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Quaternion-Kähler spaces, hyperkähler cones, and the c-map

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Abstract

Under the action of the c-map, special Kähler manifolds are mapped into a class of quaternion-Kähler spaces. We explicitly construct the corresponding Swann bundle or hyperkähler cone, and determine the hyperkähler potential in terms of the prepotential of the special Kähler geometry.

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1 Introduction

The moduli spaces of Kähler and complex structure deformations of Calabi-Yau manifolds are naturally related to Special Kähler (SK) and quaternion-Kähler (QK) geometry. Consequently, these types of manifolds arise in the low energy effective action of string theory compactifications on Calabi-Yau three-folds.

SK manifolds were discovered in the context of $N = 2$ supergravity[1] theories coupled to vector multiplets [1]. They are described by a holomorphic function $F(X)$ that is homogeneous of degree two in complex coordinates X^I. A more mathematically precise and intrinsic formulation of this special geometry was given in [2, 3].

QK manifolds arose in the context of $N = 2$ supergravity coupled to hypermultiplets [4]. QK manifolds are also described by a single function. This follows from the construction of the Swann bundle [5] over the QK space. This bundle is hyperkähler; locally, its metric is determined by a hyperkähler potential $\chi(\phi)$, where ϕ are local coordinates on the space. In [6], we called such spaces hyperkähler cones (HKC's) because they have a homothety arising from the underlying conformal symmetry. One therefore also uses the terminology conformal hyperkähler manifolds, as in [7].

In this note, we review the construction of Quaternion-Kähler (QK) manifolds from special Kähler (SK) geometry, along the lines of our recent work [8], but with more emphasis on the mathematical structure. One constructs QK spaces from SK manifolds

[1]The supersymmetry transformations have eight real components.

by using the c-map [9]. This maps extends Calabi's construction of hyperkähler metrics on cotangent bundles of Kähler manifolds [10, 11]–a construction well known in the mathematics community–to quaternionic geometry.

As we shall see, the hyperkähler cones arising from the c-map have additional symmetries: they have an equal number of commuting triholomorphic isometries as their quaternionic dimension. Hyperkähler manifolds with such isometries were classified in [12] by performing a Legendre transform on the (hyper)kähler potential [13] and writing the result in terms of a contour integral of a meromorphic[2] function H. In our case, because of the conformal symmetry of the HKC, this function H is homogeneous of degree one. As a result, the c-map induces a map from the holomorphic function F, which characterizes the SK geometry and is homogeneous of degree two, to a function H, which characterizes the QK geometry and is homogeneous of degree one. Following [8], we now describe this construction.

2 The c-map

In this section, we introduce our notation and review the c-map [9, 14].

Consider an affine (or rigid) special Kähler manifold[3] of dimension $2(n+1)$. It is characterized by a holomorphic prepotential $F(X^I)$, which is homogeneous of degree two ($I = 1, \cdots, n+1$). The Kähler potential and metric of the rigid special geometry are given by

$$K(X,\bar{X}) = i(\bar{X}^I F_I - X^I \bar{F}_I) \ , \quad \mathrm{d}s^2 = N_{IJ}\,\mathrm{d}X^I \mathrm{d}\bar{X}^J \ , \quad N_{IJ} = i(F_{IJ} - \bar{F}_{IJ}) \ , \qquad (2.1)$$

where F_I is the first derivative of F, *etc.*

The projective (or local) special Kähler geometry is then of real dimension $2n$, with complex inhomogeneous coordinates

$$Z^I = \frac{X^I}{X^1} = \{1, Z^A\} \ , \qquad (2.2)$$

where A runs over n values. Its Kähler potential is given by

$$\mathcal{K}(Z,\bar{Z}) = \ln(Z^I N_{IJ} \bar{Z}^I) \ . \qquad (2.3)$$

We further introduce the matrices [1]

$$\mathcal{N}_{IJ} = -i\bar{F}_{IJ} - \frac{(NX)_I (NX)_J}{(XNX)} \ , \qquad (2.4)$$

[2] Actually, certain branch cut singularities sometimes arise.

[3] We use the language of local coordinates, but a coordinate free description can be found in [2, 3].

where $(NX)_I \equiv N_{IJ}X^J$, *etc.*

The c-map defines a $4(n+1)$-dimensional quaternion-Kähler metric as follows: One builds a G-bundle, with $2(n+2)$ dimensional fibers coordinatized by ϕ, σ, A^I and B_I, over the projective special Kähler manifold; the real group G is a semidirect product of a Heisenberg group with $\mathbb{R}$, and acts on the coordinates by:

$$
\begin{aligned}
A^I \to \mathrm{e}^{\beta}(A^I+\epsilon^I) \quad , \qquad & B_I \to \mathrm{e}^{\beta}(B_I+\epsilon_I) \ , \\
\phi \to \phi+\beta \quad , \qquad & \sigma \to \mathrm{e}^{2\beta}(\sigma+\alpha-\tfrac{1}{2}\epsilon_I A^I+\tfrac{1}{2}\epsilon^I B_I) \ ,
\end{aligned} \tag{2.5}
$$

Then the explicit G-invariant QK metric is [14]

$$
\begin{aligned}
\mathrm{d}s^2 &= \mathrm{d}\phi^2 - \mathrm{e}^{-\phi}(\mathcal{N}+\bar{\mathcal{N}})_{IJ}W^I\bar{W}^J + \mathrm{e}^{-2\phi}\Big(\mathrm{d}\sigma - \frac{1}{2}(A^I\mathrm{d}B_I - B_I\mathrm{d}A^I)\Big)^2 \\
&\quad -4\mathcal{K}_{A\bar{B}}\,\mathrm{d}Z^A\mathrm{d}\bar{Z}^{\bar{B}} \ .
\end{aligned} \tag{2.6}
$$

The metric is only positive definite in the domain where $(ZN\bar{Z})$ is positive and hence $\mathcal{K}_{A\bar{B}}$ is negative definite. One can then show that $\mathcal{N}+\bar{\mathcal{N}}$ is negative definite [15]. The one-forms W^I are defined by

$$
W^I = (\mathcal{N}+\bar{\mathcal{N}})^{-1\,IJ}\Big(2\bar{\mathcal{N}}_{JK}\mathrm{d}A^K - i\mathrm{d}B_J\Big) \ . \tag{2.7}
$$

As shown in [14], such metrics are indeed quaternion-Kähler; they were further studied in [16], including an analysis of their isometries. There are always the $2(n+2)$ manifest isometries (2.5), of which $n+2$ are commuting, *e.g.,*

$$
B_I \to B_I+\epsilon_I \ , \qquad \sigma \to \sigma+\alpha-\tfrac{1}{2}\epsilon_I A^I \ . \tag{2.8}
$$

This is one isometry *more* than the quaternionic dimension of the QK manifold.

3 Hyperkähler cones and the Legendre transform

The Swann bundle over a QK geometry, *i.e.,* the hyperkähler cone (HKC), is a hyperkähler manifold with one extra quaternionic dimension. As for special Kähler manifolds, the geometry of the HKC is again affine. In physics terminology, this arises because one adds a compensating hypermultiplet. Adding the compensator to the original hypermultiplets that parametrize the $4(n+1)$-dimensional QK space, one obtains a cone with real dimension $4(n+2)$. This space is hyperkähler and admits a homothety as well as an $SU(2)$ isometry group that rotates the three complex structures.

The metric on the HKC can be constructed from a hyperkähler potential [5], which is a Kähler potential with respect to *any* of the complex structures. In real local coordinates ϕ^A, the metric and the hyperkähler potential $\chi(\phi)$ are related by

$$g_{AB} = D_A \partial_B \, \chi(\phi) \ , \tag{3.1}$$

where D_A is the Levi-Civita connection. As for all Kähler manifolds, in complex coordinates, the hermitian part of (3.1) defines the metric in terms of the complex hessian of the potential; however, in this case, the vanishing of the holomorphic parts of the metric is an additional constraint on the geometry.

Any QK isometry can be lifted to a triholomorphic isometry on the HKC. In the physics literature, this was shown in [6, 17]. Using the notation of the previous section, we thus have an HKC of real dimension $4(n+2)$ together with $n+2$ commuting triholomorphic isometries determined by (2.8). As mentioned before, hyperkähler manifolds of this type were classified in [12]. It is convenient to introduce complex coordinates $v^{\hat{I}}$ and $w_{\hat{I}}$, in such a way that the isometries act as imaginary shifts in $w_{\hat{I}}$. Notice that $\hat{I} = 0, 1, ..., n+1$. The hyperkähler potential is then a function $\chi(v, \bar{v}, w + \bar{w})$, and can be written as a Legendre transform of a function $\mathcal{L}(v, \bar{v}, G)$ of $3(n+2)$ variables, The Legendre transform with respect to $G^{\hat{I}}$ is defined by

$$\chi(v, \bar{v}, w, \bar{w}) \equiv \mathcal{L}(v, \bar{v}, G) - (w + \bar{w})_{\hat{I}} \, G^{\hat{I}} \ , \qquad w_{\hat{I}} + \bar{w}_{\hat{I}} = \frac{\partial \mathcal{L}}{\partial G^{\hat{I}}} \ . \tag{3.2}$$

The constraints from hyperkähler geometry can be solved by writing $\mathcal{L}$ in terms of a contour integral [18, 19, 12]

$$\mathcal{L}(v, \bar{v}, G) \equiv \mathrm{Im} \oint_{\mathcal{C}} \frac{\mathrm{d}\zeta}{2\pi i \zeta} \, H(\eta, \zeta) \ , \tag{3.3}$$

with

$$\eta^{\hat{I}} \equiv \frac{v^{\hat{I}}}{\zeta} + G^{\hat{I}} - \bar{v}^{\hat{I}} \zeta \ . \tag{3.4}$$

These objects have an interpretation in twistor space as sections of an $\mathcal{O}(2)$ bundle. In physics terminology, these are $N = 2$ tensor multiplets. Furthermore, the conditions for a homothetic Killing vector and $SU(2)$ isometries imply that H is a function homogeneous of first degree[4] (in η) and without explicit ζ dependence [6].

Since H is homogeneous of first degree in η, it follows that the hyperkähler potential is also homogeneous of first degree in v and $\bar{v}$:

$$\chi(\lambda v, \lambda \bar{v}, w, \bar{w}) = \lambda \chi(v, \bar{v}, w, \bar{w}) \ . \tag{3.5}$$

[4] Actually, quasihomogeneity up to terms of the form $\eta \ln(\eta)$ is sufficient [6], but such terms do not seem to arise in the c-map.

In addition to a homothety, hyperkähler cones also have an $SU(2)$ isometry group that rotates the sphere of complex structures. Under infinitesimal variations with respect to an element of the Lie algebra $\varepsilon^+T_+ + \varepsilon^-T_- + \varepsilon^3T_3$, with $\varepsilon^- = (\varepsilon^+)^*$, these act as [6]

$$\delta_{\vec{\varepsilon}}\, v^{\hat{I}} = -i\varepsilon^3 v^{\hat{I}} + \varepsilon^- G^{\hat{I}}(v, \bar{v}, w, \bar{w}) \ , \qquad \delta_{\vec{\varepsilon}}\, w_{\hat{I}} = \varepsilon^+ \frac{\partial \mathcal{L}}{\partial \bar{v}^{\hat{I}}} \ , \tag{3.6}$$

where $G^{\hat{I}}$ has to be understood as the function of the coordinates $v, \bar{v}, w, \bar{w}$ obtained by the Legendre transform defined in (3.2). The coordinates w_I do not transform under ε^3. One can now explicitly check that the hyperkähler potential is $SU(2)_R$ invariant,

$$\delta_{\vec{\varepsilon}}\, \chi = \mathcal{L}_{v^{\hat{I}}}\, \delta_{\vec{\varepsilon}}\, v^{\hat{I}} + \mathcal{L}_{\bar{v}^{\hat{I}}}\, \delta_{\vec{\varepsilon}}\, \bar{v}^{\hat{I}} - \delta_{\vec{\varepsilon}}\,(w_{\hat{I}} + \bar{w}_{\hat{I}})\, G^{\hat{I}} = 0 \ . \tag{3.7}$$

(The δG terms cancel identically because χ is a Legendre transform). For the generators $\varepsilon^{\pm}$ this is immediately obvious; for variations proportional to ε^3 one needs to use the invariance of $\mathcal{L}$, *i.e.*, $v^{\hat{I}}\mathcal{L}_{v^{\hat{I}}} = \bar{v}^{\hat{I}}\mathcal{L}_{\bar{v}^{\hat{I}}}$.

4 Hyperkähler cones from the c-map

The Quaternion-Kähler space in the image of the c-map has dimension $4(n+1)$. The hyperkähler cone above it has dimension $4(n+2)$. It therefore needs to be described by $n+2$ twistor variables, say η^I and η^0, where $I = 1, \cdots, n+1$. As we shall show, the result for the tree level c-map is given by

$$H(\eta^I, \eta^0) = \frac{F(\eta^I)}{\eta^0} \ , \tag{4.8}$$

where F is the prepotential of the special Kähler geometry, now evaluated on the twistor variables η. This is our main result. Note that H does not depend explicitly on ζ and, since F is homogeneous of degree two, H is homogeneous of degree one, as required by superconformal invariance.

We now give a detailed proof of (4.8) by explicit calculation [8]. To be precise, we prove that (4.8) leads to (2.6).

4.1 Gauge fixing and the contour integral

As explained in the previous section, any hyperkähler cone has an $SU(2)$ symmetry and a homothety. The generators of the homothety and $U(1) \subset SU(2)$ give a natural complexified action on the HKC; the remaining two generators of the $SU(2)$ combine to give the roots $T_{\pm}$.

To evaluate the contour integral (3.3), it is convenient to make use of the isometries. In physics terminology, one can impose gauge choices. Mathematically, the isometries fiber the total space by the orbits, and a gauge choice is just a choice of section. For the symmetries generated by $T_\pm$, whose action is given by (3.6), we choose

$$v^0 = 0 \ . \tag{4.9}$$

In this gauge, we have that $\eta^0 = G^0$ and this simplifies the pole structure in the complex ζ-plane. Then, using as well the homogeneity properties of F, the contour integral (3.3) simplifies to

$$\mathcal{L}(v, \bar{v}, G) = \frac{1}{G^0} \,\mathrm{Im} \oint \frac{\mathrm{d}\zeta}{2\pi i} \frac{F(\zeta\eta^I)}{\zeta^3} \ , \tag{4.10}$$

with

$$\zeta\eta^I = v^I + \zeta G^I - \zeta^2 \bar{v}^I \ , \qquad I = 1, \cdots, n+1 \ , \tag{4.11}$$

which, for nonzero values of v, has no zeroes at $\zeta = 0$. Therefore, since $F(\zeta\eta)$ is homogeneous of positive degree, it has no poles at $\zeta = 0$. It is now easy to evaluate the contour integral, because the residue at $\zeta = 0$ replaces all the $\zeta\eta^I$ by v^I. The result is

$$\mathcal{L}(v, \bar{v}, G) = \frac{1}{4G^0} \Big(N_{IJ} G^I G^J - 2K(v, \bar{v}) \Big) \ , \tag{4.12}$$

where $K(v, \bar{v})$ is the Kähler potential of the rigid special geometry given in (2.1), with $F_I(v)$ now the derivative with respect to v^I, *etc.* Notice that the function $\mathcal{L}$ satisfies the Laplace-like equations [18, 19, 12]

$$\mathcal{L}_{G^I G^J} + \mathcal{L}_{v^I \bar{v}^J} = 0 \ . \tag{4.13}$$

The equation is not satified for the components $\mathcal{L}_{G^0 G^0}$ and $\mathcal{L}_{G^0 G^I}$, because we have chosen the gauge $v^0 = 0$. It would be interesting to compute $\mathcal{L}$ for arbitrary values of v^0. For a special case, this was done in [20].

4.2 The hyperkähler potential

To compute the hyperkähler potential χ, we have to Legendre transform $\mathcal{L}$,

$$\chi(v, \bar{v}, w, \bar{w}) = \mathcal{L}(v, \bar{v}, G) + (w + \bar{w})_0 \, G^0 - (w + \bar{w})_I \, G^I \ , \tag{4.14}$$

The hyperkähler potential χ, computed by extremizing[5] (4.14) with respect G^0, G^I completely determines the associated hyperkähler geometry. In general, it is a function

[5] The relative minus signs between the last two terms in (4.14) is purely a matter of convention.

of the $2(n+2)$ complex coordinates v^0, v^I and w_0, w_I, but we work only on the (Kähler but not hyperkähler) submanifold $v^0 = 0$. This is sufficient for calculating the metric on the underlying QK manifold. The geometry of the HKC only depends on w through the combination $w + \bar{w}$ which makes manifest the $n+2$ commuting isometries. The Legendre transform of (4.12) gives:

$$\frac{G^I}{G^0} = 2N^{IJ}(w+\bar{w})_J \ , \qquad (G^0)^2 = \frac{K}{2\Big((w+\bar{w})_I N^{IJ}(w+\bar{w})_J - (w+\bar{w})_0\Big)} \ . \tag{4.15}$$

Up to an irrelevant overall sign, we find, using (4.12)

$$\chi\Big(v, \bar{v}, G(v, \bar{v}, w, \bar{w})\Big) = \frac{K(v, \bar{v})}{G^0} \ , \tag{4.16}$$

where G^0 is determined by (4.15). More explicitly, in terms of the HKC coordinates,

$$\chi(v, \bar{v}, w, \bar{w}) = \sqrt{2}\ \sqrt{K(v, \bar{v})}\ \sqrt{(w+\bar{w})_I N^{IJ}(w+\bar{w})_J - (w+\bar{w})_0} \ . \tag{4.17}$$

4.3 Twistor space

The twistor space above a $4(n+1)$ dimensional QK manifold has dimension two higher, and is Kähler. It can be seen as a CP^1 bundle over the QK. It can also be obtained from the HKC by a Kähler quotient with respect to $U(1) \subset SU(2)$. Equivalently, we define inhomogeneous coordinates, *e.g.*,

$$Z^I = \frac{v^I}{v^1} = \{1, Z^A\} \ , \tag{4.18}$$

where A runs over n values. As we show below, these inhomogeneous coordinates will be identified with (2.2).

The Kähler potential on the twistor space, denoted by K_T, is given by the logarithm of the hyperkähler potential restricted to the submanifold given by $v^1 = 1$ [6]:

$$K_T(Z, \bar{Z}, w, \bar{w}) = \frac{1}{2}\Big[\mathcal{K}(Z, \bar{Z}) + \ln\Big((w+\bar{w})_I N^{IJ}(w+\bar{w})_J - (w+\bar{w})_0\Big)\Big] + \ln(\sqrt{2}) \ , \tag{4.19}$$

where $\mathcal{K}(Z, \bar{Z})$ is the *same* as the special Kähler potential (2.3).

On the twistor space, there always exists a holomorphic one-form $\mathcal{X}$ which can be constructed from the holomorphic two-form that any hyperkähler manifold admits. In our case this one-form is obtained from the holomorphic HKC two-form $\Omega = \mathrm{d}w_I \wedge \mathrm{d}v^I$. Without going into details, it is given by [6]

$$\mathcal{X} = 2Z^I \mathrm{d}w_I \equiv \mathcal{X}_\alpha \mathrm{d}z^\alpha \ , \tag{4.20}$$

where the index $\alpha = 1, \cdots, 2(n+1)$ runs over the complete set of holomorphic coordinates w_I, w_0, Z^A on the submanifold[6] of the twistor space given by $v^0 = 0$. In total this gives $2(n+1) + 2 + 2n = 4(n+1)$–the (real) dimension of the QK. The metric on the QK manifold can then be computed[7]:

$$G_{\alpha\bar{\beta}} = K_{T,\alpha\bar{\beta}} - \mathrm{e}^{-2K_T} \mathcal{X}_\alpha \bar{\mathcal{X}}_{\bar{\beta}} \ . \tag{4.21}$$

4.4 The quaternionic metric

We now compute the QK metric that follows from the c-map using (4.21). To compare with (2.6) we only need to identify the coordinates w_I, w_0 with those of (2.6), since the Z^A coordinates of the special Kähler manifold can be identified with the ones above. We define

$$\begin{aligned} w_0 &= iA^I A^J F_{IJ} - i(\sigma + \frac{1}{2} A^I B_I) - \mathrm{e}^\phi \ , \\ w_I &= iF_{IJ} A^J - \frac{i}{2} B_I \ . \end{aligned} \tag{4.22}$$

The metric can be written in these coordinates; after considerable calculation [8], up to an irrelevant overall all factor of $-1/8$, we obtain precisely the result (2.6)! From (4.15), we find the following relations between the QK coordinates and the twistor variables $G^{\hat{I}}$ (see 3.4):

$$2A^I = \frac{G^I}{G^0} \ , \qquad 4\mathrm{e}^\phi = \frac{K(v,\bar{v})}{(G^0)^2} \ . \tag{4.23}$$

This concludes the proof of (4.8).

5 Summary and conclusion

We have constructed the Swann bundle over the Quaternion-Kähler manifolds that arise in the c-map. The corresponding hyperkähler potential was given in (4.17), and was first derived in [8]. Introducing coordinates

$$X^I(v,\bar{v},w,\bar{w}) \equiv \frac{v^I}{\sqrt{G^0(v,\bar{v},w,\bar{w})}} \ , \tag{5.1}$$

we can conveniently rewrite the hyperkähler potential as

$$\boxed{\chi(v,\bar{v},w,\bar{w}) = K\Big(X^I(v,\bar{v},w,\bar{w}), \bar{X}^I(v,\bar{v},w,\bar{w})\Big)} \ . \tag{5.2}$$

[6]This submanifold can be thought of invariantly as a quotient of the original HKC.

[7]Note that the constant term in K_T (4.19) enters in (4.21).

Here K is the Kähler potential $K(X,\bar{X}) = i(\bar{X}^I F_I - X^I \bar{F}_I)$ of the affine special geometry.

The special hyperkähler cones given by the c-map have as many $(n+2)$ commuting triholomorphic isometries as their quaternionic dimension. As explained before, this implies the hyperkähler potential can be Legendre transformed to a function $\mathcal{L}$ that can be written in terms of a contour integral over a function $H(\eta)$; equivalently, the twistor space of the HKC can be described in terms of sections of $(n+2)$ $\mathcal{O}(2)$-bundles. These twistor variables η were defined in (3.4) and the function H was determined in (4.8). Defining

$$X^I(\eta) \equiv \frac{\eta^I}{\sqrt{\eta^0}} \ , \tag{5.3}$$

we can write H as

$$\boxed{H(\eta^I,\eta^0) = F[X^I(\eta^I,\eta^0)]} \ . \tag{5.4}$$

The function F is well known to be related to the topological string amplitude [21, 22]. Typical examples that appear in the context of Calabi-Yau compactifications are of the form

$$F(X^I) = d_{ABC}\frac{X^A X^B X^C}{X^1} \ , \tag{5.5}$$

where $X^I = \{X^1, X^A\}$ and the constants d_{ABC} are related to the triple intersection numbers of the Calabi-Yau. To give an explicit example, one can choose specific values for these coefficients such that the local (projective) special Kähler geometry is the symmetric space

$$\frac{SU(1,1)}{U(1)} \times \frac{SO(n-1,2)}{SO(n-1)\times SO(2)} \ . \tag{5.6}$$

After the c-map, the hyperkähler cone is based on the function

$$H(\eta^{\hat{I}}) = d_{ABC}\frac{\eta^A\eta^B\eta^C}{\eta^0\eta^1} \ . \tag{5.7}$$

This corresponds to a homogeneous quaternion-Kähler manifold of the form (see e.g. appendix C in [9], and references therein)

$$\frac{SO(n+1,4)}{SO(n+1)\times SO(4)} \ . \tag{5.8}$$

Other examples were recently given in [23], were quantum effects were taken into account.

The connection of these geometries with topological strings is very profound, and has important physical implications. For instance, it was recently shown that the topological string amplitude F appears in the study of supersymmetric black holes in string theory [24, 25]. More precisely, the Legendre transform of F is related to the entropy of the black hole. It would be interesting to see if this Legendre transform is related to the one described here; speculations along these lines can be found in [8]. To make progress on this issue, one needs to evaluate the contour integral (3.3) without making use of the special coordinate system in which we can set $v^0 = 0$. We leave this for future research.

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