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Quantum Corrections to the Universal Hypermultiplet and Superspace

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ABSTRACT

We investigate quantum corrections to the effective action of the universal hypermultiplet in the language of projective superspace. We rederive the recently found one-loop correction to the universal hypermultiplet moduli space geometry. The deformed metric is described as a superspace action in terms of a single function, homogeneous of first degree. Our framework leads us to a natural proposal for the nonperturbative moduli space metric induced by five-brane instantons.

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Contents

1 Introduction

Quantum corrections to the moduli spaces of low energy effective actions of string compactifications are important for many reasons, chief among them being their implications for supersymmetry breaking and the lifting of vacuum degeneracy. However, studying them is in general quite a difficult task. Explicit computations are manageable in the case of type II compactification on a Calabi-Yau three-fold ($CY(3)$) due to the stringent restrictions imposed by $N = 2$, $D = 4$ local supersymmetry. As the geometry of the moduli space of hypermultiplets coupled to supergravity is the same in four and five dimensions, this is more than just a toy model. The five-dimensional case is relevant for heterotic M-theory [1, 2], which may have interesting implications for string phenomenology [3] and cosmology [4].

The field content of the four-dimensional low energy limit of type II string theory is determined by the Hodge numbers of the internal Calabi-Yau space; in particular for IIA there are $h_{1,2} + 1$ hypermultiplets and $h_{1,1}$ vector multiplets. The "+1" is the universal hypermultiplet, which appears in compactification on *any* $CY(3)$ [5]. It contains as bosonic fields the dilaton, an axion (the dual of the NS-NS two-form) and a complex scalar originating from the RR three-form potential. As the dilaton belongs to a hypermultiplet, the vector multiplet moduli space is protected from quantum corrections (in the string coupling constant), but the hypermultiplet moduli space is not. Loop corrections to the latter were studied in [6]. More precisely, these authors investigated the non-universal directions (*i.e.*, directions orthogonal to the universal hypermultiplet) and found a one-loop contribution proportional to the Euler number of the Calabi-Yau manifold. Perturbative corrections to the universal hypermultiplet target space itself were first considered in [7] (see also [8]). As was stressed there, this is particularly interesting since any corrections to this moduli space are gravitational in nature.[1]

Recently this issue was reconsidered in [10] and it was found, in contrast to the claim of [7], that there *is* a one-loop correction which cannot be absorbed in a field redefinition. In the present paper, we derive the form of this new term in the framework of projective superspace [11]; this leads us to propose a natural candidate for the nonperturbative metric of the universal hypermultiplet moduli space induced by five-brane instantons. Recall that although since the work of [12] it is known conceptually that instantons due to euclidean membranes and five-branes wrapping supersymmetric cycles[2] in the CY give nonperturbative corrections to the moduli space metric, explicit metrics have been proposed only for special cases (with vanishing five-brane charge) [13]. The relevance of the quantum (and in particular *nonperturbative*) corrections to the universal hypermultiplet for cosmological applications (more precisely, finding de Sitter vacua in $N = 2$ gauged supergravity) was underscored recently by the work of [14].

What enables us to use the powerful projective superspace techniques [11] is the recently developed approach [15, 16, 17] for studying hypermultiplet couplings to supergravity. Instead of the usual description in terms of $4n$-(real) dimensional quaternion-Kähler (QK) spaces [18], we use the corresponding $4(n+1)$-dimensional hyperkähler cones (HKC) [19, 20]. As hyperkähler geometry is in general simpler than quaternion-Kähler geometry, one finds that many calculations simplify; for example, in [21] a simple derivation of the scalar potential of vector multiplets and hypermultiplets in gauged $N = 2$ supergravity was given. Furthermore, the HKC approach has also turned out to be convenient for the recent construction of effective supergravity actions that incorporate the additional light degrees of freedom appearing in flop transitions of M-theory on $CY(3)$ [22]. Most relevant for our purposes, studying hypermultiplet couplings to supergravity is now translated to studying sigma models with hyperkähler target spaces, which is exactly what projective superspace is designed for [23, 24]. A particularly interesting example, somewhat comparable to our case, is the projective superspace description of the Atiyah-Hitchin metric [25] and our aim is to discuss the universal hypermultiplet and its quantum corrections

[1] The reason is that the **c**-map (or T-duality) maps the universal hypermultiplet into the gravity multiplet instead of into a vector multiplet [5, 9].

[2] As we are considering type IIA, clearly we mean D2- and NS5-branes. There are no contributions from D4-brane instantons since CY three-folds do not have nontrivial supersymmetric five-cycles.

in a similar way.

The present paper is organized as follows. In Section 2 we concentrate on the classical geometry. To recall necessary material and establish notation, in 2.1 we give the metric of the universal hypermultiplet target space in several coordinate systems along with the coordinate transformations among them, and identify an important set of isometries forming a Heisenberg algebra. In 2.2 we provide background on the HKC approach and projective superspace and explain the description of the classical moduli space in this framework. In 2.3 we elaborate on the precise relationship between four-dimensional QK metrics with $U(1) \times U(1)$ isometries and the metrics of the corresponding HKCs. Section 3 is devoted to the perturbative corrections. In 3.1 we review the essential role of the Heisenberg algebra and the result of [10] about the one-loop correction. In 3.2 we derive the projective superspace form of the perturbatively deformed metric by considering deformations that preserve the Heisenberg algebra. In 3.3 we show that the deformed HKC metric that we have found reproduces the result of [10] upon reduction to the QK space. In Section 4 we explain our proposal for the moduli space metric induced by five-brane instantons. We also comment on the possible structure of the full nonperturbative metric. Appendix A contains the derivation of the action of the QK Heisenberg algebra on the corresponding HKC. In Appendix B we give an alternate dual description of the universal hypermultiplet in projective superspace.

2 Classical aspects

The tree-level quaternion-Kähler target space for the universal hypermultiplet is [5, 9]

$$\mathcal{M}_H = \frac{\mathrm{SU}(1,2)}{\mathrm{U}(2)} \ . \tag{2.1}$$

It is obtained from a compactification of type IIA strings on a Calabi-Yau manifold with $h_{1,2} = 0$, *i.e.*, without complex structure deformations. The generic low-energy effective action is $N = 2$ Poincaré supergravity coupled to $h_{1,2} + 1$ hypermultiplets as well as $h_{1,1}$ vector multiplets that play no role in the subsequent discussion. Suppressing the Einstein-Hilbert term, the bosonic Lagrangian for the hypermultiplets takes the form of a nonlinear sigma-model,

$$e^{-1}\mathcal{L}_{\mathrm{HM}} = -\frac{1}{2}\, g_{AB}\, \partial_\mu \phi^A \partial^\mu \phi^B \ , \tag{2.2}$$

where for the universal hypermultiplet, g_{AB} is the metric on the coset space (2.1). Before we reformulate this action in superspace, we first discuss some properties of the metric in different coordinate systems.

2.1 Quaternionic description and isometries

The coset manifold (2.1) is both quaternion-Kähler and Kähler, and there exist complex coordinates S and C in which the Kähler potential is given by

$$K = -\ln(S + \bar{S} - 2C\bar{C}) \ . \tag{2.3}$$

The isometry group of this manifold is SU(1,2) and a particular subgroup of isometries is generated by the Heisenberg algebra that acts as

$$\delta S = \mathrm{i}\alpha + 2\bar{\epsilon}C \ , \qquad \delta C = \epsilon \ . \tag{2.4}$$

The parameters α and ϵ are real and complex respectively.

An alternative parametrization of the universal hypermultiplet metric in terms of four real scalars is

$$\mathrm{d}s^2 = (\mathrm{d}\phi)^2 + \mathrm{e}^{-\phi}\big((\mathrm{d}\chi)^2 + (\mathrm{d}\varphi)^2\big) + \mathrm{e}^{-2\phi}\big(\mathrm{d}\sigma + \chi\mathrm{d}\varphi\big)^2 \ . \tag{2.5}$$

The relation with the complex coordinates above is given by

$$\begin{aligned} \mathrm{e}^{\phi} &= \frac{1}{2}(S + \bar{S} - 2C\bar{C}) \qquad & \chi &= C + \bar{C} \ , \\ \sigma &= \frac{\mathrm{i}}{2}(S - \bar{S} + C^2 - \bar{C}^2) \qquad & \varphi &= -\mathrm{i}(C - \bar{C}) \ . \end{aligned} \tag{2.6}$$

In this basis, the Heisenberg algebra acts as

$$\delta\phi = 0 \ , \qquad \delta\chi = \epsilon + \bar{\epsilon} \ , \qquad \delta\sigma = -\alpha - (\epsilon + \bar{\epsilon})\varphi \ , \qquad \delta\varphi = -\mathrm{i}(\epsilon - \bar{\epsilon}) \ . \tag{2.7}$$

The generators satisfy the commutation relations,

$$[\delta_\alpha, \delta_\epsilon] = [\delta_\alpha, \delta_{\bar{\epsilon}}] = 0 \ , \qquad [\delta_\epsilon, \delta_{\bar{\epsilon}}] = 2i\delta_{\alpha(\epsilon,\bar{\epsilon})} \ , \qquad \alpha(\epsilon, \bar{\epsilon}) \equiv -\epsilon\bar{\epsilon} \ . \tag{2.8}$$

Notice that there is a subalgebra of two commuting isometries, acting as shifts on σ and φ, namely $\delta_1 = \delta_\alpha$ and $\delta_2 = -i(\delta_\epsilon - \delta_{\bar{\epsilon}})$. The Heisenberg algebra of isometries is thought to persist at the perturbative level (as discussed in Section 3.1) and hence the loop-corrected metric should still have these symmetries.

Finally there is the coordinate system used by Calderbank and Pedersen [26], who classified four-dimensional quaternion-Kähler metrics with two commuting isometries. The metric is given in terms of a single function $\tilde{F}(\eta, \rho)$ satisfying a Laplace-like equation

$$\rho^2(\partial_\rho^2 + \partial_\eta^2)\tilde{F}(\eta, \rho) = \frac{3}{4}\tilde{F}(\eta, \rho) \ . \tag{2.9}$$

Using the notation $\tilde{F}_\rho \equiv \partial_\rho\tilde{F}$ and $\tilde{F}_\eta \equiv \partial_\eta\tilde{F}$, we can introduce matrices,

$$Q = \begin{pmatrix} \frac{1}{2}\tilde{F} - \rho\tilde{F}_\rho & -\rho\tilde{F}_\eta \\ -\rho\tilde{F}_\eta & \frac{1}{2}\tilde{F} + \rho\tilde{F}_\rho \end{pmatrix} \ , \qquad N = \begin{pmatrix} 0 & 1/\sqrt{\rho} \\ \sqrt{\rho} & \eta/\sqrt{\rho} \end{pmatrix} \ . \tag{2.10}$$

We can now write the metric as [26]

$$\mathrm{d}s^2 = \frac{-2\det Q}{\tilde{F}^2\rho^2}\Big((\mathrm{d}\rho)^2 + (\mathrm{d}\eta)^2\Big) + M^{IJ}(\rho,\eta)\,\mathrm{d}\phi_I \mathrm{d}\phi_J\ , \tag{2.11}$$

where we denote the two extra coordinates by ϕ_I, which for the universal hypermultiplet are related to $\{\varphi, \sigma\}$ from (2.5). The matrix M is related to Q and N by

$$M^{IJ} = -\frac{2}{\tilde{F}^2\det Q}\left(N\,Q^2 N^t\right)^{IJ}\ . \tag{2.12}$$

Note that (2.9) is manifestly invariant under constant rescalings of $\tilde{F}$, whereas the metric is invariant only after rescaling the coordinates ϕ_I. The overall normalization that we have chosen differs from [26, 10] by a factor of 2, which is convenient for comparison with (2.5).

The tree-level universal hypermultiplet metric is obtained by taking

$$\tilde{F} = \rho^{3/2}\ , \tag{2.13}$$

which trivially satisfies (2.9). The relation between the Calderbank-Pedersen variables and the ones from (2.5) is

$$\mathrm{e}^{\phi} = \rho^2\ , \qquad \chi = 2\eta\ , \qquad \sigma = 2\phi_1\ , \qquad \varphi = \phi_2\ , \tag{2.14}$$

as can be checked by plugging (2.13) into (2.11) and changing coordinates as above.

The Calderbank-Pedersen metric has a manifest (and for the case of interest non-compact) $U(1)\times U(1)$ isometry group, which acts as shifts on the scalars ϕ_I (this is the commuting subalgebra of the Heisenberg algebra noted below (2.8)). Using the known Legendre transform techniques, one can dualize these scalars into two tensors. The resulting action of such a double tensor multiplet reads

$$e^{-1}\mathcal{L}_{\rm DT} = \frac{\det Q}{\tilde{F}^2\rho^2}\Big(\partial_\mu\rho\,\partial^\mu\rho + \partial_\mu\eta\,\partial^\mu\eta\Big) + \frac{1}{2}M_{IJ}H^{\mu I}H^J_\mu\ , \tag{2.15}$$

where the H^I are a pair of three-form field strengths, $H^{\mu I} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\partial_\nu B^I_{\rho\sigma}$, and M_{IJ} is the inverse of M^{IJ}, $M^{IK}M_{KJ} = \delta^I_J$.

In the coordinates (2.14), the dual Lagrangian of (2.5) takes the form [28, 27]

$$e^{-1}\mathcal{L}_{\rm DT} = -\frac{1}{2}\,\partial^\mu\phi\,\partial_\mu\phi - \frac{1}{2}\,\mathrm{e}^{-\phi}\,\partial^\mu\chi\,\partial_\mu\chi + \frac{1}{2}M_{IJ}H^{\mu I}H^J_\mu\ , \tag{2.16}$$

where, after taking into account factors of two from the relation $\sigma = 2\phi_1$,

$$M_{IJ} = \mathrm{e}^{\phi}\begin{pmatrix} \mathrm{e}^{\phi} + \chi^2 & -\chi \\ -\chi & 1 \end{pmatrix}\ . \tag{2.17}$$

For reasons that become clear later, we have used a slightly different notation than in [28, 27], in that we have lowered/raised and interchanged the labels "1" and "2".

The two scalars ϕ and χ parameterize the coset $\mathrm{SL}(2,\mathbb{R})/\mathrm{O}(2)$. The presence of the tensors breaks the $\mathrm{SL}(2,\mathbb{R})$ symmetries to a two-dimensional subgroup generated by a certain rescaling of the fields and by the remaining generator of the Heisenberg algebra (2.7) with real parameter $\epsilon = \gamma/2$. It acts as a shift on χ and transforms the tensors linearly into each other [28],

$$\chi \to \chi + \gamma \ , \qquad B^2 \to B^2 + \gamma B^1 \ , \tag{2.18}$$

with ϕ and B^1 invariant. Note that the combination $\hat{H}^2 = H^2 - \chi H^1$ is invariant under (2.18).

2.2 Hyperkähler/superspace description

As quaternion-Kähler (QK) geometry is rather complicated, the original description of hypermultiplet couplings to supergravity in terms of QK geometry [18] is not easy to study. As mentioned in the introduction, an alternative approach [15, 16, 17], based on the $N = 2$ superconformal tensor calculus [30], uses the one-to-one correspondence[3] between $4n$-(real) dimensional QK spaces and $4(n+1)$-dimensional hyperkähler cones (HKC) [19, 20]. In the superconformal calculus, the HKC is realized by introducing an additional hypermultiplet as a compensator. As a result the theory is invariant under local $N = 2$ superconformal transformations, where the combined set of hypermultiplet scalar fields (physical and compensator) parametrize the HKC. If one eliminates the compensator by imposing suitable gauge-fixing conditions that fix the dilatations and local SU(2) R-symmetries, one recovers the matter coupled Poincaré supergravity. The target space of the hypermultiplets of the conformal theory is the HKC, which is a hyperkähler space with a conformal homothety and an SU(2) isometry that rotates the complex structures. Having described the theory by a hyperkähler-target sigma-model, we can use the powerful techniques of projective superspace [11].

Projective superspace is a subspace of $N = 2$ superspace defined with the help of an additional complex projective coordinate ζ. For a recent detailed account of its properties and conventions, we refer to Appendix B of [16]. The $N = 2$ projective superfields that we need are all power series in the variable ζ with coefficients that project to $N = 1$ superfields. More precisely, these are the so-called polar and $\mathcal{O}(2p)$ multiplets [24, 29].

[3]This correspondence is one-to-one only locally. Globally there may exist a second HKC, the double cover of the first one, if a certain topological invariant of the quaternion-Kähler space vanishes [19].

A polar multiplet [24] has the form

$$\Upsilon = \sum_{n=0}^{\infty} \Upsilon_n \zeta^n \,. \tag{2.19}$$

As the real structure in projective superspace is given by complex conjugation composed with the antipodal map $\zeta^* \to -1/\zeta$, the conjugate is

$$\bar{\Upsilon} = \sum_{n=0}^{\infty} \bar{\Upsilon}_n (-\frac{1}{\zeta})^n \,. \tag{2.20}$$

The polar multiplets are off-shell hypermultiplets; as a product of them is again a polar multiplet, they can be used as (quaternionic) coordinates which admit a natural gauge action, and hence can realize the supersymmetric hyperkähler quotient construction of [31, 23]. Superspace actions can be written in terms of the usual $N = 1$ measure [24]

$$S = \oint_C \frac{\mathrm{d}\zeta}{2\pi i \zeta} \, \mathrm{d}^4 x \; D^2 \bar{D}^2 G(\Upsilon, \bar{\Upsilon}, \zeta) \ , \tag{2.21}$$

where C is a contour in the ζ-plane that generically depends on the form of the Lagrangian G. To make the action invariant under *local* superconformal symmetries, one simply couples the polar multiplets to the Weyl multiplet, without changing the form of G. For on-shell hypermultiplets in components, this coupling was described in [15].

In [16], the hyperkähler cone of the universal hypermultiplet was described in great detail. For our purposes, the projective superspace formulation in terms of polar multiplets, which makes the SU(1,2) symmetry of the universal hypermultiplet manifest and linear is most useful; the Lagrange density is given by

$$\hat{G}_V = \mathrm{e}^V \bar{\Upsilon}^i \, \eta_{ij} \Upsilon^j \ , \quad i = 1, 2, 3 \tag{2.22}$$

where V is a projective superspace $U(1)$ gauge potential[4], and $\eta_{ij} = (- + +)$. The factor e^V arises because the HKC is the hyperkähler quotient of the 12-dimensional flat σ-model by the diagonal $U(1)$ action: $\delta \Upsilon^i = i \Upsilon^i$. Integrating out V after a certain duality transformation as explained in [16], and fixing the gauge $\Upsilon^3 = 1$ gives the polar multiplet superspace Lagrangian for the hyperkähler cone of the universal hypermultiplet:

$$G_{\mathrm{HKC}} = 2\sqrt{-\bar{\Upsilon}^1 \Upsilon^1 + \bar{\Upsilon}^2 \Upsilon^2} \ . \tag{2.23}$$

Notice that the quotient Lagrangian has only a manifest U(1,1) rather than SU(1,2) invariance.

[4] Although e^V acts as a Lagrange multiplier in (2.22), as explained in [16], after reducing to $N = 1$ superspace or components, or after duality transformations, this Lagrangian is perfectly sensible.

The other multiplet of interest, the $\mathcal{O}(2p)$ multiplet, is real and of the form

$$\eta^{(2p)}(\zeta) = \frac{1}{\zeta^p} \sum_{n=0}^{2p} \eta_n^{(2p)} \zeta^n \,. \tag{2.24}$$

Particularly important for us will be the $\mathcal{O}(2)$ multiplet:

$$\eta^{(2)} = \frac{\bar{z}}{\zeta} + x - z\zeta \,, \tag{2.25}$$

where z projects to an $N = 1$ chiral superfield and x projects to a real linear superfield.

In Section 4, we will also need the $\mathcal{O}(4)$ multiplet:

$$\eta^{(4)} = \frac{\bar{z}}{\zeta^2} + \frac{\bar{v}}{\zeta} + x - v\zeta + z\zeta^2 \,, \tag{2.26}$$

where z is constrained as for the $\mathcal{O}(2)$ multiplet, v is a complex linear superfield which has one complex physical scalar, and x projects to an auxiliary real unconstrained $N = 1$ superfield. In total, the $\mathcal{O}(4)$ multiplet has four real physical degrees of freedom.

The $\mathcal{O}(2)$ multiplet is precisely the $N = 2$ tensor multiplet. Because dualizing scalars into tensors with respect to commuting isometries of a quaternionic geometry in Poincaré supergravity translates to dualizing with respect to triholomorphic isometries of the corresponding hyperkähler cone, the double tensor multiplet description of the universal hypermultiplet is equivalent to a description of the superconformal universal hypermultiplet (obtained by adding a compensating hypermultiplet) in terms of two $N = 2$ tensor multiplets η_1 and η_2.

In general, an HKC may have several different pairs of commuting isometries. For the case of the universal hypermultiplet, as explained in [16], there are at least three choices, yielding three tensor multiplet descriptions. The particular choice we study here is the one that corresponds to the HKC-lift of the $u(1) \times u(1)$ subalgebra inside the Heisenberg algebra (2.7) (a second choice is described in Appendix B). In [21], the relation between isometries on a quaternion-Kähler space Q and their lift to triholomorphic isometries on the HKC above Q was written down explicitly. Using the relation between the S, C coordinates of (2.3) and the HKC variables given in [16], the procedure given in [21] yields the action of the Heisenberg subalgebra on the fields Υ^i. We give more details on the derivation of the generators in Appendix A. The result is:

$$T_1 = \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad T_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} \quad , \quad T_3 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & i \\ i & -i & 0 \end{pmatrix} \quad . \tag{2.27}$$

By exponentiating these generators, one finds the linear action of the Heisenberg group on the polar multiplets Υ^i. The correspondence with (2.7) is such that T_1 and T_2 generate the shifts in σ and φ respectively. These generators obey the algebra

$$[T_2, T_3] = 2iT_1 \ , \tag{2.28}$$

with all other commutators vanishing. Note that the above matrix representation of the algebra obeys a much stronger relation: $T_1 T_m = T_m T_1 = 0$.

The dualization with respect to T_1 and T_2 in superspace was worked out in detail in [16].[5] The resulting action has two $N = 2$ tensor multiplets and the superspace Lagrange density is[6]

$$G(\eta_1, \eta_2) = \frac{(\eta_2)^2}{\eta_1} \ . \tag{2.29}$$

Notice that this is a function homogeneous of first degree and there is no explicit ζ dependence. As explained in [16] this is a consequence of the superconformal invariance. The Lagrange density (2.29) is gauge-equivalent to the double-tensor multiplet Lagrangian (2.16). The correspondence between the HKC and QK formulations is discussed further in the next subsection.

2.3 Relation between the HKC and QK metrics

In this subsection, we explain in more detail how the HKC and QK metrics are related. As known since [32, 23, 33], the metric on a $4n$-dimensional hyperkähler space with n commuting isometries can be written in the form

$$\mathrm{d}s^2 = U_{IJ}(r)\, \mathrm{d}\vec{r}^I \cdot \mathrm{d}\vec{r}^J + [U^{-1}]^{IJ}(r)\, (\mathrm{d}t_I + A_I)(\mathrm{d}t_J + A_J) \, ; \qquad I, J = 1, ..., n \, , \tag{2.30}$$

where the coordinates t_I parametrize the n U(1) isometries and $\vec{r}^I$ are n three-dimensional vectors which are the moment maps associated with ∂_{t_I}. Finally, $A_I \equiv \vec{A}_{IJ} \cdot \mathrm{d}\vec{r}^J$ are one-forms determined by the Bogomol'nyi equation

$$\mathcal{R}^{(K)}_{r^I_a r^J_b} = \sum_c \varepsilon_{abc} \nabla_{r^I_c} U_{JK} \ , \qquad a, b, c = 1, 2, 3 \ , \tag{2.31}$$

where $\mathcal{R}^{(K)}$ is the SU(2) curvature (field strength) of A_K.

An explicit solution to (2.31) was found in [31, 11, 23] by regarding the manifold as the target space of a supersymmetric nonlinear σ-model. Then the metric (2.30) is encoded in a single function F depending on $3n$ variables $x^I, z^I, \bar{z}^I$. Supersymmetry requires this function to satisfy a Laplace-like equation

$$F_{x^I x^J} + F_{z^I \bar{z}^J} = 0 \ , \tag{2.32}$$

[5] More precisely, the generators considered in that work are the transpose of T_1 and T_2 in (2.27). However this difference does not affect the end product of the dualization.

[6] For later convenience, we have interchanged the indices 1 and 2 of η_I and T_I in (2.27), as well as the roles of T_I and T'_I, relative to the conventions used in [16].

where $F_{x^I x^J}$ denote the second derivatives. The constraint (2.32) can be solved in terms of the following contour integral representation [11, 23]:

$$F = Re \oint \frac{\mathrm{d}\zeta}{2\pi i\zeta} G(\eta^I(\zeta), \zeta) \ , \qquad \eta^I = \frac{\bar{z}^I}{\zeta} + x^I - z^I \zeta \, . \tag{2.33}$$

The hyperkähler metric (2.30) is then determined by

$$U_{IJ} = -\frac{1}{2} F_{x^I x^J} \ , \qquad A_I = i(F_{z^K x^I} dz^K - F_{\bar{z}^K x^I} d\bar{z}^K) \, , \tag{2.34}$$

where

$$x^I = r_3^I \qquad z^I = \frac{r_1^I + i r_2^I}{2} \, , \qquad \bar{z}^I = \frac{r_1^I - i r_2^I}{2} \, . \tag{2.35}$$

The scalars of the universal hypermultiplet parametrize a four-dimensional quaternion-Kähler space corresponding to an eight-dimensional HKC. As discussed above, four-dimensional QK manifolds with two commuting isometries were classified by Calderbank and Pedersen in terms of a single function $\tilde{F}$ satisfying the Laplace-like equation (2.9). Given a solution $\tilde{F}$ one can construct the hyperkähler cone metric (2.30) by writing [26]

$$U_{IJ} = \frac{\tilde{F} \det Q}{|q|^2} \left[(N^{-1})^t \, Q^{-1} N^{-1} \right]_{IJ} \ , \tag{2.36}$$

where Q and N are defined in (2.10), and the 8 real coordinates of the HKC are the QK coordinates $\rho, \eta, t_I \equiv \phi_I$, $I = 1, 2$ along with the 4 real components of the quaternion q (the hypermultiplet compensator). The one forms A_I can be computed by solving (2.31), or equivalently, by finding F and using (2.34); see also [34].

The precise relations between the QK and HKC variables are further given by [26]

$$\rho = \frac{|\vec{r}^1 \times \vec{r}^2|}{|\vec{r}^1|^2} \ , \qquad \eta = \frac{\vec{r}^1 \cdot \vec{r}^2}{|\vec{r}^1|^2} \ , \tag{2.37}$$

and (correcting a misprint in [26], which was already noted in [35])

$$|\vec{r}^1 \times \vec{r}^2| = |q|^4 / \tilde{F}^2 \, . \tag{2.38}$$

Notice that all these relations are consistent with the scaling weights of the coordinates of the HKC, whereby we assign weights 2,1 and 0 to the coordinates $\vec{r}^I, q$ and t_I respectively. This assignment can be understood as follows: A generic cone metric can be put in the form $\mathrm{d}s^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2$, where r is the radial coordinate and $\mathrm{d}\Omega^2$ is the metric on the base space. The homothety that this metric has keeps $\mathrm{d}\Omega^2$ invariant and in four dimensions[7] acts on r as $r \to \alpha r$. Clearly that means that under the homothety $\mathrm{d}s^2 \to \alpha^2 \mathrm{d}s^2$ or in

[7] We note that as this action is dimension dependent so are the resulting scaling weights. For a general discussion and also the weigths of the fields in various multiplets in $d = 5$ see [36].

other words that the metric has scaling weight 2. Now, looking at (2.30) and (2.36) and keeping in mind that the QK quantities ρ, η and t_I have to be inert under this rescaling, we deduce the above weights of $\vec{r}^I$ and q.

To recapitulate: all of the above implies that comparison between the descriptions in terms of the functions F and $\tilde{F}$ amounts to comparison between the matrices U_{IJ} in (2.34) and (2.36), possibly up to a coordinate transformation which is equivalent to taking two linearly independent constant-coefficient combinations of the two commuting isometries as the new isometries.

For completeness, before concluding this section we comment on the lifting of the symmetry (2.18) to the tensor multiplet description in projective superspace. This is also helpful for identifying unambiguously the component-field content of the two η_I tensor multiplets. Combining (2.14), (2.37) and the invariance of ϕ (equivalently, ρ), we see that the residual Heisenberg transformation acts as

$$\vec{r}^2 \to \vec{r}^2 + \frac{\gamma}{2}\vec{r}^1 , \qquad \vec{r}^1 \to \vec{r}^1 . \tag{2.39}$$

Hence from (2.33), (2.35) it follows that

$$\eta_2 \to \eta_2 + \frac{\gamma}{2}\eta_1 , \qquad \eta_1 \to \eta_1 . \tag{2.40}$$

So indeed η_1 contains ϕ and B_1 whereas η_2 contains χ and B_2. Finally, note that the action determined by (2.29) is invariant under the residual Heisenberg symmetry (2.40) as an infinitesimal transformation on $G(\eta_1, \eta_2)$ yields a term linear in η_2, which vanishes under the superspace integral.

3 Perturbative corrections

3.1 One-loop metrics

The classical action of the universal hypermultiplet is invariant under eight symmetry transformations whose explicit form was found in [37]. While four of them are not expected to survive beyond the classical limit, the other four are thought to play a role at the quantum level. One of them, the classical symmetry acting as an $SO(2)$ transformation on the RR scalars χ and φ or, equivalently, on η and ϕ_2 (see (2.14)):

$$C \to e^{i\theta} C , \qquad S \to S \tag{3.1}$$

with θ real, reduces in the full quantum theory to [38]

$$ReC \to ImC , \qquad ImC \to -ReC , \tag{3.2}$$

i.e. to the case $\theta = \frac{\pi}{2}$.

The remaining three are the transformations (2.4). They are believed to be preserved by perturbative corrections [7]. It was noticed in [38] that they satisfy a Heisenberg algebra (in our notation: (2.8)). It was further argued there that membrane and fivebrane instanton corrections lead to discrete identifications due to the charge quantization conditions that must be obeyed by the corresponding field strengths. Hence the continuous Heisenberg symmetry is broken non-perturbatively to a discrete subgroup.

The perturbative corrections to the moduli space of the universal hypermultiplet were first studied in [7] with the conclusions that they arise only at one-loop and that they can be absorbed in a field-redefinition. However, recently it was found that the second conclusion is incorrect [10]. The authors of [10] performed explicit stringy one-loop computation of the three-point amplitude with external graviton/NS-NS tensor and two RR scalars and showed that this induces a genuine (unremovable) deformation of the hypermultiplet target space. This deformation is still invariant under the Heisenberg algebra of isometries, and in particular under the shift of the coordinate η (not to be confused with the projective superfields η_I). However, it does not preserve the Kähler structure of the classical moduli space. It is encoded in the following change of (2.13):

$$\tilde{F} = \rho^{3/2} - \hat{\chi}\rho^{-1/2} \,, \tag{3.3}$$

where $\hat{\chi}$ is a constant parameter. Moreover, as shown in [10], quantum corrections proportional to $\hat{\chi}$ also modify the relation between the Calderbank-Pedersen coordinate ρ and the four-dimensional dilaton ϕ. We will see in the following how to recover (3.3) from the perspective of projective superspace.

3.2 Superspace derivation

In this subsection we write the most general deformation of the HKC corresponding to the universal hypermultiplet compatible with the Heisenberg algebra, and find the dual description in terms of $\mathcal{O}(2)$ multiplets (*i.e.*, $N = 2$ tensor multiplets). In the next subsection we show that it indeed reproduces the one-loop deformation (3.3) of [10].

As derived at the end of the previous section, the Heisenberg algebra in projective superspace acts as (2.40). We should therefore look for a function $G(\eta_1, \eta_2)$ which is homogeneous of first degree (due to the conformal symmetry on the HKC), and invariant under (2.40). The classical part is given by (2.29). The deformation describing the one-loop correction should be proportional to η_1 as this is the multiplet that contains the dilaton. Up to a constant parameter $\hat{\chi}$, these constraints are solved by

$$G(\eta_1, \eta_2) = \frac{(\eta_2)^2}{\eta_1} - 2\hat{\chi}\,\eta_1 \ln \eta_1 \ . \tag{3.4}$$

We now derive this result from the hyperkähler quotient construction on the twelve-dimensional flat hyperkähler space on which the isometries act linearly. Classically, this quotient was obtained from the gauged Lagrangian density (2.22). It turns out that it is a bit tricky to obtain a deformed HKC metric. What first comes to mind is to deform the flat metric of the initial σ-model whose hyperkähler quotient is the HKC. Looking for the most general metric preserved by the Heisenberg algebra:

$$g_{ij}(T_m)^j{}_k = (T_m^\dagger)_i{}^j g_{jk} \ , \tag{3.5}$$

we find

$$g_{ij} \propto (\eta_{ij} - \epsilon \eta_{ik}(T_1)^k{}_j) \ . \tag{3.6}$$

However, substituting $\eta \to g$ in (2.22) with a constant deformation ϵ does not modify the universal hypermultiplet, as the deformation can be absorbed by a linear change of variables acting on Υ^i. Fortunately, there is another way to proceed. Namely, we deform the $U(1)$ gauge group with respect to which we take the quotient by adding a term proportional to T_1.[8] Because $(T_1)^2 = 0$, this has the effect of deforming the metric as in (3.6) with $\epsilon \propto V$. Thus the gauged Lagrange density is now:

$$\hat{G}_V = \mathrm{e}^V \bar{\Upsilon}^i (\eta_{ij} - 4\hat{\chi} V \eta_{ik}(T_1)^k{}_j)\Upsilon^j \ , \tag{3.7}$$

where $\hat{\chi}$ is a constant deformation parameter.

We now dualize with respect to two $U(1)$ isometries that commute with and are independent of the gauge symmetry; again, there are a number of choices,[9] but as explained for the classical Lagrangian, we focus on the symmetries generated by T_1 and T_2. We use the explicit relations $(T_2)^2 = -T_1$ and $T_1 T_m = T_m T_1 = 0$, to find

$$\mathrm{e}^{V_1 T_1 + V_2 T_2} = 1 + (V_1 - \tfrac{1}{2}(V_2)^2) T_1 + V_2 T_2 \ . \tag{3.8}$$

This implies that we may write down the first-order Lagrange density

$$\mathrm{e}^V \bar{\Upsilon}^i [\eta_{ij} + \eta_{ik}(V_1 - \tfrac{1}{2}(V_2)^2 - 4\hat{\chi} V)(T_1)^k{}_j + V_2 (T_2)^k{}_j)]\Upsilon^j - \eta_1 V_1 - \eta_2 V_2 \ , \tag{3.9}$$

where $\eta_I,\ I = 1,2$ are $\mathcal{O}(2)$ multiplets; if we integrate out η_I, we find that both V_I are pure gauge and recover (3.7). On the other hand, if we integrate out V_I and V, after dropping total derivative terms, we get the dual theory expressed purely in terms of η_I. Before we begin this calculation, we simplify (3.9) by shifting V_1 and obtain:

$$\mathrm{e}^V (X_0 + V_1 X_1 + V_2 X_2) - \eta_1 (V_1 + \tfrac{1}{2}(V_2)^2 + 4\hat{\chi} V) - \eta_2 V_2 \ , \tag{3.10}$$

[8] Strictly speaking, this changes $U(1)$ to a noncompact gauge symmetry but we have not investigated the consequences of this.

[9] An alternative dualization in terms of two different generators, one of which corresponds to the symmetry (3.1), is given in Appendix B.

where

$$
\begin{aligned}
X_0 &\equiv \bar{\Upsilon}^i \eta_{ij} \Upsilon^j = -\bar{\Upsilon}^1 \Upsilon^1 + \bar{\Upsilon}^2 \Upsilon^2 + \bar{\Upsilon}^3 \Upsilon^3 \ , \\
X_1 &\equiv \bar{\Upsilon}^i \eta_{ik} (T_1)^k{}_j \Upsilon^j = -(\bar{\Upsilon}^1 - \bar{\Upsilon}^2)(\Upsilon^1 - \Upsilon^2) \ , \\
X_2 &\equiv \bar{\Upsilon}^i \eta_{ik} (T_2)^k{}_j \Upsilon^j = -\bar{\Upsilon}^3 (\Upsilon^1 - \Upsilon^2) - (\bar{\Upsilon}^1 - \bar{\Upsilon}^2) \Upsilon^3 \ .
\end{aligned}
\tag{3.11}
$$

Varying (3.10) with respect to V, V_1, and V_2 respectively, we find

$$
\mathrm{e}^V (X_0 + V_1 X_1 + V_2 X_2) = 4\hat{\chi}\eta_1 \ , \quad \mathrm{e}^V X_1 = \eta_1 \ , \quad \mathrm{e}^V X_2 = \eta_2 + V_2 \eta_1 \ . \tag{3.12}
$$

We solve these for V, V_1, and V_2:

$$
V = \ln \eta_1 - \ln X_1 \quad , \quad V_2 = -\frac{\eta_2}{\eta_1} + \frac{X_2}{X_1} \quad ,
$$

$$
V_1 = 4\hat{\chi} - \frac{X_0}{X_1} + \frac{\eta_2 X_2}{\eta_1 X_1} - \left(\frac{X_2}{X_1} \right)^2 \quad . \tag{3.13}
$$

Substituting back into (3.10), and dropping numerous total derivatives of the form $\eta(\bar{\Upsilon} + \Upsilon)$, we find[10]

$$
\tfrac{1}{2} \frac{(\eta_2)^2}{\eta_1} - 4\hat{\chi}\, \eta_1 \ln \eta_1 \ . \tag{3.14}
$$

However, recall that for convenience we have redefined the canonical generator in the direction of t_1 (or equivalently, σ) by a factor of two (see (A.8)). So the dual variable η_1 is twice bigger than it should be. Rescaling $\eta_1 \to \eta_1/2$, and dropping a term linear in η_1 (which vanishes under the superspace integral) we obtain our final result

$$
\frac{(\eta_2)^2}{\eta_1} - 2\hat{\chi}\, \eta_1 \ln \eta_1 \ , \tag{3.15}
$$

as stated in the beginning of this subsection.

3.3 Comparing the one-loop metric and superspace action

In this subsection we will show that the superspace Lagrange density (3.15) gives exactly the metric determined by (3.3). We thereby make use of the general results of Section 2.3.

The matrix U_{IJ} of (2.36) with $\tilde{F}$ as in (3.3) has the form

$$
U_{IJ} = \frac{1}{|\vec{r}^{\,1}|} \begin{pmatrix} (2\eta^2 - \rho^2 - \hat{\chi}) & -2\eta \\ -2\eta & 2 \end{pmatrix} \ , \tag{3.16}
$$

[10] In particular, X_2/X_1 and $X_0/X_1 + \frac{1}{2}(X_2/X_1)^2$ both have the form $\bar{\Upsilon} + \Upsilon$.

where we have used (2.37) and (2.38).

Now let us find the matrix $-\frac{1}{2}F_{x^Ix^J}$ with $I,J=1,2$ (see (2.34)). The deformation piece in (3.15) is

$$G^{def} = -2\hat{\chi}\,\eta_1 \ln \eta_1 \qquad \Rightarrow \qquad F^{def}_{x^1x^1} = \frac{2\hat{\chi}}{\sqrt{(x^1)^2+4z^1\bar{z}^1}} = \frac{2\hat{\chi}}{|\vec{r}^{\,1}|}\,. \tag{3.17}$$

For the classical contribution we find:

$$\begin{aligned} F^{cl} &= \oint \frac{d\zeta}{2\pi i\zeta}\,\frac{\eta_2^2}{\eta_1} = \frac{(x^2)^2}{|\vec{r}^{\,1}|} \\ &+ \frac{((x^1)^2+2z^1\bar{z}^1)((z^1\bar{z}^2)^2+(z^2\bar{z}^1)^2)-2z^1\bar{z}^1(x^1x^2(z^2\bar{z}^1+z^1\bar{z}^2)-2z^1\bar{z}^1z^2\bar{z}^2)}{2(z^1\bar{z}^1)^2|\vec{r}^{\,1}|}\,, \end{aligned} \tag{3.18}$$

where the contour of integration is the same as the one in Figure 1 of [23]. Despite its unappealing form the last result gives the following elegant expressions for the second derivatives of interest:

$$F^{cl}_{x^2x^2} = -\frac{4}{|\vec{r}^{\,1}|}\,, \qquad F^{cl}_{x^1x^2} = \frac{4\vec{r}^{\,1}\cdot\vec{r}^{\,2}}{|\vec{r}^{\,1}|^3} = \frac{4\eta}{|\vec{r}^{\,1}|}\,,$$

$$F^{cl}_{x^1x^1} = -4\frac{[(\vec{r}^{\,1}\cdot\vec{r}^{\,2})^2-\frac{1}{2}(\vec{r}^{\,1}\times\vec{r}^{\,2})^2]}{|\vec{r}^{\,1}|^5} = -\frac{4}{|\vec{r}^{\,1}|}\left(\eta^2-\frac{\rho^2}{2}\right)\,, \tag{3.19}$$

where we have used (2.37). Hence we obtain the complete result

$$-\frac{1}{2}F_{x^Ix^J} = \frac{1}{|\vec{r}^{\,1}|}\begin{pmatrix} (2\eta^2-\rho^2) & -2\eta \\ -2\eta & 2 \end{pmatrix} - \frac{\hat{\chi}}{|\vec{r}^{\,1}|}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\,, \tag{3.20}$$

which agrees precisely with (3.16).

4 Proposal for instanton corrections

On general grounds the perturbative hypermultiplet moduli space metric is expected to be corrected by instanton effects. As already recalled, these are due to membranes and five-branes wrapping supersymmetric cycles in the internal space. The relevant instanton actions were calculated in [38, 39, 28]. However, these actions are only the first step in the determination of the quantum corrected moduli space metric. The main obstacle for performing explicitly such a computation is the fact that in string/M theory, unlike in second quantized field theory, the rules for calculating the one-loop fluctuation determinants have not yet been established.

Addressing this problem is beyond the scope of our paper. In the current section we have a more modest goal. Namely, we make a conjecture for the form of the nonperturbative universal hypermultiplet moduli space metric due to five-brane instantons.[11] An obvious virtue of our proposal is that it reduces to the perturbative result in the asymptotic region. Further investigation of the proposed metric is more than desirable (although technically not easy) and we hope to report on this in the near future.[12]

To explain the rationale behind our proposal we now recall the projective superspace description of the four-dimensional hyperkähler moduli space metric on the Coulomb branch of pure three-dimensional $N = 4$ Yang-Mills theory with gauge group $SU(2)$. The perturbative (one-loop corrected) moduli space metric is the Taub-NUT one (with mass parameter $m = -1$). Taking into account the instanton effects it becomes the Atiyah-Hitchin metric [41, 42]. These two metrics are determined, in the language of Sections 2.2 and 2.3, by a single superfield with

$$F_{TN} = \oint_{C_0} \frac{d\zeta}{2\pi i\zeta} \, (\eta^{(2)})^2 + m \oint_C \frac{d\zeta}{\zeta} \, \eta^{(2)}(\ln \eta^{(2)} - 1) \, , \tag{4.1}$$

and

$$F_{AH} = \oint_{C_0} \frac{d\zeta}{2\pi i\zeta} \, \eta^{(4)} - \oint_{C'} \frac{d\zeta}{\zeta} \, \sqrt{\eta^{(4)}} \, , \tag{4.2}$$

respectively [25] (notice that our normalization of the η's is different from the one in [25]). The contour C_0 encloses the origin and C (C') - all two (four) roots of $\eta^{(2)}$ ($\eta^{(4)}$).[13] Asymptotically $\eta^{(4)} \to (\eta^{(2)})^2$ [43] and so the Atiyah-Hitchin metric tends to the Taub-NUT one (with $m = -1$) as is known to be the case.

From (4.1) and (4.2) we can see that the full moduli space Lagrange density has the same form as the perturbative one, but is written in terms of an $\mathcal{O}(4)$ multiplet rather than an $\mathcal{O}(2)$ one.[14] It is tempting to try the same trick for the universal hypermultiplet; there are two $\mathcal{O}(2)$ mulitplets η_I in (3.15), but any Lagrange density involving an $\mathcal{O}(4)$ multiplet $\eta^{(4)}$ must have at least two terms involving $\eta^{(4)}$, as otherwise the auxiliary $N = 1$ superfield x contained in $\eta^{(4)}$ cannot be consistently eliminated; thus the only candidate for "promotion" to $\eta^{(4)}$ is η_1. Moreover, conformal symmetry requires no explicit ζ-dependence in the Lagrange density, and scales an $\mathcal{O}(4)$ multiplet with weight 2. Hence

[11] Explicit five-brane instanton calculations in the semiclassical and supergravity approximation are performed in [40].

[12] We note that quantum corrected metrics were also proposed in [13]. The author considered deformations of the QK metric (rather than the corresponding HKC one) for the case of vanishing five-brane charge (*i.e.* only 2-brane instantons present).

[13] For more details on the contours see [43], page 4.

[14] The log term in (4.1) seems to defy this rule, but we recall that it is present only to define the correct contour of integration. In (4.2) this goal is already achieved by the square root.

our proposal is

$$F_{UH} = \oint \frac{d\zeta}{2\pi i\zeta} \left(\frac{(\eta_2)^2}{\sqrt{\eta^{(4)}}} - 2\hat{\chi}\sqrt{\eta^{(4)}} \right) . \tag{4.3}$$

This function determines the Kähler potential of the HKC, associated to the universal hypermultiplet moduli space, via the generalized Legendre transform of [24].

The reason why (4.3) is suitable to describe the nonvanishing five-brane charge instantons is due to the symmetries that it preserves. Recall that there are three charges that a general instanton can carry, one of which descends from the 5-brane charge of M-theory and the other two from the M2-brane charge (due to the existence of two 3-cycle homology classes) [38]. These charges Q_i, $i = 1,2,3$ are related to the Noether currents corresponding to the three isometries generated by our T_1, T_2, T_3. Now, the 5-brane charge is associated precisely to T_1 and hence breaking that isometry (which is exactly what happens as a result of the substitution $\eta_1 \rightarrow \sqrt{\eta^{(4)}}$) signals the presence of five-brane instantons. On the other hand, due to the $\mathcal{O}(2)$ multiplet η_2 in (4.3), F_{UH} still has the isometry generated by T_2 and therefore does not include contributions from 2-brane instantons carrying the associated charge Q_2. Finally, going to the description in terms of one $\mathcal{O}(2)$ and one $\mathcal{O}(4)$ multiplet breaks also the isometry generated by T_3 (which at the perturbative level intertwines the two $\mathcal{O}(2)$ multiplets; see (2.40)). Thus our nonperturbative proposal also includes the contribution of 2-brane instantons with Q_3 charge. Summarizing, we propose that (4.3) describes the metric for the case of instanton corrections with two nonvanishing charges: (Q_1, Q_3).[15] We are currently investigating the explicit form and properties of the QK metric corresponding to F_{UH}.

The fully corrected universal hypermultiplet including *all* instanton corrections must also break the symmetry that comes from the remaining tensor multiplet η_2. It is not clear how to implement this. Naively promoting $\eta_2 \rightarrow \sqrt{\eta_2^{(4)}}$ clearly cannot work. However, it is possible that one needs to choose a different basis than $\eta_{1,2}$ before changing the superfields; further, the nonperturbative action may involve other forms of the hypermultiplet than just $\mathcal{O}(4)$, *e.g.*, $\eta_2 \rightarrow (\eta_2^{(6)})^{(1/3)}$, etc. Clearly, some new insights will be needed to make progress on this problem.

[15] Note that this is consistent with the explicit supergravity calculation of [28] of the five-brane instanton action, where it was found that the action contains not only a term proportional to the 5-brane charge Q_1 but also another one proportional to the asymptotic value of the field χ, which is the representation of the third charge in the double-tensor formulation.

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A Lifting the Heisenberg algebra to the HKC

In accord with [16, 21], we denote the coordinates of the eight-dimensional HKC corresponding to the universal hypermultiplet by z_+^I and z_{-I}, $I = 1, 2$, and their complex conjugates. Descending to the underlying QK space can be achieved by introducing[16]:

$$z_{-2} = \mathrm{e}^{2z}\,, \qquad z_{-1} = \mathrm{e}^{2z} w\,, \qquad z_+^2 = \frac{\xi}{2}\,, \qquad z_+^1 = v\ , \tag{A.1}$$

and taking $\xi = 0$. The coordinate z scales out, and the resulting four-dimensional QK metric is determined by the Kähler potential [16]

$$K = \ln\left(1 - w'\bar{w}' - v'\bar{v}'\right), \tag{A.2}$$

where

$$w' = w(1 - v\bar{v}) \qquad \text{and} \qquad v' = \bar{v}\,. \tag{A.3}$$

From (4.14) of [21] we see that the triholomorphic isometries of the HKC induce the following transformations of the QK coordinates:

$$\begin{aligned} \delta w &= i({T^2}_2 - {T^1}_1)w + i{T^3}_1 wv + iw^2({T^1}_2 - v{T^3}_2) - i\bar{v}\left(\frac{1}{1 - v\bar{v}} - w\bar{w}\right){T^2}_3 \\ &+ iv\bar{v}w\bar{w}{T^2}_1 - \frac{i}{1 - v\bar{v}}{T^2}_1\,, \\ \delta v &= i({T^1}_1 - v{T^3}_1)v + i{T^1}_3 - iv{T^3}_3 + i\bar{w}(1 - v\bar{v})({T^2}_3 + v{T^2}_1)\,, \end{aligned} \tag{A.4}$$

where ${T^m}_n$, $m, n = 1, 2, 3$ are the matrix elements of the symmetry generators.

We want to find out which generators correspond to the transformations in (2.4). For that purpose we note that the relationship between the metrics determined by the Kähler potentials in (2.3) and (A.2) is given by

$$w' = \frac{1 - S}{1 + S} \qquad \text{and} \qquad v' = \frac{2C}{1 + S}\,. \tag{A.5}$$

[16] We follow the notation of [21] for convenience. In [16] w is denoted by u.

Hence combining (A.3), (A.5) and $\delta S = i\alpha + 2\bar{\epsilon}C, \quad \delta C = \epsilon$ we obtain

$$\begin{aligned}\delta w &= \epsilon(1+w)wv - \bar{\epsilon}\bar{v}\left(\frac{1}{1-v\bar{v}} - w\bar{w}\right) - \frac{i\alpha}{2}\left(2w + w^2 - w\bar{w}v\bar{v} + \frac{1}{1-v\bar{v}}\right)\\ \delta v &= -\epsilon v^2 + \bar{\epsilon}[1+\bar{w}(1-v\bar{v})] + \frac{i\alpha}{2}v[1+\bar{w}(1-v\bar{v})]\,. \end{aligned} \tag{A.6}$$

Comparison of (A.4) and (A.6) yields:

$$T_\epsilon = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -i & i & 0 \end{pmatrix} \quad , \quad T_{\bar{\epsilon}} = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & -i \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad T_\alpha = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad . \tag{A.7}$$

It will be convenient for us to define

$$T_1 = 2T_\alpha\,, \qquad T_2 = -i(T_\epsilon - T_{\bar{\epsilon}})\,, \qquad T_3 = -(T_\epsilon + T_{\bar{\epsilon}}). \tag{A.8}$$

Clearly, as mentioned in the main text, T_1 and T_2 generate shifts along σ and φ respectively.

B An alternative dualization

An alternative dualization is also possible with respect to the two $U(1)$ isometries generated by T_1 and

$$T_2' = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad . \tag{B.1}$$

This is of interest as, following the same procedure as in the previous Appendix, one can verify that the generator T_2' induces precisely the symmetry (3.1). Since the latter can survive when only five-brane instanton contributions are taken into account (which means that only the isometry generated by T_1 is broken whereas the other Heisenberg symmetries are still continuous, as recalled in Section 4), it may be useful for some purposes to have a formulation in which this symmetry is explicit.

As T_2' commutes with T_1, we can write down a first order action

$$\mathrm{e}^V \bar{\Upsilon}^i[\eta_{im} + \eta_{ik}(V_1 - 4\hat{\chi}V)(T_1)^k{}_m](\mathrm{e}^{V_2 T_2'})^m{}_j \Upsilon^j - \tilde{\eta}_1 V_1 - \tilde{\eta}_2 V_2\ , \tag{B.2}$$

which, after shifting V_1 and V_2, can be rewritten as

$$\mathrm{e}^{\frac{1}{3}V_2}(X_0 - X_3 + V_1 X_1) + \mathrm{e}^{-\frac{2}{3}V_2 + 3V} X_3 - \tilde{\eta}_1(V_1 + 4\hat{\chi}V) - \tilde{\eta}_2(V_2 - 3V)\ , \quad X_3 \equiv \bar{\Upsilon}^3\Upsilon^3\ . \tag{B.3}$$

Varying with respect to V, V_1, V_2, we find

$$
\begin{aligned}
3\,\mathrm{e}^{-\frac{2}{3}V_2+3V}X_3 &= -3\tilde{\eta}_2 + 4\hat{\chi}\tilde{\eta}_1 \ , \\
e^{\frac{1}{3}V_2}X_1 &= \tilde{\eta}_1 \ , \\
\frac{1}{3}\left(\mathrm{e}^{\frac{1}{3}V_2}(X_0 - X_3 + V_1X_1)\right) - \frac{2}{3}\left(\mathrm{e}^{-\frac{2}{3}V_2+3V}X_3\right) &= \tilde{\eta}_2 \ .
\end{aligned}
\tag{B.4}
$$

Again, we can solve for V, V_1, V_2, and again we drop numerous total derivative terms to obtain:

$$
(\tilde{\eta}_2 - 4\hat{\chi}\tilde{\eta}_1)\ln(\tilde{\eta}_2 - 4\hat{\chi}\tilde{\eta}_1) - \tilde{\eta}_2\ln\tilde{\eta}_1 \ . \tag{B.5}
$$

This is a deformation of another form of the universal hypermultiplet that was written in [16]. It can be put into a slightly nicer form by the redefinition $\tilde{\eta}_2 \to \tilde{\eta}_2 + 4\hat{\chi}\tilde{\eta}_1$:

$$
G = \tilde{\eta}_2 \ln\frac{\tilde{\eta}_2}{\tilde{\eta}_1} - 2\hat{\chi}\,\tilde{\eta}_1\ln\tilde{\eta}_1 \ , \tag{B.6}
$$

where we have also taken into account the rescaling $\tilde{\eta}_1 \to \tilde{\eta}_1/2$ for the same reasons as in Section 3.2. Notice that the deformation term is of exactly the same form as the one in (3.4). It would be interesting to understand the action of the Heisenberg algebra in this basis, in particular the corresponding symmetries on the tensor multiplets $\tilde{\eta}_1$ and $\tilde{\eta}_2$.

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