

Instanton Solutions for the Universal Hypermultiplet[†]

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Abstract

We expand our previous analysis on fivebrane and membrane instanton solutions in the universal hypermultiplet, including near-extremal multi-centered solutions and mixed five-brane-membrane charged instantons. The results are most conveniently described in terms of a double-tensor multiplet.

1 Introduction

Low energy effective actions for type II strings on a Calabi-Yau (CY) threefold are determined by $D = 4$ $N = 2$ supergravity actions coupled to vector- and hypermultiplets, or multiplets that can be dualized into these. The number of such multiplets depends on the Hodge numbers of the CY, and is respectively $h_{1,1}$ ($h_{1,2}$) and $h_{1,2} + 1$ ($h_{1,1} + 1$) for type IIA(B). The hypermultiplet moduli space is quaternion-Kähler and contains the dilaton. Therefore this space receives quantum corrections, both perturbatively and non-perturbatively. In this note, we focus on rigid ($h_{1,2} = 0$) CY threefold compactifications of type IIA, or its mirror version in type IIB, understood in terms of a Landau-Ginzburg

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orbifold [1]. Then there is only a single hypermultiplet, the universal hypermultiplet, whose tree level effective action is determined by the quaternion-Kähler target space [2, 3]

$$\mathcal{M}_H = \frac{\text{SU}(1, 2)}{\text{U}(2)} . \quad (1.1)$$

This particular coset space is also Kähler, and there exist complex coordinates S and C in which the Kähler potential is given by

$$K = -\ln(S + \bar{S} - 2C\bar{C}) . \quad (1.2)$$

An alternative parametrization of the universal hypermultiplet in terms of four real scalars is

$$e^{-1}\mathcal{L}_{\text{UH}} = -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}e^{-\phi}(\partial^\mu\chi\partial_\mu\chi + \partial^\mu\varphi\partial_\mu\varphi) - \frac{1}{2}e^{-2\phi}(\partial_\mu\sigma + \chi\partial_\mu\varphi)^2 . \quad (1.3)$$

The relation with the complex coordinates above is given by

$$\begin{aligned} e^\phi &= \frac{1}{2}(S + \bar{S} - 2C\bar{C}) & \chi &= C + \bar{C} , \\ \sigma &= \frac{i}{2}(S - \bar{S} + C^2 - \bar{C}^2) & \varphi &= -i(C - \bar{C}) . \end{aligned} \quad (1.4)$$

Recently, the perturbative analysis of [4, 5] was revisited in [6], and it was shown that non-trivial perturbative quantum corrections only appear at one-loop. We will not discuss these corrections here, but only mention that they were found by studying deformations of (1.1) that preserve the Heisenberg subgroup of the classical group of $\text{SU}(1, 2)$ isometries

$$S \rightarrow S + i\alpha + 2\bar{\epsilon}C + |\epsilon|^2 , \quad C \rightarrow C + \epsilon . \quad (1.5)$$

The parameters α and ϵ are real and complex respectively and, under the assumption that these transformations do not receive any quantum corrections, generate the symmetries that are preserved in string perturbation theory [4]. In the basis of the real variables (1.4), the Heisenberg algebra of infinitesimal transformations is generated by

$$\delta\phi = 0 , \quad \delta\chi = \epsilon + \bar{\epsilon} , \quad \delta\varphi = -i(\epsilon - \bar{\epsilon}) , \quad \delta\sigma = -\alpha - (\epsilon + \bar{\epsilon})\varphi . \quad (1.6)$$

One can therefore choose a set of two commuting isometries, corresponding to the parameters α and $\epsilon = i\beta$ (with β real), to dualize the pseudoscalars φ and σ into two tensors, using the known Legendre transformation techniques. In such a double-tensor multiplet formulation, perturbative corrections are highly constrained due to the rather restrictive couplings of scalar-tensor systems with $N = 2$ supersymmetry [7].

After dualization, the resulting tree level double-tensor multiplet Lagrangian reads [8, 7]

$$e^{-1}\mathcal{L}_{\text{DT}} = -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}e^{-\phi}\partial^\mu\chi\partial_\mu\chi + \frac{1}{2}M^{IJ}H_I^\mu H_{\mu J} , \quad (1.7)$$

where the H_I are a pair of three-form field strengths, $H_I^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\partial_\nu B_{\rho\sigma I}$, and

$$M = e^\phi \begin{pmatrix} 1 & -\chi \\ -\chi & e^\phi + \chi^2 \end{pmatrix}. \quad (1.8)$$

The two scalars ϕ and χ parameterize the coset $\text{SL}(2, \mathbb{R})/\text{O}(2)$. The presence of the tensors breaks the $\text{SL}(2, \mathbb{R})$ symmetries to a two-dimensional subgroup generated by a certain rescaling of the fields and by the remaining generator of the Heisenberg algebra (1.6) with real parameter $\epsilon = \gamma/2$. It acts as a shift on χ and transforms the tensors linearly into each other [8],

$$\chi \rightarrow \chi + \gamma, \quad B_1 \rightarrow B_1 + \gamma B_2, \quad (1.9)$$

with ϕ and B_2 invariant. An invariant combination is then the tensor $\hat{H}_1 = H_1 - \chi H_2$, and we will call this the Heisenberg invariant. An off-shell superspace formulation of the Lagrangian (1.7) was given in [9], in terms of a single function satisfying a linear second order differential equation. In that paper, the superconformal calculus was used to write down the action. Upon gauge-fixing the conformal symmetries, it yields (1.7) or, equivalently, the action given in terms of the Calderbank-Pedersen variables [10].

In this note, we elaborate on [8], and investigate the non-perturbative effects that contribute to the low energy effective action. As explained in [11], these arise from Euclidean branes wrapped around supersymmetric cycles in the CY. In the case of IIA with $h_{1,2} = 0$, the NS-fivebrane can wrap the entire CY, or the D2-brane can wrap a non-trivial three-cycle inside the CY. From the four-dimensional point of view, such configurations are localized in space and time and correspond to fivebrane and membrane instantons of the $N = 2$ $D = 4$ supergravity action.

In [8], following previous work of [12], the Bogomol'nyi equations were derived from (1.7). The solutions were shown to describe fivebrane and membrane-like instantons, and we review and extend this below. The natural description of these instantons was given in terms of the Euclidean continuation of the double-tensor multiplet action. This has the advantage over the Euclidean universal hypermultiplet that the Euclidean action is semi-positive definite and hence it justifies the semi-classical approximation [8]. Perturbatively, the double-tensor multiplet guarantees $\text{U}(1) \times \text{U}(1)$ isometries in the dual hypermultiplet description. Non-perturbatively, however, the duality is expected to involve also the constant modes of the dual scalars φ and σ by means of theta-angle-like terms. Such terms break the $\text{U}(1)$ isometries to a discrete subgroup, see e.g. [13, 14].

2 Fivebrane Instantons

The first Bogomol'nyi equation that can be derived from the double-tensor multiplet action is given by [8]

$$\begin{pmatrix} H_{\mu 1} \\ H_{\mu 2} \end{pmatrix} = \pm \partial_\mu \begin{pmatrix} e^{-\phi} \chi \\ e^{-\phi} \end{pmatrix}, \quad (2.1)$$

where the plus and minus signs refer to instantons and anti-instantons, respectively. The closure of the three-form field strengths then implies Laplace-like equations for the scalars. On a flat spacetime \mathcal{M} with points $\{x_i\}$ excised from \mathbb{R}^4 , we find the multi-centered solutions in terms of two harmonic functions,

$$e^{-\phi} = e^{-\phi_\infty} + \sum_i \frac{|Q_{2i}|}{4\pi^2 (x - x_i)^2}, \quad e^{-\phi} \chi = e^{-\phi_\infty} \chi_\infty + \sum_i \frac{Q_{1i}}{4\pi^2 (x - x_i)^2}, \quad (2.2)$$

where Q_{1i} , Q_{2i} , χ_∞ , and ϕ_∞ are independent integration constants; the latter two determine the asymptotic values of the fields at infinity. We wrote the absolute value of Q_{2i} to make $e^{-\phi}$ positive everywhere in space, and we identify the string coupling constant via $g_s = e^{-\phi_\infty/2}$. Furthermore, two charges are defined by integrating the tensor field strengths $H_{\mu\nu\rho I} = -\varepsilon_{\mu\nu\rho\sigma} H_I^\sigma$ over 3-spheres at infinity,

$$Q_I = \int_{S_\infty^3} H_I, \quad I = 1, 2. \quad (2.3)$$

They are related to the constants appearing in the scalar fields through the field equation (2.1). Using $** = -1$ on a three-form in four Euclidean dimensions, we find

$$Q_2 = \mp \sum_i |Q_{2i}|, \quad Q_1 = \mp \sum_i Q_{1i}. \quad (2.4)$$

This implies that for instantons, Q_2 should be taken negative, whereas for anti-instantons, Q_2 must be positive. Note that there is no restriction on the sign of the Q_{1i} .

The (anti-) instanton action for the fivebrane can be computed using the formulas in [8]. It is finite only if χ remains finite near the excised points x_i . In the limit, we find

$$\chi_i \equiv \lim_{x \rightarrow x_i} \chi(x) = \frac{Q_{1i}}{|Q_{2i}|}, \quad (2.5)$$

which is finite whenever $Q_{2i} \neq 0$ for nonvanishing Q_{1i} . This implies that the integrated Heisenberg invariants vanish, $\hat{Q}_{1i} \equiv Q_{1i} - \chi_i |Q_{2i}| = 0$. Plugging the solution into the action, we find

$$S_{\text{inst}} = \frac{|Q_2|}{g_s^2} + \frac{1}{2} \sum_i |Q_{2i}| (\chi_\infty - \chi_i)^2. \quad (2.6)$$

The quadratic dependence on the string coupling constant is precisely what corresponds to a wrapped NS-fivebrane [11].

For a single-centered instanton around x_0 , this reduces to

$$S_{\text{inst}} = \frac{|Q_2|}{g_s^2} \left(1 + \frac{1}{2} g_s^2 (\Delta\chi)^2 \right) , \quad (2.7)$$

where $\Delta\chi = \chi_\infty - \chi_0$ is the difference between the values of the R-R scalar at the boundaries $S_\infty^3 \cup S_0^3$ of $\mathbb{R}^4 - \{x_0\}$. It would be interesting to have a better string theoretic interpretation of this term.

An important issue is whether our solutions (2.2) preserve half of the supersymmetry. This question is analyzed in detail in [15]. It turns out that the spherically symmetric solutions are BPS in the sense of preserving four supercharges. Interestingly, the general multi-centered solution satisfying the Bogomol'nyi bound (2.1) is not, and imposing the BPS condition leads to the further restriction that all the χ_i must be equal. This implies that the solution is characterized in terms of a single harmonic function, given by $e^{-\phi}$. When all χ_i are equal, the value of the action is lowered and coincides with the spherically symmetric case (2.7). A possible interpretation is that the general multi-centered solution is metastable and decays into the state where all χ_i are equal. In such a state, the points x_i can be brought together to a spherically symmetric configuration without changing the action. We call the general solution (2.2) near-extremal when the values of $\chi_\infty - \chi_i$ are all close to the lowest value $\Delta\chi$.

3 Membrane Instantons

The second Bogomol'nyi equation that can be derived from (1.7) contains an arbitrary constant χ_0 [8],

$$\begin{pmatrix} H_{\mu 1} \\ H_{\mu 2} \end{pmatrix} = \pm \frac{1}{|\tau'|} \begin{pmatrix} \chi(\chi - \chi_0) \partial_\mu e^{-\phi} + e^{-\phi} (\chi + \chi_0) \partial_\mu \chi + 2e^\phi \partial_\mu e^{-\phi} \\ (\chi - \chi_0) \partial_\mu e^{-\phi} + 2e^{-\phi} \partial_\mu \chi \end{pmatrix} , \quad (3.1)$$

where $\tau' = (\chi - \chi_0) + 2ie^{\phi/2}$. Below, for the single-centered solution, we will identify χ_0 with the value of χ at the excised point. To solve these equations, we first consider the combination

$$H_{\mu 1} - \chi_0 H_{\mu 2} = \pm \partial_\mu (e^{-\phi} |\tau'|) . \quad (3.2)$$

The Bianchi identities then imply that $h = e^{-\phi} |\tau'|$ is harmonic and positive everywhere. With χ expressed in terms of h and ϕ ,

$$\chi - \chi_0 = e^\phi \sqrt{h^2 - 4e^{-\phi}} , \quad (3.3)$$

the condition for $H_{\mu 2}$ turns into

$$H_{\mu 2} = \pm \frac{1}{\sqrt{h^2 - 4e^{-\phi}}} (2e^{-\phi} \partial_\mu h - h \partial_\mu e^{-\phi}) . \quad (3.4)$$

Notice that we have taken the positive branch in (3.3), the negative branch just changes the sign in (3.4). The Bianchi identity and the harmonic property of h now imply that

$$(h^2 - 4e^{-\phi}) \partial_\mu \partial^\mu e^{-\phi} + 2 \partial_\mu e^{-\phi} \partial^\mu e^{-\phi} - 2h \partial_\mu h \partial^\mu e^{-\phi} + 2e^{-\phi} \partial_\mu h \partial^\mu h = 0 . \quad (3.5)$$

It is unclear how to solve this equation in general, but some multi-centered solutions were constructed in [16]. To continue, we assume that the dilaton depends on the coordinates only through h . This assumption is justified if we restrict ourselves to spherically symmetric configurations,

$$h = e^{-\phi_\infty} |\tau'_\infty| + \frac{|\hat{Q}_1|}{4\pi^2 (x - x_0)^2} , \quad (3.6)$$

where $\hat{Q}_1 = Q_1 - \chi_0 Q_2$, which must be taken negative for instantons (upper sign in (3.1)). We now proceed to find the most general spherically symmetric solution of (3.5), extending the special cases of [8]. By differentiating (3.5) once more, one can solve for the dilaton in terms of three integration constants,

$$e^{-\phi} = ah^2 + bh + c . \quad (3.7)$$

Combining this with equations (3.4) and (3.5), we find that $c = -\beta^2$, where $\beta \equiv \pm Q_2/|\hat{Q}_1|$, and $b = -\beta\sqrt{1-\alpha}$, where $a = \alpha/4$. Positivity of the dilaton and reality of H_2 requires furthermore $0 \leq \alpha \leq 1$, but the case $\alpha = 0$ must be treated separately and in fact precisely coincides with the fivebrane instanton solution. Plugging (3.7) into (3.3), and evaluating it at infinity, one finds that α is fixed in terms of the charges and the asymptotic values of the fields,

$$\alpha = 1 - \frac{(\Delta\chi - 2\beta e^{\phi_\infty})^2}{|\tau'_\infty|^2} , \quad (3.8)$$

where $\Delta\chi = \chi_\infty - \chi_0$. The solution for χ can be read off from (3.3) and one can check that χ_0 is indeed the value at the origin. An important difference with the fivebrane is that χ needs a source at the excised point. Positivity of α forces, for fixed g_s and $\Delta\chi$ (positive), β to be in the interval

$$\frac{\Delta\chi - |\tau'_\infty|}{2e^{\phi_\infty}} \leq \beta \leq \frac{\Delta\chi + |\tau'_\infty|}{2e^{\phi_\infty}} . \quad (3.9)$$

The complete solution is given in terms of one harmonic function, and all the constants a , b , c are fixed in terms of the charges and the asymptotic values of the scalars. We have checked, using the supersymmetry transformation rules given in [7, 15], that the solution

preserves half of the supersymmetry (except in the limit $\alpha \rightarrow 0$), and hence is truly BPS. The instanton action can be computed using the formulas of [8], and equals

$$S_{\text{inst}} = |\tau'_\infty| \left(|\hat{Q}_1| + \frac{1}{2} \Delta\chi Q_2 \right). \quad (3.10)$$

Positivity of the action is guaranteed by the bounds (3.9). For $Q_2 = 0$, which trivially satisfies (3.9), we find instanton actions which are, for $\Delta\chi = 0$, inversely proportional to the string coupling constant, $S_{\text{inst}} = 2|Q_1|/g_s$. This is the correct g_s -dependence of a membrane instanton with charge Q_1 [11]. Adding fivebrane charge Q_2 raises the action until, for fixed g_s and $\Delta\chi$, one reaches the $\alpha \rightarrow 0$ bound (3.9). At that point, the solution is no longer stable, and should be replaced by the fivebrane instanton.

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