The identification of conformal hypercomplex and quaternionic manifolds

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Abstract

We review the map between hypercomplex manifolds that admit a closed homothetic Killing vector (i.e. 'conformal hypercomplex' manifolds) and quaternionic manifolds of 1 dimension less. This map is related to a method for constructing supergravity theories using superconformal techniques. An explicit relation between the structure of these manifolds is presented, including curvatures and symmetries. An important role is played by 'ξ transformations', relating connections on quaternionic manifolds, and a new type '$\hat{\xi}$ transformations' relating complex structures on conformal hypercomplex manifolds. In this map, the subclass of conformal hyper-Kähler manifolds is mapped to quaternionic-Kähler manifolds.

Contents

1   Introduction

The paper [1] of Dmitri Alekseevsky received a lot of attention in the physics literature as the homogeneous quaternionic-Kähler spaces that he investigated occur in $N = 2$ supergravity theories [2]. Further applications of quaternionic geometry in supersymmetric theories have been discussed in [3].

In 1996, a general framework to study hypercomplex, hyper-Kähler, quaternionic and quaternionic-Kähler manifolds was given in [4]. These are commonly denoted by 'quaternionic-like manifolds'. Our paper [5] is in many respects a continuation of [4]. We will review here the main results of this work.

In this paper, we explain the 1-to-1 correspondence[1] (locally) between conformal hypercomplex manifolds of quaternionic dimension $n_H + 1$ and quaternionic manifolds of dimension n_H. Furthermore, we show that this 1-to-1 correspondence is also applicable between the subset of hypercomplex manifolds that are hyper-Kähler and the subset of quaternionic manifolds that are quaternionic-Kähler. The map between quaternionic-Kähler and hyper-Kähler manifolds is constructed by Swann [6], and its generalization to quaternionic manifolds is treated in [7]. Here, we give explicit expressions for the complex structures and connections, curvatures and symmetries. These results are useful in the context of the conformal tensor calculus in supergravity [8–12].

Our work was initiated by an investigation of couplings of hypermultiplets in supergravity. In section 2 we will give an overview of the geometric structures in supergravity theories and indicate where quaternionic geometry finds it place. Section 3 is for a large part a review of [4]. We give more details on the vielbeins in these manifolds, which is necessary to discuss supersymmetry. We will show the need of torsionless affine connections when discussing supersymmetric theories in spacetime dimensions $D = 3, 4, 5$ and 6. We will also give results on the relations between curvature decompositions of the quaternionic-like manifolds.

The main part of this review is section 4, where the precise correspondence between conformal hypercomplex manifolds of dimension $n_H + 1$ and quaternionic manifolds of dimension n_H is explained. We start that section by explaining the relevance of closed homothetic Killing vectors. Then the general structure of the map is exhibited. We explain the relevance of the ξ transformations in the quaternionic manifolds and the existence of similar so-called $\hat{\xi}$ transformations in conformal hypercomplex manifolds. We finish that section with a pictorial representation of the map. A short treatment of the symmetries of these manifolds is given in section 5. Such symmetries are a generalization of isometries that occur in manifolds with a metric.

We finish in section 6 with conclusions and some remarks on the relevance of the signature of the extra quaternion in the hypercomplex manifold.

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[1] As will be explained below, the correspondence is actually 1-to-1 between families (or 'equivalence classes') of manifolds.

2 Hypermultiplets and hypercomplex/quaternionic manifolds

Table 1 gives an overview of theories with rigid and local supersymmetry[2] in dimensions $D = 4$ to $D = 11$. The latter is the maximal dimension for supersymmetric field theories. The top row indicates the number of real independent components of the spinors describing the supersymmetry generators. The lowest row indicates which theories exist only for supergravity, or for supersymmetry and supergravity. Supergravity is the theory of *local* supersymmetry, i.e. where there is supersymmetry invariance for transformations that can differ in each spacetime point, as opposed to *rigid* supersymmetry where the same transformation should be applied for any point of spacetime. We will concentrate on the theories with 8 supercharges for a reason that we will now explain.

Table 1: *Supersymmetry and supergravity theories in dimensions 4 to 11.* An entry represents the possibility to have supergravity theories in a specific spacetime dimension D with the number of supersymmetries indicated in the top row. At the bottom is indicated whether these theories exist only in supergravity, or also with just rigid supersymmetry.

D	32	24	20	16	12	8	4
11	M						
10	IIA | IIB			I			
9	$N=2$			$N=1$			
8	$N=2$			$N=1$			
7	$N=4$			$N=2$			
6	$(2,2)$	$(2,1)$		$(1,1)$ | $(2,0)$		$(1,0)$	
5	$N=8$	$N=6$		$N=4$		$N=2$	
4	$N=8$	$N=6$	$N=5$	$N=4$	$N=3$	$N=2$	$N=1$
	SUGRA			SUGRA/SUSY	SUGRA	SUGRA/SUSY	

Nearly all these theories have scalar fields, which are maps from spacetime to a 'target space'. These target spaces have interesting geometrical properties. These geometries in the case of more than 8 real supercharges are shown in table 2. One notices that they are all symmetric spaces. On the other hand, the theories with 4 real supersymmetries, which are the $N = 1$ theories in 4 dimensions, lead to general Kähler manifolds. In both these cases, we thus obtain geometric structures that are well known. The geometrically interesting case are the supersymmetric theories with 8 real supercharges. The type of geometries depends on the occurrence of different representations (multiplets) of the supersymmetry algebra. For our purpose here, we can restrict our attention to vector multiplets and hypermultiplets. Vector multiplets in 6 dimensions do not have scalars and thus no associated target-space geometry. Vector multiplets in 5 dimensions have real scalars that parametrize a geometry

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[2]This table is a shorter version of a table in [13].

Table 2: *Scalar geometries in theories with more than 8 supersymmetries (and dimension ≥ 4).* The theories are ordered as in table 1. For more than 16 supersymmetries, there is a unique supergravity (up to gaugings irrelevant to the geometry), while for 16 and 12 supersymmetries there is a number n indicating the number of vector multiplets that are included.

D	32	24	20	16		12
10	$SO(1,1)$ $\left|\ \frac{SU(1,1)}{U(1)}\right.$					
9	$\frac{S\ell(2)}{SO(2)} \otimes SO(1,1)$			$\frac{SO(1,n)}{SO(n)} \otimes SO(1,1)$		
8	$\frac{S\ell(3)}{SO(3)} \otimes \frac{S\ell(2)}{SO(2)}$			$\frac{SO(2,n)}{SO(2)\times SO(n)} \otimes SO(1,1)$		
7	$\frac{S\ell(5)}{SO(5)}$			$\frac{SO(3,n)}{SO(3)\times SO(n)} \otimes SO(1,1)$		
6	$\frac{SO(5,5)}{SO(5)\times SO(5)}$	$\frac{SO(5,1)}{SO(5)}$		$\frac{SO(4,n)}{SO(n)\times SO(4)} \otimes SO(1,1)$	$\frac{SO(5,n)}{SO(n)\times SO(5)}$	
5	$\frac{E_{6,6}}{USp(8)}$	$\frac{SU^*(6)}{USp(6)}$		$\frac{SO(5,n)}{SO(5)\times SO(n)} \otimes SO(1,1)$		
4	$\frac{E_{7,7}}{SU(8)}$	$\frac{SO^*(12)}{U(6)}$	$\frac{SU(1,5)}{U(5)}$	$\frac{SU(1,1)}{U(1)} \times \frac{SO(6,n)}{SO(6)\times SO(n)}$		$\frac{SU(3,n)}{U(3)\times SU(n)}$

that is denoted by 'very special real geometry'. Those in 4 dimensions have complex scalars that parametrize a restricted class of Kähler geometries denoted by 'special Kähler geometry'. Furthermore, one can have hypermultiplets in dimensions $D = 6, 5, 4$ and 3. In dimensions $D = 6, 5, 4$ their scalars parametrize a quaternionic-Kähler manifold, while in $D = 3$ one can have a direct product of 2 quaternionic-Kähler manifolds, which is essentially due to the fact that the 'would be' vector multiplets in $D = 3$ occur as independent hypermultiplets in that case.

All these geometries for the case of 8 supersymmetries exist in two different versions: one which applies to rigid supersymmetry and one to supergravity. A schematic overview of these possibilities, for $D = 4$ and $D = 5$, is given in table 3. As indicated there, the

Table 3: *Geometries from supersymmetric theories with 8 real supercharges with vector multiplets and hypermultiplets.*

	$D = 5$ vector multiplets	$D = 4$ vector multiplets	hypermultiplets
rigid (affine)	affine very special real	affine special Kähler	hyper-Kähler
local (projective)	(projective) very special real	(projective) special Kähler	quaternionic-Kähler

geometries appearing in supergravity can be considered as projective versions of the 'affine'

ones in rigid supersymmetry. When one uses the terminology 'very special real manifolds' or 'special Kähler manifolds', one usually refers to the versions in supergravity. Very special real manifolds were first found in [14] and connected to special geometry in [15], where they got their name. Special Kähler geometry was found in [16], and denoted as such in [17]. A coordinate-independent formulation was found in [18, 19]. The version in rigid supersymmetry was first investigated in [20, 21]. This is called e.g. 'rigid special Kähler' or was appropriately called 'affine special Kähler' in [22] when the supergravity version is called projective special Kähler[3]. In that sense if hyper-Kähler geometry and quaternionic-Kähler geometry would not have got these names before, appropriate names would be affine, respectively projective quaternionic-Kähler geometry. This is in fact what we will clarify in this paper, using methods that connect also the manifolds of the other two columns of table 3.

Complex structures are endomorphisms J on the tangent space that square to $-\mathbb{1}$, and are 1-integrable. A hypercomplex structure has 3 such operations with $J^1 J^2 = J^3$, which we collectively denote by $\vec{J}$. Hermitian metrics g obey $g(JX, JY) = g(X, Y)$. This leads to the following characterization of manifolds that we mentioned here, which all have an hermitian metric:

$$
\begin{array}{rcl}
\text{Kähler manifolds} & : & \text{complex structure with } \nabla J = 0, \\
\text{hyper-Kähler man.} & : & \text{hypercomplex structure with } \nabla \vec{J} = 0, \\
\text{quaternionic-Kähler} & : & \text{hypercomplex structure with } \nabla \vec{J} + 2\vec{\omega} \times \vec{J} = 0.
\end{array} \qquad (2.1)
$$

The first two involve the Levi-Civita connection, while the latter condition involves moreover an SU(2) connection 1-form $\vec{\omega}$. The $\times$ symbol in this equation is the exterior product in the 3-dimensional vector space.

Up till now all the geometries were based on a manifold with a metric, which is in the physical theory related to the existence of a Lagrangian. The dynamical equations of motion then follow from Euler-Lagrange equations. However, in supersymmetric theories the dynamical equations may also be determined by the supersymmetry algebra. This leads to theories where the dynamics is only governed by field equations rather than by an action. To illustrate the difference, consider the action

$$
S = \int \mathrm{d}t \, \mathcal{L} = \int \mathrm{d}t \, g_{ij}(\phi) \frac{\mathrm{d}\phi^i}{\mathrm{d}t} \frac{\mathrm{d}\phi^j}{\mathrm{d}t}. \qquad (2.2)
$$

The Euler-Lagrange equation becomes the geodesic equation

$$
\frac{\mathrm{d}^2 \phi^i}{\mathrm{d}t^2} + \Gamma^i_{jk}(\phi) \frac{\mathrm{d}\phi^j}{\mathrm{d}t} \frac{\mathrm{d}\phi^k}{\mathrm{d}t} = 0. \qquad (2.3)
$$

Note that while (2.2) involves a metric, the geodesic equation involves only an affine connection. In this case, the affine connection is the Levi-Civita connection, but in general

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[3] A mathematical definition of very special Kähler geometry can be found in [3]. Definitions of special Kähler manifold independent of supergravity were given in [23], and a review appeared in [24]. Other mathematical definitions of special Kähler geometries have been given in [22, 25].

one could consider (2.3) with another affine connection. In the applications that we have in mind, the closure of the supersymmetry algebra leads directly to equations similar to (2.3), which do not necessarily involve a metric. In the above equation only a torsionless connection occurs. We will show below that in the supersymmetric theories that we consider we have to require the affine connection to be torsionless.

3 Quaternionic-like manifolds

3.1 Affine connections

We will now repeat some properties of the family of quaternionic-like manifolds. As we mentioned above, hypercomplex structures are defined by 3 endomorphisms denoted as $H = \{\vec{J}\} = \{J^1, J^2, J^3\}$. A quaternionic structure is the linear space $Q = \{\vec{a} \cdot \vec{J} | \vec{a} \in \mathbb{R}^3\}$. A hermitian bilinear form is a form $F(X, Y)$ with $F(JX, JY) = F(X, Y)$. A non-singular hermitian bilinear form is a 'good metric'. As such we can define a quartet of quaternionic-like manifolds in table 4. The table is essentially taken over from [4], where even more distinction has been made between various cases. The quaternionic manifolds represent the generic case that includes all the others as special cases.

Table 4: *Quaternionic-like manifolds of real dimension $4r$, and their holonomy groups.*

	no good metric	with a good metric	
no SU(2) connection	***hypercomplex*** $\mathrm{Aut}(H) =$ $\mathrm{G}\ell(r, \mathbb{H})$	***hyper-Kähler*** $\mathrm{Aut}(H, g) =$ $\mathrm{USp}(2r)$	rigid supersymmetry
non-zero SU(2) connection	***quaternionic*** $\mathrm{Aut}(Q) =$ $\mathrm{SU}(2) \cdot \mathrm{G}\ell(r, \mathbb{H})$	***quaternionic-Kähler*** $\mathrm{Aut}(Q, g) =$ $\mathrm{SU}(2) \cdot \mathrm{USp}(2r)$	supergravity
	field equations	action	

A hypercomplex manifold is equipped with an affine connection such that

$$\nabla \vec{J} = 0. \tag{3.1}$$

Given the hypercomplex structure, this connection is unique and is in general the sum of a so-called 'Obata connection' and the Nijenhuis tensor. The last part is the torsion, and as we mentioned that we are interested in torsionless connections, it should vanish.

For a quaternionic manifold we only need

$$\nabla Q \subset Q, \qquad \text{i.e.} \qquad \nabla \vec{J} + 2\vec{\omega} \times \vec{J} = 0. \tag{3.2}$$

In this case, the connection is not unique, even not for torsionless connections to which we will restrict here. Indeed, for a solution of (3.2) and any 1-form ξ we can construct other

Table 5: *The affine connections in quaternionic-like manifolds*

hypercomplex	**hyper-Kähler**
Obata connection	Obata connection
	= Levi-Civita connection
quaternionic	**quaternionic-Kähler**
Oproiu connection or	Levi-Civita connection =
other related by ξ_X transformation	connection related to Oproiu
	by a particular choice of ξ_X

solutions of (3.2) as

$$\nabla' = \nabla + S^\xi, \qquad \vec{\omega}' = \vec{\omega} + \vec{J}^*\xi, \tag{3.3}$$

with

$$S_X^\xi Y = \xi(X)Y + \xi(Y)X - \xi(\vec{J}X)\cdot\vec{J}Y - \xi(\vec{J}Y)\cdot\vec{J}X. \tag{3.4}$$

Of course, when we are discussing hyper-Kähler or quaternionic-Kähler manifolds, the affine connections should be the Levi-Civita connections. The condition (3.2) is a weaker condition than (3.1). The condition that this can be solved is that the Nijenhuis tensor is of the form

$$N_{XY}{}^Z = -\vec{J}_{[X}{}^Z \cdot \vec{\omega}_{Y]}^{\mathrm{Op}} = -\tfrac{1}{2}\vec{J}_X{}^Z \cdot \vec{\omega}_Y^{\mathrm{Op}} + \tfrac{1}{2}\vec{J}_Y{}^Z \cdot \vec{\omega}_X^{\mathrm{Op}}, \tag{3.5}$$

where $\vec{\omega}^{\mathrm{Op}}$ is at this point an arbitrary triplet of 1-forms. X, Y, Z in this equation are indices labelling the $4r$ coordinates q^X of the manifold. E.g. $\vec{\omega} = \vec{\omega}_X \mathrm{d}q^X$. We warn the reader that X has been used also as indication of a vector field, and it should be clear from the context what is meant. $\vec{\omega}^{\mathrm{Op}}$ is an SU(2) connection such that the Nijenhuis condition (3.5) guarantees the existence of corresponding affine connection coefficients

$$\Gamma^{\mathrm{Op}}{}_{XY}{}^Z \equiv \Gamma^{\mathrm{Ob}}{}_{XY}{}^Z - \tfrac{1}{2}\vec{J}_X{}^Z \cdot \vec{\omega}_Y^{\mathrm{Op}} - \tfrac{1}{2}\vec{J}_Y{}^Z \cdot \vec{\omega}_X^{\mathrm{Op}}, \tag{3.6}$$

where the first term involves the Obata connection coefficients. This solves (3.2) and is called the Oproiu connection. The connections for the different quaternionic manifolds are schematically shown in table 5. Note that the freedom of choice for ξ disappears in quaternionic-Kähler manifolds, where the ξ is determined by the requirement that the affine connection should coincide with the Levi-Civita connection.

3.2 Supersymmetry

We will illustrate here how the hypercomplex geometry arises in supersymmetric models, and why this needs torsionless connections. This statement holds[4] for supersymmetric theories in spacetime dimensions $D = 3, 4, 5$ or 6. The essence of this part is not dependent on whether

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[4]Hyper-Kähler manifolds with torsion do appear in 2-dimensional supersymmetric theories [26, 27].

we consider any of these dimensions, but the notation is simplest for $D = 5$ or 6, to which we will restrict ourselves for convenience. The hypermultiplets consist of fields $q^X(x)$ that are maps from spacetime with coordinates x^a to the quaternionic space with coordinates q^X, with $X = 1, \ldots, 4r$, and fermionic partners, which are spinors of spacetime $\zeta^A(x)$, where $A = 1, \ldots, 2r$, and spinor indices are suppressed. They belong to the irreducible spinor modules for SO(4, 1), resp. SO(5, 1). They are 'symplectic Majorana spinors' for $D = 5$ or 'symplectic Majorana-Weyl' for $D = 6$, using the symplectic matrix ρ^{AB}, which satisfies

$$\rho^{AB} = -\rho^{BA}, \qquad \rho_{AB} = (\rho^{AB})^*, \qquad \rho^{AB}\rho_{CB} = \delta^A_C. \tag{3.7}$$

We suppress below the dependence of the fields q^X and ζ^A on spacetime coordinates.

Supersymmetry is defined also by a symplectic spinor generator Q_i (for $i = 1, 2$, related to the notation $N = 2$ in table 1), where the symplectic matrix is $\varepsilon^{ij} = \varepsilon_{ij}$, with $\varepsilon^{12} = -\varepsilon^{21} = 1$, satisfying the same relations as in (3.7). The supersymmetry transformations are denoted as

$$\delta(\epsilon) = \bar{\epsilon}^i Q_i, \tag{3.8}$$

where ϵ^i are the parameters of the supersymmetry transformations and $\bar{\epsilon}^i$ is the Majorana conjugate.

The supersymmetry algebra is

$$[\delta(\epsilon_1), \delta(\epsilon_2)] = \tfrac{1}{2}\bar{\epsilon}^i_2 \gamma^a \epsilon^j_1 \varepsilon_{ji} P_a, \qquad a = 1, \ldots D. \tag{3.9}$$

where γ^a are the matrices used for the Clifford algebra, and satisfy $\gamma^a\gamma^b + \gamma^b\gamma^a = 2\eta^{ab}$ where $\eta = \text{diag}(-1, +1, \ldots, +1)$. The generators P_a are translations that act on the q^X as $P_a q^X = \partial_a q^X$, where ∂_a are the derivatives with respect to the spacetime coordinates x^a.

The supersymmetry transformations of the q^X take the general form

$$\delta(\epsilon)q^X \equiv \bar{\epsilon}^i Q_i q^X = -\mathrm{i}\bar{\epsilon}^i \zeta^A f^X_{iA}(q), \tag{3.10}$$

This implies for consistency reality conditions on the coefficients functions $f^X_{iA}(q)$:

$$(f^X_{iA})^* = \varepsilon^{ij}\rho^{AB} f^X_{jB}, \tag{3.11}$$

To generate a translation in the commutator on q^X according to (3.9), the supersymmetry transformation of the spinor must contain a term of the form

$$\delta(\epsilon)\zeta^A = \tfrac{1}{2}\mathrm{i}\gamma^a \epsilon_i \left[\partial_a q^X\right] f^{iA}_X(q) + \ldots, \tag{3.12}$$

where

$$f^{iA}_X f^Y_{iA} = \delta^Y_X, \tag{3.13}$$

i.e. the f^{iA}_X and f^X_{iA} are each others inverse as $4r \times 4r$ matrices. Note that we also suppress the dependence on the coordinates q^X. However, when calculating the commutator on q^X we have to take into account the q-dependence of f^X_{iA}, leading to a term

$$[\delta(\epsilon_1), \delta(\epsilon_2)] q^X = \ldots - \mathrm{i}\bar{\epsilon}^i_2 \zeta^A \left[\partial_Y f^X_{iA}\right] \left[\delta(\epsilon_1)q^Y\right] - (\epsilon_1 \leftrightarrow \epsilon_2). \tag{3.14}$$

To remove this term, we can modify the transformation of ζ^A by a term proportional to another ζ^B and a supersymmetry transformation of q^X, i.e. we complete (3.12) to a form

$$\delta(\epsilon)\zeta^A = \tfrac{1}{2}\mathrm{i}\gamma^a\epsilon_i\left[\partial_a q^X\right]f_X^{iA} - \zeta^B\omega_{XB}{}^A(q)\left[\delta(\epsilon)q^X\right], \tag{3.15}$$

where $\omega_{XB}{}^A(q)$ has to be determined. Using (3.10) in (3.14) and adding the contribution of the last term of (3.15), leads to

$$[\delta(\epsilon_1),\delta(\epsilon_2)]\,q^X \quad = \tfrac{1}{4}\bar{\epsilon}_2^i\gamma^a\epsilon_1^j\varepsilon_{ji}\partial_a q^X \quad +f_Z^{iA}\left[\delta(\epsilon_2)q^Z\right]\partial_Y f_{iA}^X\left[\delta(\epsilon_1)q^Y\right] \tag{3.16}$$
$$- \left[\delta(\epsilon_2)q^Z\right]f_Z^{iB}f_{iA}^X\omega_{YB}{}^A\left[\delta(\epsilon_1)q^Y\right] - (\epsilon_1 \leftrightarrow \epsilon_2).$$

In order that the latter terms do not contribute to the commutator, they should add to a symmetric expression in (YZ) for any X, which we denote as Γ_{YZ}^X:

$$f_Z^{iA}\partial_Y f_{iA}^X - f_Z^{iB}f_{iA}^X\omega_{YB}{}^A = -\Gamma_{YZ}^X. \tag{3.17}$$

This equation is equivalent to the requirement

$$\partial_Y f_{iA}^X - \omega_{YA}{}^B f_{iB}^X + f_{iA}^Z\Gamma_{YZ}^X = 0. \tag{3.18}$$

This is the condition of covariant constancy of a vielbein f_{iA}^X in the quaternionic manifold with a torsionless connection $\Gamma_{YZ}^X = \Gamma_{ZY}^X$ and $\omega_{XA}{}^B$ is a $\mathrm{G}\ell(r,\mathbb{H})$ connection (written as $2r \times 2r$ complex matrices with reality properties determined by the consistency in the transformation (3.15)).

We want to remark that in the case of local supersymmetry, this condition can be relaxed. Indeed, in this case the algebra can contain a field-dependent supersymmetry transformation. This means that (3.17) can include in the right-hand side a term of the form $-f_{iA}^X f_Z^{jA}\omega_{Yj}{}^i(q)$, where $\omega_{Yj}{}^i(q)$ is arbitrary and defines an SU(2) connection. This possibility does not apply to rigid supersymmetry as it involves a field-dependent supersymmetry transformation.

3.3   Vielbeins

The supersymmetry analysis of the previous subsection leads to the result that for rigid supersymmetry one needs a vielbein that satisfies the integrability condition (3.18), including a torsionless connection and a connection for $\mathrm{G}\ell(r,\mathbb{H})$. The vielbeins determine the hypercomplex structure as

$$\vec{J}_X{}^Y = -\mathrm{i}\vec{\sigma}_i{}^j f_X^{iA} f_{jA}^Y. \tag{3.19}$$

The factors $f_X^{iA} f_{jA}^Y$ define for any X, Y, a 2×2 matrix with trace δ_X^Y. The $\vec{\sigma}$ are the traceless Hermitian Pauli matrices and project the other 3 components of the 2×2 matrix. These expressions automatically satisfy the relations for a hypercomplex structure. Moreover, the covariant constancy of the vierbein according to (3.18) implies the covariant constancy (3.1) of the hypercomplex structure.

The condition (3.18) can always be solved for the connection $\omega_{XA}{}^B$ once we know the vielbein and the Γ_{YZ}^X. As mentioned above, the torsionless connection can be found if the Nijenhuis tensor vanishes, and is in that case uniquely determined.

For quaternionic manifolds, the vielbeins should satisfy[5]

$$\partial_X f_Y^{iA} - \Gamma_{XY}^Z f_Z^{iA} + f_Y^{jA} \omega_{Xj}{}^i + f_Y^{iB} \omega_{XB}{}^A = 0. \tag{3.20}$$

Also here, this equation determines $\omega_{XA}{}^B$ once we know the other connections. But now the latter are not uniquely defined by (3.2), but allow the ξ-transformations (3.3).

3.4 Curvature decompositions

The relation (3.20) has as integrability condition a relation between curvatures:

$$\begin{aligned} R_{XYW}{}^Z &= R^{\mathrm{SU}(2)}{}_{XYW}{}^Z &+ R^{\mathrm{G}\,\ell(r,\mathbb{H})}{}_{XYW}{}^Z \\ &= -\vec{J}_W{}^Z \cdot \vec{\mathcal{R}}_{XY} &+ L_W{}^Z{}_A{}^B \, \mathcal{R}_{XYB}{}^A. \end{aligned} \tag{3.21}$$

The left-hand side is the curvature defined by the affine connection Γ_{YZ}^X, while $\vec{\mathcal{R}}_{XY}$ is the curvature determined by the SU(2) connection $\vec{\omega}_X$ and $\mathcal{R}_{XYB}{}^A$ is determined by $\omega_{XB}{}^A$. The object $L_W{}^Z{}_A{}^B$ is defined similar to the complex structures (3.19), but with contraction over indices i rather than A:

$$L_W{}^Z{}_A{}^B \equiv f_{iA}^Z f_W^{iB}. \tag{3.22}$$

The relation (3.21) holds for general quaternionic manifolds. In the case of hypercomplex or hyper-Kähler manifolds, the SU(2) term is absent. The Ricci tensor Ric determined by the curvature $R_{XYW}{}^Z$ can in general have an antisymmetric part if the trace of the $\mathrm{G}\,\ell(r,\mathbb{H})$ curvature is non-vanishing:

$$Ric_{[XY]} = R_{Z[XY]}{}^Z = -\mathcal{R}_{XY}^{\mathbb{R}} \equiv -\mathcal{R}_{XYA}{}^A. \tag{3.23}$$

As the Ricci tensor associated to a Levi-Civita connection is symmetric, this has to vanish in the case of hyper-Kähler or quaternionic-Kähler manifolds.

An unnatural feature of the splitting (3.21) is that the individual terms do not satisfy the first Bianchi identity. An alternative splitting is

$$R_{XYZ}{}^W = R^{\mathrm{Ric}}{}_{XYZ}{}^W - \frac{1}{2} f_X^{Ai} \varepsilon_{ij} f_Y^{jB} f_W^{kC} f_{kD}^Z \mathcal{W}_{ABC}{}^D. \tag{3.24}$$

Here both curvature tensors do satisfy the first Bianchi identity. The first term is called the Ricci part because it is determined only by the Ricci tensor. The second part is called the Weyl part. The Ricci tensor of the Weyl part vanishes. It is determined by a tensor $\mathcal{W}_{ABC}{}^D$ that is symmetric in its lower indices and traceless. The proofs of these statements are reviewed in Appendix B of [28].

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[5]One can make the transition from doublet to vector notation by using the sigma matrices, $\omega_{Xi}{}^j = \mathrm{i}\vec{\sigma}_i{}^j \cdot \vec{\omega}_X$, and similarly $\vec{\omega}_X = -\frac{1}{2}\mathrm{i}\vec{\sigma}_i{}^j \omega_{Xj}{}^i$. This transition between doublet and triplet notation is valid for any triplet object as e.g. the complex structures.

We can summarize these curvature decompositions in the following scheme:

$$R_{XYZ}{}^W = \; (R^{\text{Ric}}_{\text{symm}} \quad + \qquad\qquad R^{\text{Ric}}_{\text{antis}} \qquad\qquad + \quad R^{(\text{W})})_{XYZ}{}^W \tag{3.25}$$

$$= \; (R^{\text{SU}(2)} \quad + \qquad\qquad R^{\mathbb{R}} \qquad\qquad + \quad R^{\text{S}\,\ell(r,\mathbb{H})})_{XYZ}{}^W.$$

In this decomposition we made two further splits. The Ricci curvature has been separated in a part determined by the symmetric part of the Ricci tensor, and a part determined by its antisymmetric part. On the other hand, the $\text{G}\,\ell(r,\mathbb{H})$ has been split into $\mathbb{R} \times \text{S}\,\ell(r,\mathbb{H})$.

The terms in the second line depend only on specific terms of the first line as indicated by the arrows. This is the general scheme and thus applicable for quaternionic manifolds, which is the general case. For hypercomplex manifolds, we mentioned already that there is no $\text{SU}(2)$ part, and in the upper decomposition there is no symmetric part. In this case, the full curvature can be written as

$$R_{XYZ}{}^W = -\frac{1}{2} f_X^{Ai} \varepsilon_{ij} f_Y^{jB} f_W^{kC} f_{kD}^Z W_{ABC}{}^D, \tag{3.26}$$

where the trace $W_{ABC}{}^C$ determines the antisymmetric Ricci tensor, and the traceless part of $W_{ABC}{}^D$ is $\mathcal{W}_{ABC}{}^D$ that appears in (3.24).

On the other hand, in quaternionic-Kähler manifolds there can be no antisymmetric Ricci part and no $\mathbb{R}$-curvature. For hyper-Kähler manifolds, both restrictions apply, and one has only the right-most terms in both decompositions.

The ξ-transformation can always be used to choose connections such that the $\mathbb{R}$-curvature vanishes.

Finally, we want to make a remark about the curvature of quaternionic-Kähler manifolds. These are Einstein manifolds with

$$Ric_{XY} = \nu(r+2)g_{XY}, \qquad \vec{\mathcal{R}}_{XY} = \tfrac{1}{2}\nu\vec{J}_{XY}, \tag{3.27}$$

where ν is an arbitrary real number. In supergravity this number is related to Newton's gravitational constant:

$$\nu = -\kappa^2 = -8\pi G_N. \tag{3.28}$$

Therefore only negative values of ν appear in supergravity, and the scalar curvature is negative. This implies that the manifold is non-compact (if there is at least one isometry).

4 Conformal symmetry and the map

Constructing supergravity theories is more complicated than constructing rigid supersymmetric theories. There exists a method to construct supergravity theories that starts from the rigid theories. These rigid theories should be invariant under superconformal transformations. Then one can gauge the superconformal group, and afterwards break explicitly the

symmetries that are extra with respect to the super-Poincaré group. Indeed, the final goal is only to obtain theories that are invariant under this latter group. However, the construction via the superconformal group simplifies the calculations due to the larger amount of symmetry.

The gauge-fixing procedure consists in choosing a parametrization such that for every extra symmetry there is a unique field that transforms under it. Whenever this happens, the statement of symmetry is just that this field is irrelevant. The remaining fields are then the physical fields, and they do not feel the extra symmetries. The main extra symmetry is the dilatation. We will choose one field that describes the scale, which is the field that is fixed in the procedure described above. The superconformal group includes for our case also an SU(2) group which we will use to eliminate a further 3 fields. Mathematically this means that we consider first a projective version of the theory that we want to describe.

A conformal symmetry amounts to the presence of a vector k such that the Lie derivative of the metric is proportional to the metric: $\mathcal{L}_k g = w\,g$ for a constant (positive) w. Such a vector is called a homothetic Killing vector. Special conformal transformations need an extra condition, namely that the one-form $g(k, \cdot)$ is closed. The combination of these two conditions can be written as

$$\nabla_X k = \tfrac{1}{2} w X. \tag{4.1}$$

A vector that satisfies this condition is 'a closed homothetic Killing vector'. We use the normalization $w = 3$. It is important to notice that the condition (4.1) is independent of a metric. Therefore we can use the concept of closed homothetic Killing vectors for manifolds without a good metric, despite the fact that homothetic Killing vectors are only defined with respect to a metric.

The presence of the hypercomplex structure implies that the vectors

$$\vec{k} = \tfrac{1}{3} \vec{J} k \tag{4.2}$$

generate an SU(2), which is the subgroup of the superconformal group mentioned above.

The general strategy is thus to start with a hypercomplex manifold that has a closed homothetic Killing vector. We will denote such a manifold as a 'conformal hypercomplex manifold'. Assume that the dimension of this manifold is $4(n_H+1)$. Then we will isolate 4 directions in this manifold that transform under the dilatation and the SU(2) transformations, and the orthogonal $4n_H$ dimensional manifold that is invariant. This invariant submanifold then inherits the property of being quaternionic. Some of these steps have already been performed in [10]. Recently, we [5] clarified in this way the general structure of the map, especially showing its one to one character. This we will review below.

4.1 Conformal hypercomplex manifolds

Following the previous ideas, it is appropriate to choose coordinates adapted to the conformal structure. As such we define a first coordinate that we will denote by z^0 such that the closed homothetic Killing vector points in this direction. Denoting the coordinates of this hypercomplex manifold by $q^{\hat{X}}$, we choose

$$k^{\hat{X}} = \delta_0^{\hat{X}} k^0 = 3\delta_0^{\hat{X}} z^0. \tag{4.3}$$

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The factor 3 is purely a matter of normalization. Then we choose 3 more coordinates such that the vectors $\vec{k}$ in (4.2) only point in these three directions. We denote these directions with an index $\alpha = 1, 2, 3$, and thus the vectors $\vec{k}$ have only nonzero components $\vec{k}^\alpha$. All other components of $q^{\hat{X}}$ are denoted by q^X:

$$q^{\hat{X}} = \left\{ z^0, z^\alpha, q^X \right\}. \tag{4.4}$$

The strategy is to fix z^0 by a gauge choice for dilatations, and z^α by fixing the SU(2) symmetries. The gauge-fixed manifold thus contains only the directions q^X, and will be the quaternionic manifold.

This choice of coordinates and the hypercomplex algebra imply that the hypercomplex structures decompose as

$$
\begin{aligned}
\widehat{\vec{J}}_0{}^0 &= 0, & \widehat{\vec{J}}_\alpha{}^0 &= -z^0 \vec{m}_\alpha, & \widehat{\vec{J}}_X{}^0 &= z^0 \vec{A}_X, \\
\widehat{\vec{J}}_0{}^\beta &= \tfrac{1}{z^0} \vec{k}^\beta, & \widehat{\vec{J}}_\alpha{}^\beta &= \vec{k}^\beta \times \vec{m}_\alpha, & \widehat{\vec{J}}_X{}^\beta &= \vec{A}_X \times \vec{k}^\beta + \vec{J}_X{}^Z \left(\vec{A}_Z \cdot \vec{k}^\beta \right), \\
\widehat{\vec{J}}_0{}^Y &= 0, & \widehat{\vec{J}}_\alpha{}^Y &= 0, & \widehat{\vec{J}}_X{}^Y &= \vec{J}_X{}^Y.
\end{aligned}
\tag{4.5}
$$

In this equation $\vec{m}_\alpha$ are the inverse of $\vec{k}^\alpha$ as 3×3 matrices:

$$\vec{k}^\alpha \cdot \vec{m}_\beta = \delta_\beta^\alpha. \tag{4.6}$$

Note that (4.5) depends on z^0, $\vec{k}^\alpha$, $\vec{A}_X$ and $\vec{J}_X{}^Y$. We mentioned already that z^0 is a scale variable and $\vec{k}^\alpha$ are the SU(2) Killing vectors. Furthermore, there is the triplet of one-forms $\vec{A} = \vec{A}_X \mathrm{d} q^X$, which are arbitrary up to this point and the $\vec{J}_X{}^Y$, which satisfy the hypercomplex algebra by itself. The latter span the quaternionic structure on the $4n_H$-dimensional submanifold.

Up to now, the matrices (4.5) define an almost hypercomplex structure. In order to become an hypercomplex structure we need to impose the vanishing of the Nijenhuis tensor $N_{\hat{X}\hat{Y}}{}^{\hat{Z}} = 0$. This leads to two further conditions:

- The curvature of the triplet $\vec{A}$ is related to the complex structure:

$$\left(2\mathrm{d}\vec{A} - \vec{A} \times \vec{A} \right)(X, Y) = h(\vec{J}X, Y) - h(\vec{J}Y, X), \tag{4.7}$$

  where the wedge product between the one forms $\vec{A}$ is understood. h is an arbitrary symmetric bilinear form. Here the X, Y denote vectors of the $4n_H$-dimensional subspace.

- The subspace is quaternionic. This means that the Nijenhuis tensor of the $J_X{}^Y$ satisfies (3.5), with

$$\vec{\omega}_X^{\mathrm{Op}} = -\tfrac{1}{6} \left(2\vec{A}_X + \vec{A}_Y \times \vec{J}_X{}^Y \right). \tag{4.8}$$

This form of the SU(2) connection gives the Oproiu connection. We know already that we can use the freedom of ξ-transformations in (3.3) to obtain other forms of the SU(2) connections. In this way, we can simplify it to

$$\vec{\omega}_X = -\tfrac{1}{2}\vec{A}_X.$$
(4.9)

This choice has further advantages: the $\mathbb{R}$-curvature of the quaternionic manifold is the same as the $\mathbb{R}$-curvature of the hypercomplex manifold. As a consequence, in this ξ-choice, the $\mathbb{R}$-curvatures of the quaternionic manifold is Hermitian, because $\mathbb{R}$-curvatures of hypercomplex manifolds are always Hermitian [4].

4.2 The $\hat{\xi}$-transformations

The above analysis of the general form of the hypercomplex structures for conformal hypercomplex manifolds leads to a new transformation [5], which is similar to the ξ-transformations discussed in section 3.1. Indeed, we can consider changes of the triplet 1-forms $\vec{A}$ in (4.5), such that (4.7) remains satisfied. Such changes are determined by a 1-form on the quaternionic space, $\hat{\xi}$:

$$\delta(\widehat{\xi})\vec{A} = 2\vec{J}^*\widehat{\xi}.$$
(4.10)

These induce therefore transformations on the hypercomplex structures, preserving the hypercomplex algebra. Notice that this implies that the $\hat{\xi}$ transformations have a different meaning than the ξ-transformations. The latter do not transform the complex structures, but only the connections.

We can use these $\hat{\xi}$ transformations to eliminate the $\mathbb{R}$-curvature of the conformal hypercomplex manifold.

4.3 Metric spaces

If the conformal hypercomplex manifold allows a good metric, i.e. when it is hyper-Kähler, one can show that it should be of the form

$$\mathrm{d}\widehat{s}^2 = -\frac{(\mathrm{d}z^0)^2}{z^0} + z^0\Big\{h_{XY}\mathrm{d}q^X\mathrm{d}q^Y$$
$$-\vec{m}_\alpha\cdot\vec{m}_\beta[\mathrm{d}z^\alpha - \vec{A}_X\cdot\vec{k}^\alpha\mathrm{d}q^X][\mathrm{d}z^\beta - \vec{A}_Y\cdot\vec{k}^\beta\mathrm{d}q^Y]\Big\},$$
(4.11)

where h_{XY} are the components of the bilinear form that was introduced in (4.7).

We find furthermore that the large space is a hyper-Kähler manifold if and only if the submanifold is quaternionic-Kähler. This is the case if h_{XY} is Hermitian and invertible. Then

$$g_{XY} = z^0 h_{XY}$$
(4.12)

is the metric on the quaternionic-Kähler manifold. We see that its scale is determined by the choice of z^0. Indeed, it determines the value ν in (3.27):

$$\nu = -\frac{1}{z^0}.$$
(4.13)

The SU(2) connection of the quaternionic-Kähler manifold is still given by (4.9). It can be shown that this corresponds precisely to the ξ-gauge for which the affine connection coincides with the Levi-Civita connection.

4.4 Curvature mapping

We can now compare the curvature decompositions discussed in (3.4) for the large and small manifolds. This leads to the following scheme:

$$
\begin{array}{ccccccc}
\widehat{R} & = & & \widehat{R}^{\mathrm{Ric}}_{\mathrm{antis}} & + & \widehat{R}^{(\mathrm{W})} \\
& & & \uparrow & \nearrow & \uparrow \\
& h_{XY} & & \widehat{W}_{ABC}{}^{C} & & \mathcal{W}_{ABC}{}^{D} \\
& \downarrow & & \downarrow & & \downarrow \\
R & = & R^{\mathrm{Ric}}_{\mathrm{symm}} & + & R^{\mathrm{Ric}}_{\mathrm{antis}} & + & R^{(\mathrm{W})}
\end{array}
\tag{4.14}
$$

The upper line and lower line are respectively the curvature decompositions of the hypercomplex and quaternionic manifolds. One remarks that h determines the symmetric Ricci tensor. On the other hand the trace $\widehat{W}_{ABC}{}^{C}$ determines the antisymmetric Ricci tensors of as well the hypercomplex as the quaternionic space, while the traceless tensor $\mathcal{W}_{ABC}{}^{D}$ contributes to the Weyl curvature of both manifolds. For hyper-Kähler and quaternionic-Kähler manifolds, $\widehat{W}_{ABC}{}^{C} = 0$, and $\mathcal{W}_{ABC}{}^{D} = \widehat{W}_{ABC}{}^{D}$, such that there are no antisymmetric Ricci tensors.
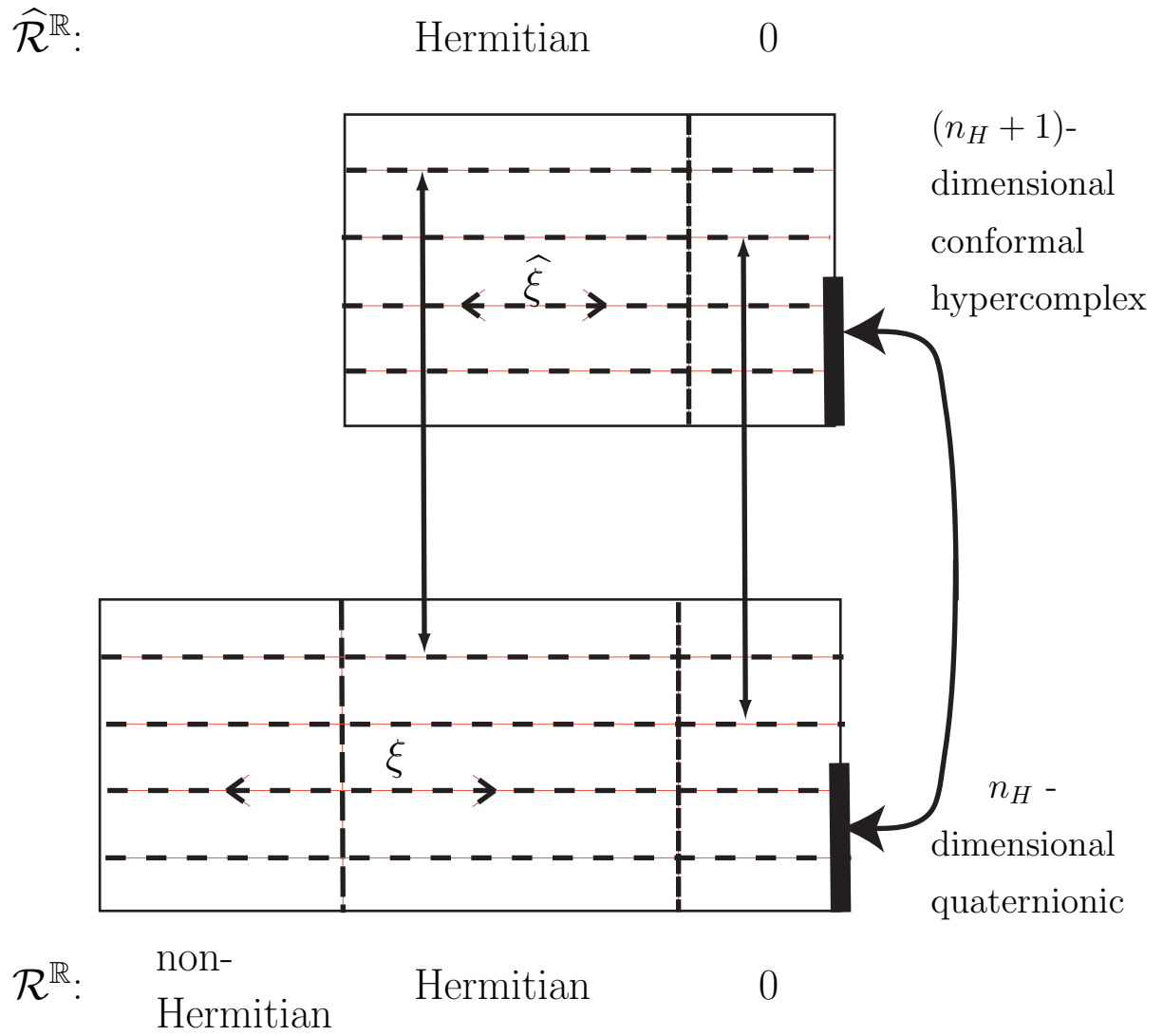
4.5 The picture of the map

The figure 1 gives a schematic overview of our results. The two blocks represent the families of large (upper block) and small spaces (lower block), where the horizontal lines indicate how they are related by $\widehat{\xi}$, resp. ξ, transformations. They connect parametrizations of the same manifold with different complex structures for hypercomplex manifolds, and different affine and SU(2) connections for the quaternionic manifolds. On the far right, the spaces have no $\mathbb{R}$ curvature, and part of these are hyper-Kähler, resp. quaternionic-Kähler. The latter two classes are indicated by the thick lines. The vertical arrows represent the map described in this review, connecting the manifolds with similar parametrizations. The map between hypercomplex and quaternionic manifolds is not one point to one point on this picture, as a hypercomplex manifold is represented by a full horizontal line where each point is a particular parametrization. The same holds for the quaternionic manifolds. The vertical lines are a representation of the map between the horizontal lines. For some manifolds there is a representation as a hyper-Kähler (or quaternionic-Kähler) manifold. The thick arrow indicates the map between hyper-Kähler and quaternionic-Kähler spaces.

5 Symmetries

In this section we introduce symmetries as generalizations of isometries. Indeed, we do not have necessarily a metric and thus no Killing vectors. We will comment on the moment maps

$\widehat{\mathcal{R}}^{\mathbb{R}}$:

Hermitian   0

$(n_H + 1)$-
dimensional
conformal
hypercomplex

$\widehat{\xi}$

ξ

n_H -
dimensional
quaternionic

$\mathcal{R}^{\mathbb{R}}$:

non-
Hermitian   Hermitian   0

of quaternionic manifolds and show how they originate from the map between conformal hypercomplex and quaternionic spaces.

If there is a metric, symmetries are generated from Killing vectors k_I^X, where I labels the different generators. From the Killing equation one derives that

$$\nabla_X \nabla_Y k_I^Z = R_{XWY}{}^Z k_I^W. \tag{5.1}$$

This condition is independent of a metric. It turns out that a shift of the coordinates in the Euler-Lagrange equations (2.3) by an amount k_I leaves the set of these equations invariant if (5.1) is satisfied. Hence this becomes the defining equation for symmetries when there is no good metric available. In the presence of a quaternionic structure, we also demand that the Lie derivative of the quaternionic structure rotates them, i.e.

$$\mathcal{L}_{k_I} \vec{J} = \vec{r}_I \times \vec{J}, \tag{5.2}$$

for some triplet of functions $\vec{r}_I$. These are called quaternionic symmetries. For hypercomplex (and hyper-Kähler) manifolds, the hypercomplex structures should be invariant, which means that $\vec{r}_I = 0$ in the above equation, and the symmetries are then called triholomorphic.

It can be shown that this equation is equivalent to the existence of a decomposition of the derivatives of k_I in an $SU(2)$ part and a $G\ell(r, \mathbb{H})$ part, similar to the decomposition of the curvature in (3.21):

$$\nabla_X k_I^Y = \nu \vec{J}_X{}^Y \cdot \vec{P}_I + L_X{}^Y{}_A{}^B t_{IB}{}^A. \tag{5.3}$$

This defines the triplet 'moment map' $\vec{P}_I$. Its value is related to the triplet $\vec{r}_I$ in (5.2):

$$\nu \vec{P}_I \equiv -\tfrac{1}{2} \vec{r}_I - \vec{\omega}(k_I). \tag{5.4}$$

Since connections can change by ξ-transformations, so will the symmetry condition depend on the choice of ξ. We have shown [5] that a symmetry is preserved only under ξ transformations that satisfy

$$\mathcal{L}_{k_I} \xi = 0. \tag{5.5}$$

Under such a transformation the moment map transforms as

$$\nu \widetilde{\vec{P}}_I = \nu \vec{P}_I - \xi(\vec{J} k_I). \tag{5.6}$$

How do symmetries relate under the map between conformal hypercomplex and quaternionic manifolds? First it is important to realize that the closed homothetic Killing vector is itself a triholomorphic symmetry according to the definition (5.1) and (5.2). For any other 'symmetry' of the conformal hypercomplex manifold we demand that it commutes with the closed homothetic Killing vector. For such symmetries there is then a one-to-one mapping with symmetries of the quaternionic submanifold. The precise relation is expressed by giving the components of this symmetry vector in the large space:

$$\widehat{k}_I^0 = 0, \qquad \widehat{k}_I^\alpha = \vec{k}^\alpha \cdot \vec{r}_I, \qquad \widehat{k}_I^X = k_I^X(q). \tag{5.7}$$

The last relations says that the symmetry vector of the small space is just the symmetry vector of the large space projected to the small space. The components $\widehat{k}_I^\alpha$ that disappear after this projection are related to the moment maps, in the sense that they define the $\vec{r}_I$.

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Table 6: *The 1-to-1 map in quaternionic-like manifolds*

CONFORMAL hypercomplex	*CONFORMAL* hyper-Kähler
$\updownarrow$	$\updownarrow$
quaternionic	*quaternionic-Kähler*

6   Conclusions and final remarks

We have illustrated how the picture of quaternionic-like manifolds can be extended by a mapping as in table 6. To obtain this map we started with a conformal hypercomplex manifold of dimension $4n_H + 4$, i.e. a manifold admitting a closed homothetic Killing vector k. This vector defines slices of the manifold $z^0 =$ constant, where z^0 is the coordinate such that k has only the non-zero component k^0. This slice defines a $4n_H + 3$ dimensional manifold which is a tri-Sasakian manifold. These manifolds still possess an SU(2) symmetry, which can be divided out such that we are left with a $4n_H$ dimensional manifold, which turns out to be quaternionic. Inversely, any quaternionic manifold can locally be embedded in such a conformal hypercomplex manifold of dimension 4 higher.

For quaternionic-Kähler manifolds, the curvature satisfies a relation (3.27) depending on a number ν that sets the scale. We find here that in this picture, ν depends on the slice, i.e. $\nu = -(z^0)^{-1}$. In supergravity the value of z^0 is the square of the Planck mass M_{Planck}^2. On the other hand, the sign of z^0 determines the signature of the extra 4 dimensions in uplifting a quaternionic-Kähler manifold to a conformal hyper-Kähler manifold. E.g. if the quaternionic-Kähler manifold has a completely positive signature, and z^0 is positive (ν negative), the signature of the hyper-Kähler manifold is

$$(- - - - + + + + \cdots + + + +). \tag{6.1}$$

The construction that we presented can be applied to any signature of the quaternionic space and ν positive or negative. This gives then an arbitrary signature $(4p, 4q)$ for the hyper-Kähler manifold. With ν negative as in supergravity the quaternionic-Kähler manifold has negative scalar curvature and this includes non-compact symmetric spaces. But the construction can thus easily be applied to compact symmetric spaces as well.

The construction of the map that we presented uses heavily the ξ-transformations of connections in the quaternionic manifolds. We have shown that there are analogous $\hat{\xi}$ transformations in conformal hypercomplex manifolds. The latter transformations act on the hypercomplex structure, and are in this respect different from the ξ transformations. But the map that we have constructed is compatible with both these transformations.

Finally, we have discussed the map between triholomorphic symmetries of the conformal hypercomplex manifold commuting with dilatations and quaternionic symmetries of the quaternionic manifolds. It was shown that, apart form the dilatation symmetry, they relate

one-to-one. The moment maps of quaternionic symmetries are related to components of the symmetry vector of the hypercomplex manifolds orthogonal to the quaternionic manifold.

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