants for the 4- and 3-dimensional theories, denoted by g_0 and e_0 , respectively, are related by

$$g_0^2 = 2\pi R e_0^2 . \quad (22)$$

The one-loop contribution to the coupling constant e at finite R is given by [16, 15]

$$\frac{1}{e^2} = \frac{1}{e_0^2} + 4i \sum_{n=-\infty}^{+\infty} \int \frac{d^3 p}{(2\pi)^3} \left[p^2 - M^2 - \left(A + \frac{n}{R} \right)^2 \right]^{-2} = \frac{1}{e_0^2} - \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} \left[M^2 + \left(A + \frac{n}{R} \right)^2 \right]^{-\frac{1}{2}} , \quad (23)$$

where we assume a constant background of the scalar fields, X and A . Here $M^2 = 2X\bar{X}$ denotes the mass of the charged particles in the 4-dimensional theory. Note the manifest periodicity of A in units of $1/R$. In the decompactification limit, the above expression yields

$$\frac{1}{e^2} = \frac{1}{e_0^2} + 8i\pi R \int \frac{d^4 p}{(2\pi)^4} \frac{1}{[p^2 - M^2]^2} = \frac{1}{e_0^2} + \frac{R}{2\pi} \log \frac{X\bar{X}}{\Lambda^2} , \quad (24)$$

where we have cut off the momentum integral at a scale Λ , chosen such as to coincide with the cut-off in (21). Observe that the A -dependence has disappeared, which can be understood from the fact that in the uncompactified case, a constant vector potential is gauge equivalent to zero.

For arbitrary value of R we can further evaluate (23) by means of a Poisson resummation [17],

$$\frac{1}{e^2} = \frac{1}{e_0^2} + \frac{R}{2\pi} \log \frac{X\bar{X}}{\Lambda^2} - \frac{2R}{\pi} \sum_{n=1}^{+\infty} K_0(2\pi R M n) \cos(2\pi R A n) , \quad (25)$$

where we have again adjusted the cut-off Λ such as to make contact with (21). For large R , the modified Bessel function in the infinite sum vanishes exponentially, so that we indeed recover the result (24). Note that $2\pi R A$ equals the flux through the α -cycle of the torus.

In the limit $R \rightarrow 0$, the S^1 -modes become infinitely heavy and decouple (up to some renormalisation effect), except for the $n = 0$ mode. One is left with the 3-dimensional result and (23) yields a charge renormalisation given by $1/e^2 = 1/e_0^2 - 1/(2\pi\sqrt{2XX + A^2})$ [15]. Here we did absorb an infinite renormalisation into the definition of e_0 .

The above results demonstrate that the holomorphic structure that is characteristic for special geometry, is lost. The equivalence transformations of the 4-dimensional theories, which take the form of symplectic reparametrizations, are therefore no longer manifest in the S^1 -compactification. An intriguing question is, whether these equivalence transformations can still remain in some modified form, for instance, after combining with T-duality. This question deserves further study.

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