

YANG-MILLS- AND D-INSTANTONS

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Abstract

In these lectures, which are written at an elementary and pedagogical level, we discuss general aspects of (single) instantons in $SU(N)$ Yang-Mills theory, and then specialize to the case of $\mathcal{N} = 4$ supersymmetry and the large N limit. We show how to determine the measure of collective coordinates and compute instanton corrections to certain correlation functions. We then relate this to D-instantons in type IIB supergravity. By taking the D-instantons to live in an $AdS_5 \times S^5$ background, we perform explicit checks of the AdS/CFT correspondence.

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1 Introduction.

In the last decade we have seen an enormous progress in understanding non-perturbative effects both in supersymmetric field theories and superstring theories. When we talk about non-perturbative effects, we usually mean solitons and instantons, whose masses and actions, respectively, are inversely proportional to the coupling constant. Therefore, these effects become important in the strongly coupled regime. The typical examples of solitons are the kink and the magnetic monopole in field theory, and the D-branes in supergravity or superstring theories. In the context of supersymmetry, these solutions preserve one half of the supersymmetry and are therefore BPS. As for instantons, we have the Yang-Mills (YM) instantons [1], and there are various kinds of instantons in string theory, of which the D-instantons [2] are the most important for these lectures.

In more general terms, without referring to supersymmetry, instantons are solutions to the field equations in euclidean space with finite action, and describe tunneling processes in Minkowski space-time from one vacuum at time t_1 to another vacuum at time t_2 . The simplest model to consider is a quantum mechanical system with a double well potential having two vacua. Classically there is no trajectory for a particle to interpolate between the two vacua, but quantum mechanically tunneling occurs. The tunneling amplitude can be computed in the WKB approximation, and is typically exponentially suppressed. In the euclidean picture, after performing a Wick rotation, the potential is turned upside down, and it is possible for a particle to propagate between the two vacua, as described by the classical solution to the equations of motion (see e.g. [3]).

Also in YM theories, instantons are known to describe tunneling processes between different vacua, labeled by an integer winding number, and lead to the introduction of the CP-violating θ -term [4, 5]. It was hoped that instantons could shed some light on the mechanism of quark confinement. Although this was successfully shown in three-dimensional gauge theories [8], the role of instantons in relation to confinement in four dimensions is much more obscure. Together with the non-perturbative chiral $U(1)$ anomaly in the instanton background, which led to baryon

number violation and the solution to the $U(1)$ problem [6, 7], instantons have shown their relevance to phenomenological models like QCD and the Standard Model. To avoid confusion, note that the triangle chiral anomalies in perturbative field theories in Minkowski space-time are canceled by choosing suitable multiplets of fermions. There are, however, also chiral anomalies at the non-perturbative level. It is hard to compute the non-perturbative terms in the effective action which lead to a breakdown of the chiral symmetry by using methods in Minkowski space-time. However, by using instantons in euclidean space, one can relatively easily determine these terms. As we shall see, due to the presence of instantons there are fermionic zero modes (and also bosonic zero modes) which appear in the path integral measure. One must saturate these integrals and this leads to correlation functions of composite operators with fermion fields which do violate the chiral $U(1)$ symmetry. The new non-perturbative terms are first computed in euclidean space, but then continued to Minkowski space where they give rise to new physical effects [7]. They have the following form in the effective action

$$S_{\text{eff}} \propto \exp \left\{ -\frac{8\pi^2}{g^2} (1 + \mathcal{O}(g^2)) + i\theta \right\} (\lambda\lambda)^n , \quad (1.1)$$

where n depends on the number of fermionic zero modes. The prefactor is due to the classical instanton action and is clearly non-perturbative. The terms indicated by $\mathcal{O}(g^2)$ are due to standard radiative corrections computed by using Feynman graphs in an instanton background. The term $(\lambda\lambda)^n$ involving the chiral spinor λ comes from saturating the integration in the path integral over the fermionic collective coordinates and violates in general the chiral symmetry. On top of (1.1) we have to add the contributions from anti-instantons, generating $(\bar{\lambda}\bar{\lambda})^n$ terms in the effective action. As we shall discuss, in euclidean space the chiral and anti-chiral spinors are independent, but in Minkowski space-time they are related by complex conjugation, and one needs the sum of instanton and anti-instanton contributions to obtain a hermitean action.

In these lectures we will mainly concentrate on supersymmetric YM theories, especially on the $\mathcal{N} = 4$ $SU(N)$ SYM theory, and its large N limit. Instantons in $\mathcal{N} = 1, 2$ models have been extensively studied in the past, and still are a topic of current research. For the $\mathcal{N} = 1$ models, one is mainly interested in the calculation of the superpotential and the gluino condensate [9]. In some specific models, instantons also provide a mechanism for supersymmetry breaking [10], see [11] for a recent review on these issues. In the case of $\mathcal{N} = 2$, there are exact results for the prepotential [12], which acquires contributions from all multi-instanton sectors. These predictions were successfully tested in the one-instanton sector in [13], and for two-instantons in [14].

Our interest in $\mathcal{N} = 4$ SYM is twofold. On the one hand, since this theory is believed to be S-dual [15], one expects that the complete effective action, including all instanton and anti-instanton effects, organizes itself into an $SL(2, \mathbf{Z})$ invariant expression. It would be important to test this

explicitly using standard field-theoretical techniques. On the other hand, not unrelated to the previous, we are motivated by the AdS/CFT correspondence [16]. In this picture, D-instantons in type IIB supergravity are related to YM-instantons in the large N limit [17, 18]. By making use of the work by Green et al. on D-instantons [19], definite predictions come out for the large N SYM theory, which were successfully tested to leading order in the coupling constant, in the one-instanton sector in [20], and for multi-instantons in [21]. Although these calculations are already sufficiently complicated, it is nevertheless desirable to go beyond leading order, such that one can obtain exact results for certain correlation functions at the non-perturbative level. This will be the guideline for our subsequent investigation.

The lectures are set up as follows. In section 2, we discuss the bosonic YM instanton solution for $SU(N)$ and relate the counting of bosonic collective coordinates to the index of the Dirac operator. Section 3 deals with fermionic collective coordinates, parametrizing the solutions of the Dirac equation in the background of an instanton. We write down explicit formulae for these solutions in the one-instanton sector and elaborate further on the index of the Dirac operator. For multi-instantons, one must use the ADHM construction [22], which is beyond the scope of these lectures. There are already comprehensive reviews on this topic [23, 21]. Section 4 gives a treatment of the zero modes and the one-instanton measure on the moduli space of collective coordinates. We explain in detail the normalization of the zero modes since it is crucial for the construction of the measure. In section 5 we discuss the one-loop determinants in the background of an instanton, arising from integrating out the quantum fluctuations. We then apply this to supersymmetric theories, and show that the determinants for all supersymmetric YM theories cancel each other [24].

Starting from section 6, we apply the formalism to $\mathcal{N} = 4$ SYM theory. We explicitly construct the euclidean action, and discuss in detail the reality conditions on the bosonic and fermionic fields. In section 7, we set up an iteration procedure in Grassmann collective coordinates to solve the equations of motion. For the gauge group $SU(2)$ this iteration amounts to applying successive ordinary and conformal supersymmetry transformations on the fields. However, in the case of $SU(N)$ not all solutions can be obtained by means of supersymmetry, and we solve the equations of motions explicitly. Then, in section 8, we show how to compute correlation functions, and discuss the large N limit. Finally, in section 9, we briefly discuss D-instantons in type IIB supergravity, both in flat space and in an $AdS_5 \times S^5$ curved background, and perform checks on the AdS/CFT correspondence.

After a short outlook we present a few appendices where we set up our conventions and give a detailed derivation of some technical results in order to make the paper self-contained.

2 Classical euclidean solutions and collective coordinates.

2.1 Generalities.

We start with some elementary facts about instantons in $SU(N)$ Yang-Mills theories. The action, continued to euclidean space, is

$$S = -\frac{1}{2g^2} \int d^4x \operatorname{tr} F_{\mu\nu} F_{\mu\nu} . \quad (2.1)$$

We have chosen traceless anti-hermitean N by N generators satisfying $[T_a, T_b] = f_{abc} T_c$ with real structure constants and $\operatorname{tr} \{T_a T_b\} = -\frac{1}{2} \delta_{ab}$. For instance, for $SU(2)$, one has $T_a = -\frac{i}{2} \tau_a$, where τ_a are the Pauli matrices. Notice that the action is positive. Further conventions are $\mathcal{D}_\mu Y = \partial_\mu Y + [A_\mu, Y]$ for any Lie algebra valued field Y , and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$, such that $F_{\mu\nu} = [\mathcal{D}_\mu, \mathcal{D}_\nu]$. The euclidean metric is $\delta_{\mu\nu} = \operatorname{diag}(+, +, +, +)$. In (2.1), the only appearance of the coupling constant is in front of the action.

By definition, a Yang-Mills instanton is a solution to the euclidean equations of motion with finite action. The equations of motion read

$$\mathcal{D}_\mu F_{\mu\nu} = 0 . \quad (2.2)$$

To find solutions with finite action, we require that the field strength tends to zero at infinity, hence the gauge fields asymptotically approaches a pure gauge ¹

$$A_\mu \stackrel{x^2 \rightarrow \infty}{\equiv} U^{-1} \partial_\mu U , \quad (2.3)$$

for some $U \in SU(N)$.

There is actually a way of classifying fields which satisfy this boundary condition. It is known from homotopy theory (the Pontryagin class) that all gauge fields with vanishing field strength at infinity can be classified into sectors characterized by an integer number

$$k = -\frac{1}{16\pi^2} \int d^4x \operatorname{tr} F_{\mu\nu} {}^* F_{\mu\nu} , \quad (2.4)$$

where ${}^* F_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma}$ is the dual field strength, and $\epsilon_{1234} = 1$. The derivation of this result can be found in Appendix B. As a part of the proof, one can show that the integrand in (2.4) is the

¹ Another way of satisfying the finite action requirement is to first formulate the theory on a compactified \mathbf{R}^4 , by adding and identifying points at infinity. Then the topology is that of the four-sphere, since $\mathbf{R}^4 \cup \infty \simeq S^4$. The stereographic map from $\mathbf{R}^4 \cup \infty$ to S^4 preserves the angles, and is therefore conformal. Also the YM action is conformally invariant, implying that the field equations on $\mathbf{R}^4 \cup \infty$ are the same as on S^4 . The finiteness requirement is satisfied when the gauge potentials can be smoothly extended from \mathbf{R}^4 to S^4 . The action is then finite because S^4 is compact and A_μ is smooth on the whole of the four-sphere.

divergence of the current

$$K_\mu = -\frac{1}{8\pi^2}\epsilon_{\mu\nu\rho\sigma}\text{tr} A_\nu \left(\partial_\rho A_\sigma + \frac{2}{3}A_\rho A_\sigma\right) . \quad (2.5)$$

The four-dimensional integral in (2.4) then reduces to an integral over a three-sphere at infinity, and one can use (2.3) to show that the integer k counts how many times this sphere covers the gauge group three-sphere $S^3 \approx SU(2) \subset SU(N)$. In more mathematical terms, the integer k corresponds to the third homotopy group $\pi_3(SU(2)) = \mathbf{Z}$.

Since we require instantons to have finite action, they satisfy the above boundary conditions at infinity, and hence they are classified by an integer number k , called the instanton number or topological charge. Gauge potentials leading to field strengths with different instanton number can not be related by gauge transformations. This follows from the fact that the instanton number is a gauge invariant quantity.

We now show that, in a given topological sector, there is a unique solution to the field equations, the (anti-)instanton, that minimizes the action. This is the field configuration which has (anti-)selfdual field strength

$$F_{\mu\nu} = \pm {}^*F_{\mu\nu} = \pm \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma} . \quad (2.6)$$

This equation is understood in euclidean space, where $(*)^2 = 1$. In Minkowski space there are no solutions to the selfduality equations since $(*)^2 = -1$. So, as seen from (2.4), instantons (with selfdual field strength) have $k > 0$ whereas anti-instantons (with anti-selfdual field strength) have $k < 0$. To see that this configuration is indeed the unique minimum of the action, we perform a trick similar to the one used for deriving the BPS bound for solitons:

$$\begin{aligned} S &= -\frac{1}{2g^2}\int d^4x \text{tr} F^2 = -\frac{1}{4g^2}\int d^4x \text{tr} (F \mp {}^*F)^2 \mp \frac{1}{2g^2}\int d^4x \text{tr} F {}^*F \\ &\geq \mp \frac{1}{2g^2}\int d^4x \text{tr} F {}^*F = \frac{8\pi^2}{g^2}|k| , \end{aligned} \quad (2.7)$$

where the equality is satisfied if and only if the field strength is (anti-)selfdual. The action is then $S_{\text{cl}} = (8\pi^2/g^2)|k|$, and has the same value for the instanton as well as for the anti-instanton. However, in euclidean space, we can also add a theta-angle term to the action, which reads

$$S_\theta = -i\frac{\theta}{16\pi^2}\int d^4x \text{tr} F_{\mu\nu} {}^*F^{\mu\nu} = \pm i\theta|k| . \quad (2.8)$$

The plus or minus sign corresponds to the instanton and anti-instanton respectively, so the theta angle distinguishes between them.

It is worth mentioning that the energy-momentum tensor for a selfdual field strength is always zero

$$\Theta_{\mu\nu} = \frac{2}{g^2}\text{tr} \left\{ F_{\mu\rho}F_{\nu\rho} - \frac{1}{4}\delta_{\mu\nu}F_{\rho\sigma}F_{\rho\sigma} \right\} = 0 . \quad (2.9)$$

This follows from the observation that the instanton action $\int d^4x F^2 = \int d^4x *FF$ is metric independent. The vanishing of the energy-momentum tensor is consistent with the fact that instantons are topological in nature. It also implies that instantons do not curve euclidean space, as follows from the Einstein equations.

Note that we have not shown that all the solutions of (2.2) with finite action are given by instantons, i.e. by selfdual field strengths. In principle there could be configurations which are local minima of the action, but are neither selfdual nor anti-selfdual. No such examples of exact solutions have been found in the literature so far.

2.2 The $k = 1$ instanton in $SU(2)$.

An explicit construction of finite action solutions of the euclidean classical equations of motion was given by Belavin et al. [1]. The gauge configuration for one-instanton ($k = 1$) in $SU(2)$ is

$$A_\mu^a(x; x_0, \rho) = 2 \frac{\eta_{\mu\nu}^a (x - x_0)_\nu}{(x - x_0)^2 + \rho^2}, \quad (2.10)$$

where x_0 and ρ are arbitrary parameters called collective coordinates. They correspond to the position and the size of the instanton. The above expression solves the selfduality equations for any value of the collective coordinates. Notice that it is regular for $x = x_0$, as long as $\rho \neq 0$. The antisymmetric eta-symbols are defined as (see Appendix A for more of their properties)

$$\begin{aligned} \eta_{\mu\nu}^a &= \epsilon_{\mu\nu}^a & \mu, \nu &= 1, 2, 3, & \eta_{\mu 4}^a &= -\eta_{4\mu}^a = \delta_\mu^a, \\ \bar{\eta}_{\mu\nu}^a &= \epsilon_{\mu\nu}^a & \mu, \nu &= 1, 2, 3, & \bar{\eta}_{\mu 4}^a &= -\bar{\eta}_{4\mu}^a = -\delta_\mu^a. \end{aligned} \quad (2.11)$$

The η and $\bar{\eta}$ -tensors are selfdual and anti-selfdual respectively, for each index a . They form a basis for the antisymmetric four by four matrices, and we have listed their properties in Appendix A. The gauge transformation corresponding to (2.3) is simply $U(x) = \sigma_\mu x_\mu / \sqrt{x^2}$, where the sigma matrices are given by $\sigma_\mu = (\vec{\tau}, i)$.

The field strength corresponding to this gauge potential is

$$F_{\mu\nu}^a = -4\eta_{\mu\nu}^a \frac{\rho^2}{[(x - x_0)^2 + \rho^2]^2}, \quad (2.12)$$

and it is selfdual. Notice that the special point $\rho = 0$, called zero size instantons, leads to zero field strength and corresponds to pure gauge. This point must be excluded from the instanton moduli space of collective coordinates, since it is singular. Finally one can compute the action by integrating the density

$$\text{tr } F_{\mu\nu} F^{\mu\nu} = -96 \frac{\rho^4}{[(x - x_0)^2 + \rho^2]^4}. \quad (2.13)$$

Using the integral given at the end of Appendix A, one finds that this indeed corresponds to $k = 1$.

One can also consider the instanton in singular gauge, for which

$$A_\mu^a = 2 \frac{\rho^2 \bar{\eta}_{\mu\nu}^a (x - x_0)_\nu}{(x - x_0)^2 [(x - x_0)^2 + \rho^2]} = -\bar{\eta}_{\mu\nu}^a \partial_\nu \ln \left\{ 1 + \frac{\rho^2}{(x - x_0)^2} \right\}. \quad (2.14)$$

This gauge potential is singular for $x = x_0$, where it approaches a pure gauge configuration $A_\mu \stackrel{x \rightarrow x_0}{\equiv} U \partial_\mu U^{-1}$ with $U(x - x_0)$ given before. Moreover this gauge group transformation relates the regular gauge instanton (2.10) to the singular one (2.14) at all points. The field strength in singular gauge is then (taking the instanton position zero, $x_0 = 0$, otherwise replace $x \rightarrow x - x_0$)

$$F_{\mu\nu}^a = -\frac{4\rho^2}{(x^2 + \rho^2)^2} \left\{ \bar{\eta}_{\mu\nu}^a - 2\bar{\eta}_{\mu\rho}^a \frac{x_\rho x_\nu}{x^2} + 2\bar{\eta}_{\nu\rho}^a \frac{x_\rho x_\mu}{x^2} \right\}. \quad (2.15)$$

Notice that, despite the anti-selfdual eta-tensors, the field strength is still selfdual, as can be seen by using the properties of the eta-tensors given in Appendix A. Singular gauge is frequently used, because, as we will see later, the instanton measure can be computed most easily in this gauge. One can compute the winding number again in singular gauge. Then one finds that there is no contribution coming from infinity. Instead, all the winding is coming from the singularity at the origin.

At first sight it seems there are five collective coordinates. There are however extra collective coordinates corresponding to the gauge orientation. In fact, one can act with an $SU(2)$ matrix on the solution (2.10) to obtain another solution,

$$A_\mu(x; x_0, \rho, \vec{\theta}) = U(\vec{\theta}) A_\mu(x; x_0, \rho) U^\dagger(\vec{\theta}), \quad U \in SU(2). \quad (2.16)$$

One might think that, since this configuration is gauge equivalent to the expression given above, it should not be considered as a new solution. This is not true however, the reason is that, after we fix the gauge, we still have left a rigid $SU(2)$ symmetry which acts as in (2.16). So in total there are eight collective coordinates, also called moduli.

In principle, one could also act with the (space-time) rotation matrices $SO(4)$ on the instanton solution, and construct new solutions. However, as was shown by Jackiw and Rebbi [25], these rotations can be undone by suitably chosen gauge transformations. If one puts together the instanton and anti-instanton in a four by four matrix, the gauge group $SU(2)$ can be extended to $SO(4) = (SU(2) \times SU(2))/Z_2$, which is the same as the euclidean rotation group. A similar analysis holds for the other generators of the conformal group. In fact, Jackiw and Rebbi showed that for the (euclidean) conformal group $SO(5, 1)$, the subgroup $SO(5)$ consisting of rotations and combined special conformal transformation with translations ($R^\mu \equiv K^\mu + P^\mu$), leaves the instanton invariant, up to gauge transformations. This leads to a 5 parameter instanton moduli space $SO(5, 1)/SO(5)$, which is the euclidean version of the five-dimensional anti-de Sitter space

AdS_5 . The coordinates on this manifold correspond to the four positions and the size ρ of the instanton. On top of that, there are still three gauge orientation collective coordinates, yielding a total of eight moduli.

2.3 The $k = 1$ instanton in $SU(N)$.

Instantons in $SU(N)$ can be obtained by embedding $SU(2)$ instantons into $SU(N)$. For instance, a particular embedding is given by the following N by N matrix

$$A_\mu^{SU(N)} = \begin{pmatrix} 0 & 0 \\ 0 & A_\mu^{SU(2)} \end{pmatrix}. \quad (2.17)$$

Of course this is not the most general solution, as one can choose different embeddings. One could act with a general $SU(N)$ element on the solution (2.17) and obtain a new one. Some of them correspond to a different embedding² inside $SU(N)$. Not all elements of $SU(N)$ generate a new solution. There is a stability group that leaves (2.17) invariant, acting only on the zeros, or commuting trivially with the $SU(2)$ embedding. Such group elements should be divided out, so we consider, for $N > 2$,

$$A_\mu^{SU(N)} = U \begin{pmatrix} 0 & 0 \\ 0 & A_\mu^{SU(2)} \end{pmatrix} U^\dagger, \quad U \in \frac{SU(N)}{SU(N-2) \times U(1)}. \quad (2.18)$$

One can now count the number of collective coordinates. From counting the dimension of the coset space in (2.18), one finds there are $4N - 5$ angles. Together with the position and the scale of the $SU(2)$ solution, we find in total $4N$ collective coordinates.

It is instructive to work out the example of $SU(3)$. Here we use the eight Gell-Mann matrices $\{\lambda_\alpha\}$, $\alpha = 1, \dots, 8$. The first three λ_a , $a = 1, 2, 3$, form an $SU(2)$ algebra and are used to define the $k = 1$ instanton by contracting (2.10) with λ_a . The generators $\lambda_4, \dots, \lambda_7$ form two doublets under this $SU(2)$, and can be used to generate new solutions. Then there is λ_8 , which is a singlet, corresponding to the $U(1)$ factor in (2.18). It commutes with the $SU(2)$ subgroup spanned by λ_a , and so it belongs to the stability group leaving the instanton invariant. For $SU(3)$ there are seven gauge orientation zero modes.

The question then arises whether or not these are all the solutions. To find this out, one can study deformations of the solution (2.17), $A_\mu + \delta A_\mu$, and see if they preserve selfduality. Expanding to first order in the deformation, this leads to the condition

$$\mathcal{D}_\mu \delta A_\nu - \mathcal{D}_\nu \delta A_\mu = *(\mathcal{D}_\mu \delta A_\nu - \mathcal{D}_\nu \delta A_\mu), \quad (2.19)$$

²There are also embeddings which can not be obtained by $SU(N)$ or any other similarity transformations. They are completely inequivalent, but correspond to higher instanton numbers k [27]. Since we do not cover multi-instantons in these lectures, these embeddings are left out of the discussion here.

where the covariant derivative depends only on the original classical solution.

In addition we require that the new solution is not related to the old one by a gauge transformation. This can be achieved by requiring that the deformations are orthogonal to any gauge transformation $\mathcal{D}_\mu\Lambda$, for any function Λ , i.e.

$$\int d^4x \operatorname{tr} \mathcal{D}_\mu\Lambda\delta A_\mu = 0 . \quad (2.20)$$

After partial integration the orthogonality requirement leads to the usual background field gauge condition

$$\mathcal{D}_\mu\delta A^\mu = 0 . \quad (2.21)$$

Bernard et al. in [26] have studied the solutions of (2.19) subject to the condition (2.21) using the Atiyah-Singer index theorem. Index theory turns out to be a useful tool when counting the number of solutions to a certain linear differential equation of the form $\hat{D}T = 0$, where \hat{D} is some differential operator and T is a tensor. We will elaborate on this in the next subsection and also when studying fermionic collective coordinates. The ultimate result of [26] is that there are indeed $4Nk$ solutions, leading to the above constructed $4N$ (for $k = 1$) collective coordinates. An assumption required to apply index theorems is that the space has to be compact. One must therefore compactify euclidean space to a four-sphere S^4 , as was also mentioned in footnote 1.

2.4 Bosonic collective coordinates and the Dirac operator.

In this section we will make more precise statements on how to count the number of solutions to the selfduality equations, by relating it to the index of the Dirac operator. A good reference on this topic is [28].

The problem is to study the number of solutions to the (anti-)selfduality equations with topological charge k . As explained in the last subsection, we study deformations of a given classical solution $A_\mu^{\text{cl}} + \delta A_\mu$. Let us denote $\phi_\mu \equiv \delta A_\mu$ and $f_{\mu\nu} \equiv \mathcal{D}_\mu\phi_\nu - \mathcal{D}_\nu\phi_\mu$. The covariant derivative here contains only A_μ^{cl} . The constraints on the deformations of an anti-instanton ³ can then be written as

$$\eta_{\mu\nu}^a f_{\mu\nu} = 0 , \quad \mathcal{D}_\mu\phi_\mu = 0 . \quad (2.22)$$

The first of this equation says that the selfdual part of $f_{\mu\nu}$ must vanish. It also means that the deformation cannot change the anti-instanton into an instanton. It will prove convenient to use

³**Note on conventions:** We are switching here, and in the remainder, from instantons to anti-instantons. The reason has to do with the conventions for the σ_μ and $\bar{\sigma}_\mu$ matrices as defined in the text (see also Appendix A). Our conventions are different from most of the instanton literature, but are in agreement with the literature on supersymmetry. Due to this, we obtain the somewhat unfortunate result that $\sigma_{\mu\nu}$ is anti-selfdual, while $\eta_{\mu\nu}^a$ is selfdual. For this reason, we will choose to study anti-instantons.

quaternionic notation

$$\Phi^{\alpha\beta'} = \phi_\mu \sigma_\mu^{\alpha\beta'} , \quad (2.23)$$

with $\sigma_\mu = (\vec{\tau}, i)$, and $\bar{\sigma}_\mu = (\vec{\tau}, -i)$. These sigma matrices satisfy $\sigma_\mu \bar{\sigma}_\nu + \sigma_\nu \bar{\sigma}_\mu = 2\delta_{\mu\nu}$ and

$$\sigma_{\mu\nu} \equiv \frac{1}{2} (\sigma_\mu \bar{\sigma}_\nu - \sigma_\nu \bar{\sigma}_\mu) = i\bar{\eta}_{\mu\nu}^a \tau^a , \quad \bar{\sigma}_{\mu\nu} \equiv \frac{1}{2} (\bar{\sigma}_\mu \sigma_\nu - \bar{\sigma}_\nu \sigma_\mu) = i\eta_{\mu\nu}^a \tau^a , \quad (2.24)$$

so $\sigma_{\mu\nu}$ and $\bar{\sigma}_{\mu\nu}$ are anti-selfdual and selfdual respectively. The constraints on the deformations can then be rewritten as a quaternion valued Dirac equation

$$\bar{\mathcal{D}}\Phi = 0 , \quad (2.25)$$

with $\bar{\mathcal{D}} = \bar{\sigma}^\mu \mathcal{D}_\mu$. We can represent the quaternion by

$$\Phi = \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix} , \quad (2.26)$$

with a and b complex adjoint-valued functions. Then (2.25) reduces to two adjoint spinor equations, one for

$$\lambda = \begin{pmatrix} a \\ b \end{pmatrix} \quad \bar{\mathcal{D}}\lambda = 0 , \quad (2.27)$$

and one for $-i\sigma^2\lambda^*$. Conversely, for each spinor solution λ to the Dirac equation, one shows that also $-i\sigma^2\lambda^*$ is a solution. Therefore, the number of solutions for Φ is twice the number of solutions for a single two-component adjoint spinor. So, the problem of counting the number of bosonic collective coordinates is now translated to the computation of the Dirac index, which we will discuss in the next section.

3 Fermionic collective coordinates and the index theorem.

Both motivated by the counting of bosonic collective coordinates, as argued in the last section, and by the interest of coupling YM theory to fermions, we study the Dirac equation in the presence of an anti-instanton. We start with a Dirac fermion ψ , in an arbitrary representation (adjoint, fundamental, etc) of the gauge group, and consider the Dirac equation in the presence of an anti-instanton background

$$\gamma_\mu \mathcal{D}_\mu^{\text{cl}} \psi = \mathcal{D}^{\text{cl}} \psi = 0 . \quad (3.1)$$

The Dirac spinor can be decomposed into its chiral and anti-chiral components

$$\lambda \equiv \frac{1}{2} (1 + \gamma^5) \psi , \quad \bar{\chi} \equiv \frac{1}{2} (1 - \gamma^5) \psi . \quad (3.2)$$

A euclidean representation for the Clifford algebra is given by ⁴

$$\gamma^\mu = \begin{pmatrix} 0 & -i\sigma^{\mu\alpha\beta'} \\ i\bar{\sigma}^{\mu}_{\alpha'\beta} & 0 \end{pmatrix}, \quad \gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3.3)$$

The Dirac equation then becomes

$$\bar{\mathcal{D}}^{\text{cl}}\lambda = 0, \quad \mathcal{D}^{\text{cl}}\bar{\chi} = 0, \quad (3.4)$$

where \mathcal{D} is a two by two matrix.

3.1 The index of the Dirac operator.

We now show that in the presence of an anti-instanton (recall the footnote 3), (3.4) has solutions for λ , but not for $\bar{\chi}$. Conversely, in the background of an instanton, \mathcal{D} has zero modes, but $\bar{\mathcal{D}}$ has not. The argument goes as follows. Given a zero mode $\bar{\chi}$ for \mathcal{D} , it also satisfies $\bar{\mathcal{D}}\bar{\mathcal{D}}\bar{\chi} = 0$. In other words, the kernel $\ker \mathcal{D} \subset \ker \{\bar{\mathcal{D}}\mathcal{D}\}$. Now we evaluate

$$\bar{\mathcal{D}}\bar{\mathcal{D}} = \bar{\sigma}_\mu\sigma_\nu\mathcal{D}_\mu\mathcal{D}_\nu = \mathcal{D}^2 + \bar{\sigma}_{\mu\nu}F_{\mu\nu}, \quad (3.5)$$

where we have used $\bar{\sigma}_\mu\sigma_\nu + \bar{\sigma}_\nu\sigma_\mu = 2\delta_{\mu\nu}$, and $\bar{\sigma}_{\mu\nu}$ is defined as in (2.24). But notice that the anti-instanton field strength is anti-selfdual whereas the tensor $\bar{\sigma}_{\mu\nu}$ is selfdual, so the second term vanishes. From this it follows that $\bar{\chi}$ satisfies $\mathcal{D}^2\bar{\chi} = 0$. Now we can multiply with its conjugate $\bar{\chi}^*$ and integrate to get, after partial integration and assuming that the fields go to zero at infinity, $\int d^4x |\mathcal{D}_\mu\bar{\chi}|^2 = 0$. From this it follows that $\bar{\chi}$ is covariantly constant, and so $F_{\mu\nu}^a T_a \bar{\chi} = 0$. This implies that $\eta_{\mu\nu}^a T_a \bar{\chi} = 0$, and hence $T_a \bar{\chi} = 0$ for all T_a . We conclude that $\bar{\chi} = 0$. Stated differently, $-\mathcal{D}^2$ is a positive definite operator and has no zero modes (with vanishing boundary conditions). This result is independent of the representation of the fermion.

For the λ -equation, we have $\mathcal{D}\bar{\mathcal{D}}\lambda = 0$, i.e. $\ker \bar{\mathcal{D}} \subset \ker \{\mathcal{D}\bar{\mathcal{D}}\}$, and compute

$$\mathcal{D}\bar{\mathcal{D}} = \mathcal{D}^2 + \sigma_{\mu\nu}F_{\mu\nu}. \quad (3.6)$$

This time the second term does not vanish in the presence of an anti-instanton, so zero modes are possible. Knowing that \mathcal{D} has no zero modes, one easily concludes that $\ker \bar{\mathcal{D}} = \ker \{\mathcal{D}\bar{\mathcal{D}}\}$. Now we can count the number of solutions using index theorems. The index of the Dirac operator is defined as

$$\text{Ind } \bar{\mathcal{D}} = \dim \{\ker \bar{\mathcal{D}}\} - \dim \{\ker \mathcal{D}\}. \quad (3.7)$$

⁴In euclidean space the Lorentz group decomposes according to $SO(4) = SU(2) \times SU(2)$. The spinor indices α and α' correspond to the doublet representations of these two $SU(2)$ factors. As opposed to Minkowski space, α and α' are not in conjugate representations.

This index will give us the relevant number, since the second term is zero. There are several ways to compute its value, and we represent it by

$$\text{Ind } \bar{\mathcal{D}} = \text{tr} \left\{ \frac{M^2}{-\bar{\mathcal{D}} \bar{\mathcal{D}} + M^2} - \frac{M^2}{-\bar{\mathcal{D}} \bar{\mathcal{D}} + M^2} \right\}, \quad (3.8)$$

where M is an arbitrary parameter. The trace stands for a sum over group indices, spinor indices, and includes an integration over space-time. We can in fact show that this expression is independent of M . The reason is that the operators $\bar{\mathcal{D}} \bar{\mathcal{D}}$ and $\bar{\mathcal{D}} \bar{\mathcal{D}}$ have the same spectrum for non-zero eigenvalues. Indeed, if ψ is an eigenfunction of $\bar{\mathcal{D}} \bar{\mathcal{D}}$, then $\bar{\mathcal{D}}\psi$ is an eigenfunction of $\bar{\mathcal{D}} \bar{\mathcal{D}}$ with the same eigenvalue and $\bar{\mathcal{D}}\psi$ does not vanish. Conversely, if ψ is an eigenfunction of $\bar{\mathcal{D}} \bar{\mathcal{D}}$, then $\bar{\mathcal{D}}\psi$ does not vanish and is an eigenfunction of $\bar{\mathcal{D}} \bar{\mathcal{D}}$ with the same eigenvalue. This means that there is a pairwise cancellation in (3.8) coming from the sum over eigenstates with non-zero eigenvalues. So the only contribution is coming from the zero modes, for which the first term simply gives one for each zero mode, and the second term vanishes because there are no zero modes. The result is then clearly independent of M , and moreover, it is an integer, namely $\dim \{ \ker \bar{\mathcal{D}} \}$.

In the basis of the four-dimensional Dirac matrices, the index can be written as

$$\text{Ind } \bar{\mathcal{D}} = \text{tr} \left\{ \frac{M^2}{-\bar{\mathcal{D}}^2 + M^2} \gamma_5 \right\}. \quad (3.9)$$

Because this expression is independent of M^2 , we might as well evaluate it in the large M^2 limit. The calculation is then identical to the calculation of the chiral anomaly, and we will not repeat it here. The results are well known, and depend on the representation of the generators,

$$\text{Ind } \bar{\mathcal{D}} = -\frac{1}{16\pi^2} \int d^4x F_{\mu\nu}^a * F_{\mu\nu}^b \text{tr} (T_a T_b), \quad (3.10)$$

which yields

$$\begin{aligned} \text{Ind}_{\text{adj}} \bar{\mathcal{D}} &= 2Nk, & \text{adjoint}, \\ \text{Ind}_{\text{fund}} \bar{\mathcal{D}} &= k, & \text{fundamental}. \end{aligned} \quad (3.11)$$

This also proves the fact that there are $4Nk$ bosonic collective coordinates, as mentioned in the last subsection.

3.2 Construction of the fermionic instanton.

In this subsection we will construct the solutions to the Dirac equation explicitly. Because we only know the gauge field for $k = 1$ explicitly, we can only construct the fermionic zero modes for the single anti-instanton case. For an $SU(2)$ adjoint fermion, there are 4 zero modes, and these can be written as [37]

$$\lambda^\alpha = -\frac{1}{2} \sigma_{\mu\nu}{}^\alpha{}_\beta \left(\xi^\beta - \bar{\eta}_{\gamma'} \bar{\sigma}_\rho^{\gamma'\beta} (x - x_0)^\rho \right) F_{\mu\nu}. \quad (3.12)$$

Actually, this expression also solves the Dirac equation for higher order k , but there are additional solutions, $4k$ in total for $SU(2)$. The four fermionic collective coordinates are denoted by ξ^α and $\bar{\eta}_{\gamma'}$, where $\alpha, \gamma' = 1, 2$ are spinor indices in euclidean space. They can somehow be thought of as the fermionic partners of the translational and dilatational collective coordinates in the bosonic sector. These solutions take the same form in any gauge, one just takes the corresponding gauge for the field strength. For $SU(N)$, there are a remaining of $2 \times (N - 2)$ zero modes, and their explicit form depends on the chosen gauge. In regular gauge, with color indices $u, v = 1, \dots, N$ explicitly written, the gauge field is (setting $x_0 = 0$, otherwise replace $x \rightarrow x - x_0$)

$$A_\mu{}^u{}_v = A_\mu^a (T_a)^u{}_v = -\frac{\sigma_{\mu\nu}{}^u{}_v x_\nu}{x^2 + \rho^2}, \quad \sigma_{\mu\nu}{}^u{}_v = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_{\mu\nu}{}^\alpha{}_\beta \end{pmatrix}. \quad (3.13)$$

Then the corresponding fermionic instanton reads

$$\lambda^{\alpha u}{}_v = \frac{\rho}{\sqrt{(x^2 + \rho^2)^3}} (\mu^u \delta^\alpha{}_v + \epsilon^{\alpha u} \bar{\mu}_v). \quad (3.14)$$

Here we have introduced Grassmann collective coordinates (GCC)

$$\mu^u = (\mu^1, \dots, \mu^{N-2}, 0, 0), \quad \epsilon^{\alpha u} = (0, \dots, 0, \epsilon^{\alpha\beta'}) \quad \text{with} \quad N - 2 + \beta' = u, \quad (3.15)$$

and similarly for $\bar{\mu}$. The canonical dimension of μ is chosen to be $-1/2$.

In singular gauge, the gauge field is

$$A_{\mu u}{}^v = -\frac{\rho^2}{x^2(x^2 + \rho^2)} \bar{\sigma}_{\mu\nu}{}^u{}_v x_\nu. \quad (3.16)$$

Notice that the position of the color indices is different from that in regular gauge. This is due to the natural position of indices on the sigma matrices. The fermionic anti-instanton in singular gauge reads [38]

$$\lambda^{\alpha}{}^u{}_v = \frac{\rho}{\sqrt{x^2(x^2 + \rho^2)^3}} (\mu_u x^{\alpha v} + x^{\alpha}{}^u \bar{\mu}^v), \quad (3.17)$$

where for fixed α , the N -component vectors μ_u and $x^{\alpha v}$ are given by

$$\mu_u = (\mu_1, \dots, \mu_{N-2}, 0, 0), \quad x^{\alpha v} = (0, \dots, 0, x^\mu \sigma_\mu^{\alpha\beta'}) \quad \text{with} \quad N - 2 + \beta' = v. \quad (3.18)$$

Further, $x^{\alpha}{}^u = x^{\alpha v} \epsilon_{vu}$ and $\bar{\mu}^v$ also has $N - 2$ nonvanishing components. The particular choice of zeros in the last two entries corresponds to the chosen embedding of the $SU(2)$ instanton inside $SU(N)$. Notice that the adjoint field λ is indeed traceless in its color indices. This follows from the observation that λ only has non-zero entries on the off-diagonal blocks inside $SU(N)$.

Depending on whether or not there is a reality condition on λ in euclidean space, the μ and $\bar{\mu}$ are related by complex conjugation. We will illustrate this in a more concrete example when we discuss instantons in $\mathcal{N} = 4$ SYM theory.

We should also mention that while the bosonic collective coordinates are related to the rigid symmetries of the theory, this is not obviously true for the fermionic collective coordinates, although, as we will see later, the ξ and $\bar{\eta}$ collective coordinates can be obtained by supersymmetry and superconformal transformations in SYM theories.

A similar construction holds for a fermion in the fundamental representation. Now there is only one fermionic collective coordinate, which we denote by \mathcal{K} . The explicit expression, in singular gauge, is

$$(\lambda^\alpha)_u = \frac{\rho}{\sqrt{x^2(x^2 + \rho^2)^3}} x^\alpha_u \mathcal{K} . \quad (3.19)$$

4 Zero modes and the measure.

In the following two sections we will show how to set up and do (one-loop) perturbation theory around an (anti)-instanton. As a first step, in this section, we will discuss the zero mode structure and show how to reduce the path integral measure over instanton field configurations to an integral over the moduli space of collective coordinates, closely following [29]. In the next section, we compute the fluctuations around an anti-instanton background.

4.1 Normalization of the zero modes.

In order to construct the zero modes and discuss perturbation theory, we first decompose the fields into a background part and quantum fields

$$A_\mu = A_\mu^{\text{cl}}(\gamma) + A_\mu^{\text{qu}} . \quad (4.1)$$

Here γ_i denote a set of collective coordinates, and, for gauge group $SU(N)$, $i = 1, \dots, 4Nk$. Before we make the expansion in the action, we should also perform gauge fixing and introduce ghosts, c , and anti-ghosts, b . We choose the background gauge condition

$$\mathcal{D}_\mu^{\text{cl}} A_\mu^{\text{qu}} = 0 . \quad (4.2)$$

Then the action, expanded up to quadratic order in the quantum fields, is of the form

$$S = \frac{8\pi^2}{g^2} + \frac{1}{g^2} \text{tr} \int d^4x \left\{ A_\mu^{\text{qu}} M_{\mu\nu}^{\text{cl}} A_\nu^{\text{qu}} - 2b M^{\text{gh}} c \right\} , \quad (4.3)$$

with $M^{\text{gh}} = \mathcal{D}^2$ and

$$\begin{aligned} M_{\mu\nu} &= \mathcal{D}^2 \delta_{\mu\nu} + 2F_{\mu\nu} \\ &= \left(\mathcal{D}^2 \delta_{\mu\nu} - \mathcal{D}_\nu \mathcal{D}_\mu + F_{\mu\nu} \right) + \mathcal{D}_\mu \mathcal{D}_\nu \equiv M_{\mu\nu}^1 + M_{\mu\nu}^2 , \end{aligned} \quad (4.4)$$

where we have dropped the subscript cl. Here, M^1 stands for the quadratic operator coming from the classical action, and M^2 corresponds to the gauge fixing term.

In making an expansion as in (4.3), we observe the existence of zero modes (i.e. eigenfunctions of the operator $M_{\mu\nu}$ with zero eigenvalues),

$$Z_\mu^{(i)} \equiv \frac{\partial A_\mu^{\text{cl}}}{\partial \gamma_i} + \mathcal{D}_\mu^{\text{cl}} \Lambda^i, \quad (4.5)$$

where the Λ^i -term is chosen to keep Z_μ in the background gauge, so that

$$\mathcal{D}_\mu^{\text{cl}} Z_\mu^{(i)} = 0. \quad (4.6)$$

The first term in (4.5) is a zero mode of M^1 , as follows from taking the derivative with respect to γ_i of the field equation. The $\mathcal{D}_\mu \Lambda$ term is also a zero mode of M^1 , since it is a pure gauge transformation. The sum of the two terms is also a zero mode of M^2 , because Λ is chosen such that Z_μ is in the background gauge.

Due to these zero modes, we cannot integrate the quantum fluctuations, since the corresponding determinants would give zero and yield divergences in the path integral. They must therefore be extracted from the quantum fluctuations, in a way we will describe in a more general setting in the next subsection. It will turn out to be important to compute the matrix of inner products

$$U^{ij} \equiv \langle Z^{(i)} | Z^{(j)} \rangle \equiv -\frac{2}{g^2} \int d^4x \text{tr} \left\{ Z_\mu^{(i)} Z^{\mu(j)} \right\}. \quad (4.7)$$

We now evaluate this matrix for the anti-instanton. For the four translational zero modes, one can easily keep the zero mode in the background gauge by choosing $\Lambda^\mu = A_\mu^{\text{cl}}$. Indeed,

$$Z_\mu^{(\nu)} = \frac{\partial A_\mu^{\text{cl}}}{\partial x_{0\nu}} + \mathcal{D}_\mu A_\nu^{\text{cl}} = -\partial_\nu A_\mu^{\text{cl}} + \mathcal{D}_\mu A_\nu^{\text{cl}} = F_{\mu\nu}^{\text{cl}}, \quad (4.8)$$

which satisfies the background gauge condition. The norm of this zero mode is

$$U^{\mu\nu} = \frac{8\pi^2 |k|}{g^2} \delta^{\mu\nu} = S_{\text{cl}} \delta^{\mu\nu}. \quad (4.9)$$

This result actually holds for any k , and arbitrary gauge group.

Now we consider the dilatational zero mode corresponding to ρ and limit ourselves to $k = 1$. Taking the derivative with respect to ρ leaves the zero mode in the background gauge, so we can set $\Lambda^\rho = 0$. In singular gauge we have

$$Z_\mu^{(\rho)} = -2 \frac{\rho \bar{\sigma}_{\mu\nu} x_\nu}{(x^2 + \rho^2)^2}. \quad (4.10)$$

Using the integral given in Appendix A, one easily computes that

$$U^{\rho\rho} = \frac{16\pi^2}{g^2} = 2S_{\text{cl}}. \quad (4.11)$$

The gauge orientation zero modes can be obtained from (2.16). By expanding⁵ $U(\theta) = \exp(-2\theta^a T_a)$ infinitesimally in (2.16) we get

$$\frac{\partial A_\mu}{\partial \theta^a} = 2 [A_\mu, T_a] , \quad (4.12)$$

which is not in the background gauge. To satisfy (4.6) we have to add appropriate gauge transformations, which differ for different generators of $SU(N)$. First, for the $SU(2)$ subgroup corresponding to the instanton embedding, we add

$$\Lambda^a = -2 \frac{\rho^2}{x^2 + \rho^2} T_a , \quad (4.13)$$

and find that

$$Z_\mu^{(a)} = 2\mathcal{D}_\mu \left[\frac{x^2}{x^2 + \rho^2} T_a \right] . \quad (4.14)$$

We have given the zero mode by working infinitesimally in θ^a . One should be able to redo the analysis for finite θ , and we expect the result to be an $SU(2)$ rotation on (4.14), which drops out under the trace in the computation of the zero mode norms. One can now show, using (A.9) that the zero mode (4.14) is in the background gauge, and its norm reads

$$U^{ab} = \delta^{ab} 2\rho^2 S_{\text{cl}} . \quad (4.15)$$

It is also fairly easy to prove that there is no mixing between the different modes, i.e. $U^{\mu(\rho)} = U^{\mu a} = U^{(\rho)a} = 0$.

The matrix U^{ij} for $SU(2)$ is eight by eight, with non-vanishing entries

$$U^{ij} = \begin{pmatrix} \delta^{\mu\nu} S_{\text{cl}} & & \\ & 2S_{\text{cl}} & \\ & & 2\delta^{ab} \rho^2 S_{\text{cl}} \end{pmatrix}_{[8] \times [8]} , \quad (4.16)$$

and the square root of the determinant is

$$\sqrt{U} = 2^2 S_{\text{cl}}^4 \rho^3 = \frac{2^{14} \pi^8 \rho^3}{g^8} . \quad (4.17)$$

Now we consider the remaining generators of $SU(N)$ by first analyzing the example of $SU(3)$. There are seven gauge orientation zero modes, three of which are given in (4.14) by taking for T_a ($-i/2$) times the first three Gell-Mann matrices $\lambda_1, \lambda_2, \lambda_3$. For the other four zero modes,

⁵Note the factor 2 in the exponent. This is to make the normalization the same as in [29]. In that paper, the generators are normalized as $\text{tr} T_a T_b = -2\delta_{ab}$ (versus $-1/2$ in our conventions), and there is no factor of two in the exponent. If we leave out the factor of 2 in the exponent, then subsequent formula for the norms of the gauge orientation zero modes will change, but this would eventually be compensated by the integration over the angles θ^a , such that the total result remains the same. See Appendix C for more details.

corresponding to $\lambda_4, \dots, \lambda_7$, the formula (4.12) still holds, but we have to change the gauge transformation in order to keep the zero mode in background gauge,

$$\Lambda^k = 2 \left[\sqrt{\frac{x^2}{x^2 + \rho^2}} - 1 \right] T_k, \quad k = 4, 5, 6, 7, \quad (4.18)$$

with $T_k = (-i/2)\lambda_k$. The difference in x -dependence of the gauge transformations (4.13) and (4.18) is due to the change in commutation relations. Namely, $\sum_{a=1}^3 [\lambda_a, [\lambda_a, \lambda_\beta]] = -(3/4)\lambda_\beta$ for $\beta = 4, 5, 6, 7$, whereas it is $-2\lambda_\beta$ for $\beta = 1, 2, 3$. As argued before, there is no gauge orientation zero mode associated with λ_8 , since it commutes with the $SU(2)$ embedding. The zero modes are then

$$Z_\mu^{(k)} = 2\mathcal{D}_\mu \left[\sqrt{\frac{x^2}{x^2 + \rho^2}} T_k \right], \quad k = 4, 5, 6, 7, \quad (4.19)$$

with norms

$$U^{kl} = \delta^{kl} \rho^2 S_{\text{cl}}, \quad (4.20)$$

and are orthogonal to (4.14), such that $U^{ka} = 0$.

This construction easily generalizes to $SU(N)$. One first chooses an $SU(2)$ embedding, and this singles out 3 generators. The other generators can then be split into $2(N-2)$ doublets under this $SU(2)$ and the rest are singlets. There are no zero modes associated with the singlets, since they commute with the chosen $SU(2)$. For the doublets, each associated zero mode has the form as in (4.19), with the same norm $\rho^2 S_{\text{cl}}$. This counting indeed leads to $4N - 5$ gauge orientation zero modes. Straightforward calculation for the square-root of the complete determinant then yields

$$\sqrt{U} = \frac{2^{6N+2}}{\rho^5} \left(\frac{\pi\rho}{g} \right)^{4N}. \quad (4.21)$$

This ends the discussion about the (bosonic) zero mode normalization.

4.2 Measure of collective coordinates.

We now construct the measure on the moduli space of collective coordinates, and show how the matrix U plays the role of a Jacobian. We first illustrate the idea for a generic system without gauge invariance, with fields ϕ^A , and action $S[\phi]$. We expand around the instanton solution

$$\phi^A(x) = \phi_{\text{cl}}^A(x, \gamma) + \phi_{\text{qu}}^A(x, \gamma). \quad (4.22)$$

The collective coordinate is denoted by γ and, for notational simplicity, we assume there is only one. At this point the fields ϕ_{qu}^A can still depend on the collective coordinate, as it can include zero modes. The action, up to quadratic terms in the quantum fields is

$$S = S_{\text{cl}} + \frac{1}{2} \phi_{\text{qu}}^A M_{AB}(\phi_{\text{cl}}) \phi_{\text{qu}}^B. \quad (4.23)$$

The operator M has zero modes given by

$$Z^A = \frac{\partial \phi_{\text{cl}}^A}{\partial \gamma}, \quad (4.24)$$

since MZ is just the derivative of the field equation with respect to the collective coordinate. More generally, if the operator M is hermitean, it has a set of eigenfunctions F_α with eigenvalues ϵ_α ,

$$MF_\alpha = \epsilon_\alpha F_\alpha. \quad (4.25)$$

One of the solutions is of course the zero mode $Z = F_0$ (we are suppressing the index A) with $\epsilon_0 = 0$. Any function can be written in the basis of eigenfunctions, in particular the quantum fields,

$$\phi_{\text{qu}}^A = \sum_\alpha \xi_\alpha F_\alpha^A, \quad (4.26)$$

with some coefficients ξ_α . The eigenfunctions have norms, determined by their inner product

$$\langle F_\alpha | F_\beta \rangle = \int d^4x F_\alpha(x) F_\beta(x). \quad (4.27)$$

The eigenfunctions can always be chosen orthogonal, such that $\langle F_\alpha | F_\beta \rangle = \delta_{\alpha\beta} u_\alpha$. The action then becomes

$$S = S_{\text{cl}} + \frac{1}{2} \sum_\alpha \xi_\alpha \xi_\alpha \epsilon_\alpha u_\alpha. \quad (4.28)$$

If there would be a coupling constant in the action (4.23), then we rescale the inner product with the coupling, such that (4.28) still holds. This is done in (4.7), where also a factor of 2 is brought in, but it cancels after taking the trace. The measure is defined as

$$[d\phi] \equiv \prod_{\alpha=0} \sqrt{\frac{u_\alpha}{2\pi}} d\xi_\alpha. \quad (4.29)$$

We now perform the gaussian integration over the ξ_α and get

$$\int [d\phi] e^{-S[\phi]} = \int \sqrt{\frac{u_0}{2\pi}} d\xi_0 e^{-S_{\text{cl}}} (\det' M)^{-1/2}. \quad (4.30)$$

One sees that if there would be no zero modes, it produces the correct result, namely the determinant of M . In the case of zero modes, the determinant of M is zero, and the path integral would be ill-defined. Instead, we must leave out the zero mode in M , take the amputated determinant (denoted by \det'), and integrate over the mode ξ_0 .

The next step is to convert the ξ_0 integral to an integral over the collective coordinate γ . This can be done by inserting unity into the path integral [39]. Consider the identity

$$1 = \int d\gamma \delta(f(\gamma)) \frac{\partial f}{\partial \gamma}, \quad (4.31)$$

which holds for any (invertible) function $f(\gamma)$. Taking $f(\gamma) = -\langle \phi - \phi_{\text{cl}}(\gamma) | Z \rangle$, (recall that the original field ϕ is independent of γ), we get

$$1 = \int d\gamma \left(u_0 - \left\langle \phi_{\text{qu}} \left| \frac{\partial Z}{\partial \gamma} \right. \right\rangle \right) \delta(\langle \phi_{\text{qu}} | Z \rangle) = \int d\gamma \left(u_0 - \left\langle \phi_{\text{qu}} \left| \frac{\partial Z}{\partial \gamma} \right. \right\rangle \right) \delta(\xi_0 u_0) . \quad (4.32)$$

This trick is somehow similar to the Faddeev-Popov trick for gauge fixing. In the semiclassical approximation, the second term between the brackets is subleading and we will neglect it⁶. This leads to

$$\int [d\phi] e^{-S} = \int d\gamma \sqrt{\frac{u_0}{2\pi}} e^{-S_{\text{cl}}} (\det' M)^{-1/2} . \quad (4.33)$$

For a system with more zero modes Z^i with norms-squared U^{ij} , the result is

$$\int [d\phi] e^{-S} = \int \prod_{i=1} \frac{d\gamma^i}{\sqrt{2\pi}} (\det U)^{1/2} e^{-S_{\text{cl}}} (\det' M)^{-1/2} . \quad (4.34)$$

Notice that this result is invariant under rescalings of Z , which can be seen as rescalings on the collective coordinates. More generally, the matrix U^{ij} can be interpreted as the metric on the moduli space of collective coordinates. The measure is then invariant under general coordinate transformations on the moduli space.

This expression for the measure also generalizes to systems with fermions. The only modifications are dropping the factors of $\sqrt{2\pi}$ (because gaussian integration over fermions does not produce this factor), and inverting the determinants.

One can repeat the analysis for gauge theories to show that (4.34) also holds for Yang-Mills instantons in singular gauge. For regular gauges, there are some modifications due to the fact that the gauge orientation zero mode functions Λ^a do not fall off fast enough at infinity. This is explained in [29], and we will not repeat it here. For this reason, it is more convenient to work in singular gauge.

The collective coordinate measure for $k = 1$ $SU(N)$ YM theories, without the determinant from integrating out the quantum fluctuations which will be analyzed in the next section, is now

$$\frac{2^{4N+2} \pi^{4N-2}}{(N-1)!(N-2)! g^{4N}} \int d^4 x_0 \frac{d\rho}{\rho^5} \rho^{4N} . \quad (4.35)$$

This formula contains the square-root of the determinant of U , $4N$ factors of $1/\sqrt{2\pi}$, and we have also integrated out the gauge orientation zero modes. This may be done only if we are evaluating

⁶It will appear however as a two loop contribution. To see this, one first writes this term in the exponential, where it enters without \hbar , so it is at least a one loop effect. Then, ϕ_{qu} has a part proportional to the zero mode, which drops out by means of the delta function insertion. The other part of ϕ_{qu} is genuinely quantum and contains a power of \hbar (which we have suppressed). Therefore, it contributes at two loops [40, 41].

gauge invariant correlation functions. The result of this integration follows from the volume of the coset space

$$\text{Vol} \left\{ \frac{SU(N)}{SU(N-2) \times U(1)} \right\} = \frac{\pi^{2N-2}}{(N-1)!(N-2)!} . \quad (4.36)$$

The derivation of this formula can be found in Appendix C, and is based on [29].

4.3 The fermionic measure.

Finally we construct the measure on the moduli space of fermionic collective coordinates. For the ξ zero modes (3.12), one finds

$$Z_{(\beta)}^\alpha = \frac{\partial \lambda^\alpha}{\partial \xi^\beta} = -\frac{1}{2} \sigma_{\mu\nu}{}^\alpha{}_\beta F_{\mu\nu} . \quad (4.37)$$

The norms of these two zero modes are given by

$$(U_\xi)_{\beta^\gamma} = -\frac{2}{g^2} \int d^4x \text{tr} \left\{ Z_{(\beta)}^\alpha Z_{\alpha^\gamma} \right\} = 4S_{\text{cl}} \delta_{\beta^\gamma} , \quad (4.38)$$

where we have used the expression (4.7). This produces a term in the measure ⁷

$$\int d\xi^1 d\xi^2 (4S_{\text{cl}})^{-1} , \quad (4.39)$$

So the Jacobian for the ξ zero modes is given by $U_\xi = 4S_{\text{cl}}$, and the result (4.39) actually holds for any k .

For the $\bar{\eta}$ zero modes, we obtain, using some algebra for the σ -matrices,

$$(U_{\bar{\eta}})_{\alpha'\beta'} = 8S_{\text{cl}} \delta_{\alpha'\beta'} , \quad (4.40)$$

so that the corresponding measure is

$$\int d\bar{\eta}_1 d\bar{\eta}_2 (8S_{\text{cl}})^{-1} , \quad (4.41)$$

which only holds for $k = 1$.

Finally we compute the Jacobian for the fermionic ‘‘gauge orientation’’ zero modes. For convenience, we take the solutions in regular gauge (the Jacobian is gauge invariant anyway), and find

$$\left(Z_{(\mu^w)}^\alpha \right)_v^u = \frac{\rho}{\sqrt{(x^2 + \rho^2)^3}} \delta_v^\alpha \Delta^u_w , \quad \left(Z_{(\bar{\mu}^w)}^\alpha \right)_v^u = \frac{\rho}{\sqrt{(x^2 + \rho^2)^3}} \epsilon^{\alpha u} \Delta^w_v , \quad (4.42)$$

where the N by N matrix Δ is the unity matrix in the $(N-2)$ by $(N-2)$ upper diagonal block, and zero elsewhere. The norms of Z_μ and $Z_{\bar{\mu}}$ are easily seen to be zero, but the nonvanishing inner product is

$$(U_{\mu\bar{\mu}})^u_v = -\frac{2}{g^2} \int d^4x \text{tr} Z_{(\bar{\mu}^u)}^\alpha Z_{\alpha(\mu^v)} = \frac{2\pi^2}{g^2} \Delta^u_v , \quad (4.43)$$

⁷Sometimes one finds in the literature that $U_\xi = 2S_{\text{cl}}$. This is true when one uses the conventions for Grassmann integration $\int d^2\xi \xi^\alpha \xi^\beta = \frac{1}{2} \epsilon^{\alpha\beta}$. In our conventions $d^2\xi \equiv d\xi^1 d\xi^2$.

where we have used the integral (A.24). It also follows from the index structure that the ξ and $\bar{\eta}$ zero modes are orthogonal to the μ zero modes, so there is no mixing in the Jacobian.

Putting everything together, the fermionic measure for \mathcal{N} adjoint fermions coupled to $SU(N)$ YM theory, with $k = 1$, is

$$\int \prod_{A=1}^{\mathcal{N}} d^2 \xi^A \left(\frac{g^2}{32\pi^2} \right)^{\mathcal{N}} \int \prod_{A=1}^{\mathcal{N}} d^2 \bar{\eta}^A \left(\frac{g^2}{64\pi^2 \rho^2} \right)^{\mathcal{N}} \int \prod_{A=1}^{\mathcal{N}} \prod_{u=1}^{N-2} d\mu^{A,u} d\bar{\mu}_u^A \left(\frac{g^2}{2\pi^2} \right)^{\mathcal{N}(N-2)}. \quad (4.44)$$

Similarly, one can include fundamental fermions, for which the Jacobian factor is

$$U_{\mathcal{K}} \equiv \frac{1}{g^2} \int d^4 x Z^\alpha{}_u Z_\alpha{}^u = \frac{\pi^2}{g^2}, \quad (4.45)$$

for each specie.

5 One loop determinants.

After having determined the measure on the collective coordinate moduli space, we now compute the determinants that arise after Gaussian integration over the quantum fluctuations. Before doing so, we extend the model by adding real scalar fields in the adjoint representation. The action is

$$S = -\frac{1}{g^2} \int d^4 x \text{tr} \left\{ \frac{1}{2} F_{\mu\nu} F_{\mu\nu} + (\mathcal{D}_\mu \phi) (\mathcal{D}_\mu \phi) - i \bar{\lambda} \bar{\mathcal{D}} \lambda - i \lambda \mathcal{D} \bar{\lambda} \right\}. \quad (5.1)$$

Here, λ is a two-component Weyl spinor which we take in the adjoint representation. Generalization to fundamental fermions is straightforward. In Minkowski space, $\bar{\lambda}$ belongs to the conjugate representation of the Lorentz group, but in euclidean space it is unrelated to λ .

The anti-instanton solution which we will expand around is

$$A_\mu^{\text{cl}}, \quad \phi_{\text{cl}} = 0, \quad \lambda_{\text{cl}} = 0, \quad \bar{\lambda}_{\text{cl}} = 0, \quad (5.2)$$

where A_μ^{cl} is the anti-instanton. Although this background represents an exact solution to the field equations, it does not include the fermionic zero modes, which are the solutions to the Dirac equation. In this approach, one should treat these zero modes in perturbation theory. As will become clearer in later sections, we would like to include the fermionic zero modes in the classical anti-instanton background and treat them exactly. This is also more compatible with supersymmetry and the ADHM construction for (supersymmetric) multi-instantons. But then one would have to redo the following analysis, which, to our knowledge, has not been done so far. We comment on this issue again at the end of this section.

After expanding $A_\mu = A_\mu^{\text{cl}} + A_\mu^{\text{qu}}$, and similarly for the other fields, we add gauge fixing terms

$$S_{\text{gf}} = \frac{1}{g^2} \int d^4 x \text{tr} \left\{ \left(\mathcal{D}_\mu^{\text{cl}} A_\mu^{\text{qu}} \right)^2 - 2 b \mathcal{D}_{\text{cl}}^2 c \right\}, \quad (5.3)$$

such that the total gauge field action is given by (4.3). The integration over A_μ gives

$$[\det' \Delta_{\mu\nu}]^{-1/2} , \quad \Delta_{\mu\nu} = -\mathcal{D}^2 \delta_{\mu\nu} - 2F_{\mu\nu} , \quad (5.4)$$

where the prime stands for the amputated determinant, with zero eigenvalues left out. We have suppressed the subscript ‘cl’ and Lie algebra indices.

Integration over the scalar fields results in

$$[\det \Delta_\phi]^{-1/2} , \quad \Delta_\phi = -\mathcal{D}^2 , \quad (5.5)$$

and the ghost system yields similarly

$$[\det \Delta_{\text{gh}}] , \quad \Delta_{\text{gh}} = -\mathcal{D}^2 . \quad (5.6)$$

For the fermions λ and $\bar{\lambda}$, we give a bit more explanation. Since neither \mathcal{D} nor $\bar{\mathcal{D}}$ is hermitean, we can not evaluate the determinant in terms of its eigenvalues. But both products

$$\Delta_- = -\mathcal{D} \bar{\mathcal{D}} = -\mathcal{D}^2 - \sigma_{\mu\nu} F_{\mu\nu} , \quad \Delta_+ = -\bar{\mathcal{D}} \mathcal{D} = -\mathcal{D}^2 , \quad (5.7)$$

which still have unwritten spinor indices, are hermitean. Hence we can expand λ in terms of eigenfunctions F_i of Δ_- with coefficients ξ_i , and $\bar{\lambda}$ in terms of eigenfunctions \bar{F}_i of Δ_+ with coefficients $\bar{\xi}_i$. We have seen in section 3 that both operators have the same spectrum of non-zero eigenvalues, and the relation between the eigenfunctions is $\bar{F}_i = \bar{\mathcal{D}} F_i$. Defining the path integral over λ and $\bar{\lambda}$ as the integration over ξ_i and $\bar{\xi}_i$, one gets the determinant over the nonzero eigenvalues. The result for the integration over the fermions gives

$$[\det' \Delta_-]^{1/4} [\det \Delta_+]^{1/4} . \quad (5.8)$$

As stated before, since all the eigenvalues of both Δ_- and Δ_+ are the same, the determinants are formally equal. This result can also be obtained by writing the spinors in terms of Dirac fermions, the determinant we have to compute is then

$$[\det' \Delta_D^2]^{1/2} , \quad \Delta_D = \begin{pmatrix} 0 & \mathcal{D} \\ \bar{\mathcal{D}} & 0 \end{pmatrix} . \quad (5.9)$$

Now we notice that the determinants for the bosons are related to the determinants of Δ_- and Δ_+ . For the ghosts and adjoint scalars this is obvious,

$$\det \Delta_\phi = \det \Delta_{\text{gh}} = [\det \Delta_+]^{1/2} . \quad (5.10)$$

For the vector fields, we use the identity

$$\Delta_{\mu\nu} = \frac{1}{2} \text{tr} \{ \bar{\sigma}_\mu \Delta_- \sigma_\nu \} = \frac{1}{2} \bar{\sigma}_{\mu\alpha'\beta} \Delta_-^\beta{}_\gamma \delta^{\alpha'\delta'} \sigma_\nu^{\gamma\delta'} , \quad (5.11)$$

to prove that

$$\det' \Delta_{\mu\nu} = [\det' \Delta_-]^2 . \quad (5.12)$$

Now we can put everything together. The determinant for a YM system coupled to n adjoint scalars and \mathcal{N} Weyl spinors is

$$[\det' \Delta_-]^{-1+\mathcal{N}/4} [\det' \Delta_+]^{\frac{1}{4}(2+\mathcal{N}-n)} . \quad (5.13)$$

This expression simplifies to the ratio of the determinants when $\mathcal{N} - \frac{n}{2} = 1$. Particular cases are

$$\begin{aligned} \mathcal{N} = 1 \quad n = 0 & \rightarrow \left[\frac{\det \Delta_+}{\det' \Delta_-} \right]^{3/4} , \\ \mathcal{N} = 2 \quad n = 2 & \rightarrow \left[\frac{\det \Delta_+}{\det' \Delta_-} \right]^{1/2} , \\ \mathcal{N} = 4 \quad n = 6 & \rightarrow \left[\frac{\det \Delta_+}{\det' \Delta_-} \right]^0 . \end{aligned} \quad (5.14)$$

These cases correspond to supersymmetric models with \mathcal{N} -extended supersymmetry. Notice that for $\mathcal{N} = 4$, the determinants between bosons and fermions cancel, so there is no one-loop contribution. For $\mathcal{N} = 1, 2$, the determinants give formally unity since the non-zero eigenvalues are the same. As we will explain below, however, we must first regularize the theory to define the determinants properly and this may yield different answers. In all other cases, we will not get this ratio of these particular determinants.

All of the above manipulations are a bit formal. We know that as soon as we do perturbation theory, one must first choose a regularization scheme in order to define the quantum theory. After that, the renormalization procedure must be carried out and counterterms must be added. The counterterms are the same as in the theory without instantons and their finite parts must be specified by physical normalization conditions. The ratios of products of non-zero eigenvalues have the meaning of a mass correction to the instanton (seen as a five-dimensional soliton). One can write this ratio as the exponent of the difference of two infinite sums

$$\frac{\det \Delta_+}{\det' \Delta_-} = \exp \left(\sum_n \omega_n^{(+)} - \sum_n \omega_n^{(-)} \right) , \quad (5.15)$$

where the eigenvalues $\lambda_n = \exp \omega_n$. The frequencies $\omega_n^{(+)}$ and $\omega_n^{(-)}$ are discretized by putting the system in a box of size R and imposing suitable boundary conditions on the quantum fields at R (for example, $\phi(R) = 0$, or $\frac{d}{dR} \phi(R) = 0$, or a combination of thereof [6]). These boundary conditions may be different for different fields. The sums over $\omega_n^{(+)}$ and $\omega_n^{(-)}$ are divergent; their difference is still divergent (although less divergent than each sum separately) but after adding counterterms ΔS one obtains a finite answer. The problem is that one can combine the terms

in both series in different ways, possibly giving different answers. By combining $\omega_n^{(+)}$ with $\omega_n^{(-)}$ for each fixed n , one would find that the ratio $(\det \Delta_+ / \det' \Delta_-)$ equals unity. However, other values could result by using different ways to regulate these sums. It is known that in field theory the results for the effective action due to different regularization schemes differ at most by a local finite counterterm. In the background field formalism we are using, this counterterm must be background gauge invariant, and since we consider only vacuum expectation values of the effective action, only one candidate is possible: it is proportional to the gauge action $\int d^4x \operatorname{tr} F^2$ and multiplied by the one-loop beta-function for the various fields which can run in the loop,

$$\Delta S \propto \beta(g) \int d^4x \operatorname{tr} F^2 \ln \frac{\mu^2}{\mu_0^2} . \quad (5.16)$$

The factor $\ln(\mu^2/\mu_0^2)$ parametrizes the freedom in choosing different renormalization schemes.

A particular regularization scheme used in [6] is Pauli-Villars regularization. In this case 't Hooft used first x -dependent regulator masses to compute the ratios of the one-loop determinants Δ in the instanton background and $\Delta^{(0)}$ in the trivial vacuum. Then he argued that the difference between using the x -dependent masses and using the more usual constant masses, was of the form ΔS given above. The final result for pure YM $SU(N)$ in the $k = 1$ sector is [6, 29]

$$\left[\frac{\det' \Delta_-}{\det \Delta_-^{(0)}} \right]^{-1} \left[\frac{\det \Delta_+}{\det \Delta_+^{(0)}} \right]^{1/2} = \mu^{4N} \exp \left\{ -\frac{1}{3} N \ln(\mu\rho) - \alpha(1) - 2(N-2)\alpha\left(\frac{1}{2}\right) \right\} . \quad (5.17)$$

Here we have normalized the determinants against the vacuum, indicated by the superscript (0). Note that Pauli-Villars regulator fields contribute one factor of μ for each zero mode of the original fields. The numerical values of the function $\alpha(t)$ are related to the Riemann zeta function, and take the values $\alpha\left(\frac{1}{2}\right) = 0.145873$ and $\alpha(1) = 0.443307$. Notice that this expression for the determinant depends on ρ , and therefore changes the behaviour of the ρ integrand in the collective coordinate measure. Combined with (4.35) one correctly reproduces the β -function of $SU(N)$ YM theory.

Let us briefly come back to the point of expanding around a background which includes the fermionic zero modes. Upon expanding around this classical configuration, one finds mixed terms between the gauge field and fermion quantum fluctuations, e.g. terms like $\bar{\lambda}_{\text{qu}} A_{\mu}^{\text{qu}} \lambda_{\text{cl}}$. Integrating out the quantum fields yields a superdeterminant in the space of all the fields, which will in general depend on the Grassmann collective coordinates (GCC) appearing in λ_{cl} . It remains to be seen if this superdeterminant will still give unity in the supersymmetric cases, and if not, one would like to find its dependence on the GCC. We hope to report on this in a future publication.

6 $\mathcal{N} = 4$ supersymmetric Yang-Mills theory.

For reasons explained in the introduction, we now focus on the $\mathcal{N} = 4$ model [30]. The action is of course well known in Minkowski space, but instantons require, however, the formulation of the $\mathcal{N} = 4$ euclidean version. Due to absence of a real representation of Dirac matrices in four-dimensional euclidean space, the notion of Majorana spinor is absent. This complicates the construction of euclidean Lagrangians for supersymmetric models [31, 32, 33]. For $\mathcal{N} = 2, 4$ theories, one can replace the Majorana condition by the so-called symplectic Majorana condition and consequently construct real supersymmetric Lagrangians [34, 35].

In the following subsection we write down the action in Minkowski space-time and discuss the reality conditions on the fields. Next we construct the hermitean $\mathcal{N} = 4$ euclidean model via the dimensional reduction of 10D $\mathcal{N} = 1$ super-Yang-Mills theory along the time direction. Using this, we study in the consequent section the solutions of the classical equations of motion, using an iteration procedure in the Grassmann collective coordinates.

6.1 Minkowskian $\mathcal{N} = 4$ SYM.

The $\mathcal{N} = 4$ action in Minkowski space-time with the signature $\eta^{\mu\nu} = \text{diag}(-, +, +, +)$ is given by

$$S = \frac{1}{g^2} \int d^4x \text{tr} \left\{ \frac{1}{2} F_{\mu\nu} F^{\mu\nu} - i \bar{\lambda}_A^{\dot{\alpha}} \bar{\mathcal{D}}_{\dot{\alpha}\beta} \lambda^{\beta,A} - i \lambda_{\dot{\alpha}}^A \mathcal{D}^{\alpha\dot{\beta}} \bar{\lambda}_{A\dot{\beta}} + \frac{1}{2} (\mathcal{D}_\mu \bar{\phi}_{AB}) (\mathcal{D}^\mu \phi^{AB}) \right. \\ \left. - \sqrt{2} \bar{\phi}_{AB} \{ \lambda^{\alpha,A}, \lambda_\alpha^B \} - \sqrt{2} \phi^{AB} \{ \bar{\lambda}_{\dot{\alpha},A}, \bar{\lambda}_{\dot{\alpha},B} \} + \frac{1}{8} [\phi^{AB}, \phi^{CD}] [\bar{\phi}_{AB}, \bar{\phi}_{CD}] \right\}. \quad (6.1)$$

The on-shell $\mathcal{N} = 4$ supermultiplet consists out of a real gauge field, A_μ , four complex Weyl spinors $\lambda^{\alpha,A}$ and an antisymmetric complex scalar ϕ^{AB} with labels $A, B = 1, \dots, 4$ of internal R symmetry group $SU(4)$.

The reality conditions on the components of this multiplet are ⁸ the Majorana conditions $(\lambda^{\alpha,A})^* = -\bar{\lambda}_A^{\dot{\alpha}}$ and $(\lambda_{\dot{\alpha}}^A)^* = \bar{\lambda}_{\dot{\alpha},A}$ and

$$\bar{\phi}_{AB} \equiv (\phi^{AB})^* = \frac{1}{2} \epsilon_{ABCD} \phi^{CD}. \quad (6.2)$$

The sigma matrices are defined by $\sigma^{\mu\alpha\dot{\beta}} = (-1, \tau^i)$, $\bar{\sigma}_{\dot{\alpha}\beta}^\mu = (1, \tau^i)$ and complex conjugation gives $(\sigma_\mu^{\alpha\dot{\beta}})^* = \sigma_\mu^{\beta\dot{\alpha}} = \epsilon^{\dot{\alpha}\gamma} \epsilon^{\beta\delta} \bar{\sigma}_{\mu\gamma\delta}$, with $\epsilon^{\dot{\alpha}\beta} = \epsilon_{\dot{\alpha}\beta} = -\epsilon^{\alpha\beta} = -\epsilon_{\alpha\beta}$.

Since ϕ^{AB} is antisymmetric, one can express it in a basis spanned by the eta-matrices (see Appendix A)

$$\phi^{AB} = \frac{1}{\sqrt{2}} \{ S^i \eta^{iAB} + iP^i \bar{\eta}^{iAB} \}, \quad \bar{\phi}_{AB} = \frac{1}{\sqrt{2}} \{ S^i \eta_{AB}^i - iP^i \bar{\eta}_{AB}^i \}, \quad (6.3)$$

⁸Unless specified otherwise, equations which involve complex conjugation of fields will be understood as not Lie algebra valued, i.e. they hold for the components $\lambda^{a,\alpha,A}$, etc.

in terms of real scalars S^i and pseudoscalars P^i , $i = 1, 2, 3$, so that reality condition is automatically fulfilled. Then the kinetic terms for the (S, P) fields take the standard form.

The action (6.1) is invariant under the supersymmetry transformation laws

$$\begin{aligned}\delta A_\mu &= -i\bar{\zeta}_A^{\dot{\alpha}}\bar{\sigma}_{\mu\dot{\alpha}\beta}\lambda^{\beta,A} + i\bar{\lambda}_{\dot{\beta},A}\sigma_\mu^{\alpha\dot{\beta}}\zeta_\alpha^A, \\ \delta\phi^{AB} &= \sqrt{2}\left(\zeta^{\alpha,A}\lambda_\alpha^B - \zeta^{\alpha,B}\lambda_\alpha^A + \epsilon^{ABCD}\bar{\zeta}_C^{\dot{\alpha}}\bar{\lambda}_{\dot{\alpha},D}\right), \\ \delta\lambda^{\alpha,A} &= -\frac{1}{2}\sigma^{\mu\nu\alpha}{}_\beta F_{\mu\nu}\zeta^{\beta,A} - i\sqrt{2}\bar{\zeta}_{\dot{\alpha},B}\mathcal{D}^{\alpha\dot{\alpha}}\phi^{AB} + \left[\phi^{AB}, \bar{\phi}_{BC}\right]\zeta^{\alpha,C},\end{aligned}\tag{6.4}$$

which are consistent with the reality conditions. Let us turn now to the discussion of the euclidean version of this model and discuss the differences with the Minkowski theory.

6.2 Euclidean $\mathcal{N} = 4$ SYM.

To find out the $\mathcal{N} = 4$ supersymmetric YM model in euclidean $d = (4, 0)$ space, we follow the same procedure as in [30]. We start with the $\mathcal{N} = 1$ SYM model in $d = (9, 1)$ Minkowski space-time, but contrary to the original paper we reduce it on a six-torus with one time and five space coordinates [34, 35]. As opposed to action (6.1) with the $SU(4) = SO(6)$ R -symmetry group, this reduction leads to the internal non-compact $SO(5, 1)$ R -symmetry group in euclidean space. As we will see, the reality conditions on bosons and fermions will both use an internal metric for this non-compact internal symmetry group.

The $\mathcal{N} = 1$ Lagrangian with $d = (9, 1)$ reads

$$\mathcal{L}_{10} = \frac{1}{g_{10}^2}\text{tr}\left\{\frac{1}{2}F_{MN}F^{MN} + \bar{\Psi}\Gamma^M\mathcal{D}_M\Psi\right\},\tag{6.5}$$

with the field strength $F_{MN} = \partial_M A_N - \partial_N A_M + [A_M, A_N]$ and the Majorana-Weyl spinor Ψ defined by the conditions

$$\Gamma^{11}\Psi = \Psi, \quad \Psi^T C_{10}^- = \Psi^\dagger i\Gamma^0 \equiv \bar{\Psi}.\tag{6.6}$$

Here the hermitean matrix $\Gamma^{11} \equiv *\Gamma$ is a product of all Dirac matrices, $\Gamma^{11} = \Gamma^0 \dots \Gamma^9$, normalized to $(*\Gamma)^2 = +1$. The Γ -matrices obey the Clifford algebra $\{\Gamma^M, \Gamma^N\} = 2\eta^{MN}$ with metric $\eta^{MN} = \text{diag}(-, +, \dots, +)$. The Lagrangian is a density under the standard transformation rules

$$\delta A_M = \bar{\zeta}\Gamma_M\Psi, \quad \delta\Psi = -\frac{1}{2}F_{MN}\Gamma^{MN}\zeta,\tag{6.7}$$

with $\Gamma^{MN} = \frac{1}{2}[\Gamma^M\Gamma^N - \Gamma^N\Gamma^M]$ and $\bar{\zeta} = \zeta^T C_{10}^- = \zeta^\dagger i\Gamma^0$.

To proceed with the dimensional reduction we choose a particular representation of the gamma matrices in $d = (9, 1)$, namely

$$\Gamma^M = \left\{\hat{\gamma}^a \otimes \gamma^5, \mathbb{1}_{[8]\times[8]} \otimes \gamma^\mu\right\}, \quad \Gamma^{11} = \Gamma^0 \dots \Gamma^9 = \hat{\gamma}^7 \otimes \gamma^5,\tag{6.8}$$

where the 8×8 Dirac matrices $\hat{\gamma}^a$ and $\hat{\gamma}^7$ of $d = (5, 1)$ with $a = 1, \dots, 6$ can be conveniently defined by means of 't Hooft symbols as follows

$$\hat{\gamma}^a = \begin{pmatrix} 0 & \Sigma^{a,AB} \\ \bar{\Sigma}_{AB}^a & 0 \end{pmatrix}, \quad \hat{\gamma}^7 = \hat{\gamma}^1 \dots \hat{\gamma}^6 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (6.9)$$

with the notations $\Sigma^{a,AB} = \{-i\eta^{1,AB}, \eta^{2,AB}, \eta^{3,AB}, i\bar{\eta}^{k,AB}\}$, $\bar{\Sigma}_{AB}^a = \{i\eta_{AB}^1, -\eta_{AB}^2, -\eta_{AB}^3, i\bar{\eta}_{AB}^k\}$ so that $\frac{1}{2}\epsilon_{ABCD}\Sigma^{a,CD} = -\bar{\Sigma}_{AB}^a$. Meanwhile γ^μ and γ^5 are the usual of $d = (4, 0)$ introduced in (3.3). Note that in this construction we implicitly associated one of the Dirac matrices, namely η^1 , in 6 dimensions with the time direction and thus it has square -1 ; all other (as well as all $d = (4, 0)$) are again hermitean with square $+1$.

Let us briefly discuss the charge conjugation matrices in $d = (9, 1)$, $d = (5, 1)$ and $d = (4, 0)$. One can prove by means of finite group theory [36] that all their properties are representation independent. In general there are two charge conjugation matrices C^+ and C^- in even dimensions, satisfying $C^\pm \Gamma^\mu = \pm (\Gamma^\mu)^T C^\pm$, and $C^+ = C^- * \Gamma$. These charge conjugation matrices do not depend on the signature of space-time and obey the relation $C^- * \Gamma = \pm (*\Gamma)^T C^-$ with $-$ sign in $d = 10, 6$ and $+$ sign in $d = 4$. The transposition depends on the dimension and leads to $(C^\pm)^T = \pm C^\pm$ in $d = 10$, $(C^\pm)^T = \mp C^\pm$ in $d = 6$, and finally $(C^\pm)^T = -C^\pm$ for $d = 4$. Explicitly, the charge conjugation matrix C_{10}^- is given by $C_6^- \otimes C_4^-$ where

$$C_4^- = \gamma^4 \gamma^2 = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\alpha'\beta'} \end{pmatrix}, \quad C_6^- = i\hat{\gamma}^4 \hat{\gamma}^5 \hat{\gamma}^6 = \begin{pmatrix} 0 & \delta_A^B \\ \delta_B^A & 0 \end{pmatrix}. \quad (6.10)$$

Upon compactification to euclidean $d = (4, 0)$ space the 10-dimensional Lorentz group $SO(9, 1)$ reduces to $SO(4) \times SO(5, 1)$ with compact space-time group $SO(4)$ and R -symmetry group $SO(5, 1)$. In these conventions a 32-component chiral Weyl spinor Ψ decomposes as follows into 8 and 4 component chiral-chiral and antichiral-antichiral spinors

$$\Psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} \lambda^{\alpha,A} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ \bar{\lambda}_{\alpha',A} \end{pmatrix}, \quad (6.11)$$

where $\lambda^{\alpha,A}$ ($\alpha = 1, 2$) transforms only under the first $SU(2)$ in $SO(4) = SU(2) \times SU(2)$, while $\bar{\lambda}_{\alpha',A}$ changes only under the second $SU(2)$. Furthermore, $\bar{\lambda}_{\alpha',A}$ transforms in the complex conjugate of the $SO(5, 1)$ representation of $\lambda^{\alpha,A}$, namely, $(\lambda^*)^{\alpha,B} \eta_{BA}^1$ transforms like $\bar{\lambda}_{\alpha,A}$, and the two spinor representation of $SO(5, 1)$ are pseudoreal, i.e. $[\hat{\gamma}^a, \hat{\gamma}^b]_L^* \eta^1 = \eta^1 [\hat{\gamma}^a, \hat{\gamma}^b]_R$ where L (R) denotes the upper (lower) 4-component spinor.

Substituting these results, the Lagrangian reduces to

$$\begin{aligned} \mathcal{L}_E^{N=4} = & \frac{1}{g^2} \text{tr} \left\{ \frac{1}{2} F_{\mu\nu} F_{\mu\nu} - i \bar{\lambda}_A^{\alpha'} \bar{\mathcal{D}}_{\alpha'\beta} \lambda^{\beta,A} - i \lambda_\alpha^A \mathcal{D}_\mu^{\alpha\beta'} \bar{\lambda}_{\beta',A} + \frac{1}{2} (\mathcal{D}_\mu \bar{\phi}_{AB}) (\mathcal{D}_\mu \phi^{AB}) \right. \\ & \left. - \sqrt{2} \bar{\phi}_{AB} \{ \lambda^{\alpha,A}, \lambda_\alpha^B \} - \sqrt{2} \phi^{AB} \{ \bar{\lambda}_A^{\alpha'}, \bar{\lambda}_{\alpha',B} \} + \frac{1}{8} [\phi^{AB}, \phi^{CD}] [\bar{\phi}_{AB}, \bar{\phi}_{CD}] \right\}, \quad (6.12) \end{aligned}$$

where we still use the definition for $\bar{\phi}_{AB} \equiv \frac{1}{2}\epsilon_{ABCD}\phi^{CD}$. These scalars come from the ten-dimensional gauge field, and can be grouped into $\phi^{AB} = \frac{1}{\sqrt{2}}\Sigma^{aAB}A_a$, where A_a are the first six real components of the ten dimensional gauge field A_M . Writing the action in terms of these 6 scalars, one finds however that one of the fields, say A_0 , has a different sign in the kinetic term, which reflects the $SO(5, 1)$ symmetry of the theory. In the basis with the ϕ^{AB} fields, we obtain formally the same action for the Minkowski case by reducing on a torus with 6 space coordinates, but the difference hides in the reality conditions which we will discuss in the next subsection.

The action is invariant under the dimensionally reduced supersymmetry transformation rules

$$\begin{aligned}
\delta A_\mu &= -i\bar{\zeta}_A^{\alpha'}\bar{\sigma}_{\mu\alpha'\beta}\lambda^{\beta,A} + i\bar{\lambda}_{\beta',A}\sigma_\mu^{\alpha\beta'}\zeta_\alpha^A, \\
\delta\phi^{AB} &= \sqrt{2}\left(\zeta^{\alpha,A}\lambda_\alpha^B - \zeta^{\alpha,B}\lambda_\alpha^A + \epsilon^{ABCD}\bar{\zeta}_C^{\alpha'}\bar{\lambda}_{\alpha',D}\right), \\
\delta\lambda^{\alpha,A} &= -\frac{1}{2}\sigma^{\mu\nu\alpha}{}_\beta F_{\mu\nu}\zeta^{\beta,A} - i\sqrt{2}\bar{\zeta}_{\alpha',B}\mathcal{P}^{\alpha\alpha'}\phi^{AB} + [\phi^{AB}, \bar{\phi}_{BC}]\zeta^{\alpha,C}, \\
\delta\bar{\lambda}_{\alpha',A} &= -\frac{1}{2}\bar{\sigma}^{\mu\nu}{}_{\alpha'}{}^{\beta'}F_{\mu\nu}\bar{\zeta}_{\beta',A} + i\sqrt{2}\zeta^{\alpha,B}\bar{\mathcal{P}}_{\alpha'\alpha}\bar{\phi}_{AB} + [\bar{\phi}_{AB}, \phi^{BC}]\bar{\zeta}_{\alpha',C}.
\end{aligned} \tag{6.13}$$

Again, these rules are formally the same as in (6.4).

6.3 Involution in euclidean space.

The Majorana-Weyl condition (6.6) on Ψ leads in 4D euclidean space to reality conditions on λ^α which are independent of those on $\bar{\lambda}_{\alpha'}$, namely,

$$\left(\lambda^{\alpha,A}\right)^* = -\lambda^{\beta,B}\epsilon_{\beta\alpha}\eta_{BA}^1, \quad \left(\bar{\lambda}_{\alpha',A}\right)^* = -\bar{\lambda}_{\beta',B}\epsilon^{\beta'\alpha'}\eta^{1,BA}. \tag{6.14}$$

These reality conditions are consistent and define a symplectic Majorana spinor in euclidean space. The $SU(2) \times SU(2)$ covariance of (6.14) is obvious from the pseudoreality of the $\mathbf{2}$ of $SU(2)$, but covariance under $SO(5, 1)$ can also be checked (use $[\eta^a, \bar{\eta}^b] = 0$). Since the first Σ matrix has an extra factor i in order that $(I^0)^2 = -1$, see (6.8), the reality condition on ϕ^{AB} involves η_{AB}^1

$$\left(\phi^{AB}\right)^* = \eta_{AC}^1\phi^{CD}\eta_{DB}^1. \tag{6.15}$$

The euclidean action in (6.12) is hermitean under the reality conditions in (6.14) and (6.15). For the σ -matrices, we have under complex conjugation

$$\left(\sigma_\mu^{\alpha\beta'}\right)^* = \sigma_{\mu\alpha\beta'}, \quad \left(\bar{\sigma}_{\mu\alpha'\beta}\right)^* = \bar{\sigma}_\mu^{\alpha'\beta}. \tag{6.16}$$

Obviously due to the nature of the Lorentz group the involution cannot change one type of indices into another as opposed to the minkowskian case.

7 Instantons in $\mathcal{N} = 4$ SYM.

One can easily derive the euclidean equations of motion from (6.12)

$$\begin{aligned}
\mathcal{D}_\nu F_{\nu\mu} - i \left\{ \bar{\lambda}_A^{\alpha'} \bar{\sigma}_{\mu\alpha'\beta}, \lambda^{\beta,A} \right\} - \frac{1}{2} \left[\bar{\phi}_{AB}, \mathcal{D}_\mu \phi^{AB} \right] &= 0, \\
\mathcal{D}^2 \phi^{AB} + \sqrt{2} \left\{ \lambda^{\alpha,A}, \lambda_\alpha^B \right\} + \frac{1}{\sqrt{2}} \epsilon^{ABCD} \left\{ \bar{\lambda}_C^{\alpha'}, \bar{\lambda}_{\alpha',D} \right\} - \frac{1}{2} \left[\bar{\phi}_{CD}, \left[\phi^{AB}, \phi^{CD} \right] \right] &= 0, \\
\bar{\mathcal{D}}_{\alpha'\beta} \lambda^{\beta,A} + i\sqrt{2} \left[\phi^{AB}, \bar{\lambda}_{\alpha',B} \right] = 0, \quad \mathcal{D}^{\alpha\beta'} \bar{\lambda}_{\beta',A} - i\sqrt{2} \left[\bar{\phi}_{AB}, \lambda^{\alpha,B} \right] &= 0.
\end{aligned} \tag{7.1}$$

An obvious solution is the configuration $\Phi = \{A_\mu = A_\mu^{\text{cl}}, \phi^{AB} = \lambda^{\alpha,A} = \bar{\lambda}_{\alpha',A} = 0\}$. However, as we have seen in previous sections, in the background of an anti-instanton the Dirac operator has zero eigenvalues λ satisfying the Dirac equation $\bar{\mathcal{D}}\lambda = 0$. There are two equivalent though formally different approaches to account for these new configurations.

According to the first, one starts with the above mentioned purely bosonic configuration Φ . Then one must treat the fermionic collective coordinates in perturbation theory, because they would appear in the quantum fluctuations as the zero eigenvalue modes of the Dirac operator. Although it is legitimate to do so, it is inconvenient for the reason that usually in perturbation theory, one restricts to quadratic order in the fluctuations (Gaussian approximation). This would however be insufficient for the fermionic collective coordinates, as we want to construct its effective action to all orders. In other words, we want to treat them exactly, and not in perturbation theory. The second approach is to include the fermionic instanton in the classical configuration. Doing this, we automatically treat them exactly and to all orders, as long as we can find exact solutions to the equations of motion. This procedure is also more consistent with supersymmetry and the ADHM construction for multi-instantons. For these reasons, we choose the second approach.

Now we describe the procedure for constructing the solution to the classical equations of motion. It is obvious that for a system with uncoupled scalars and fermions (5.1), the configuration $\{A_\mu^{\text{cl}}, \lambda^{\text{cl}}, \bar{\lambda}^{\text{cl}} = \phi^{\text{cl}} = 0\}$, with λ^{cl} a solution of the Dirac equation, is an exact solution of the field equations. As soon as the Yukawa couplings are turned on, the situation changes drastically as it happens for the case at hand with the system (7.1). The point we want to emphasize here is that the above instanton configuration with non-zero fermion mode no longer satisfies the field equations. Indeed, since λ^A is turned on, by looking at the equation for the scalar field in (7.1), we conclude that ϕ^{AB} cannot be taken to be zero at quadratic order in Grassmann collective coordinates (GCC) due to nonvanishing $\{\lambda^{\alpha,A}, \lambda_\alpha^B\}$. Knowing this, then also $\bar{\lambda}$ is turned on at cubic order in GCC, as follows from its field equation. This leads to an iteration procedure which yields a solution as an expansion in the GCC which stops (for finite N) after a finite number of steps. We will now demonstrate this more explicitly, first in the case of $SU(2)$, and in subsection 7.2 for $SU(N)$.

7.1 Iterative solution in case of $SU(2)$ group.

Let us consider first the gauge group $SU(2)$. This is an exceptional case in the sense that all fermionic zero modes can be generated by means of supersymmetric and superconformal transformation. E.g. if we denote, suppressing indices, by Q the supersymmetry generators, then a new solution is given by

$$\Phi(\xi) = e^{i\xi \cdot Q} \Phi e^{-i\xi \cdot Q} . \quad (7.2)$$

We start generating solutions of the above equations of motion iteratively in Grassmann parameters from the purely bosonic anti-instanton configuration $\Phi = \{A_\mu = A_\mu^{\bar{I}}, \phi^{AB} = \lambda^{\alpha,A} = \bar{\lambda}_{\alpha',A} = 0\}$. The exact solution can be expanded as

$$\Phi(\xi) = \sum_{n=0}^{\infty} \frac{1}{n!} \delta^n \Phi . \quad (7.3)$$

Explicitly we produce from the anti-instanton potential

$${}^{(0)}A_{\mu u}{}^v = A_{\mu u}^{\bar{I} v} = -\frac{\rho^2}{x^2(x^2 + \rho^2)} \bar{\sigma}^{\mu\nu}{}_u{}^v x_\nu , \quad (7.4)$$

the already known fermionic zero mode

$${}^{(1)}\lambda^{\alpha,A} = -\frac{1}{2} \sigma_{\mu\nu}{}^\alpha{}_\beta \xi^{\beta,A} {}^{(0)}F_{\mu\nu} . \quad (7.5)$$

It is obvious that we can only use the ζ supersymmetry transformation rules, because $\bar{\zeta}$ leaves the bosonic anti-instanton invariant. The left superscript on each field indicates how many GCC it contains. Given this solution for λ , we use the supersymmetry transformations, first on ϕ^{AB} , then on $\bar{\lambda}_A$, to determine

$${}^{(2)}\phi^{AB} = \frac{1}{\sqrt{2}} (\xi^A \sigma_{\mu\nu} \xi^B) {}^{(0)}F_{\mu\nu} , \quad (7.6)$$

$${}^{(3)}\bar{\lambda}_{\alpha',A} = \frac{i}{6} \epsilon_{ABCD} \bar{\sigma}_{\nu\alpha'\beta} \xi^{\beta,B} (\xi^C \sigma_{\rho\sigma} \xi^D) {}^{(0)}\mathcal{D}_\nu {}^{(0)}F_{\rho\sigma} . \quad (7.7)$$

When we suppress spinor indices, we understand that they appear in their natural position, dictated by the sigma-matrices. One can check that these are indeed the solutions of the field equations to the required order in the Grassmann parameters, i.e.

$$\begin{aligned} {}^{(0)}\mathcal{D}_\nu {}^{(0)}F_{\nu\mu} = 0 , & \quad {}^{(0)}\mathcal{D}^2 {}^{(2)}\phi^{AB} + \sqrt{2} \left\{ {}^{(1)}\lambda^{\alpha,A}, {}^{(1)}\lambda^B_\alpha \right\} = 0 , \\ {}^{(0)}\bar{\mathcal{D}}^{\alpha\beta'} {}^{(1)}\lambda^{\beta,A} = 0 , & \quad {}^{(0)}\bar{\mathcal{D}}^{\alpha\beta'} {}^{(3)}\bar{\lambda}_{\beta',A} - i\sqrt{2} \left[{}^{(2)}\bar{\phi}_{AB}, {}^{(1)}\lambda^{\alpha,B} \right] = 0 . \end{aligned} \quad (7.8)$$

To check, for example that the field equation for $\bar{\lambda}$ is indeed satisfied, one can use the field equation for the gauge field, the Bianchi identity and the Fierz relation for two-component spinors

$$\xi_{(1)}^\alpha \xi_{(2)\beta} = \frac{1}{2} \delta^\alpha{}_\beta (\xi_{(1)}^\gamma \xi_{(2)\gamma}) - \frac{1}{8} \sigma_{\mu\nu}{}^\alpha{}_\beta \left(\xi_{(1)\gamma} \sigma_{\mu\nu}{}^\gamma{}_\delta \xi_{(2)}^\delta \right) , \quad (7.9)$$

together with the self-duality properties of the sigma-matrices, like e.g. $\epsilon_{\nu\theta\rho\sigma}\sigma_{\mu\theta} = -\delta_{\mu\nu}\sigma_{\rho\sigma} - \delta_{\mu\rho}\sigma_{\sigma\nu} - \delta_{\mu\sigma}\sigma_{\nu\rho}$ stemming from the anti-selfduality of $\sigma_{\mu\nu}$.

Proceeding along these line, we start generating corrections to already non-vanishing fields. To fourth order in ξ , we find for the gauge field

$${}^{(4)}A_\mu = \frac{1}{24}\epsilon_{ABCD} \left(\xi^A \sigma_{\mu\nu} \xi^B \right) \left(\xi^C \sigma_{\rho\sigma} \xi^D \right) {}^{(0)}\mathcal{D}_\nu {}^{(0)}F_{\rho\sigma} . \quad (7.10)$$

From this it follows that the field strength constructed from ${}^{(0)}A_\mu + {}^{(4)}A_\mu$ is no longer selfdual.

It is obvious that due to algebraic nature of the procedure one can easily continue it to higher orders in ξ , but for present purposes (to compute the correlation function of four stress-tensors) it is sufficient to stop at fourth order. Using the superconformal supersymmetry transformation laws, one can similarly construct a solution in $\bar{\eta}$, with equation (3.12) being the first term in the expansion.

It is instructive to compare this method, which is sometimes called the sweeping procedure, with the explicit solution of the above equations of motion. For instance, for ϕ^{AB} one finds by direct integration of (7.8)

$${}^{(2)}\phi_{u \ v}^{AB} = - \left\{ \frac{8\sqrt{2}}{(x^2 + \rho^2)^2} + \frac{C_1}{x^2(x^2 + \rho^2)} + \frac{C_2}{\rho^4} \left(1 + \frac{x^4}{3\rho^2(x^2 + \rho^2)} \right) \right\} \xi_u^{[A} \xi^{B],v} , \quad (7.11)$$

where C_1 and C_2 are integration constants, $\xi_u = (\xi_1, \xi_2)$, and antisymmetrization is done with weight one, i.e. $[A, B] = \frac{1}{2}(AB - BA)$. The SUSY procedure generates only the first term here. The second one is a solution everywhere except for origin, so it requires a delta function source. Since we require regularity of the solution everywhere, this term must be dropped. The last term has a rising asymptotic solution in the infrared region. Thus this sweeping procedure gives only solutions with well defined asymptotical properties.

7.2 Extension to $SU(N)$ group.

Let us now turn to the construction of the solution to the equation of motion in the theory with $SU(N)$ gauge group. On top of the ξ^A and $\bar{\eta}^A$, we have $8(N-2)$ extra zero modes denoted by μ_u^A and $\bar{\mu}^{Au}$. As we know the Dirac operator develops zero modes in the anti-instanton background

$${}^{(1)}\lambda^{\alpha, A}_{u \ v} = \frac{\rho}{\sqrt{x^2(x^2 + \rho^2)^3}} \left(\mu_u^A x^{\alpha v} + x^\alpha_{\ u} \bar{\mu}^{A, v} \right) . \quad (7.12)$$

Recall that in writing down this expression, we have chosen a particular embedding of the $SU(2)$ singular gauge anti-instanton in the right bottom corner of the $SU(N)$ matrix.

We want to find the solution of the equations of motion as an expansion in the Grassmann collective coordinates. In the previous subsection this was done making use of the broken supersymmetry transformations for the ξ collective coordinates. Unfortunately, there is no known

symmetry that generates μ and $\bar{\mu}$ fermionic zero modes. Thus we are forced to change our strategy and solve the equations explicitly. After having obtained the fermionic zero mode (7.12), the next step is to solve the scalar field equation of motion.

By computing the fermion bilinear term in the scalar equation (7.8), we find that the scalar field takes the form

$${}^{(2)}\phi^{AB}{}_u{}^v = f(x^2, \rho^2) \left(\mu_u^{[A} \bar{\mu}^{B],v} - \frac{1}{2} \left(\mu_w^{[A} \bar{\mu}^{B],w} \right) \tilde{\delta}_u{}^v \right), \quad (7.13)$$

where we have denoted

$$\tilde{\delta}_u{}^v = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}_{[2] \times [2]} \end{pmatrix}. \quad (7.14)$$

This ansatz can now be plugged into the equation of motion for ϕ^{AB} and leads to a second order differential equation for the function f ,

$$x^2 f'' + 2f' = -\frac{\sqrt{2}\rho^2}{(x^2 + \rho^2)^3}, \quad (7.15)$$

where the prime stands for the derivative w.r.t. x^2 . Note that in the covariant derivative in (7.8) the connection drops out since in the $SU(2)$ subspace the colour structure of ϕ^{AB} is simply a unit matrix. The solution of (7.15) is given by

$$f(x^2, \rho^2) = \frac{1}{\sqrt{2}} \frac{1}{x^2 + \rho^2} + \frac{C_1}{\rho^2} + \frac{C_2}{x^2}, \quad (7.16)$$

where C_1 and C_2 are integration constants corresponding to the homogeneous solutions. The third term is however not a solution in the origin, it is rather the scalar Green's function since it satisfies $\partial^2 \frac{1}{x^2} = -4\pi^2 \delta^{(4)}(x)$. We should therefore drop it for the anti-instanton configuration. The second term involves the constant C_1 , which at this point is not specified.

To demonstrate further the procedure, we solve for ${}^{(3)}\bar{\lambda}$ and ${}^{(4)}A_\mu$. To this order, the equations to be solved read

$$\begin{aligned} {}^{(0)}\mathcal{D}^{\alpha\beta'} {}^{(3)}\bar{\lambda}_{\beta',A} &= i\sqrt{2} \left[{}^{(2)}\bar{\phi}_{AB}, {}^{(1)}\lambda^{\alpha,B} \right], \\ {}^{(0)}\mathcal{D}^2 {}^{(4)}A_\mu - {}^{(0)}\mathcal{D}_\mu {}^{(0)}\mathcal{D}_\nu {}^{(4)}A_\nu - 2 \left[{}^{(0)}F_{\nu\mu}, {}^{(4)}A_\nu \right] &= i \left\{ {}^{(3)}\bar{\lambda}_A^{\alpha'} \bar{\sigma}_{\mu\alpha'\beta}, {}^{(1)}\lambda^{\beta,A} \right\}. \end{aligned} \quad (7.17)$$

In the second equation, we have made use of the fact that the commutator of the scalar field with its derivative is zero. Calculating the remaining commutators shows the following structure of the solutions in collective coordinates,

$${}^{(3)}\bar{\lambda}_{\alpha',A}{}^v = ig(x^2, \rho^2) \epsilon_{ABCD} \left(\mu_w^C \bar{\mu}^{D,w} \right) \left(\mu_u^B \delta_{\alpha'}{}^v - \epsilon_{\alpha'u} \bar{\mu}^{B,v} \right), \quad (7.18)$$

$${}^{(4)}A_{\mu u}{}^v = h(x^2, \rho^2) \epsilon_{ABCD} \left(\mu_w^A \bar{\mu}^{B,w} \right) \left(\mu_z^C \bar{\mu}^{D,z} \right) \bar{\sigma}_{\mu\nu u}{}^v x_\nu. \quad (7.19)$$

Notice that $\bar{\lambda}$ is off-diagonal in colour space and the correction to the anti-instanton gauge field lives in the $SU(2)$ lower diagonal block only. Introducing $\tilde{g} = \rho^3 g$, depending only on the dimensionless variable $y = x^2/\rho^2$ we find the following differential equation

$$2\tilde{g}'(y) + \frac{3}{y(1+y)}\tilde{g}(y) = \frac{3}{4}\frac{1}{\sqrt{y(1+y)^5}} + \frac{3}{4}C_1\frac{1}{\sqrt{y(1+y)^3}}. \quad (7.20)$$

From this we see that the homogeneous solution for the scalar field (corresponding to C_1) now enters in the inhomogeneous part for the fermion equation. This equation can be easily solved,

$$\tilde{g} = -\frac{1}{16}\frac{1+3y}{\sqrt{y^3(1+y)^3}} - \frac{3}{16}C_1\frac{1+2y}{\sqrt{y^3(1+y)}} + C_3\left(\frac{1+y}{y}\right)^{3/2}. \quad (7.21)$$

C_3 is the new integration constant. In order to have a solution without delta function singularities (which corresponds to a behaviour of $y^{-3/2}$ at the origin), we have to choose the integration constant $C_3 = \frac{1}{2^4}(1+3C_1)$. Then the solution reduces to

$$\tilde{g} = \frac{1}{16}\left\{(3+y)\sqrt{\frac{y}{(1+y)^3}} + 3C_1\sqrt{\frac{y}{(1+y)}}\right\}. \quad (7.22)$$

To complete our analysis, we determine the function h appearing in the gauge field. Defining $\tilde{h} = \rho^4 h$, we find it must satisfy

$$4y\tilde{h}''(y) + 12\tilde{h}'(y) + \frac{24}{(1+y)^2}\tilde{h}(y) = \frac{2\tilde{g}}{\sqrt{y(1+y)^3}}. \quad (7.23)$$

The solution reads

$$\begin{aligned} \tilde{h}(y) &= -\frac{1}{128}\frac{1+4y}{y^2(1+y)^2} - \frac{C_1(2+10y-3y^2-6y^2\ln y)}{64y^2(1+y)^2} + \frac{C_3(1+6y-7y^2-2y^3-6y^2\ln y)}{4y^2(1+y)^2} \\ &+ \frac{C_4}{(1+y)^2} - \frac{C_5(1+8y-8y^3-y^4-12y^2\ln y)}{y^2(1+y)^2}. \end{aligned} \quad (7.24)$$

Since the differential equation is of second order, there appear two new integration constants, C_4 and C_5 . We have still written the constant C_3 explicitly, but its value is related to C_1 as discussed above. In order to get rid of unwanted singularity at the origin, coming from the y^{-2} dependence, we have to choose $C_5 = \frac{1+2C_1}{2^7}$. As a surprise, both the next-to-leading y^{-1} -asymptotic terms and the logarithms vanish. Taking the above values for the constants, the total solution reduces to

$$\tilde{h}(y) = -\frac{1}{128}\frac{1}{(1+y)^2}\left(2(7+18C_1-64C_4) - 4(1+C_1)y - (1+2C_1)y^2\right). \quad (7.25)$$

It is actually easy to find the structure of the all-order solutions in the colour space. Simple analysis reveals that

$$\begin{aligned} A_\mu &= \begin{pmatrix} 0 & 0 \\ 0 & \star_{[2]\times[2]} \end{pmatrix}, & \phi^{AB} &= \begin{pmatrix} \star_{[N-2]\times[N-2]} & 0 \\ 0 & -\frac{1}{2}\mathbb{1}_{[2]\times[2]}\text{tr}[\star] \end{pmatrix}, \\ \lambda^{A,\alpha} &= \begin{pmatrix} 0 & \star_{[N-2]\times[2]} \\ \star_{[2]\times[N-2]} & 0 \end{pmatrix}, \end{aligned} \quad (7.26)$$

and the same for $\bar{\lambda}$. This means that the scalar fields are uncharged with respect to the instanton, i.e. they commute trivially with the gauge fields. At each step in the iteration, one generates new integration constants, some of which are determined by requiring the absence of delta function sources at the origin. A detailed analysis of these constants and the asymptotic behaviour of the fields will be given in a future publication [42].

7.3 Four-fermion instanton action.

We can now evaluate the instanton action for the background solution constructed in the previous section. First of all, we should mention that there is no ξ or $\bar{\eta}$ dependence, since these zero modes are protected by supersymmetry and superconformal symmetry. Therefore, we only concentrate on the μ dependence. Now we can use the field equations for the fermions to see that their kinetic energy cancels against the Yukawa terms, and this works to all orders in GCC. The resulting instanton action is therefore

$$\mathcal{L}_{\text{inst}}^{\mathcal{N}=4} = \frac{1}{g^2} \text{tr} \left\{ \frac{1}{2} F_{\mu\nu} F_{\mu\nu} + \frac{1}{2} (\partial_\mu \bar{\phi}_{AB}) (\partial_\mu \phi^{AB}) + \frac{1}{8} [\phi^{AB}, \phi^{CD}] [\bar{\phi}_{AB}, \bar{\phi}_{CD}] \right\}. \quad (7.27)$$

The first non-trivial correction to the instanton action, of order four in GCC, comes from

$$\mathcal{L}_{\text{quart}} = \frac{1}{g^2} \text{tr} \left\{ 2 {}^{(0)}F_{\mu\nu} {}^{(0)}\mathcal{D}_\mu {}^{(4)}A_\nu + \frac{1}{2} (\partial_\mu {}^{(2)}\bar{\phi}_{AB}) (\partial_\mu {}^{(2)}\phi^{AB}) \right\}, \quad (7.28)$$

since the potential does not contribute to this order. Now we can plug in the solutions for the gauge and scalar fields into the action $S_{\text{quart}} = - \int d^4x \mathcal{L}_{\text{quart}}$. Taking into account the integration constants, we find

$$S_{\text{quart}} = \left\{ -\frac{1}{4} + \frac{3}{8}(1 + 2C_1) \right\} \frac{\pi^2}{g^2 \rho^2} \epsilon_{ABCD} (\mu_u^A \bar{\mu}^{u,B}) (\mu_v^C \bar{\mu}^{v,D}). \quad (7.29)$$

The first term inside the brackets is the contribution coming from the scalar fields, and is independent of C_1 . The second term, proportional to $(1 + 2C_1)$, is the contribution from the gauge fields. It is independent of C_4 and is entirely determined by the value of the function h at infinity. This can be seen by realizing that the first term in (7.28) can be written as a total derivative by using the field equations for A_μ , so there is only a contribution from the boundary at spatial infinity.

For the moment, we will determine the constant C_1 such that there is no contribution from the gauge fields, i.e. we will set $C_1 = -\frac{1}{2}$. Again, a careful analysis of these constants and a discussion about surface terms that can contribute to the action will be given in [42]. Notice that for $C_1 = -\frac{1}{2}$, the total prefactor is then $-\frac{1}{4}$ and is the same as in [20] (up to a sign!).

We conclude this section by discussing the reality conditions on the fermionic collective coordinates. These follow of course from the reality conditions on the spinors (6.14). Straightforward

substitution of the solutions yields the following results

$$\begin{aligned} (\xi^{A,\alpha})^* &= -\xi^{\beta,B} \epsilon_{\beta\alpha} \eta_{BA}^1, & (\bar{\eta}_{\alpha'}^A)^* &= -\bar{\eta}_{\beta'}^B \epsilon^{\beta'\alpha'} \eta_{BA}^1, \\ (\mu_u^A)^* &= \bar{\mu}^{B,u} \eta_{BA}^1, & (\bar{\mu}^{A,u})^* &= -\mu_u^B \eta_{BA}^1. \end{aligned} \quad (7.30)$$

The effective action (7.29) is then hermitean w.r.t. these relations since $\epsilon_{ABCD} \eta_{AA'}^1 \eta_{BB'}^1 \eta_{CC'}^1 \eta_{DD'}^1 = (\det \eta^1) \epsilon_{A'B'C'D'}$ and $\det \eta_{AB}^1 = 1$. The rules given in (7.30) sometimes simplify calculations. For instance, in (7.18) one can determine the $\bar{\mu}$ dependence by only computing the μ dependence and using the reality conditions.

8 Correlation functions.

Having discussed the zero mode structure, the measure of collective coordinates and the instanton action, we can now finally turn to the computation of correlation functions. We recall that the one-anti-instanton measure, coming from bosonic and fermionic zero modes, for the $\mathcal{N} = 4$ model is given by

$$\begin{aligned} \int d\mathcal{M}_{k=-1} &\equiv \frac{2^{4N+2} \pi^{4N-2}}{(N-1)!(N-2)!} g^{-4N} e^{-\left(\frac{8\pi^2}{g^2} - i\theta\right)} \int d^4 x_0 \int \frac{d\rho}{\rho^5} \rho^{4N} \\ &\times \int \prod_{A=1}^4 d^2 \xi^A \left(\frac{g^2}{32\pi^2}\right)^4 \int \prod_{A=1}^4 d^2 \bar{\eta}^A \left(\frac{g^2}{64\pi^2 \rho^2}\right)^4 \int \prod_{A=1}^4 \prod_{u=1}^{N-2} d\mu_u^A d\bar{\mu}^{A,u} \left(\frac{g^2}{2\pi^2}\right)^{4(N-2)}, \end{aligned} \quad (8.1)$$

as can easily be seen by combining (4.35) with (4.44). We remind the reader that, as discussed in previous sections, there can be extra corrections to the measure, proportional to the μ and $\bar{\mu}$ collective coordinates. These corrections, which are subleading in the coupling constant, have to our knowledge never been computed, and are currently under study [42].

The measure (8.1) appears in the path integral, and in order to find a nonvanishing answer, we must insert some fermion fields to saturate the ξ^A and $\bar{\eta}^A$ zero modes, otherwise these integrals would yield zero. This is a generic feature of instanton calculations, and applies as well to $\mathcal{N} = 2, 1$ and non-supersymmetric theories. The other zero modes, μ_u^A and $\bar{\mu}^{A,u}$ can be saturated by bringing down enough powers of the instanton action in the exponential $\exp(-S_{\text{quart}})$. We will evaluate the $\mu^A, \bar{\mu}^A$ integration below. The total action is $S_{\text{tot}} = S_{\text{inst}} + S_{\text{quart}}$ with the usual one-anti-instanton contribution $S_{\text{inst}} = \frac{8\pi^2}{g^2} - i\theta$. Higher order terms in the instanton action (starting from eighth order in GCC) are suppressed. This is because one must bring down less powers of the instanton action, and since the action has a $1/g^2$ in front, one has less powers of the coupling constant as compared with the leading quartic term. We also repeat that we take the value $C_1 = -\frac{1}{2}$, such that

$$S_{\text{quart}} = -\frac{\pi^2}{4g^2 \rho^2} \epsilon_{ABCD} \left(\mu_u^A \bar{\mu}^{u,B}\right) \left(\mu_v^C \bar{\mu}^{v,D}\right). \quad (8.2)$$

Notice that in order to saturate the fermionic zero modes, we have to expand the exponential up to $2N - 4$ powers of the instanton action. This will bring down a factor of ρ^{-4N+8} , such that the total ρ -dependence of the measure is independent of N , namely $d\rho/\rho^5$. Combining this with the instanton positions, this is just the measure of a five-dimensional anti-de-Sitter space, see the next section.

We will now analyze two correlators. The first one involves the insertion of sixteen fermion fields. This correlator was computed in [20], and we briefly repeat it below. The second one is the four point function of energy-momentum tensors. We show how the fermionic zero modes are saturated and we outline the computation of the full correlator.

8.1 $\langle \Lambda^{16} \rangle$ correlator.

Because we have integrated out the gauge orientation zero modes in the measure (8.1), we must compute correlation functions of gauge invariant objects. Since there are 16 fermionic zero modes protected from lifting by super(conformal) symmetry, they have to be saturated by inserting appropriate fermionic operators. The gluinos λ^A are not gauge invariant, but a suitable gauge invariant composite operator is the expression

$$\Lambda_\alpha^A = \frac{1}{2g^2} \sigma_{\mu\nu}{}^{\beta\alpha} \text{tr} \{ F_{\mu\nu} \lambda_\beta^A \} . \quad (8.3)$$

We could equally well have considered fermionic bilinears contracted to gauge invariant Lorentz scalars as was originally done in [6, 7], and in $\mathcal{N} = 4$ in [18, 21]. We can write explicitly how this operator looks like in the anti-instanton background. Using (3.12) and (2.13), we then find

$$\Lambda^{\alpha,A}(x) = -\frac{96}{g^2} \left(\xi^{\alpha,A} - \bar{\eta}_{\alpha'}^A \bar{\sigma}_\mu^{\alpha'\alpha} (x^\mu - x_0^\mu) \right) \frac{\rho^4}{[(x - x_0)^2 + \rho^2]^4} . \quad (8.4)$$

This is actually an exact formula, and there is no contribution from the μ GCC to all orders. This follows from the fact that the field strength is diagonal and the gluino is off-diagonal in colour space. Due to this, all higher point functions, with more than sixteen Λ 's, will be zero. The only contribution to $\langle \Lambda^{16} \rangle$ comes from taking Λ to be linear in the ξ or $\bar{\eta}$ zero mode. Putting this all together, we now insert 16 copies of the composite fermion, at 16 space points x_i , into the one-anti-instanton measure (8.1), and multiply by $\exp(-S_{\text{quart}})$.

To evaluate this correlator, we first discuss the integration over the lifted zero modes $\mu_u^A, \bar{\mu}^{Au}$. Define I_N to be the (un-normalized) contribution of the lifted modes to the correlator

$$I_N \equiv \int \prod_{A=1}^4 \prod_{u=1}^{N-2} d\mu_u^A d\bar{\mu}^{Au} e^{-S_{\text{quart}}} . \quad (8.5)$$

Explicit calculation for $N = 3$ shows $I_3 = \frac{3\pi^4}{\rho^4 g^4}$. To evaluate I_N for arbitrary N , it is helpful to rewrite the quadrilinear term S_{quart} as a quadratic form. To this end we introduce six independent auxiliary bosonic variables $\chi_{AB} = -\chi_{BA}$, and substitute into (8.5) the integral representation [20]

$$e^{-S_{\text{quart}}} = -\frac{\rho^6 g^6}{2^9 \pi^6} \int \prod_{1 \leq A' < B' \leq 4} d\chi_{A'B'} \exp \left(\frac{-\rho^2 g^2}{32\pi^2} \epsilon_{ABCD} \chi_{AB} \chi_{CD} + \frac{1}{2} \chi_{AB} \Lambda_N^{AB}(\mu, \bar{\mu}) \right), \quad (8.6)$$

where we have introduced the object $\Lambda_N^{AB}(\mu, \bar{\mu}) = \frac{1}{2\sqrt{2}} (\mu_u^A \bar{\mu}^{B,u} - \mu_u^B \bar{\mu}^{A,u})$ to denote the fermion bilinear that couples to χ_{AB} . Our strategy is to perform only the integrations over μ_{N-2}^A and $\bar{\mu}^{A,N-2}$, and thereby deduce a recursion relation, i.e. an equation between I_N and I_{N-1} . Accordingly we break out these terms from Λ^N :

$$\Lambda_N^{AB}(\mu, \bar{\mu}) = \Lambda_{N-1}^{AB}(\mu, \bar{\mu}) + \frac{1}{2\sqrt{2}} (\mu_{N-2}^A \bar{\mu}^{B,N-2} - \mu_{N-2}^B \bar{\mu}^{A,N-2}). \quad (8.7)$$

Now, the $\{\mu_{N-2}^A, \bar{\mu}^{A,N-2}\}$ integration in (8.6) brings down a factor of $\frac{1}{64} \det \chi_{AB}$. Next we exploit the fact that the determinant of a four-dimensional antisymmetric matrix is a total square,

$$\det \chi_{AB} = \left(\frac{1}{8} \epsilon_{ABCD} \chi_{AB} \chi_{CD} \right)^2. \quad (8.8)$$

Since the right-hand side of this determinant equation is proportional to the square of the first term in (8.6), the result of the $\{\mu_{N-2}^A, \bar{\mu}^{A,N-2}\}$ integration can be rewritten as a parametric second derivative relating I_N to I_{N-1} , namely [20]

$$I_N = \frac{1}{4} \pi^4 \rho^6 g^6 \frac{\partial^2}{\partial (\rho^2 g^2)^2} \left((\rho^2 g^2)^{-3} I_{N-1} \right). \quad (8.9)$$

The insertion of $(\rho^2 g^2)^{-3}$ inside the parentheses ensures the derivatives to act only on the exponent of (8.6), and not on the prefactor. This recursion relation, combined with the initial condition for I_3 , finally gives

$$I_N = \frac{1}{2} (2N - 2)! \left(\frac{\pi^2}{2\rho^2 g^2} \right)^{2N-4}. \quad (8.10)$$

Combining everything together, we find for the one-anti-instanton contribution to the 16-fermion correlator in $\mathcal{N} = 4$ SYM theory:

$$\begin{aligned} \langle \Lambda_{\alpha_1}^1(x_1) \cdots \Lambda_{\alpha_{16}}^4(x_{16}) \rangle &= C_N e^{-\left(\frac{8\pi^2}{g^2} - i\theta\right)} \int d^4 x_0 \frac{d\rho}{\rho^5} \int \prod_{A=1}^4 d^2 \xi^A d^2 \bar{\eta}^A \\ &\times \left(\xi_{\alpha_1}^1 - \bar{\eta}_{\alpha'}^1 \bar{\sigma}_{\mu \alpha_1}^{\alpha'} (x_1^\mu - x_0^\mu) \right) K_4(x_0, \rho; x_1, 0) \\ &\times \cdots \times \left(\xi_{\alpha_{16}}^4 - \bar{\eta}_{\alpha'}^4 \bar{\sigma}_{\mu \alpha_{16}}^{\alpha'} (x_{16}^\mu - x_0^\mu) \right) K_4(x_0, \rho; x_{16}, 0), \end{aligned} \quad (8.11)$$

where we have denoted

$$K_4(x_0, \rho; x_i, 0) = \frac{\rho^4}{[(x_i - x_0)^2 + \rho^2]^4}, \quad (8.12)$$

and the overall constant C_N is given by

$$C_N = g^{-24} \frac{2^{-2N}(2N-2)!}{(N-1)!(N-2)!} 2^{57} 3^{16} \pi^{-10}. \quad (8.13)$$

The large N limit gives by means of Stirling's formula

$$C_N \rightarrow g^{-24} \sqrt{N} 2^{55} 3^{16} \pi^{-21/2}. \quad (8.14)$$

In principle we would have to do the integrations over the ξ and $\bar{\eta}$ zero modes. This would just give a numerical tensor in spinor indices and sigma matrices, but we refrain from giving its explicit expression. Also, we should do the ρ integration. This is not straightforward, but one can check by power counting that the result is convergent and finite. The integration over x_0 is left over. The final expression for the correlator can then be seen to induce a sixteen fermion vertex in the effective Lagrangian, which is integrated over x_0 .

8.2 $\langle \Theta^4 \rangle$ correlator.

In this section we study the four-point Green's function of energy-momentum tensors $\Theta_{\mu\nu}$. The improved (traceless and symmetric) energy momentum tensor for the model is

$$\begin{aligned} \Theta_{\mu\nu} = & \frac{1}{g^2} \text{tr} \left\{ 2 \left(F_{\mu\rho} F_{\nu\rho} - \frac{1}{4} \delta_{\mu\nu} F_{\rho\sigma} F_{\rho\sigma} \right) - i \bar{\lambda}_A^{\alpha'} \bar{\sigma}_{(\mu} \alpha'_{\beta} \mathcal{D}_{\nu)} \lambda^{\beta, A} - i \lambda_\alpha^A \sigma_{(\mu}^{\alpha\beta'} \mathcal{D}_{\nu)} \bar{\lambda}_{\beta', A} \right. \\ & + \left(\mathcal{D}_\mu \bar{\phi}_{AB} \right) \left(\mathcal{D}_\nu \phi^{AB} \right) - \frac{1}{2} \delta_{\mu\nu} \left(\mathcal{D}_\rho \bar{\phi}_{AB} \right) \left(\mathcal{D}_\rho \phi^{AB} \right) - \frac{1}{8} \delta_{\mu\nu} \left[\phi^{AB}, \phi^{CD} \right] \left[\bar{\phi}_{AB}, \bar{\phi}_{CD} \right] \\ & \left. - \frac{1}{6} \left(\partial_\mu \partial_\nu - \delta_{\mu\nu} \partial^2 \right) \bar{\phi}_{AB} \phi^{AB} \right\}, \end{aligned} \quad (8.15)$$

where the last term is an improvement [43] stemming from the addendum $R\bar{\phi}\phi$ to the Lagrangian of $\mathcal{N} = 4$ super-Yang-Mills coupled selfconsistently to conformal supergravity [44]. Symmetrization is done with weight one, $(\mu, \nu) = \frac{1}{2} \{\mu\nu + \nu\mu\}$. We dropped the equations of motion for fermions and gauge fields in (8.15). This tensor is conserved and traceless upon using the equations of motion.

We now evaluate this expression in the anti-instanton background. First we concentrate on the μ -zero modes, and we will show that there is no μ -dependence, as was also the case for the field Λ in the previous subsection. This can be seen by the following argument. Since the only possible tensor structure of $\Theta_{\mu\nu}$ which may contain the collective coordinates is the traceless tensor $\Delta_{\mu\nu} \equiv \frac{x^2}{4} \delta_{\mu\nu} - x_\mu x_\nu$, it must take the form

$$\Theta_{\mu\nu} = t(x) \Delta_{\mu\nu} \epsilon_{ABCD} (\mu^A \bar{\mu}^B) (\mu^C \bar{\mu}^D). \quad (8.16)$$

Now, from the conservation of energy-momentum tensor we derive a differential equation for the x -dependent function, $t(x)$, which is solved by $t(x) = c/x^6$, with c an arbitrary (ρ -dependent)

constant. By looking at the explicit form of the instanton solution, it is simple to see that such an x -dependence can never be produced. Therefore, the only possibility is that $c = 0$. Explicit calculation confirms this observation, as there is a subtle cancellation of the μ -dependence between the bosons and fermions, showing that indeed $\Theta_{\mu\nu}$ is μ and $\bar{\mu}$ independent.

The story is different for the ξ and $\bar{\eta}$ GCC, as we now can construct different possible tensor structures which are both traceless and conserved. There are three different classes of terms, one which has four ξ 's, one with four $\bar{\eta}$'s and one with mixed ξ and $\bar{\eta}$. The stress tensor is obtained by taking the derivative of the action with respect to the metric. One expects that in curved space, the action does depend on the ξ and $\bar{\eta}$ GCC, in other words, in a general curved space, these fermion zero modes are not protected by supersymmetry. An explicit calculation supports this, and using the results of section 7.1, we find for the ξ mode contribution

$$\Theta_{\mu\nu}^{(\xi)} = \frac{1}{g^2} \frac{3 \cdot 2^8 \cdot \rho^4}{[(x-x_0)^2 + \rho^2]^6} \epsilon_{ABCD} (x-x_0)_\rho (x-x_0)_\sigma \left(\xi^A \sigma_{\mu\rho} \xi^B \right) \left(\xi^C \sigma_{\nu\sigma} \xi^D \right). \quad (8.17)$$

We repeat that when spinor indices are not explicitly written, they are in their natural position dictated by the sigma matrices. To obtain this result, we have made use of the Fierz relation (7.9) and the identity $\epsilon_{ABCD} (\xi^A \sigma_{\mu\rho} \xi^B) (\xi^C \sigma_{\rho\nu} \xi^D) = 0$ which follows from anti-selfduality of $\sigma_{\mu\nu}$. The expression (8.17) is then easily seen to be traceless and conserved.

A similar analysis can be made for the terms involving $\bar{\eta}$. One would first have to compute the $\bar{\eta}$ dependence of all the fields using the superconformal supersymmetry transformations, along the same lines as in section 7.1. For present illustrative purposes we do not need it. Having the four GCC in the energy-momentum we can saturate the ξ and $\bar{\eta}$ measure by computing the four-point function. The μ and $\bar{\mu}$ coordinates are integrated out in the same way as was done in the previous section and finally we get the result

$$\begin{aligned} \langle \Theta_{\mu_1\nu_1}(x_1) \dots \Theta_{\mu_4\nu_4}(x_4) \rangle &= \tilde{C}_N e^{-\left(\frac{8\pi^2}{g^2} - i\theta\right)} \int d^4x_0 \frac{d\rho}{\rho^5} \int \prod_{A=1}^4 d^2\xi^A d^2\bar{\eta}^A \prod_{j=1}^4 \frac{\rho^4}{[(x_j - x_0)^2 + \rho^2]^6} \\ &\times \epsilon_{ABCD} \left\{ (x_j - x_0)_{\rho_j} (x_j - x_0)_{\sigma_j} \left(\xi^A \sigma_{\mu_j\rho_j} \xi^B \right) \left(\xi^C \sigma_{\nu_j\sigma_j} \xi^D \right) + \dots \right\}, \quad (8.18) \end{aligned}$$

where the dots stand for terms proportional to $\bar{\eta}$. The normalization constant \tilde{C}_N reads in the large N limit

$$\tilde{C}_N = \text{const} \cdot g^0 \frac{2^{-2N} (2N-2)!}{(N-1)! (N-2)!} \rightarrow g^0 \sqrt{N}, \quad (8.19)$$

up to an N -independent constant. Thus in the large N limit the four-point correlation function of energy-momentum tensors has the same scaling in N as the sixteen fermion correlator discussed in the previous section.

9 D-instantons in IIB supergravity.

After having studied instantons in (supersymmetric) Yang-Mills theories, we will now discuss instantons in a (particular) supergravity theory, which is related to YM theories via the AdS/CFT correspondence [16]. We consider IIB supergravity in ten dimensions, where the bosonic fields are given by the ten-dimensional metric $g_{\mu\nu}$, the dilaton ϕ and axion a , scalar and pseudoscalar respectively, and some tensor fields which we shall specify later [45]. The ten-dimensional action for these fields is

$$S_M^{\text{boson}} = \int d^{10}x \sqrt{g} \left\{ \mathcal{R} - \frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi) - \frac{1}{2} e^{2\phi} (\partial_\mu a) (\partial^\mu a) \right\} . \quad (9.1)$$

This action is written down in Einstein frame with minkowskian signature. Our goal is now to discuss instantons in this system, which requires the euclidean formulation of IIB supergravity. In distinction to $\mathcal{N} = 4$ SYM, IIB supergravity can not be obtained by dimensional reduction (over the time coordinate) of yet another theory in higher dimensions. In fact, in euclidean space, there is no action which is supersymmetric and real at the same time. This is in agreement with the fact that there is no real form of the supersymmetry algebra $OSp(1|32)$ with signature $(10, 0)$, see e.g. [46]. The way to proceed then is to make all the fields complex, keeping the action formally the same as in minkowskian space-time. This action will not be real or hermitean, but depends holomorphically on all the fields.

To obtain (the bosonic part of) euclidean IIB supergravity one must flip the sign in front of the kinetic energy for the axion field. This sign change is explained by the argument [33] that a pseudoscalar receives a factor of i after the Wick rotation from minkowskian to euclidean space, $t \rightarrow \tau = -it$. Since pseudoscalars can always be written in terms of scalars S_i , $i = 0, \dots, 9$ as

$$a = \epsilon^{\mu_0 \mu_1 \dots \mu_9} (\partial_{\mu_0} S_0) \dots (\partial_{\mu_9} S_9) . \quad (9.2)$$

It becomes clear that the derivative associated with the time coordinate picks up a factor of i after the Wick rotation. As a consequence, in the euclidean theory, the sign in front of the axion kinetic Lagrangian changes,

$$-S_E^{\text{boson}} = \int d^{10}x \sqrt{g} \left\{ \mathcal{R} - \frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi) + \frac{1}{2} e^{2\phi} (\partial_\mu a) (\partial^\mu a) \right\} . \quad (9.3)$$

This prescription is consistent with the procedure of making all the fields complex. The sign change is then explained by taking only the real part of the dilaton and the imaginary part of the axion to be non-zero.

The field equations that follow from (9.3) are (neglecting the fermionic sector)

$$\mathcal{R}_{\mu\nu} = \frac{1}{2} (\partial_\mu \phi) (\partial_\nu \phi) - \frac{1}{2} e^{2\phi} (\partial_\mu a) (\partial_\nu a) , \quad \nabla_\mu (e^{2\phi} \partial^\mu a) = 0 , \quad \nabla^2 \phi + e^{2\phi} (\partial_\mu a) (\partial^\mu a) = 0 . \quad (9.4)$$

The action contains further two-tensors $A_{\mu\nu} = -A_{\nu\mu}$ and a selfdual rank-five field strength, $F_{\mu_1\dots\mu_5}$. The latter contributes only to the first of the above field equations,

$$\mathcal{R}_{\mu\nu} = \frac{1}{2} (\partial_\mu\phi) (\partial_\nu\phi) - \frac{1}{2} e^{2\phi} (\partial_\mu a) (\partial_\nu a) + \frac{1}{6} F_{\mu\mu_1\dots\mu_4} F^{\mu_1\dots\mu_4}{}_\nu . \quad (9.5)$$

As we will see in subsection 9.2, this $F_{\mu_1\dots\mu_5}$ becomes relevant when discussing instanton solutions. Again, in euclidean space, one must take notice of the fact that there exist no real selfdual five-forms in ten dimensions, so we take the strategy of working with a complex field strength.

The aim is now to discuss solutions of these equations by choosing a particular background for the ten dimensional space-time. We will discuss two examples which preserve the maximal number of supersymmetries, the first one will be flat \mathbf{R}^{10} , and the other one contains anti-de Sitter (AdS) space, namely $AdS_5 \times S^5$. The D-instanton solution was found by Gibbons et al. in [2], see also Green and Gutperle in [19], on which the remainder of this section is heavily based. We will however concentrate purely on the bosonic sector of the theory. An analysis of the fermionic zero modes can be found in [2, 19, 18].

9.1 D-instantons in flat \mathbf{R}^{10} .

In flat euclidean space, we set the rank-five field strength equal to zero. The field equations then become

$$(\partial_\mu\phi) (\partial_\nu\phi) = e^{2\phi} (\partial_\mu a) (\partial_\nu a) , \quad \partial_\mu (e^{2\phi} \partial_\mu a) = 0 , \quad \partial^2 \phi = -e^{2\phi} (\partial_\mu a) (\partial_\mu a) . \quad (9.6)$$

By taking the trace of the first equation and comparing it with the third one, we find

$$(\partial_\mu\phi)^2 = -\partial^2 \phi \quad \Rightarrow \quad \partial^2 (e^\phi) = 0 . \quad (9.7)$$

A spherically symmetric solution to this equation is

$$e^\phi = e^{\phi_\infty} + \frac{c}{r^8} . \quad (9.8)$$

Here, $g_s \equiv \exp(\phi_\infty)$ and c are integration constants, the first one being identified with the string coupling, which is the value of the dilaton at infinity. Obviously (9.7) is a solution everywhere except of the origin where it solves (9.7) with a delta function source, $\delta^{(10)}(x)$. One may add another term to the action of the form $\int d^{(10)}x \delta^{(10)}(x) e^{-\phi}$. This term cancels the singularity with $\delta^{(10)}(x)$ in the field equations. The physical meaning of this term is that we now have a new object in the theory at $x = 0$, namely the ‘‘D-instanton’’. This is different from YM-instantons, where we require regularity of the solution everywhere. D-instantons, or D-branes in general, as solutions of the supergravity equations of motion typically have delta function sources, but these singularities may be resolved in string theory.

The solution for the axion field equation $\partial_\mu a = \mp \partial_\mu \exp(-\phi)$ is

$$a - a_\infty = \mp \left(e^{-\phi} - e^{-\phi_\infty} \right) , \quad (9.9)$$

where a_∞ is again an integration constant, namely the value of the axion at infinity. The plus or minus sign refers to the D-instanton and D-anti-instanton respectively. One can actually write down a more general solution for the dilaton equation

$$e^\phi = e^{\phi_\infty} + \frac{c}{|x - x_0|^8} , \quad (9.10)$$

and similarly for the axion field. The coordinates x^μ are just the coordinates of \mathbf{R}^{10} . Then there are ten bosonic collective coordinates x_0^μ , denoting the position of the D-instanton in \mathbf{R}^{10} .

Now we want to determine the instanton action. By plugging the solution given above into the action, one immediately sees that it gives zero (also for the δ -function term), hence the instanton action vanishes. This is usually not the case for instantons, since they have finite but non-zero action. The resolution is that we should have taken a boundary term into account, of the form

$$S_{\text{surf}} = \int d^{10}x \partial_\mu \{ \sqrt{g} g^{\mu\nu} \exp(2\phi) (a \partial_\nu a) \} . \quad (9.11)$$

Its origin was discussed in [2, 19] and we will not repeat it here. This surface term is non-zero when evaluated in the instanton background, and is proportional to the constant c , namely $S_{\text{cl}} = 8c \text{Vol } S^9$. There is a conserved current $j_\mu = e^{2\phi} \partial_\mu a$, whose charge is $Q = \int j_\mu d\Omega^\mu = \mp 8c g_s \text{Vol } S^9$. Hence $S = |Q|/g_s$. Since D-instantons are -1 p -branes (which correspond to antisymmetric tensors with $p + 1$ indices), their duals (which are field strengths with 9 indices) define gauge potentials with 8 indices. These can couple to curls of D7-branes. Considering the “magnetic” D7-brane in the presence of the “electric” D-instanton one obtains the usual Dirac quantization condition

$$Q = 2\pi k, \quad k \in \mathbf{Z} . \quad (9.12)$$

With this quantization condition, the action becomes

$$S_{\text{D-inst}} = \frac{2\pi |k|}{g_s} . \quad (9.13)$$

This is the same as for the YM-instanton action (without the θ angle) upon identifying

$$g_s = \frac{g^2}{4\pi} , \quad (9.14)$$

with YM-coupling constant g .

9.2 D-instantons in $AdS_5 \times S^5$.

In this section we discuss D-instantons in IIB supergravity in a background different from flat ten-dimensional space. Instead, we choose the space $AdS_5 \times S^5$, and we want to solve the equations of motion for the dilaton and axion in this background.

We start with some elementary facts about AdS_5 , which is defined as the hypersurface embedded in six-dimensional flat space-time by the equation

$$-(X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 - (X^5)^2 = -R^2 . \quad (9.15)$$

This surface defines a five-dimensional non-compact space with “radius” R . It has an isometry group $SO(4, 2)$ which is the same as the conformal group in four dimensional Minkowski space-time. The AdS_5 metric can be obtained from the six dimensional flat metric with signature

$$ds^2 = -(dX^0)^2 + (dX^1)^2 + (dX^2)^2 + (dX^3)^2 + (dX^4)^2 - (dX^5)^2 , \quad (9.16)$$

upon using the constraint (9.15). Defining the coordinates

$$U = X^4 + X^5 , \quad x^\mu = \frac{X^\mu R}{U} , \quad \text{with} \quad \mu = 0, 1, 2, 3 , \quad (9.17)$$

the AdS_5 metric can be written as

$$ds^2 = \frac{U^2}{R^2} \eta_{\mu\nu} dx^\mu dx^\nu + \frac{R^2}{U^2} (dU)^2 . \quad (9.18)$$

In this expression, we have used the minkowskian metric $\eta_{\mu\nu}$ to contract the indices. But in fact, since instantons live in euclidean space, we should take the euclidean version of the above metric and replace $\eta_{\mu\nu}$ by $\delta_{\mu\nu}$. Defining the variable $\rho \equiv R^2/U$, the metric takes the form

$$ds^2 = \frac{R^2}{\rho^2} \left\{ \delta_{\mu\nu} dx^\mu dx^\nu + d\rho^2 \right\} . \quad (9.19)$$

Notice that now there is just an overall factor in front of a five-dimensional flat metric. Spaces with such a metric are called conformally flat. We can now construct the invariant volume element of AdS_5 , which is

$$\int d^4x d\rho \sqrt{g} = R^5 \int d^4x \frac{d\rho}{\rho^5} . \quad (9.20)$$

This is precisely the same expression (up to the prefactor R^5) as obtained from the collective coordinate measure in $\mathcal{N} = 4$ SYM theory after integrating out the $\mu^A, \bar{\mu}^A$ fermionic collective coordinates, combining (4.35), (4.44) and (8.9).

The metric on the full ten-dimensional $AdS_5 \times S^5$ space is

$$\begin{aligned} ds^2 &= \frac{R^2}{\rho^2} \left\{ dx_\mu dx_\mu + d\rho^2 \right\} + R^2 dS^5 = \frac{R^2}{\rho^2} dx_\mu dx_\mu + \frac{R^2}{\rho^2} \left\{ d\rho^2 + \rho^2 dS^5 \right\} \\ &= \frac{R^2}{\rho^2} \left\{ dx_\mu dx_\mu + dy^i dy^i \right\} , \end{aligned} \quad (9.21)$$

which is conformally equivalent to \mathbf{R}^{10} . We have taken the radius of the five-sphere to be the same as that of AdS_5 , and in the last equation we have introduced coordinates $\{y^i; i = 1, \dots, 6\}$ on \mathbf{R}^6 , with $\rho^2 = y^i y^i$.

Computing the Ricci tensor of $AdS_5 \times S^5$, one finds that the only non-zero components are given by

$$\mathcal{R}_{MN} = -\frac{4}{R^2} g_{MN} , \quad \mathcal{R}_{mn} = \frac{4}{R^2} g_{mn} , \quad (9.22)$$

where $M, N = 1, \dots, 5$ run over the coordinates (x^μ, ρ) , and $m, n = 1, \dots, 5$ label the angular coordinates of the five-sphere. Our aim is to solve the field equations in this background. This is different from the case of flat \mathbf{R}^{10} in the sense that now the Ricci tensor does not vanish in the field equations. However, as mentioned above, there is also a rank-five selfdual field strength $F_{\mu_1 \dots \mu_5}$ which can compensate this effect. Indeed, if we choose the non-vanishing components of this tensor to be (with flat indices to avoid having to write determinants of vielbeins)

$$F_{MNPQS} = i \frac{1}{R} \epsilon_{MNPQS} , \quad F_{mnpqs} = \frac{1}{R} \epsilon_{mnpqs} , \quad (9.23)$$

we find a cancellation between the Ricci tensor and this five-form in (9.5). Notice the factor i in the AdS part, which is due to the Wick rotation of the time coordinate X^0 .

The field equations now reduce to

$$(\partial_\mu \phi) (\partial_\nu \phi) = e^{2\phi} (\partial_\mu a) (\partial_\nu a) , \quad \nabla_\mu (e^{2\phi} \partial^\mu a) = 0 , \quad \nabla^2 \phi + e^{2\phi} (\partial_\mu a) (\partial^\mu a) = 0 , \quad (9.24)$$

where now all indices are contracted with the ten-dimensional metric (9.21). Taking the trace of the first equation and combining it with the third one, we get

$$g^{\mu\nu} \nabla_\mu \partial_\nu e^\phi = \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} g^{\mu\nu} \partial_\nu e^\phi) = 0 . \quad (9.25)$$

The solution to this equation is a rescaled version of the one in flat space, namely [18]

$$e^\phi = e^{\phi_\infty} + \frac{\rho_0^4 \rho^4}{R^8} \frac{c}{|X - X_0|^8} , \quad (9.26)$$

where now $X = (x^\mu, y^i)$ and the collective coordinates $X_0 = (x_0^\mu, y_0^i)$ denote the position of the D-instanton in $AdS_5 \times S^5$ and $\rho_0^2 \equiv y_0^i y_0^i$. To find the solution for the axion, one proceeds along the same lines as in the flat case. It is given by (9.9) with the dilaton profile from (9.26). Finally one has to determine the instanton action. Again the contribution will come from a surface term and the instanton action is the same as in \mathbf{R}^{10} [18]. Although we have not discussed the fermionic sector of the theory, it turns out that the D-instanton in $AdS_5 \times S^5$ preserves half of the supersymmetries, hence this background is on equal footing with the Minkowski background.

9.3 Supergravity scattering amplitudes and AdS/CFT.

In the context of the D-branes there an exact correspondence was conjectured between type IIB string theory on $AdS_5 \times S^5$ space and four-dimensional $\mathcal{N} = 4$ SYM theory living on the boundary of the anti-de Sitter space [16]. It is expressed as an equivalence between certain scattering amplitudes in ten-dimensional superstring theory on $AdS_5 \times S^5$ and Green's functions of composite operators on the field theory side,

$$\exp(-S_{\text{IIB}}[\Phi(J)]) = \int [d\varphi] \exp(-S_{\text{SYM}}[\varphi] + \mathcal{O}[\varphi] \cdot J) . \quad (9.27)$$

Here S_{IIB} is the type IIB superstring effective action with Φ being solutions of the massless sugra or the massive Kaluza-Klein field equations with boundary values J . On the SYM side the latter couple to composite operators \mathcal{O} which are functions of the quantum fields φ . For instance, the graviton couples to the YM energy-momentum tensor and one can compare correlations functions of the stress tensors with multi-graviton scattering amplitudes.

The precise dictionary between the coupling constants in IIB sugra and SYM theories is

$$g^2 = 4\pi g_s = 4\pi e^{\phi_\infty} , \quad \theta = 2\pi a_\infty , \quad \frac{R^2}{\alpha'} = g\sqrt{N} , \quad (9.28)$$

where α' is the string tension and appears in front of the supergravity action in the string frame, see below. From this it follows that large 't Hooft coupling $g^2 N$ corresponds to a large radius R and hence small curvature of the AdS background. This is precisely the regime where the supergravity approximation is valid.

Particular terms in the supergravity effective action which are of order $(\alpha')^3$ relative to the leading Einstein-Hilbert term were derived in [19]. In the string frame they read

$$S_{\text{IIB}} = (\alpha')^{-4} \int d^{10}x \sqrt{g} \left\{ e^{-2\phi} \mathcal{R} + \frac{(\alpha')^3}{2^{11} \pi^7} f_4(\tau, \bar{\tau}) e^{-\phi/2} \mathcal{R}^4 + (\alpha')^3 f_{16}(\tau, \bar{\tau}) e^{-\phi/2} \Lambda^{16} + \text{h.c.} \right\} , \quad (9.29)$$

where \mathcal{R}^4 in the second term is a particular contraction of Riemann tensors and Λ is a complex chiral $SO(9, 1)$ spinor (the dilatino). The functions f_i are $SL(2, \mathbf{Z})$ modular forms in the complex parameter $\tau \equiv ie^{-\phi} + a$. They have the following weak coupling $g_s = e^\phi$ expansion [19, 47]

$$f_m = a_m \zeta(3) e^{-3\phi/2} + b_m e^{\phi/2} + e^{\phi/2} \sum_{k=1}^{\infty} \left(\sum_{d|k} \frac{1}{d^2} \right) (ke^{-\phi})^{m-7/2} \exp(-2\pi k(e^{-\phi} - ia)) \left(1 + \sum_{j=1}^{\infty} c_{j,m}^k (ke^{-\phi})^{-j} \right) . \quad (9.30)$$

The first two terms with coefficients a_m and b_m correspond to tree and one-loop results in target space. The last term can be viewed as the k -anti-instanton contribution. There is a similar term coming from k -instantons. Both terms contain the perturbative fluctuations around the

D-instanton with certain coefficients $c_{j,m}^k$. For our present purposes we concentrate only on the leading part (without the fluctuations) in the one-anti-instanton sector.

This conjecture can now be checked non-perturbatively making use of the results obtained for the two correlation functions discussed in the previous section. Let us first concentrate on the $\langle \Lambda^{16} \rangle$ -correlator on the SYM side, evaluated in the semi-classical approximation, i.e. in the weak coupling region. According to [16] the operator (8.3) is identified with the (boundary value of) dilatino on the supergravity side. The bulk-to-boundary scattering amplitude obtained from the 16 dilatino vertex should therefore correctly reproduce (8.11). The dependence on the coupling constants

$$(\alpha')^{-1} e^{-\phi/2} f_{16} \sim g^{-24} \sqrt{N}, \quad (9.31)$$

can be seen to match with the large N limit of SYM prediction (8.14). Note that also the instanton actions on both sides are equal. Moreover the bulk-to-boundary propagators for the dilatinos can be shown to coincide with (8.12) implying complete agreement between the two pictures [18, 20], despite of the fact that we are not in the strongly coupled regime. This indicates that this correlator is protected from quantum corrections [48].

Now let us turn to the $\langle \Theta^4 \rangle$ correlator (8.18). Again, the coupling constant dependence of the four-graviton scattering amplitude matches the large N behaviour of (8.19), as can be seen from

$$(\alpha')^{-1} e^{-\phi/2} f_4 \sim g^0 \sqrt{N}. \quad (9.32)$$

As for the x -dependence, there is no obvious agreement with bulk-to-boundary propagator for the graviton [49]. This is not to be expected since this correlator is not protected, contrary to the previous case, by any non-renormalization theorems.

10 Discussion.

In these lectures we have reviewed the general properties of single YM-instantons, and have given tools to compute non-perturbative effects in (non-) supersymmetric gauge theories. As an application we have only considered the $\mathcal{N} = 4$ SYM theory in relation to the AdS/CFT correspondence, but our analysis can be used for a much larger class of models.

On the other hand, because of lack of space and time, we have omitted a few topics which are relevant for instanton applications in more realistic models like QCD [54, 55, 56] and spontaneously broken gauge theories such as the Standard Model [6]. Let us address some of these issues here.

- **Constrained instantons:** in spontaneously broken gauge theories, or any other non-conformal model with a scale, exact instanton solutions to the equations of motion do not exist. This can be proved by means of Derrick's theorem [50]. The way to proceed in these cases is to

construct approximate solutions which are still dominating the path integral [51]. The technique to find these configurations is somewhat similar to the method we described in section 7.2, but instead of expanding in Grassmann collective coordinates, constrained instantons are expanded in the dimensionless parameter $v^2\rho^2$, where v is the vacuum expectation value of the scalar fields. For a recent detailed discussion on this procedure, see [52]. As was shown by 't Hooft long ago [6], the integration over the size of the instanton collective coordinate diverges for large ρ . In electroweak theories this divergence can be cured by the Higgs fields of the constrained instanton, providing a factor $-\rho^2v^2$ in the classical action which cuts the ρ integral off above $\rho^2 = 1/v^2$. With this cut-off mechanism one can compute correlation functions of fermions along the same lines as in section 8. Constrained instantons are also relevant for $\mathcal{N} = 1, 2$ SYM theories, as mentioned in the introduction. They can also be studied in the context of topological YM theories, as was recently discussed in [53].

- Vacuum tunneling: instantons can be used to describe tunneling processes between different vacua of the Minkowski theory. We have not really discussed the structure of the vacuum in QCD-like theories and how tunneling occurs. This is in particular related to the presence of the theta angle term, and consequences thereof (CP violation, instanton anti-instanton interactions etc.). For a discussion on this, we refer to the original literature [4, 5] or to other reviews [3].
- Perturbation theory around the instanton: the methods described here enable us to compute non-perturbative effects in the semi-classical approximation where the coupling constant is small. It is in many cases important to go beyond this limit, and to study subleading corrections that arise from higher order perturbation theory around the instanton [40, 41]. Apart from a brief discussion about the one-loop determinants in section 5, we have not really addressed these issues. Not unrelated to this, subleading corrections also arise from treating the fermionic zero modes exactly, as explained in section 7. We have there indicated how one might compute such corrections, but a detailed investigation remains to be done [42].
- Multi-instantons: we have completely omitted a discussion on multi-instantons. These can be constructed using the ADHM formalism [22]. The main difficulty lies in the explicit construction of the instanton solution and of the measure of collective coordinates beyond instanton number $k = 2$. However, it was recently demonstrated that certain simplifications occur in the large N limit of $\mathcal{N} = 4$ SYM theories [21], where one can actually sum over all multi-instantons to get exact results for certain correlation functions. The same techniques were later applied for $\mathcal{N} = 2, 1$ SYM [57, 58], and it would be interesting to study the

consequences of multi-instantons for large N non-supersymmetric theories.

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A 't Hooft symbols and spinor algebra.

In this appendix we give a list of conventions and formulae useful in instanton calculus.

Let us first discuss the structure of Lorentz algebra $so(3,1)$ in Minkowski space-time. The generators can be represented by $L_{\mu\nu} = -i(x_\mu\partial_\nu - x_\nu\partial_\mu)$ and form the algebra $[L_{\mu\nu}, L_{\rho\sigma}] = i\eta_{\mu\rho}L_{\nu\sigma} + i\eta_{\nu\sigma}L_{\mu\rho} - i\eta_{\mu\sigma}L_{\nu\rho} - i\eta_{\nu\rho}L_{\mu\sigma}$, with the signature $\eta_{\mu\nu} = \text{diag}(-, +, +, +)$. The spatial rotations $J_i \equiv \frac{1}{2}\epsilon_{ijk}L_{jk}$ and boosts $K_i \equiv L_{0i}$ are self-adjoint operators $J_i^\dagger = J_i$, $K_i^\dagger = K_i$, i.e.

$$\int d^4x \Psi^* \mathcal{O} \Psi = \int d^4x (\mathcal{O} \Psi)^* \Psi, \quad \text{for } \mathcal{O} = J, K, \quad (\text{A.1})$$

and upon introduction of combinations $N_i \equiv \frac{1}{2}(J_i + iK_i)$, $M_i \equiv \frac{1}{2}(J_i - iK_i)$ they form two commuting $SU(2)$ algebras,

$$[N_i, M_j] = 0, \quad [N_i, N_j] = i\epsilon_{ijk}N_k, \quad [M_i, M_j] = i\epsilon_{ijk}M_k. \quad (\text{A.2})$$

However, the two $SU(2)$ algebras are related by complex conjugation $(su(2)_N)^* = su(2)_M$, and spatial inversion $\mathcal{P}su(2)_N = su(2)_M$.

The situation differs for euclidean space ($\delta_{\mu\nu} = \text{diag}(+, +, +, +)$) with $SO(4)$ Lorentz group. For the present case the linear combinations of (ij) and $(4, i)$ -plane rotations

$$N_i \equiv \frac{1}{2}(J_i + K_i), \quad M_i \equiv \frac{1}{2}(J_i - K_i), \quad (\text{A.3})$$

where obviously $J_i \equiv \frac{1}{2}\epsilon_{ijk}L_{jk}$ and boosts $K_i \equiv L_{4i}$, give the algebras of independent $SU(2)$ subgroups of $SO(4) = SU(2) \times SU(2)$ in view of hermiticity $N_i^\dagger = N_i$, $M_i^\dagger = M_i$. It is an easy exercise to check that

$$N_i = -\frac{i}{2}\bar{\eta}_{i\mu\nu}x_\mu\partial_\nu, \quad \text{and} \quad M_i = -\frac{i}{2}\eta_{i\mu\nu}x_\mu\partial_\nu, \quad (\text{A.4})$$

where we introduced 't Hooft symbols [6]

$$\begin{aligned}\eta_{a\mu\nu} &\equiv \epsilon_{a\mu\nu} + \delta_{a\mu}\delta_{\nu 4} - \delta_{a\nu}\delta_{4\mu} , \\ \bar{\eta}_{a\mu\nu} &\equiv \epsilon_{a\mu\nu} - \delta_{a\mu}\delta_{\nu 4} + \delta_{a\nu}\delta_{4\mu} ,\end{aligned}\tag{A.5}$$

and $\bar{\eta}_{a\mu\nu} = (-1)^{\delta_{4\mu} + \delta_{4\nu}} \eta_{a\mu\nu}$. They form a basis of anti-symmetric 4 by 4 matrices and are (anti-)selfdual in vector indices ($\epsilon_{1234} = 1$)

$$\eta_{a\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\eta_{a\rho\sigma} , \quad \bar{\eta}_{a\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\bar{\eta}_{a\rho\sigma} .\tag{A.6}$$

The η -symbols obey the following relations

$$\epsilon_{abc}\eta_{b\mu\nu}\eta_{c\rho\sigma} = \delta_{\mu\rho}\eta_{a\nu\sigma} + \delta_{\nu\sigma}\eta_{a\mu\rho} - \delta_{\mu\sigma}\eta_{a\nu\rho} - \delta_{\nu\rho}\eta_{a\mu\sigma} ,\tag{A.7}$$

$$\eta_{a\mu\nu}\eta_{a\rho\sigma} = \delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho} + \epsilon_{\mu\nu\rho\sigma} ,\tag{A.8}$$

$$\eta_{a\mu\rho}\eta_{b\mu\sigma} = \delta_{ab}\delta_{\rho\sigma} + \epsilon_{abc}\eta_{c\rho\sigma} ,\tag{A.9}$$

$$\epsilon_{\mu\nu\rho\theta}\eta_{a\sigma\theta} = \delta_{\sigma\mu}\eta_{a\nu\rho} + \delta_{\sigma\rho}\eta_{a\mu\nu} - \delta_{\sigma\nu}\eta_{a\mu\rho} ,\tag{A.10}$$

$$\eta_{a\mu\nu}\eta_{a\mu\nu} = 12 , \quad \eta_{a\mu\nu}\eta_{b\mu\nu} = 4\delta_{ab} , \quad \eta_{a\mu\rho}\eta_{a\mu\sigma} = 3\delta_{\rho\sigma} .\tag{A.11}$$

The same holds for $\bar{\eta}$ except for

$$\bar{\eta}_{a\mu\nu}\bar{\eta}_{a\rho\sigma} = \delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho} - \epsilon_{\mu\nu\rho\sigma} .\tag{A.12}$$

Obviously $\eta_{a\mu\nu}\bar{\eta}_{b\mu\nu} = 0$ due to different duality properties.

The spinor representation of the euclidean Lorentz algebra is defined by

$$\sigma_{\mu\nu} \equiv \frac{1}{2}[\sigma_\mu\bar{\sigma}_\nu - \sigma_\nu\bar{\sigma}_\mu] , \quad \bar{\sigma}_{\mu\nu} = \frac{1}{2}[\bar{\sigma}_\mu\sigma_\nu - \bar{\sigma}_\nu\sigma_\mu] ,\tag{A.13}$$

in terms of euclidean matrices

$$\sigma_\mu^{\alpha\beta'} = (\tau^a, i) , \quad \bar{\sigma}_{\mu\alpha'\beta} = (\tau^a, -i) , \quad \sigma_\mu^{\alpha\alpha'} = \bar{\sigma}_\mu^{\alpha'\alpha} = \epsilon^{\alpha'\beta'}\bar{\sigma}_{\mu\beta'\beta}\epsilon^{\alpha\beta} ,\tag{A.14}$$

obeying the Clifford algebra $\sigma_\mu\bar{\sigma}_\nu + \sigma_\nu\bar{\sigma}_\mu = 2\delta_{\mu\nu}$. The spinor and vector representations of the $su(2)$ algebra are related precisely via the 't Hooft symbols,

$$\bar{\sigma}_{\mu\nu} = i\eta_{a\mu\nu}\tau^a , \quad \sigma_{\mu\nu} = i\bar{\eta}_{a\mu\nu}\tau^a .\tag{A.15}$$

Some frequently used identities are

$$\bar{\sigma}_\mu\sigma_{\nu\rho} = \delta_{\mu\nu}\bar{\sigma}_\rho - \delta_{\mu\rho}\bar{\sigma}_\nu - \epsilon_{\mu\nu\rho\sigma}\bar{\sigma}_\sigma , \quad \sigma_\mu\bar{\sigma}_{\nu\rho} = \delta_{\mu\nu}\sigma_\rho - \delta_{\mu\rho}\sigma_\nu + \epsilon_{\mu\nu\rho\sigma}\sigma_\sigma ,\tag{A.16}$$

$$\sigma_{\mu\nu}\sigma_\rho = \delta_{\nu\rho}\sigma_\mu - \delta_{\mu\rho}\sigma_\nu + \epsilon_{\mu\nu\rho\sigma}\sigma_\sigma , \quad \bar{\sigma}_{\mu\nu}\bar{\sigma}_\rho = \delta_{\nu\rho}\bar{\sigma}_\mu - \delta_{\mu\rho}\bar{\sigma}_\nu - \epsilon_{\mu\nu\rho\sigma}\bar{\sigma}_\sigma .\tag{A.17}$$

The Lorentz generators are antisymmetric in vector and symmetric in spinor indices

$$\sigma_{\mu\nu\alpha\beta} = -\sigma_{\nu\mu\alpha\beta} , \quad \sigma_{\mu\nu\alpha\beta} = \sigma_{\mu\nu\beta\alpha} , \quad (\text{A.18})$$

and obey the algebra

$$[\sigma_{\mu\nu}, \sigma_{\rho\sigma}] = -2 \{ \delta_{\mu\rho}\sigma_{\nu\sigma} + \delta_{\nu\sigma}\sigma_{\mu\rho} - \delta_{\mu\sigma}\sigma_{\nu\rho} - \delta_{\nu\rho}\sigma_{\mu\sigma} \} , \quad (\text{A.19})$$

$$\{ \sigma_{\mu\nu}, \sigma_{\rho\sigma} \} = -2 \{ \delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho} - \epsilon_{\mu\nu\rho\sigma} \} . \quad (\text{A.20})$$

The same relations hold for $\bar{\sigma}$ but with $+\epsilon_{\mu\nu\rho\sigma}$. In spinor algebra the following contractions are useful

$$\sigma_{\mu}^{\alpha\alpha'} \bar{\sigma}_{\mu\beta'\beta} = 2\delta_{\alpha}^{\beta}\delta_{\alpha'}^{\beta'} , \quad \sigma_{\rho\sigma}^{\alpha}{}_{\beta}\sigma_{\rho\sigma}^{\gamma}{}_{\delta} = 4 \left\{ \delta_{\beta}^{\alpha}\delta_{\delta}^{\gamma} - 2\delta_{\delta}^{\alpha}\delta_{\beta}^{\gamma} \right\} . \quad (\text{A.21})$$

We use everywhere the north-west conventions for raising and lowering the spinor indices

$$\epsilon^{\alpha\beta}\xi_{\beta} = \xi^{\alpha} , \quad \bar{\xi}^{\beta'}\epsilon_{\beta'\alpha'} = \bar{\xi}_{\alpha'} , \quad (\text{A.22})$$

with $\epsilon_{\alpha\beta} = -\epsilon^{\alpha'\beta'}$, $\epsilon_{\alpha\beta} = \epsilon^{\alpha\beta}$, so that $\xi_{(1)}^{\alpha}\xi_{(2)\alpha} = \xi_{(2)}^{\alpha}\xi_{(1)\alpha}$. For hermitean conjugation we define $(\xi_{(1)}^{\alpha}\xi_{(2)\alpha})^{\dagger} = \xi_{(2)\alpha}^{\dagger}\xi_{(1)}^{\alpha\dagger}$ and the sigma matrices satisfy

$$(\sigma_{\mu}^{\alpha\beta'})^* = \sigma_{\mu\alpha\beta'} , \quad (\bar{\sigma}_{\mu\alpha'\beta})^* = \bar{\sigma}_{\mu}^{\alpha'\beta} . \quad (\text{A.23})$$

Throughout the paper we frequently use the following integral formula

$$\int d^4x \frac{(x^2)^n}{(x^2 + \rho^2)^m} = \pi^2 (\rho^2)^{n-m+2} \frac{\Gamma(n+2)\Gamma(m-n-2)}{\Gamma(m)} , \quad (\text{A.24})$$

which converges for $m - n > 2$.

B Winding number.

For a gauge field configuration with finite action the field strength must tend to zero faster than x^{-2} at large x . For vanishing $F_{\mu\nu}$, the potential A_{μ} becomes a pure gauge, $A_{\mu} \xrightarrow{x \rightarrow \infty} U^{-1}\partial_{\mu}U$. All configurations of A_{μ} which become pure gauge at infinity fall into equivalence classes, where each class has a definite winding number. As we now show, the winding number is given by

$$k = -\frac{1}{16\pi^2} \int d^4x \text{tr}^* F_{\mu\nu} F_{\mu\nu} , \quad (\text{B.1})$$

where the generators T_a satisfy $\text{tr} T_a T_b = -\frac{1}{2}\delta_{ab}$. This is the normalization we adopt for the fundamental representation. The key observation is that ${}^*F_{\mu\nu}F_{\mu\nu}$ is a total derivative of a gauge variant current⁹

$$\text{tr}^* F_{\mu\nu} F_{\mu\nu} = 2\partial_{\mu} \text{tr} \epsilon_{\mu\nu\rho\sigma} \left\{ A_{\nu}\partial_{\rho}A_{\sigma} + \frac{2}{3}A_{\nu}A_{\rho}A_{\sigma} \right\} . \quad (\text{B.2})$$

⁹Note that ${}^*F_{\mu\nu}F_{\mu\nu}$ equals to $2\epsilon_{\mu\nu\rho\sigma} \{ \partial_{\mu}A_{\nu}\partial_{\rho}A_{\sigma} + 2\partial_{\mu}A_{\nu}A_{\rho}A_{\sigma} + A_{\mu}A_{\nu}A_{\rho}A_{\sigma} \}$ but the last term vanishes in the trace due to cyclicity of the trace.

According to Stokes' theorem, the space-time integral becomes an integral over the three-dimensional boundary at infinity if one uses the regular gauge in which there are no singularities at the origin. Since A_μ becomes a pure gauge at large x , one obtains

$$k = \frac{1}{24\pi^2} \oint_{S^3(\text{space})} d\Omega_\mu \epsilon_{\mu\nu\rho\sigma} \text{tr} \left\{ (U^{-1}\partial_\nu U) (U^{-1}\partial_\rho U) (U^{-1}\partial_\sigma U) \right\} , \quad (\text{B.3})$$

where the integration is over a large three-sphere, $S^3(\text{space})$, in Euclidean space. To each point x^μ on this large three-sphere in space corresponds a group element U in the gauge group G . If $G = SU(2)$, the group manifold is also a three-sphere¹⁰ $S^3(\text{group})$. Then $U(x)$ maps $S^3(\text{space})$ into $S^3(\text{group})$, and as we now show, k is an integer which counts how many times $S^3(\text{space})$ is wrapped around $S^3(\text{group})$.

Choose a parametrization of the group elements of $SU(2)$ in terms of group parameters¹¹ $\xi^a(x)$ ($a = 1, 2, 3$). Hence the functions $\xi^a(x)$ map x into $SU(2)$. Consider a small surface element of $S^3(\text{space})$. According to the chain rule

$$\text{tr} \left\{ (U^{-1}\partial_\nu U) (U^{-1}\partial_\rho U) (U^{-1}\partial_\sigma U) \right\} = \frac{\partial \xi^i}{\partial x_\nu} \frac{\partial \xi^j}{\partial x_\rho} \frac{\partial \xi^k}{\partial x_\sigma} \text{tr} \left\{ (U^{-1}\partial_i U) (U^{-1}\partial_j U) (U^{-1}\partial_k U) \right\} . \quad (\text{B.4})$$

Using¹²

$$\Delta\Omega_\mu = \frac{1}{6} \epsilon_{\mu\nu\rho\sigma} \Delta x_\nu \Delta x_\rho \Delta x_\sigma \quad (\text{B.5})$$

we obtain, from $\frac{1}{6} \epsilon_{\mu\nu\rho\sigma} \epsilon_{\mu\alpha\beta\gamma} = \delta_{[\alpha\beta\gamma]}^{\nu\rho\sigma}$ and $d\xi^{[i} d\xi^j d\xi^k] = \epsilon^{ijk} d^3\xi$, for the contribution Δk of the small surface element to k

$$\Delta k = \frac{1}{24\pi^2} \epsilon^{ijk} \text{tr} \left\{ (U^{-1}\partial_i U) (U^{-1}\partial_j U) (U^{-1}\partial_k U) \right\} d^3\xi , \quad (\text{B.6})$$

with $k = \oint_{S^3(\text{space})} \Delta k$. The elements $(U^{-1}\partial_i U)$ lie in the Lie algebra, and define the group vielbein e_i^a by

$$(U^{-1}\partial_i U) = e_i^a T_a . \quad (\text{B.7})$$

With $\epsilon^{ijk} e_i^a e_j^b e_k^c = (\det e) \epsilon^{abc}$, we obtain for the contribution to k from a surface element $\Delta\Omega_\mu$

$$\Delta k = \frac{1}{24\pi^2} (\det e) \text{tr} \left(\epsilon^{abc} T_a T_b T_c \right) d^3\xi = -\frac{1}{16\pi^2} (\det e) d^3\xi . \quad (\text{B.8})$$

As we demonstrated, the original integral over the physical space is reduced to one over the group with measure $(\det e) d^3\xi$. The volume of a surface element of $S^3(\text{group})$ with coordinates

¹⁰The elements of $SU(2)$ can be written in the fundamental representation as $U = a_0 \mathbb{1} + i \sum_k a_k \tau_k$ where a_0 and a_k are real coefficients satisfying the condition $a_0^2 + \sum_k a_k^2 = 1$. This defines a sphere $S^3(\text{group})$.

¹¹For example, Euler angles, or Lie parameters $U = a_0 \mathbb{1} + i \sum_k a_k \tau_k$ with $a_0 = \sqrt{1 - \sum_k a_k^2}$.

¹²For example, if the surface element points in the x -direction we have $\Delta\Omega = \Delta\tau \Delta y \Delta z$.

$d\xi^i$ is proportional to $(\det e) d^3\xi$. Since this expression is a scalar in general relativity, we know that the value of the volume does not depend on which coordinates one uses except for an overall normalization. We fix this overall normalization of the group volume such that near $\xi = 0$ the volume is $d^3\xi$. Since $e_i^a = \delta_i^a$ near $\xi = 0$, we have the usual euclidean measure $d^3\xi$. Each small patch on $S^3(\text{space})$ corresponds to a small patch on $S^3(\text{group})$. Since the U 's fall into homotopy classes, integrating once over $S^3(\text{space})$ we cover $S^3(\text{group})$ an integer number of times. To check the proportionality factor in $\Delta k \sim \text{Vol}(d^3\xi)$, we consider the fundamental map

$$U(x) = x_\mu \sigma_\mu / \sqrt{x^2}, \quad U^{-1}(x) = x_\mu \bar{\sigma}_\mu / \sqrt{x^2}. \quad (\text{B.9})$$

This is clearly a one-to-one map from $S^3(\text{space})$ to $S^3(\text{group})$ and should yield $|k| = 1$. Direct calculation gives

$$U^{-1} \partial_\mu U = -\bar{\sigma}_{\mu\nu} x_\nu / x^2, \quad (\text{B.10})$$

and substitution into (B.3) leads to $k = -\frac{1}{2\pi^2} \oint d\Omega_\mu x_\mu / x^4 = -1$ making use of (A.19) and (A.20). To obtain $k = 1$ one has to make the change $\sigma \leftrightarrow \bar{\sigma}$ or $x \leftrightarrow -x$ in Eq. (B.9).

Let us comment on the origin of the winding number of the instanton in the singular gauge. In this case A_μ^{sing} vanishes fast at infinity, but becomes pure gauge near $x = 0$. In the region between a small sphere in the vicinity of $x = 0$ and a large sphere at $x = \infty$ we have an expression for k in terms of the total derivative, but now for A_μ^{sing} the only contribution to the topological charge comes from the boundary near $x = 0$:

$$k = -\frac{1}{24\pi^2} \oint_{S_{x \rightarrow 0}^3(\text{space})} d\Omega_\mu \epsilon_{\mu\nu\rho\sigma} \text{tr} \left\{ (U^{-1} \partial_\nu U) (U^{-1} \partial_\rho U) (U^{-1} \partial_\sigma U) \right\}. \quad (\text{B.11})$$

The extra minus sign is due to the fact that the normal to the $S^3(\text{space})$ at $x = 0$ points inward. Furthermore, $A_\mu^{\text{sing}} \sim U^{-1} \partial_\mu U = -\bar{\sigma}_{\mu\nu} x_\nu / x^2$ near $x = 0$, while $A_\mu^{\text{reg}} \sim U \partial_\mu U^{-1} = -\sigma_{\mu\nu} x_\nu / x^2$ for $x \sim \infty$. There is a second minus sign in the evaluation of k from the trace of Lorentz generators in both solutions. As a result $k_{\text{sing}} = k_{\text{reg}}$, as it should be since k is a gauge invariant object. The gauge transformation which maps A_μ^{reg} to A_μ^{sing} transfers the winding from a large to a small $S^3(\text{space})$.

C The volume of the gauge orientation moduli space.

The purpose of this appendix is to prove the equation (4.36). Let us consider an instanton in an $SU(N)$ gauge theory. Deformations of this configuration which are still self-dual and not a gauge transformation are parametrized by collective coordinates. Constant gauge transformations $A_\mu \rightarrow U^{-1} A_\mu U$ preserve self-duality and transversality, $\partial_\mu A_\mu = 0$, but not all constant $SU(N)$

matrices U change A_μ . Those U which keep A_μ fixed form the stability subgroup H of the instanton, hence we want to determine the volume of the coset space $SU(N)/H$.

If the instanton is embedded in the lower-right 2×2 submatrix of the $N \times N$ $SU(N)$ matrix, then H contains the $SU(N-2)$ in the left-upper part, and the $U(1)$ subgroup with elements $\exp(\theta A)$ where A is the diagonal matrix

$$A = i\sqrt{\frac{N-2}{N}} \text{diag} \left(\frac{2}{2-N}, \dots, \frac{2}{2-N}, 1, 1 \right). \quad (\text{C.1})$$

All generators of $SU(N)$ (only for the purposes of the present appendix, and also all generators of $SO(N)$ discussed below) are normalized according to $\text{tr} T_a T_b = -2\delta_{ab}$ in the defining $N \times N$ matrix representation. Note that in the main text we use $\text{tr} T_a T_b = -\frac{1}{2}\delta_{ab}$.

At first sight one might expect the range of θ to be such that all entries cover the range 2π an integer number of times. However, this is incorrect: only for the last two entries of $\exp(\theta A)$ we must require periodicity, because whatever happens in the other $N-2$ diagonal entries is already contained in the $SU(N-2)$ part of the stability subgroup. Thus all elements h in H are of the form [29]

$$h = e^{\theta A} g, \quad \text{with} \quad g \in SU(N-2) \quad \text{and} \quad 0 \leq \theta \leq \theta_{\max} = 2\pi\sqrt{\frac{N}{N-2}}. \quad (\text{C.2})$$

For $N = 3, 4$ this range of θ corresponds to periodicity of all entries, but for $N \geq 5$ the range of θ is less than required for periodicity. Thus $H \neq SU(N) \times U(1)$ for $N \geq 5$. The first $N-2$ entries of $\exp(k\theta_{\max} A)$ with integer k are given by $\exp\left(-ik\frac{4\pi}{N-2}\right)$ and lie therefore in the center Z_N of $SU(N-2)$. Note that the $SU(N)$ group elements $h = \exp(\theta A) g$ with $0 \leq \theta \leq \theta_{\max}$ form a subgroup. We shall denote H by $SU(N-2) \times "U(1)"$ where " $U(1)$ " denotes the part of the $U(1)$ generated by A which lies in H .

We now use three theorems to evaluate the volume of $SU(N)/H$:

$$\begin{aligned} \text{(I)} \quad & \text{Vol} \frac{SU(N)}{SU(N-2) \times "U(1)"} = \frac{\text{Vol} (SU(N)/SU(N-2))}{\text{Vol} "U(1)"} , \\ \text{(II)} \quad & \text{Vol} \frac{SU(N)}{SU(N-2)} = \text{Vol} \frac{SU(N)}{SU(N-1)} \text{Vol} \frac{SU(N-1)}{SU(N-2)} , \\ \text{(III)} \quad & \text{Vol} \frac{SU(N)}{SU(N-1)} = \frac{\text{Vol} SU(N)}{\text{Vol} SU(N-1)} . \end{aligned} \quad (\text{C.3})$$

It is, in fact, easiest to first compute $\text{Vol} (SU(N)/SU(N-1))$ and then to use this result for the evaluation of $\text{Vol} G/H$ (with $\text{Vol} SU(N)$ as a bonus).

In general the volume of a coset manifold G/H is given by $V = \int \prod_\mu dx^\mu \det e_\mu^m(x)$ where x^μ are the coordinates on the coset manifold and $e_\mu^m(x)$ are the coset vielbeins. One begins with "coset representatives" $L(x)$ which are group elements $g \in G$ such that every group element can

be decomposed as $g = kh$ with $h \in H$. We denote the coset generators by K_μ and the subgroup generators by H_i . Then $L^{-1}(x)\partial_\mu L(x) = e_\mu^m(x)K_m + \omega_\mu^i(x)H_i$. Under a general coordinate transformation from x^μ to $y^\mu(x)$, the vielbein transforms as a covariant vector with index μ but also as a contra-covariant vector with index m at $x = 0$. Hence V does (only) depend on the choice of the coordinates at the origin. Near the origin, $L(x) = \mathbb{1} + dx^\mu K_\mu$, and we fix the normalization of $T_a = \{K_\mu, H_i\}$ by $\text{tr} T_a T_b = -2\delta_{ab}$ for $T_a \in SU(N)$.

To find the volume of $SU(N)/SU(N-1)$ we note that the group elements of $SU(N)$ have a natural action on the space \mathbf{C}^N and map a vector $(z^1, \dots, z^N) \in \mathbf{C}^N$ on the complex hypersphere $\sum_{i=1}^N |z^i|^2 = 1$ into another vector on the complex hypersphere. The ‘‘south-pole’’ $(0, \dots, 0, 1)$ is kept invariant by the subgroup $SU(N-1)$, and points on the complex hypersphere are in one-to-one correspondence with the coset representatives $L(z)$ of $SU(N)/SU(N-1)$. We use as generators for $SU(N)$ the generators for $SU(N-1)$ in the upper-left block, and further the following coset generators: $N-1$ pairs T_{2k} and T_{2k+1} each of them containing only two non-zero elements

$$\begin{pmatrix} 0 & \dots & 0 \\ \vdots & & 1 \\ & \ddots & \vdots \\ 0 & -1 & \dots & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & i \\ & \ddots & \vdots \\ 0 & i & \dots & 0 \end{pmatrix}, \quad (\text{C.4})$$

and further one diagonal generator

$$T_{N^2-1} = i\sqrt{\frac{2}{N(N-1)}} \text{diag}(-1, \dots, -1, N-1). \quad (\text{C.5})$$

(For instance, for $SU(3)$ there are two pairs, the usual λ_4 and λ_5 and λ_6 and λ_7 and the diagonal hypercharge generator λ_8 .)

The idea now is to establish a natural one-to-one correspondence between points in \mathbf{C}^N and points in \mathbf{R}^{2N} , namely we write all points (x^1, \dots, x^{2N}) in \mathbf{R}^{2N} as points in \mathbf{C}^N as $(ix^1 + x^2, \dots, ix^{2N-1} + x^{2N})$. In particular the south pole in \mathbf{R}^{2N} corresponds to the south pole in \mathbf{C}^N and the sphere $\sum_{i=1}^{2N} (x^i)^2 = 1$ in \mathbf{R}^{2N} corresponds to the hypersphere $\sum_{i=1}^N |z^i|^2 = 1$. Points on the sphere S^{2N-1} in \mathbf{R}^{2N} correspond one-to-one to coset generators of $SO(2N)/SO(2N-1)$. The coset generators of $SO(2N)/SO(2N-1)$ are antisymmetric $2N \times 2N$ matrices A_I ($I = 1, \dots, 2N-1$) with the entry $+1$ in the last column and -1 in the last row. The coset element $1 + \delta g = 1 + dt^I A_I$ maps the south pole $s = (0, \dots, 0, 1)$ in \mathbf{R}^{2N} to a point $s + \delta s$ in \mathbf{R}^{2N} where $\delta s = (dt^1, \dots, dt^{2N-1}, 0)$. In \mathbf{C}^N the action of this same coset element is defined as follows: it maps $s = (0, \dots, 0, 1)$ to $s + \delta s$ with $\delta s = (idt^1 + dt^2, \dots, idt^{2N-1})$. The coset generators of $SU(N)/SU(N-1)$ also act in \mathbf{C}^N , but $g = 1 + dx^\mu K_\mu$ maps s to $s + \delta s$ where now $\delta s = (idx^1 + dx^2, \dots, i\sqrt{\frac{2(N-1)}{N}} dx^{2N-1})$.

We can cover S^{2N-1} with small patches. Each patch can be brought by the action of a suitable coset element to the south pole, and then we can use the inverse of this group element to map this

patch back into the manifold $SU(N)/SU(N-1)$. In this way both S^{2N-1} and $SU(N)/SU(N-1)$ are covered by patches which are in a one-to-one correspondence. Each pair of patches has the same ratio of volumes since both patches can be brought to the south pole by the same group element and at the south pole the ratio of their volumes is the same. To find the ratio of the volumes of S^{2N-1} and $SU(N)/SU(N-1)$, it is then sufficient to consider a small patch near the south pole.

Consider then a small patch at the south pole of S^{2N-1} with coordinates (dt^1, \dots, dt^{2N-1}) and volume $dt^1 \dots dt^{2N-1}$. The corresponding patch at the south pole of \mathbf{C}^N has coordinates $(idt^1 + dt^2, \dots, idt^{2N-1})$ and the same volume $dt^1 \dots dt^{2N-1}$. The same patch at the south pole in \mathbf{C}^N has coordinates dx^μ where $(idt^1 + dt^2, \dots, idt^{2N-1}) = (idx^1 + dx^2, \dots, i\sqrt{\frac{2(N-1)}{N}}dx^{2N-1})$. The volume of a patch in $SU(N)/SU(N-1)$ with coordinates dx^1, \dots, dx^{2N-1} is $dx^1 \dots dx^{2N-1}$. It follows that the volume of $SU(N)/SU(N-1)$ equals the volume of S^{2N-1} times $\sqrt{\frac{N}{2(N-1)}}$,

$$\text{Vol} \frac{SU(N)}{SU(N-1)} = \sqrt{\frac{N}{2(N-1)}} \text{Vol} S^{2N-1} . \quad (\text{C.6})$$

From here the evaluation of $\text{Vol} SU(N)/H$ is straightforward. Using

$$\text{Vol} S^{2N-1} = \frac{2\pi^N}{(N-1)!} , \quad (\text{C.7})$$

we obtain

$$\text{Vol} SU(N) = \sqrt{N} \prod_{k=2}^N \frac{\sqrt{2}\pi^k}{(k-1)!} , \quad (\text{C.8})$$

and

$$\begin{aligned} \text{Vol} H &= \text{Vol} SU(N-2) \text{Vol} \text{“}U(1)\text{”} , & \text{Vol} \text{“}U(1)\text{”} &= 2\pi \sqrt{\frac{N}{N-2}} , \\ \text{Vol} SU(N)/H &= \frac{\pi^{2N-2}}{(N-1)!(N-2)!} . \end{aligned} \quad (\text{C.9})$$

As an application and check of this analysis let us demonstrate a few relations between the volumes of different groups. Let us check that $\text{Vol} SU(2) = 2\text{Vol} SO(3)$, $\text{Vol} SU(4) = 2\text{Vol} SO(6)$ and $\text{Vol} SO(4) = \frac{1}{2} (\text{Vol} SU(2))^2$ (the latter will follow from $SO(4) = SU(2) \times SU(2)/Z_2$).

We begin with the usual formula for the surface of a sphere with unit radius (given already above for odd N)

$$\text{Vol} S^N = \frac{2\pi^{(N+1)/2}}{\Gamma\left(\frac{N+1}{2}\right)} . \quad (\text{C.10})$$

In particular

$$\begin{aligned} \text{Vol} S^2 &= 4\pi , & \text{Vol} S^3 &= 2\pi^2 , & \text{Vol} S^4 &= \frac{8}{3}\pi^2 , \\ \text{Vol} S^5 &= \pi^3 , & \text{Vol} S^6 &= \frac{16}{15}\pi^3 , & \text{Vol} S^7 &= \frac{1}{3}\pi^4 . \end{aligned} \quad (\text{C.11})$$

Furthermore $\text{Vol } SO(2) = 2\pi$ since the $SO(2)$ generator is $T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\exp(\theta T)$ is an ordinary rotations $\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$ for which $0 \leq \theta \leq 2\pi$. Using $\text{Vol } SO(N) = \text{Vol } S^{N-1} \text{Vol } SO(N-1)$ we obtain

$$\begin{aligned} \text{Vol } SO(2) &= 2\pi, & \text{Vol } SO(3) &= 8\pi^2, & \text{Vol } SO(4) &= 16\pi^4, \\ \text{Vol } SO(5) &= \frac{128}{3}\pi^6, & \text{Vol } SO(6) &= \frac{128}{3}\pi^9. \end{aligned} \quad (\text{C.12})$$

Now consider $SU(2)$. In the normalization $T_1 = -i\tau_1$, $T_2 = -i\tau_2$ and $T_3 = -i\tau_3$ (so that $\text{tr } T_a T_b = -2\delta_{ab}$) we find by direct evaluation using Euler angles $\text{Vol } SU(2) = 2\pi^2$. This also agrees with Eq. (C.6) for $N = 2$, using $\text{Vol } SU(1) = 1$. For higher N we get

$$\text{Vol } SU(2) = 2\pi^2, \quad \text{Vol } SU(3) = \sqrt{3}\pi^5, \quad \text{Vol } SU(4) = \frac{\sqrt{2}}{3}\pi^9. \quad (\text{C.13})$$

The group elements of $SU(2)$ can be written as $g = x^4 + i\vec{\tau} \cdot \vec{x}$ with $(x^4)^2 + (\vec{x})^2 = 1$ which defines a sphere S^3 . Since near the unit element $g \approx 1 + \vec{\tau} \cdot \delta\vec{x}$, the normalization of the generators is as before, and hence for this parametrization $\text{Vol } SU(2) = 2\pi^2$. This is indeed equal to $\text{Vol } S^3$.

However, $\text{Vol } SU(2)$ is not yet equal to $2\text{Vol } SO(3)$. The reason is that in order to compare properties of different groups we should normalize the generators such that the structure constants are the same (the Lie algebras are the same, although the group volumes are not). In other words, we should use the normalization that the adjoint representations have the same $\text{tr } T_a T_b$.

For $SU(2)$ the generators which lead to the same commutators as the usual $SO(3)$ rotation matrices (with entries $+1$ and -1) are $T_a = \left\{ -\frac{i}{2}\tau_1, -\frac{i}{2}\tau_2, -\frac{i}{2}\tau_3 \right\}$. Then $\text{tr } T_a T_b = -\frac{1}{2}\delta_{ab}$. In this normalization, the range of each group coordinate is multiplied by 2, leading to $\text{Vol } SU(2) = 2^3 \cdot 2\pi^2 = 16\pi^2$. Now indeed $\text{Vol } SU(2) = 2\text{Vol } SO(3)$.

For $SU(4)$ the generators with the same Lie algebra as $SO(6)$ are the generators $\frac{1}{4}(\gamma_m \gamma_n - \gamma_n \gamma_m)$, $i\gamma_m$ and $i\gamma_5$, where γ_m and γ_5 are the 4×4 matrices obeying the Clifford algebra $\{\gamma_m, \gamma_n\} = 2\delta_{mn}$. Now, $\text{tr } T_a T_b = -\delta_{ab}$ (for example, $\text{tr} \left\{ \left(\frac{1}{2}\gamma_1 \gamma_2 \right) \left(\frac{1}{2}\gamma_1 \gamma_2 \right) \right\} = -1$). Recall that originally we had chosen the normalization $\text{tr } T_a T_b = -2\delta_{ab}$. We must thus multiply the range of each coordinate by a factor $\sqrt{2}$, and hence we must multiply our original result for $\text{Vol } SU(4)$ by a factor $(\sqrt{2})^5$. We find that indeed the relation $\text{Vol } SU(4) = 2\text{Vol } SO(6)$ is fulfilled.

Finally, we consider the relation $SO(4) = SU(2) \times SU(2)/Z_2$. (The vector representation of $SO(4)$ corresponds to the representation $\left(\frac{1}{2}, \frac{1}{2}\right)$ of $SU(2) \times SU(2)$, but representations like $\left(\frac{1}{2}, 0\right)$ and $\left(0, \frac{1}{2}\right)$ are not representations of $SO(4)$ and hence we must divide by Z_2 . The reasoning is the same as for $SU(2)$ and $SO(3)$, or $SU(4)$ and $SO(6)$.) We choose the generators of $SO(4)$ as follows

$$T_1^{(+)} = \frac{1}{\sqrt{2}}(L_{14} + L_{23}), \quad T_2^{(+)} = \frac{1}{\sqrt{2}}(L_{31} + L_{24}), \quad T_3^{(+)} = \frac{1}{\sqrt{2}}(L_{12} + L_{34}), \quad (\text{C.14})$$

and the same but with minus sign denoted by $T_i^{(-)}$. Here L_{mn} equals +1 in the m^{th} column and n^{th} row, and is antisymmetric. Clearly $\text{tr } T_a T_b = -2\delta_{ab}$. The structure constants follow from

$$\left[\frac{1}{\sqrt{2}}(L_{12} + L_{34}), \frac{1}{\sqrt{2}}(L_{14} + L_{23}), \right] = -(L_{31} + L_{24}) , \quad (\text{C.15})$$

thus

$$[T_i^{(+)}, T_j^{(+)}] = -\sqrt{2}\epsilon_{ijk}T_k^{(+)} , \quad [T_i^{(-)}, T_j^{(-)}] = -\sqrt{2}\epsilon_{ijk}T_k^{(-)} , \quad [T_i^{(+)}, T_j^{(-)}] = 0 . \quad (\text{C.16})$$

We choose for the generators of $SU(2) \times SU(2)$ the representation

$$T_i^{(+)} = \frac{i\tau_i}{\sqrt{2}} \otimes \mathbb{1} , \quad T_i^{(-)} = \mathbb{1} \otimes \frac{i\tau_i}{\sqrt{2}} . \quad (\text{C.17})$$

Then we get the same commutation relations as for $SO(4)$ generators (C.16); however, the generators are normalized differently, namely $\text{tr } T_a T_b = -2\delta_{ab}$ for $SO(4)$ but $\text{tr } T_a T_b = -\delta_{ab}$ for $SU(2)$. With the normalization $\text{tr } T_a T_b = -2\delta_{ab}$ we found $\text{Vol } SU(2) = 2\pi^2$. In the present normalization we find $\text{Vol } SU(2) = 2\pi^2 (\sqrt{2})^3$. The relation $\text{Vol } SO(4) = \frac{1}{2} (\text{Vol } SU(2))^2$ is now indeed satisfied

$$\text{Vol } SO(4) = 16\pi^4 = \frac{1}{2} (\text{Vol } SU(2))^2 = \frac{1}{2} \left(2\pi^2 (\sqrt{2})^3 \right)^2 . \quad (\text{C.18})$$

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