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Gauging Isometries on Hyperkähler Cones and Quaternion-Kähler Manifolds

Bernard de Wit[a], **Martin Roček**[b] **and Stefan Vandoren**[b]

[a] *Institute for Theoretical Physics* and *Spinoza Institute*
Utrecht University, Utrecht, The Netherlands
B.deWit@phys.uu.nl

[b] *C.N. Yang Institute for Theoretical Physics*
SUNY, Stony Brook, NY 11794-3840, USA
rocek@insti.physics.sunysb.edu
vandoren@insti.physics.sunysb.edu

ABSTRACT

We extend our previous results on the relation between quaternion-Kähler manifolds and hyperkähler cones and we describe how isometries, moment maps and scalar potentials descend from the cone to the quaternion-Kähler space. As an example of the general construction, we discuss the gauging and the corresponding scalar potential of hypermultiplets with the unitary Wolf spaces as target spaces. This class includes the universal hypermultiplet.

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1 Introduction

The scalar fields of $N = 2$ hypermultiplets coupled to supergravity parametrize a quaternion-Kähler (QK) manifold [1]. These manifolds appear as target spaces in the low-energy effective actions of type-II superstrings compactified on a Calabi-Yau three-fold, or heterotic strings compactified on K3. The geometry of QK manifolds is best understood in terms of their hyperkähler cones (HKC's) [2, 3, 4, 5]. This one-to-one correspondence is based on the $N = 2$ superconformal quotient [4, 5], and it can be exploited in a variety of applications, ranging from supersymmetry breaking in gauged supergravity, domain walls and supergravity flows, to quantum corrections to hypermultiplet moduli spaces of Calabi-Yau string compactifications.

In this paper, we discuss the gauging of isometries on QK spaces. The strategy is to first gauge the corresponding isometries on the HKC, where the computations are relatively easy, and subsequently project to the QK space. We construct the moment maps and the resulting scalar potential, and explain how the gauge fields of the isometries contribute to R-symmetry connections. As the scalars of the vector multiplets live on special Kähler manifolds and contribute to the scalar potential, we review some aspects of special Kähler geometry.

Finally, as a concrete example, we consider the unitary Wolf spaces, and in particular, the universal hypermultiplet. We point out certain subtleties in performing the QK quotient [6] which can elegantly be avoided by first taking a hyperkähler quotient followed by an $N = 2$ superconformal quotient of the HKC.

We first summarize some of the results of [4, 5] about $4n$-dimensional HKC's and their corresponding $4(n-1)$-dimensional QK metrics. The HKC metric is determined by a hyperkähler potential χ, satisfying

$$D_A \partial_B \chi = g_{AB} \ , \qquad A = 1, \ldots, 4n \ . \tag{1.1}$$

The associated $N = 2$ sigma model is therefore scale invariant, with a homothetic conformal Killing vector $\chi^A = g^{AB}\chi_B$, where $\chi_A \equiv \partial_A \chi$. Moreover, using the complex structures $\vec{J}$ of the HKC, one can construct the Killing vectors associated with an Sp(1) isometry group,

$$\vec{k}^A = \vec{J}^A{}_B \, \chi^B \ . \tag{1.2}$$

This implies that the corresponding hypermultiplet action is rigidly $N = 2$ superconformally invariant.

The $N = 2$ superconformal quotient amounts to gauging the superconformal algebra (hence the sigma model couples to superconformal gravity), and eliminating the dilatation and $\mathrm{Sp}(1) = \mathrm{SU}(2)$ gauge fields which are not part of the Poincaré supergravity multiplet. The resulting sigma model has a quaternion-Kähler metric [4],

$$G_{AB} = \frac{1}{\chi}\left(g_{AB} - \frac{1}{2\chi}\Big[\chi_A\chi_B + \vec{k}_A\cdot\vec{k}_B\Big]\right) \ , \tag{1.3}$$

and quaternionic two-forms

$$\vec{\mathcal{Q}}_{AB} = G_{AC}\vec{J}^C{}_B \ . \tag{1.4}$$

A QK manifold also has an Sp(1) connection $\vec{\mathcal{V}}_A = \chi^{-1}\vec{k}_A$. Its curvature is proportional to the quaternionic two-forms[1],

$$\vec{\mathcal{R}}_{AB} \equiv \partial_A\vec{\mathcal{V}}_B - \partial_B\vec{\mathcal{V}}_A - \vec{\mathcal{V}}_A\times\vec{\mathcal{V}}_B = -2\vec{\mathcal{Q}}_{AB} \ . \tag{1.5}$$

The constant on the right hand side is fixed by the normalization of the metric (1.3).

The metric and two-forms (1.3) and (1.4) still carry indices in the $4n$-dimensional HKC, but they are horizontal in the sense that they are orthogonal to the conformal Killing vector χ^A and the Sp(1) Killing vectors $\vec{k}^A$. These vectors are associated with a cone over an Sp(1) fibration of the QK space. The fibration over the QK space is a $(4n-1)$-dimensional 3-Sasakian manifold. To descend to the $4(n-1)$-dimensional QK space, we choose appropriate gauge conditions corresponding to the scale and Sp(1) symmetries [5]. This is most easily done by first choosing $2n$ holomorphic coordinates z^a on the HKC for which $J^{3a}{}_b = i\delta^a{}_b$. The vector field χ^a is then holomorphic and one can single out a coordinate z (with remaining coordinates $u^i,\ i = 1,\ldots,2n-1$) by defining

$$\chi^a(u,z)\,\frac{\partial}{\partial z^a} \equiv \frac{\partial}{\partial z} \ . \tag{1.6}$$

In this basis, (1.1) can be solved:

$$\chi(u,\bar{u};z,\bar{z}) = \mathrm{e}^{z+\bar{z}+K(u,\bar{u})} \ , \tag{1.7}$$

where $K(u,\bar{u})$ is the Kähler potential of a complex $(2n-1)$-dimensional Einstein-Kähler space, which is the twistor space $\mathcal{Z}$ above the QK manifold [7, 2]. The metric (1.3) takes the following form in these coordinates,

$$G_{i\bar{\jmath}} = K_{i\bar{\jmath}} - \mathrm{e}^{2K}X_iX_{\bar{\jmath}} \ . \tag{1.8}$$

All other components vanish consistent with horizontality.

In the coordinates (u^i, z), the homothety and Sp(1) Killing vectors have components

$$\begin{aligned} \chi^a &= -ik^{3a} = (0,\ldots,0,1) \ , \qquad & \chi_a &= ik^3_a = \partial_a\chi = \chi(K_i,1) \ , \\ k^+_a &= \mathrm{e}^{2z}(X_i(u),0) \ , & k^+_{\bar{a}} &= 0 \ , \qquad k^-_{\bar{a}} = (k^+_a)^* \ , \end{aligned} \tag{1.9}$$

[1]Since the curvature of the Sp(1) connection is nonzero, we cannot trivialize the Sp(1) bundle. All quantities that tranform under Sp(1), such as $\vec{\mathcal{Q}}$, are defined on this bundle, and not just on the QK base space itself, and consequently are subject to local Sp(1) gauge transformations.

where $X_i(u)$ is a holomorphic one-form on the twistor space. Its curl defines a holomorphic two-form $\omega_{ij} = -\partial_{[i}X_{j]}$; the pair (X, ω) is called a contact structure. The two-form ω lives in odd dimensions and has a unique (up to normalization) holomorphic zero-eigenvalue vector field $Y^i(u) : \omega_{ij}Y^j = 0$. Using $Y^i(u)$, we can distinguish a second special coordinate ζ (with remaining coordinates v^α, $\alpha = 1, \ldots, 2n-2$) by defining

$$Y^i(v,\zeta)\,\frac{\partial}{\partial u^i} \equiv \frac{\partial}{\partial \zeta}\ . \tag{1.10}$$

To descend to the QK subspace, we adopt the gauge-fixing conditions $z = \zeta = 0$. These conditions are imposed on all quantities of interest, such as the QK metric,

$$G_{\alpha\bar\beta} = K_{\alpha\bar\beta} - \mathrm{e}^{-2K} X_\alpha X_{\bar\beta}\ , \tag{1.11}$$

the quaternionic structure,

$$\mathcal{Q}^3_{\alpha\bar\beta} = -iG_{\alpha\bar\beta}\ , \qquad \mathcal{Q}^+_{\alpha\beta} = \mathrm{e}^{-K}\Big(\omega_{\alpha\beta} + 2K_{[\alpha}X_{\beta]}\Big)\ , \tag{1.12}$$

and the Sp(1) connection ($\mathcal{Q}^-$ and $\mathcal{V}^-$ follow by complex conjugation)

$$\mathcal{V}^3_\alpha = -iK_\alpha\ , \qquad \mathcal{V}^+_\alpha = \mathrm{e}^{-K} X_\alpha\ . \tag{1.13}$$

Our conventions are such that for compact QK manifolds, both the HKC and QK metrics are positive definite. For the non-compact QK spaces that one encounters in supergravity, the HKC metric g is pseudo-Riemannian (with a mostly minus signature), and $-G$ is positive definite. This feature will be important in sect. 3.

2 Isometries and moment maps

In this section, we show how triholomorphic isometries and their moment maps descend from the HKC to the QK space. The isometries that commute with supersymmetry generate precisely the group of triholomorphic isometries; they can be gauged by coupling the model to $N = 2$ vector multiplets [8, 9]. A triholomorphic Killing vector k^A on a hyperkähler manifold obeys $\mathcal{L}_k\vec\Omega = 0$, where $\vec\Omega_{AB} = g_{AC}\vec J^C{}_B$ denotes the triplet of closed two-forms on the hyperkähler space. This implies there exists a triplet of moment maps $\vec\mu$, such that

$$\vec\Omega_{AB}k^B = \partial_A\vec\mu \qquad \Leftrightarrow \qquad -ig_{a\bar b}k^{\bar b} = \partial_a\mu^3\ , \quad \Omega^+_{ab}k^b = \partial_a\mu^+\ . \tag{2.1}$$

By taking the covariant derivative of these equations, and using the quaternion algebra of complex structures, one shows that the moment maps are harmonic functions on the hyperkähler manifold:

$$\Delta\vec\mu = 0\ . \tag{2.2}$$

HKC's also have Sp(1) isometries (1.2); they are not triholomorphic but rotate the complex structures and supercharges. These isometries can only be gauged by coupling to the superconformal gravity multiplet, which contains Sp(1) gauge fields. A consistent gauging of both the triholomorphic and the Sp(1) isometries requires them to mutually commute, and to commute with dilatations. This leads to the identities [4]

$$\chi_A \vec{k}^A = 0 \ , \qquad \chi_A k^A = 0 \ , \qquad \vec{k}_A k^A = -2\vec{\mu} \ . \tag{2.3}$$

Note that the last equation defines the moment maps $\vec{\mu}$ algebraically in terms of the Killing vectors; this is a special feature of HKC spaces that has no counterpart on general hyperkähler manifolds. Not surprisingly, it turns out that precisely these isometries descend to isometries of the underlying twistor and QK spaces.

A general analysis of the HKC Killing equation for isometries that commute with the homothety and k^3 shows that they are independent of the coordinate z, and take the form,

$$k^i = -i\mu^i \ , \qquad k^z = i(K_i\mu^i - \mu) \ , \tag{2.4}$$

where $\mu(u,\bar{u})$ is a real function on $\mathcal{Z}$, with $\mu_i = \partial_i \mu$ and $\mu^i = \mu_{\bar{j}} K^{\bar{j}i}$, satisfying

$$D_i \partial_j \mu = 0 \ . \tag{2.5}$$

This implies that the vector (2.4) is holomorphic. The hyperkähler potential χ is invariant and $K(u,\bar{u})$ changes by a Kähler transformation under the isometry, so that the twistor space $\mathcal{Z}$ admits an isometry generated by

$$k_i = i\,\partial_i \mu \, , \tag{2.6}$$

and its complex conjugate. Note that μ is the Kähler moment map of the isometry on $\mathcal{Z}$; the case of constant μ corresponds to the k^3 isometry, which acts trivially on $\mathcal{Z}$. Because the HKC isometry (2.4) is triholomorphic, there is an extra constraint on μ,

$$\mathcal{L}_k X_i \equiv -i\mu^j \partial_j X_i - i\partial_i \mu^j \, X_j = -2i(K_j\mu^j - \mu)X_i \ . \tag{2.7}$$

Using (2.4)-(2.7) one shows that the moment map equations (2.1) are solved explicitly by

$$\mu^3(z,\bar{z},u,\bar{u}) = \chi\,\mu(u,\bar{u}) \ , \qquad \mu^+(z,u) = \tfrac{1}{2} i\,\mathrm{e}^{2z} X_i(u)\,\mu^i(u) \ . \tag{2.8}$$

The integration constant is set to zero, since in a conformally invariant theory, the hyperkähler moment maps must scale with weight two.

The existence of moment maps on a generic hyperkähler space is only guaranteed if the corresponding isometries are triholomorphic. On a QK space, *all* isometries have moment maps. To see this, we introduce real indices $\mathcal{A},\mathcal{B}=1,\ldots,4(n-1)$ on the QK space, and define, for an *arbitrary* Killing vector [6]

$$\vec{\hat{\mu}} \equiv -\frac{1}{4(n-1)}\vec{\mathcal{Q}}^{\mathcal{AB}}D_{\mathcal{A}}\hat{k}_{\mathcal{B}} \ . \tag{2.9}$$

It then follows, by taking the Sp(1) covariant derivative, that

$$\vec{\mathcal{Q}}_{\mathcal{AB}}\,\hat{k}^{\mathcal{B}} = \mathcal{D}_{\mathcal{A}}\,\vec{\hat{\mu}} \ . \tag{2.10}$$

Here we have used the Sp(1) covariant constancy of the quaternionic structure $\mathcal{D}_{\mathcal{A}}\vec{\mathcal{Q}}_{\mathcal{BC}} \equiv D_{\mathcal{A}}\vec{\mathcal{Q}}_{\mathcal{BC}} - \vec{\mathcal{V}}_{\mathcal{A}}\times\vec{\mathcal{Q}}_{\mathcal{BC}} = 0$, the identity $D_{\mathcal{A}}D_{\mathcal{B}}\hat{k}_{\mathcal{C}} = R_{\mathcal{BCAD}}\hat{k}^{\mathcal{D}}$, which holds for any Killing vector, and the well known fact that the Sp(1) holonomy is equal to the Sp(1) curvature: $R_{\mathcal{ABC}}{}^{\mathcal{D}}\,\vec{\mathcal{J}}^{\mathcal{C}}{}_{\mathcal{D}} = 2(n-1)\,\vec{\mathcal{R}}_{\mathcal{AB}}$.

Equation (2.10) is the generalization of the moment map equation for hyperkähler spaces (2.1); it implies that the Killing vector $\hat{k}$ rotates the quaternionic structure according to

$$\mathcal{L}_{\hat{k}}\vec{\mathcal{Q}} = \left(2\vec{\hat{\mu}} + \hat{k}^{\mathcal{A}}\vec{\mathcal{V}}_{\mathcal{A}}\right)\times\vec{\mathcal{Q}} \ . \tag{2.11}$$

Hence, *any* isometry on a QK space has a triplet of moment maps.

Similar manipulations as those leading to (2.2) yield [6]

$$\hat{\Delta}\vec{\hat{\mu}} = -4(n-1)\vec{\hat{\mu}} \ , \tag{2.12}$$

where $\hat{\Delta}$ is now the Sp(1) covariantized Laplacian. Recently, (2.9) and (2.12) were discussed in [10] and [11], respectively.

The HKC Killing vector k descends to the QK space provided we add a compensating Sp(1) transformation to preserve the gauge $\zeta = 0$. The isometry of the QK space is then, in the basis of complex coordinates, [5]

$$\hat{k}^{\alpha} = k^{\alpha} - \frac{k^{\zeta}}{X^{\zeta}}X^{\alpha} = -i\Big[\mu^{\alpha} - \mu^{\zeta}\,\frac{X^{\alpha}}{X^{\zeta}}\Big] \ , \qquad \hat{k}^{\zeta} = 0 \ , \tag{2.13}$$

where $X^i = (X^{\alpha}, X^{\zeta}) = K^{i\bar{\jmath}}X_{\bar{\jmath}}$. In the same coordinates, the moment map equations are

$$\begin{aligned}
\mathcal{Q}^3_{\alpha\bar{\beta}}\,\hat{k}^{\bar{\beta}} &= \partial_{\alpha}\hat{\mu}^3 - 2i\mathcal{V}^+_{\alpha}\,\hat{\mu}^- \ , \\
\mathcal{Q}^+_{\alpha\beta}\,\hat{k}^{\beta} &= \partial_{\alpha}\hat{\mu}^+ + i(\mathcal{V}^+_{\alpha}\,\hat{\mu}^3 - \mathcal{V}^3_{\alpha}\,\hat{\mu}^+) \ , \\
0 &= \partial_{\bar{\alpha}}\hat{\mu}^+ - i\mathcal{V}^3_{\bar{\alpha}}\,\hat{\mu}^+ \ ,
\end{aligned} \tag{2.14}$$

and their complex conjugates. Using some of the identities in (3.17) from [5], (2.7), as well as the explicit form of the Sp(1) connection (1.13) and the quaternionic structure, we find the QK moment maps in terms of the twistor space moment map,

$$\hat{\mu}^3 = \mu \ , \qquad \hat{\mu}^+ = \tfrac{1}{2}i\,\mathrm{e}^{-K}X_i\,\mu^i \ . \tag{2.15}$$

The relation between the HKC and QK moment maps is then simply

$$\mu^3 = \chi\,\hat{\mu}^3\;, \qquad \mu^+ = \mathrm{e}^{z-\bar{z}}\chi\,\hat{\mu}^+\;. \tag{2.16}$$

One can verify that a set of HKC Killing vectors associated with a Lie algebra **g** induce corresponding QK Killing vectors that generate the same algebra **g**.

3 Gauging and the scalar potential

We now turn to the scalar potential that arises after gauging the isometries of a hypermultiplet action. This potential depends on the spacetime dimension because the field content of the vector multiplets associated with the gauging varies with the dimension. We focus on the case of $d = 4$, but generalizations to $d = 5$ are straightforward. The idea is to start from the scalar potential of the rigidly supersymmetric hypermultiplet action with a HKC target space and to derive the corresponding potential after coupling to supergravity; in this coupling the hypermultiplet target space is converted into a QK space.

On the HKC, the scalar potential arises after gauging triholomorphic isometries by introducing $N = 2$ vector multiplets coupled superconformally to the hypermultiplets. We give the scalar kinetic terms and the auxiliary field terms of the rigidly superconformally invariant vector multiplet Lagrangian [12, 13] (we use the notation of [14]),

$$\mathcal{L} = i(\mathcal{D}_\mu F_I\,\mathcal{D}^\mu \bar{X}^I - \mathcal{D}_\mu \bar{F}_I\,\mathcal{D}^\mu X^I) + \tfrac{1}{16} N_{IJ}\,\vec{Y}^I\cdot\vec{Y}^J\,. \tag{3.1}$$

We remind the reader that the action is expressed in terms of a holomorphic function $F(X)$ in the complex scalar fields X^I; superconformal invariance requires $F(X)$ to be homogeneous of second degree. The index I labels the vector multiplets and subscripts I, J denote derivatives with respect to X^I, X^J. Because the vector multiplet action itself plays an ancillary role in what follows, we convert from special coordinates to holomorphic sections only at the end of this section. The derivatives $\mathcal{D}_\mu$ are gauge covariant derivatives associated with a nonabelian gauge group. The scalars parametrize a Kähler manifold with metric

$$N_{IJ} = -iF_{IJ} + i\bar{F}_{IJ}\;, \tag{3.2}$$

(so that the scalar kinetic terms take the form $-N_{IJ}\,\mathcal{D}_\mu X^I\,\mathcal{D}^\mu \bar{X}^J$). The vector multiplets contain SU(2) triplets of auxiliary fields denoted by $\vec{Y}^I$. For a nonabelian gauge group, the action also contains a scalar potential that is quadratic in both X^I and $\bar{X}^I$, which we include below.

The hypermultiplets have a kinetic term and a potential equal to (see *e.g.* [4], where one can also find the corresponding expressions in terms of sections of the $\mathrm{Sp}(n)\times\mathrm{Sp}(1)$

bundle; this is convenient when including the fermions),

$$\mathcal{L} = -\tfrac{1}{2} g_{AB} \mathcal{D}_\mu \phi^A \, \mathcal{D}^\mu \phi^B - 2g^2 \, g_{AB} \, k_I^A k_J^B \, X^I \bar{X}^J + \tfrac{1}{2} g \, \vec{Y}^I \cdot \vec{\mu}_I \,, \tag{3.3}$$

where g denotes the coupling constant and N^{IJ} is the inverse of (3.2). This form of the Lagrangian holds for arbitrary hyperkähler target spaces. In six dimensions the $X^I \bar{X}^J$ term is absent because there are no scalar fields in the vector multiplet. In five dimensions, the $X^I \bar{X}^J$ fields are replaced by the real scalar fields of the $d = 5$ vector multiplet. The potentials in various dimensions are of course related by dimensional reduction [8].

After eliminating the auxiliary fields of the vector multiplets we obtain the scalar potential,

$$\mathcal{L}^{\rm scalar} = -g^2 \Big(2 g_{AB} \, k_I^A k_J^B \, X^I \bar{X}^J + N^{IJ} \vec{\mu}_I \cdot \vec{\mu}_J - N_{IJ} \, f_{KL}^I X^K \bar{X}^L \, f_{MN}^J X^M \bar{X}^N \Big) \,, \tag{3.4}$$

where, for completeness, we include the potential for the nonabelian vector multiplets; the structure constants of the nonabelian gauge group are denoted by f_{JK}^I. Here and henceforth we assume that the holomorphic function $F(X)$ is gauge invariant. The nonabelian gauge transformations define holomorphic isometries of the Kähler space; their corresponding Kähler moment maps are equal to $\nu_I = f_{IK}^J (F_J \bar{X}^K + \bar{F}_J X^K)$. Observe that when g_{AB} and N_{IJ} are positive definite, the potential (3.4) is nonnegative, as it should be.

When we couple the vector multiplets and hypermultiplets to conformal supergravity, some of their component fields will act as compensating fields and can be gauged away. Furthermore the $\mathrm{SU}(2) \times \mathrm{U}(1)$ gauge fields of conformal supergravity are integrated out. For the vector multiplets this implies that one is taking a Kähler quotient upon which one obtains a *special* Kähler space; for the hypermultiplets the procedure amounts to performing the $N = 2$ superconformal quotient, so that the resulting target space is QK. However, to end up with a Lagrangian that has the proper signs for the Einstein-Hilbert Lagrangian and for the kinetic terms of the various fields, in particular for the scalars of the vector multiplets and hypermultiplets, the Kähler and hyperkähler metrics that we start from cannot be positive definite. The compensating scalar fields (two from the vector multiplets and four from the hypermultiplets) should appear with opposite sign. With our conventions, the easiest thing is to change the overall sign of the full Lagrangian. The resulting special Kähler and the QK metrics are then negative definite.

First we consider the kinetic terms of the combined Lagrangian. We covariantize the derivatives with respect to the (bosonic) superconformal symmetries,

$$\begin{aligned} \mathcal{D}_\mu X^I &= \partial_\mu X^I - (b_\mu - i A_\mu) X^I - g \, f_{JK}^I \, W_\mu^J X^K \,, \\ \mathcal{D}_\mu \phi^A &= \partial_\mu \phi^A - \chi^A \, b_\mu + \tfrac{1}{4} \vec{k}^A \cdot \vec{V}_\mu - g \, k_I^A \, W_\mu^I \,, \end{aligned} \tag{3.5}$$

where A_μ and $\vec{V}_\mu$ are the gauge fields associated with $\mathrm{SU}(2)\times\mathrm{U}(1)$ and b_μ is the gauge field associated with local scale transformations. Eliminating the gauge fields A_μ and $\vec{V}_\mu$ by their field equations (explicit solutions will be presented below) and setting $b_\mu = 0$ by a gauge transformation, one obtains the result,

$$\begin{aligned} e^{-1}\mathcal{L}^{\rm kin} \;=\;& N_{KL}X^K\bar{X}^L\,\mathcal{M}_{I\bar{J}}\,\mathcal{D}_\mu X^I\,\mathcal{D}^\mu\bar{X}^J + \tfrac{1}{2}\chi\,G_{AB}\mathcal{D}_\mu\phi^A\,\mathcal{D}^\mu\phi^B \\ & -X^I N_{IJ}\bar{X}^J\left[\tfrac{1}{6}R - \tfrac{1}{4}(\partial_\mu \ln[X^I N_{IJ}\bar{X}^J])^2\right] - \chi\left[\tfrac{1}{6}R - \tfrac{1}{4}(\partial_\mu\ln\chi)^2\right] \\ & +D\Big[X^I N_{IJ}\bar{X}^J - \tfrac{1}{2}\chi\Big]\,, \end{aligned} \tag{3.6}$$

where the terms proportional to the Ricci scalar originate from the covariantization with respect to conformal boosts and D is an auxiliary field which acts as a Lagrange multiplier and imposes the condition,

$$\chi = 2X^I N_{IJ}\bar{X}^J\ . \tag{3.7}$$

The covariant derivatives depend only on the gauge fields W_μ^I of the vector multiplets. The horizontal metric associated with the special Kähler space is

$$\mathcal{M}_{I\bar{J}} = \frac{1}{[N_{MN}X^M\bar{X}^N]^2}\Big[N_{IJ}N_{KL} - N_{IK}N_{JL}\Big]\bar{X}^K X^L \tag{3.8}$$

Subsequently we absorb the factor χ into the vierbein by rescaling $e_\mu^{\,a}\to\sqrt{2/\chi}\,e_\mu^{\,a}$. Imposing the gauge conditions for the QK target space, the result for (3.6) reads,

$$e^{-1}\mathcal{L}^{\rm kin} = -\tfrac{1}{2}R + \mathcal{M}_{I\bar{J}}\,\mathcal{D}_\mu X^I\,\mathcal{D}^\mu\bar{X}^J + 2G_{\alpha\bar{\beta}}\mathcal{D}_\mu\phi^\alpha\,\mathcal{D}^\mu\phi^{\bar{\beta}}\,, \tag{3.9}$$

where the derivatives are covariant with respect to the gauged isometry group. Note that the fields ϕ^α coincide with the QK coordinates v^α introduced above (1.10).

The potential can now be evaluated straightforwardly, using the results derived in sect. 2. Absorbing the factor χ into the vierbein as before leads to the following expression for $\mathcal{L}^{\rm scalar}$,

$$\begin{aligned} e^{-1}\mathcal{L}^{\rm scalar} \;=\;& 4g^2\left[2\,G_{\alpha\bar{\beta}}\,\hat{k}^\alpha_{(I}\hat{k}^{\bar{\beta}}_{J)} + 3\,\vec{\hat{\mu}}_I\cdot\vec{\hat{\mu}}_J\right]\,\frac{X^I\bar{X}^J}{N_{MN}X^M\bar{X}^N} \\ & +g^2\,N_{MN}X^M\bar{X}^N\,\mathcal{M}_{I\bar{J}}\,\left[4\,N^{IK}N^{JL}\vec{\hat{\mu}}_K\cdot\vec{\hat{\mu}}_L - \frac{f_{KL}^I X^K\bar{X}^L}{N_{PQ}X^P\bar{X}^Q}\,\frac{f_{MN}^J X^M\bar{X}^N}{N_{PQ}X^P\bar{X}^Q}\right]\,. \end{aligned} \tag{3.10}$$

The resulting potential is no longer positive definite. Using that $G_{\alpha\bar{\beta}}$ and $\mathcal{M}_{I\bar{J}}$ are negative definite, as explained earlier, we see that the term proportional to $|X^I\,\vec{\hat{\mu}}_I|^2$ is negative whereas the others are positive (note that χ must be positive). For an early discussion of this, see *e.g.*, [12].

Our form of the potential (3.10) is equivalent to earlier results [15], obtained by other methods. To appreciate the relation, we briefly discuss some aspects of special Kähler geometry (coordinatized by the vector multiplet scalars X^I). The Lagrangians (3.9) and (3.10) do not depend on the X^I but rather on their ratios, and hence the geometry is projective. Consequently, we can replace the X^I by holomorphic sections $X^I(z)$, defined modulo multiplication by an arbitrary holomorphic function of the coordinates $z^{\hat{I}}$, *i.e.*, $X^I(z) \to \exp[f(z)]\, X^I(z)$. Observe that this projective invariance includes the U(1) invariance, for which we have, so far, not imposed a gauge condition. At this point one may convert to the formulation in terms of these holomorphic sections of the complex line bundle of special geometry [16, 17]. The number of independent holomorphic coordinates $z^{\hat{I}}$ is one less than the number of X^I, and this difference can be traced back to the scale and U(1) invariance of the original formulation. The special Kähler metric based on the coordinates z has a Kähler potential[2]

$$\mathcal{K}(z,\bar{z}) = -\ln\Big[i\bar{X}^I(\bar{z})\, F_I(X(z)) - i\bar{F}_I(\bar{X}(\bar{z}))\, X^I(z)\Big]\,, \tag{3.11}$$

which, under the projective transformations, changes by a Kähler transformation: $\mathcal{K} \to \mathcal{K} - f(z) - \bar{f}(\bar{z})$. The potential (3.10) is then expressed in terms of projective invariants, such as $X^I(z)\,\bar{X}^J(\bar{z})\,\exp[\mathcal{K}(z,\bar{z})]$, and in terms of the (inverse) special Kähler metric $\mathcal{K}^{\hat{I}\hat{\bar{J}}}$ via the relation (see, *e.g.* [19])

$$N^{KL} = \mathrm{e}^{\mathcal{K}}\Big[\mathcal{K}^{\hat{I}\hat{\bar{J}}}(\partial_{\hat{I}} + \partial_{\hat{I}}\mathcal{K})X^K(z)\,(\partial_{\hat{\bar{J}}} + \partial_{\hat{\bar{J}}}\mathcal{K})\bar{X}^L(\bar{z}) - X^K(z)\bar{X}^L(\bar{z})\Big]\ . \tag{3.12}$$

We note that for the hypermultiplet sector there also exists a formulation in terms of local sections, in this case of an $\mathrm{Sp}(1)\times\mathrm{Sp}(n)$ bundle. The existence of this associated quaternionic bundle is known from general arguments [2] and was explained for supersymmetric models in [4].

Let us briefly return to the expressions for the $\mathrm{SU}(2)\times\mathrm{U}(1)$ gauge fields, which pick up certain corrections associated with the gauging. These corrections show up in the coupling to the gravitini, which are related by supersymmetry to the fermion mass terms. The corresponding expressions read,

$$\begin{aligned} A_\mu &= -\tfrac{1}{2}[N_{MN}X^M\bar{X}^N]^{-1}(\bar{F}_I \overleftrightarrow{\partial}_\mu X^I - \bar{X}^I \overleftrightarrow{\partial}_\mu F_I) + g\, W_\mu^I\, \hat{\nu}_I\,, \\ \vec{V}_\mu &= -2\,\partial_\mu\phi^{\mathcal{A}}\,\vec{\mathcal{V}}_{\mathcal{A}} - 4g\, W_\mu^I\, \vec{\hat{\mu}}_I\,, \end{aligned} \tag{3.13}$$

[2] As is well known, the sections $X^I(z)$ can be combined with $F_I(z)$ into sections that transform covariantly under electric-magnetic duality. However, the duality is affected by the gauging, as can be seen from the fact that the various terms in the potential and in the moment maps (see below) are not symplectically invariant. Although we started from a holomorphic function $F(X)$, it is sufficient that only symplectic sections $(X^I(z), F_I(z))$ exist in a way that is consistent with the requirements of special geometry [18].

where we have imposed the gauge conditions on the HKC and used the definition for the special Kähler moment map, $\hat{\nu}_I = \nu_I [N_{MN} X^M \bar{X}^N]^{-1}$. Observe that this expression is indeed consistent with the projective invariance associated with the sections $X^I(z)$ (but it is *not* consistent with electric-magnetic duality). The HKC and QK moment maps can be expressed explicitly in terms of the associated bundles. The relevant expressions are given in eqs. (3.35) and (5.7) of [4].

4 The unitary Wolf spaces

As an example we construct the moment maps for the QK unitary Wolf spaces,

$$X(n-1) = \frac{\mathrm{U}(n-1,2)}{\mathrm{U}(n-1)\times \mathrm{U}(2)} \ . \tag{4.1}$$

These spaces have real dimension $4(n-1)$. The universal hypermultiplet corresponds to $n=2$; moment maps and gaugings of abelian isometries were discussed in four dimensions in [20], whereas a more general gauging in five dimensions appeared in [21, 10]. Our construction not only gives a more uniform treatment of all moment maps, but is also applicable for $n \geq 2$ and other QK spaces.

As explained in [5], the HKC above $X(n-1)$ can be constructed from a hyperkähler quotient [22] of $\mathbf{C}^{2n+2}$ with respect to a U(1) isometry which acts with opposite phases on the homogeneous coordinates z_+^M, z_{-M}, with $M=1,\ldots,n+1$, of $\mathbf{C}^{2n+2}$. On these coordinates there is a *linear* action of $\mathrm{SU}(n-1,2)$:

$$z_+^M \to U^M{}_N \, z_+^N \ , \qquad z_{-M} \to (U^{-1})^N{}_M \, z_{-N} \ . \tag{4.2}$$

Clearly, these transformations define triholomorphic isometries. The reality condition on the $\mathrm{SU}(n-1,2)$ matrices is

$$\bar{U}^{\bar{M}}{}_{\bar{N}} = \eta^{\bar{M}M} (U^{-1})^N{}_M \, \eta_{N\bar{N}} \ , \tag{4.3}$$

where $\eta_{M\bar{N}} = \mathrm{diag}(-\cdots-++)$.

The U(1) hyperkähler quotient imposes the holomorphic moment map condition $z_{-M} z_+^M = 0$; we solve this in a specific gauge:

$$z_+^{n+1} = 1 \ , \qquad z_{-n+1} = -z_+^a z_{-a} \ . \tag{4.4}$$

One then finds the hyperkähler potential for the HKC of $X(n-1)$ [5]:

$$\chi_{2n} = 2\chi_+ \chi_- \ , \tag{4.5}$$

where

$$\chi_+ = \sqrt{\eta_{M\bar{N}} z_+^M \bar{z}_+^{\bar{N}}} \ , \qquad \chi_- = \sqrt{\eta^{M\bar{N}} z_{-M} \bar{z}_{-\bar{N}}} \ . \tag{4.6}$$

The triholomorphic SU$(n-1{,}2)$ transformations on the HKC above $X(n-1)$ were given in [5]; in infinitesimal form, with $U = \mathbf{1} + i\theta^I T_I$; $I = 1, \ldots, n(n+2)$, they determine the HKC Killing vectors $(\delta z_+^M, \delta z_{-M}) = \theta^I(k_{I+}^M, k_{I\,-M})$,

$$\begin{aligned} k_{I+}^M &= i(T_I)^M{}_N\, z_+^N - iz_+^M (T_I)^{n+1}{}_N\, z_+^N \ , \\ k_{I\,-M} &= -i(T_I)^N{}_M\, z_{-N} + iz_{-M}(T_I)^{n+1}{}_N\, z_+^N \ . \end{aligned} \tag{4.7}$$

Notice the compensating terms needed to preserve the conditions (4.4).

The moment maps on the HKC now simply follow from (2.3) and (1.9) (with the holomorphic one-form $k^+ = \mathrm{e}^{2z}X = 2z_{-M}\mathrm{d}z_+^M$):

$$\begin{aligned} \mu_I^3 &= \frac{2i}{\chi}\Big[\chi_-^2(\bar{z}_+^{\bar{M}}\eta_{N\bar{M}}k_{I+}^N) + \chi_+^2(\bar{z}_{-\bar{M}}\eta^{\bar{M}N}k_{I-N})\Big] \ , \\ &= -\frac{2}{\chi}\Big[\chi_-^2(z_+^M(T_I)^N{}_M\,\eta_{N\bar{M}}\,\bar{z}_+^{\bar{M}}) - \chi_+^2(z_{-M}(T_I)^M{}_N\,\eta^{N\bar{M}}\bar{z}_{-\bar{M}})\Big] \ , \\ \mu_I^+ &= -z_{-M}k_{I+}^M \ = \ -iz_{-M}(T_I)^M{}_N\, z_+^N \ , \end{aligned} \tag{4.8}$$

subject to the constraints (4.4).

Finally, we determine the moment maps and Killing vectors on $X(n-1)$; we set

$$z_{-n} = \mathrm{e}^{2z} \ , \qquad z_{-i} = \mathrm{e}^{2z}w_i \ , \qquad 2z_+^n = \zeta \ , \qquad z_+^i = v^i \ , \tag{4.9}$$

where $a = (i, n)$ with $i = 1, \cdots, n-1$. The coordinates on $X(n-1)$ are then $v^\alpha = \{w_i, v^i\}$. The moment maps follow from the HKC moment maps (4.8) using (2.16). The QK Killing vectors are obtained from (2.13), so we must first compute the compensating Sp(1) transformation associated with X^α/X^ζ. These follow from the Kähler potential on the twistor space

$$K(v,w,\zeta,\bar{v},\bar{w},\bar{\zeta}) = \ln\Big[2\,\chi_+(v,\zeta,\bar{v},\bar{\zeta})\ \chi_-(v,w,\zeta,\bar{v},\bar{w},\bar{\zeta})\Big] \ , \tag{4.10}$$

with

$$\chi_+ = \sqrt{1 + \tfrac{1}{4}\zeta\bar{\zeta} + \eta_{i\bar{j}}v^i\bar{v}^j} \ , \qquad \chi_- = \sqrt{1 + |\tfrac{1}{2}\zeta + v^i w_i|^2 + \eta^{i\bar{j}}w_i\bar{w}_j} \ , \tag{4.11}$$

where $\eta_{i\bar{j}} = -\delta_{i\bar{j}}$. Applying the general formula (4.16) from [5] yields

$$\frac{X^{w_i}}{X^\zeta} = -\tfrac{1}{2}\eta_{i\bar{j}}\,\bar{v}^j\,\frac{\chi_-^2}{\chi_+^2} \ , \qquad \frac{X^{v^i}}{X^\zeta} = \tfrac{1}{2}(\eta^{i\bar{j}}\bar{w}_j + \bar{w}_k\bar{v}^k v^i) \ . \tag{4.12}$$

Moreover, using (henceforth we suppress the index I for the generators)

$$\begin{aligned} k^\zeta|_{z=\zeta=0} &= 2i\left[T^n{}_{n+1} + T^n{}_i\, v^i\right] \ , \\ k^z|_{z=\zeta=0} &= \tfrac{1}{2}i\left[T^{n+1}{}_n\,(w_i v^i) - T^n{}_n - T^i{}_n\, w_i + T^{n+1}{}_{n+1} + T^{n+1}{}_i\, v^i\right] \ , \end{aligned} \tag{4.13}$$

we get the following expression for the QK Killing vectors:

$$
\begin{aligned}
\hat{k}^{v^i} &= \left[iT^i{}_j - iv^iT^{n+1}{}_j\right] v^j + iT^i{}_{n+1} - iv^iT^{n+1}{}_{n+1} - \tfrac{1}{2}k^\zeta\left(\eta^{i\bar{j}}\bar{w}_j + v^i\bar{w}_k\bar{v}^k\right) \ , \\
\hat{k}^{w_i} &= -\left[iT^j{}_i - iv^jT^{n+1}{}_i\right] w_j - iT^n{}_i + w_i\left[iT^{n+1}{}_{n+1} + iT^{n+1}{}_j\, v^j\right] \\
&\quad -2k^z\, w_i + \frac{\chi_-^2}{2\chi_+^2}k^\zeta\, \eta_{i\bar{j}}\,\bar{v}^j \ .
\end{aligned}
\tag{4.14}
$$

As a specific illustration, we gauge a sample isometry for the universal hypermultiplet. Its QK metric is given by

$$
\begin{aligned}
G_{w\bar{w}} &= -\frac{1-v\bar{v}}{2\left[1-w\bar{w}(1-v\bar{v})\right]^2} \ , \\
G_{w\bar{v}} &= \frac{\bar{w}v}{2\left[1-w\bar{w}(1-v\bar{v})\right]^2} \ , \\
G_{v\bar{v}} &= -\frac{1-w\bar{w}(1-v\bar{v})^2}{2\left[1-w\bar{w}(1-v\bar{v})\right]^2[1-v\bar{v}]^2} \ .
\end{aligned}
\tag{4.15}
$$

We choose the generator

$$
T = \tfrac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ . \tag{4.16}
$$

On the HKC we then find,

$$
k^z = \tfrac{1}{2}i \ , \qquad k^\zeta = -i\zeta \ , \qquad k^v = 0 \ , \qquad k^w = -iw \ , \tag{4.17}
$$

with corresponding moment maps

$$
\mu^3 = \frac{1}{\chi}\left[\tfrac{1}{2}\chi_-^2\,\zeta\bar{\zeta} - 2\chi_+^2\,\mathrm{e}^{2(z+\bar{z})}\right] \ , \qquad \mu^+ = \tfrac{1}{2}i\,\mathrm{e}^{2z}\zeta \ . \tag{4.18}
$$

Because $k^\zeta|_{\zeta=0} = 0$ in this case, the QK Killing vector is

$$
\hat{k}^v = 0 \ , \qquad \hat{k}^w = -iw \ . \tag{4.19}
$$

The fact that this vector is an isometry of the metric (4.15) can easily be checked by direct calculation. The corresponding QK moment maps are

$$
\hat{\mu}^3 = -\tfrac{1}{2}\frac{1}{\left[1-w\bar{w}(1-v\bar{v})\right]} \ , \qquad \hat{\mu}^+ = 0 \ . \tag{4.20}
$$

Note that, due to the gauge choice $\zeta = 0$, $\hat{\mu}^+$ vanishes on the QK space, but not on the HKC. This is not surprising; because the QK moment maps are sections of the Sp(1) bundle, one can always employ a local Sp(1) transformation to make $\hat{\mu}^\pm$ vanish. This is in contrast with triholomorphic isometries on hyperkähler spaces, where the

Sp(1) bundle is trivial and a vanishing moment map would imply that the Killing vector vanishes. This observation clarifies the supersymmetric low-energy effective dynamics of the Higgs branch (at scales below the masses of the gauge fields W_μ^I): when some of the moment maps vanish, one might worry that the QK quotient [6] becomes degenerate. However, since we can lift everything to the HKC where this problem does not arise, it is guaranteed that this will not happen. Therefore, the geometry of the light scalar-field manifold is best understood from the perspective of the HKC.

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