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SYMMETRY ALGEBRAS OF LAGRANGIAN LIOUVILLE-TYPE SYSTEMS

© **A. V. Kiselev[*] and J. W. van de Leur[*]**

We calculate the generators and commutation relations explicitly for higher symmetry algebras of a class of hyperbolic Lagrangian systems of Liouville type, in particular, for two-dimensional Toda chains associated with semisimple complex Lie algebras.

Keywords: symmetry, two-dimensional Toda chain, Liouville-type system, Hamiltonian hierarchy, bracket

1. Introduction

We describe the generators and relations in higher symmetry algebras for a class of Darboux-integrable hyperbolic Lagrangian systems of Liouville type [1]–[3]. There exist many nonequivalent definitions of this type of PDEs [1], [3], [4]; we investigate the systems $\mathcal{E}_{\mathrm{L}}$ that admit as many first integrals for each of the characteristic equations $D_y(w) \doteq 0$ and $D_x(\overline{w}) \doteq 0$ on $\mathcal{E}_{\mathrm{L}}$ as there are unknown functions. The two-dimensional Toda chains $\boldsymbol{u}_{xy} = e^{K\boldsymbol{u}}$ associated with semisimple complex Lie algebras are the best-studied example of such equations [1], [2], [5]–[7]. The systems in this class are known to have higher symmetries $\varphi = \square(\boldsymbol{\phi})$ that depend on free functional parameters $\boldsymbol{\phi} = {}^{\mathrm{t}}\big(\phi_1(x,[w]),\dots,\phi_r(x,[w])\big)$ and belong to the image of matrix total differential operators $\square$ (linear operators in total derivatives) [6], [8]–[10]. The existence of such operators $\square$ for Liouville-type systems was noted in [1], [6] and [3], [10], where the importance of the linearizations $\ell_w^{(u)}$ of the first integrals w in the construction of $\square$ was revealed. In [9], we proved that with the additional assumption that $\mathcal{E}_{\mathrm{L}}$ is Lagrangian, the known results can be strengthened and the operators $\square$ can be obtained explicitly (see formula (3) below).

Here, we establish the transformation rules for the operators $\square$ under unrelated coordinate changes in their domains and images. We show that under natural assumptions on the geometry of $\mathcal{E}_{\mathrm{L}}$, the images of these operators are closed under commutation, whence the Lie algebra structure on their domains appears. We calculate the brackets on the domains explicitly, which, by the push forward of the Lie algebra structure, yields the commutation relations in the symmetry algebras $\operatorname{sym}\mathcal{E}_{\mathrm{L}}$. For this, we introduce auxiliary Hamiltonian operators that have the same domain as $\square$.

Remark 1. We do not assume the presence of a symmetry $x \leftrightarrow y$ in $\mathcal{E}_{\mathrm{L}}$. We work with the "x-half" of the algebra $\operatorname{sym}\mathcal{E}_{\mathrm{L}}$ related to the first integrals $w^i \in \ker D_y|_{\mathcal{E}_{\mathrm{L}}}$; the reasonings hold for the corresponding "y-half" of $\operatorname{sym}\mathcal{E}_{\mathrm{L}}$, and the two subalgebras commute with each other. For the Lagrangian systems $\mathcal{E}_{\mathrm{L}}$ under consideration, the integrals $\overline{w}^{\bar\imath} \in \ker D_x|_{\mathcal{E}_{\mathrm{L}}}$ are not used in the proofs, unlike in [10], where the Liouville-type systems were arbitrary.

The full list of assumptions on the systems $\mathcal{E}_{\mathrm{L}}$ and their integrals is given in our main Theorem 2 (also see Remark 4). But we note that the reasonings in Sec. 2 hold under less restrictive conditions. In

[*]Mathematical Institute, University of Utrecht, Utrecht, The Netherlands, e-mail: A.V.Kiselev@uu.nl, J.W.vandeLeur@uu.nl.

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particular, the number of first integrals $w^1, \dots, w^r$ for the characteristic equation on $\mathcal{E}_{\mathrm{L}}$ can be less than the number of the unknowns $u^1, \dots, u^m$ in $\mathcal{E}_{\mathrm{L}}$. In that case, the auxiliary $(r\times r)$-matrix operators $\hat{A}_k$ defined in (7) become smaller in size but remain Hamiltonian; an example is the second Poisson structure for the Korteweg–de Vries (KdV) equation constructed from the two-dimensional Toda chains with a unique integral, as presented in [9].

This paper is organized as follows. In Sec. 2, we define the operators $\square$ used to construct symmetry generators for the systems $\mathcal{E}_{\mathrm{L}}$, and we introduce auxiliary Hamiltonian operators. We again obtain the higher Poisson structures for the Drinfel'd–Sokolov hierarchies [11] on two-dimensional Toda chains related to semisimple complex Lie algebras; we give an example for the A_2 Toda chain. In Sec. 3, we then establish the commutation closure for images of the operators $\square$ and calculate the structural relations in the algebras $\operatorname{sym}\mathcal{E}_{\mathrm{L}}$; we give an illustration for the Kaup–Boussinesq equation. Finally, in Sec. 4, we discuss some properties of the operators that yield symmetries of Liouville-type systems that are not necessarily Lagrangian.

All notions and constructions from the geometry of PDEs are standard [12]–[14]. We use the notation in [9], [15], [16]. This paper further develops the ideas formulated in [9].

2. Symmetry generators for $\mathcal{E}_{\mathrm{L}}$

Definition. A system $\mathcal{E} = \{\boldsymbol{u}_{xy} = f(\boldsymbol{u}, \boldsymbol{u}_x, \boldsymbol{u}_y; x, y)\}$ of m hyperbolic equations on the unknowns $\boldsymbol{u} = (u^1, \dots, u^m)$ is a *Liouville-type system* if there exist nontrivial first integrals

$$w^1, \dots, w^r \in \ker D_y|_{\mathcal{E}}, \qquad \overline{w}^1, \dots, \overline{w}^{\bar{r}} \in \ker D_x|_{\mathcal{E}}, \quad 0 < r, \bar{r} \le m,$$

for the linear first-order characteristic equations $D_y|_{\mathcal{E}}(w^i) \doteq 0$ and $D_x|_{\mathcal{E}}(\overline{w}^{\bar{j}}) \doteq 0$ that hold by virtue of $\mathcal{E}$ ($\doteq$).

Example 1. It was proved in [2] that the two-dimensional Toda chains [5] $u^i_{xy} = e^{K^i_j u^j}$ associated with semisimple complex Lie algebras with the Cartan matrices K admit maximal ($r{=}\bar{r}{=}m$) sets of the integrals. Various methods for reconstructing w^i and $\overline{w}^{\bar{j}}$ for these exponentially nonlinear Toda chains were proposed in [3], [4], [7]. After a shift by -1, the differential orders (with respect to $\boldsymbol{u}$) of the integrals $w^1, \dots, w^r$ are equal to the exponents of the corresponding semisimple complex Lie algebras of rank r (see p. 21 in [2]).

For instance, below, we consider the Lagrangian two-dimensional Toda system $\mathcal{E}_{\mathrm{Toda}}$ associated with the simple Lie algebra $\mathfrak{sl}_3(\mathbb{C})$ (see [1], [5], [7]),

$$\mathcal{E}_{\mathrm{Toda}} = \left\{ u_{xy} = e^{2u-v},\ v_{xy} = e^{-u+2v},\ K = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \right\}. \tag{1}$$

The integrals (of the second and third orders) for system (1) are (see, e.g., [17])

$$w^1 = u_{xx} + v_{xx} - u_x^2 + u_x v_x - v_x^2,$$
$$w^2 = u_{xxx} - 2u_x u_{xx} + u_x v_{xx} + u_x^2 v_x - u_x v_x^2.$$

The generating sections $\varphi = \square\big(\boldsymbol{\phi}(x, [w])\big)$ of higher symmetry algebras for Liouville-type equations are given by matrix total differential operators $\square$ (see [3], [6]). If the Liouville-type systems are Lagrangian, i.e., $\mathcal{E}_{\mathrm{L}} = \{F \equiv \mathbf{E}(\mathcal{L}) = 0\}$ (see [8], [9], [18]), then the existence of certain factorizations for at least some of the symmetries is rigorous and can be readily verified as follows. For the integrals w, there exists an operator ∇ such that $D_y(w) = \nabla(F)$ vanishes on the differential ideal $\{F = 0\}^\infty$. Indeed, the generating section $\psi_I = [\nabla^* \circ (\ell_w^{(u)})^* \circ (\ell_I^{(w)})^*](1)$ for a conservation law $\int I\,dx$ solves the equations $\ell^*_{\mathbf{E}(\mathcal{L})}(\psi_I) \doteq 0$ on $\mathcal{E}_{\mathrm{L}}$ for any $I(x,[w])$ (see [12]–[14]). The Helmholtz condition $\ell_{\mathbf{E}(\mathcal{L})} = \ell^*_{\mathbf{E}(\mathcal{L})}$ for the linearization (the Frechét derivative) implies that the vector

$$\varphi[\boldsymbol{\phi}] = [\nabla^* \circ (\ell_w^{(u)})^*](\boldsymbol{\phi}(x,[w])) \in \ker \ell_{\mathbf{E}(\mathcal{L})}|_{\mathcal{E}_{\mathrm{L}}} \tag{2}$$

is a symmetry of $\mathcal{E}_{\mathrm{L}}$ for any $\boldsymbol{\phi} = (\ell_I^{(w)})^*(1) = \mathbf{E}_w(I\,dx)$. A standard homological reasoning (see Chap. 5 in [13] or Sec. 7.8 in [14]) shows that formula (2) yields symmetries of the Lagrangian system $\mathcal{E}_{\mathrm{L}}$ even if the sections $\boldsymbol{\phi}$ do not belong to the image of the variational derivative $\mathbf{E}_w$ with respect to w.

In this section, we recall the procedure for constructing the operators $\square$ that give the symmetries for a class of Lagrangian Liouville-type systems. We everywhere assume that the integrals w are minimal, i.e., $I \in \ker D_y|_{\mathcal{E}_{\mathrm{L}}}$ implies $I = I(x,[w])$.

Proposition 1 [9]**.** *Let κ be an invertible symmetric constant real $m\times m$ matrix. Further, let $\mathcal{L} = \int L\,dx\,dy$ with the density*

$$L = -\frac{1}{2}\sum_{i,j}\kappa_{ij}u^i_x u^j_y - H_{\mathrm{L}}(u;x,y)$$

be the Lagrangian of a Liouville-type equation $\mathcal{E}_{\mathrm{L}} = \{\mathbf{E}(\mathcal{L}) = 0\}$. We introduce the canonical momenta $\mathfrak{m} = \partial L/\partial u_y$ and assume that $w[\mathfrak{m}] = (w^1,\ldots,w^r)$ is the minimal set of integrals for $\mathcal{E}_{\mathrm{L}}$ that belong to the kernel of $D_y|_{\mathcal{E}_{\mathrm{L}}}$. Then the operator

$$\square = (\ell_w^{(\mathfrak{m})})^*, \tag{3}$$

adjoint to the linearization of the integrals over the momenta, yields Noether symmetries $\varphi_{\mathcal{L}}$ of $\mathcal{E}_{\mathrm{L}}$ by the formula

$$\varphi_{\mathcal{L}} = \square\left(\frac{\delta\mathcal{H}}{\delta w}\right) \quad \forall\,\mathcal{H} = \int H(x,[w])\,dx. \tag{4}$$

Corollary. *Under the assumptions and with the notation in Proposition 1, the section*

$$\varphi = \square\bigl(\boldsymbol{\phi}(x,[w])\bigr) \tag{5}$$

is a symmetry of the Liouville-type equation $\mathcal{E}_{\mathrm{L}}$ for any r-tuple $\boldsymbol{\phi} = {}^{\mathrm{t}}(\phi_1,\ldots,\phi_r)$.

Proof. We consider the jet bundle $J^\infty(\xi)$ over the fiber bundle $\xi\colon \mathbb{R}^r\times\mathbb{R}\to\mathbb{R}$ with the base $\mathbb{R}\ni x$ and the fibers $\mathbb{R}^r$ with the coordinates $w^1,\ldots,w^r$. By Proposition 1, the statement holds for any $\boldsymbol{\phi}$ in the image of the variational derivative $\mathbf{E}_w$. Obviously, its image contains all variational covectors $\boldsymbol{\phi}$ whose components $\phi_i(x) \in C^\infty(\mathbb{R})$ are functions on the base of the new bundle ξ. The prolongation of the substitution $w = w\bigl[\mathfrak{m}[u]\bigr]\colon J^\infty(\pi)\to\Gamma(\xi)$ converts the components of sections $\boldsymbol{\phi}$ to smooth differential functions in u (one letter denotes the entire set of m coordinates in the fiber bundle π over the same base $\mathbb{R}\ni x$). We now recall that $\square$ is an operator in total derivatives D_x whose action on differential functions $f[u]$ is defined by

$$j_\infty(s)\bigl(D_x(f)\bigr) := \frac{\partial}{\partial x}\bigl(j_\infty(s)(f)\bigr),$$

i.e., through the restrictions $j_\infty(s)(f)$ of f onto the jets $j_\infty(s)$ of sections $u = s(x)$. Hence, we obtain $\phi_i(x) = \phi_i\bigl(x,\bigl[w\bigl[\mathfrak{m}[s(x)]\bigr]\bigr]\bigr)$, whence the assertion follows.

Remark 2. This proof combines taking the variational derivatives with respect to w on one jet space with calculating the total derivatives of differential functions in u on the other jet space over the same base. If these two spaces coincide, then the reasoning reduces to the definition of D_x. The proof scheme is then called the *substitution principle* ([13]; also see a detailed discussion in [14]).

Theorem 1. *Under differential reparameterizations $\widetilde{w} = \widetilde{w}[w]$ and $\tilde{u} = \tilde{u}[u]$ of the coordinates $w^1, \dots, w^r$ and $u^1, \dots, u^m$ in the infinite jet bundles over ξ and π that specify the respective domain and image of the operator $\Box$, that operator transforms according to the formula*

$$\Box \mapsto \widetilde{\Box} = \ell_{\tilde{u}}^{(u)} \circ \Box \circ (\ell_{\widetilde{w}}^{(w)})^* \Big|_{\substack{w=w[u] \\ u=u[\tilde{u}]}}. \tag{6}$$

Proof. The velocity transformation $\tilde{\varphi} = \ell_{\tilde{u}}^{(u)}(\varphi)$ is obvious. Under differential reparameterizations $w = w[\widetilde{w}]$ of the integrals, the sections $\boldsymbol{\phi} = \delta\mathcal{H}/\delta w$ transform by $\boldsymbol{\phi} = (\ell_{\widetilde{w}}^{(w)})^*(\tilde{\boldsymbol{\phi}})$; hence, $\Box$ becomes well defined on $\operatorname{im} \mathbf{E}_w$. Namely, it maps variational covectors for the fiber bundle ξ to evolutionary derivations in the jet space over the other fiber bundle π. Repeating the reasoning used to prove Corollary 1, we establish transformation rule (6) for $\Box$ on the entire domain.

In Theorem 1, we showed that sections in the domain of the operator $\Box$ transform by the same rule as the arguments of Hamiltonian operators. There is indeed a deep reason for this.

The integrals $w[\mathfrak{m}]$ of Lagrangian Liouville-type systems $\mathcal{E}_{\mathrm{L}}$ determine the Miura substitutions from commutative Hamiltonian hierarchies $\mathfrak{B}$ (of the same type as the modified KdV equation) of Noether symmetries for $\mathcal{E}_{\mathrm{L}}$ to completely integrable KdV-type hierarchies $\mathfrak{A}$ of higher symmetries of the multicomponent wave equations $\mathcal{E}_{\varnothing} = \{s_{xy} = 0\}$ (see below). A natural example is the potential modified KdV equation

$$u_t = -\frac{1}{2}u_{xxx} + u_x^3,$$

which transforms into the KdV equation

$$w_t = -\frac{1}{2}w_{xxx} + 3ww_x$$

by $w = u_x^2 - u_{xx}$. This example was studied in detail in [9]. Such a method for generating relevant differential substitutions by the integrals w of Liouville-type systems was discovered in [19] (see the discussion in [3]). This fact was used in [20] to classify the first-order differential substitutions.

The hierarchies $\mathfrak{A}$ and $\mathfrak{B}$ share the Hamiltonians $\mathcal{H}_i[\mathfrak{m}] = \mathcal{H}_i\bigl[w[\mathfrak{m}]\bigr]$ through the Miura substitution $w[\mathfrak{m}]$. The Hamiltonian structures for the Magri schemes of $\mathfrak{A}$ and $\mathfrak{B}$ are correlated by the operators $\Box$, which map the "cosymmetries" ϕ_i for the hierarchy $\mathfrak{A}$ to the symmetries φ_{i+1} of the modified hierarchy $\mathfrak{B}$. We stress that the first Hamiltonian operator $\widehat{B}_1 = (\ell_{\mathfrak{m}}^{(u)})^*$ for $\mathfrak{B}$ originates from the differential constraint $\mathfrak{m} = \partial L/\partial u_y$ between the coordinates u and the momenta $\mathfrak{m}$ for $\mathcal{E}_{\mathrm{L}}$. Using the explicit form of the "junior" operator $\widehat{B}_1$ and the differential-functional closure in $[w]$ for the velocities of the evolution of the integrals along the symmetries (see [3]), we realize the classical scheme for generating higher Poisson structures via Miura substitutions [21].

Lemma 1. *We introduce the linear differential operator*

$$\hat{A}_k = \Box^* \circ \widehat{B}_1 \circ \Box \tag{7}$$

mapping variational covectors for the jet bundle $J^\infty(\xi)$ over ξ to evolutionary vector fields on it, i.e.,

$$\hat{A}_k\colon \Gamma(\hat{\xi}) \bigotimes_{C^\infty(\mathbb{R})} C^\infty\bigl(J^\infty(\xi)\bigr) \to \Gamma(\xi) \bigotimes_{C^\infty(\mathbb{R})} C^\infty\bigl(J^\infty(\xi)\bigr).$$

Then operator (7) *is Hamiltonian and determines a Poisson structure*[1] *for the KdV-type hierarchy $\mathfrak{A}$. The coefficients of $\hat{A}_k$ are differential functions in w.*

Proof. By construction, the Poisson bracket

$$\{\mathcal{H}_1, \mathcal{H}_2\}_{\hat{A}_k} = \bigl\langle \mathbf{E}_w(\mathcal{H}_1), \hat{A}_k\bigl(\mathbf{E}_w(\mathcal{H}_2)\bigr)\bigr\rangle$$

satisfies the equality

$$\{\mathcal{H}_1[w], \mathcal{H}_2[w]\}_{\hat{A}_k} = \bigl\{\mathcal{H}_1\bigl[w[\mathfrak{m}]\bigr], \mathcal{H}_2\bigl[w[\mathfrak{m}]\bigr]\bigr\}_{\widehat{B}_1}. \tag{8}$$

Therefore, the left-hand side of (8) is bilinear, is skew-symmetric, and satisfies the Jacobi identity. Moreover, it measures the velocity of the integrals w along a Noether symmetry of $\mathcal{E}_{\mathrm{L}}$. Because evolutionary derivations are permutable with the total derivative D_y, the velocity $\{\mathcal{H}_1, \mathcal{H}_2\}_{\hat{A}_k}$ lies in $\ker D_y|_{\mathcal{E}_{\mathrm{L}}}$, and its density is hence a differential function of the minimal integrals w.

The multicomponent wave equation $\mathcal{E}_\varnothing = \{s_{xy} = 0\}$, the symmetries of which contain the hierarchy $\mathfrak{A}$ and such that $\hat{A}_1 = (\ell_w^{(s)})^*$ encodes the differential constraint between the coordinates s and momenta w for $\mathcal{E}_\varnothing$, is chosen as follows. The first structure $A_1 = \hat{A}_1^{-1}$ for $\mathfrak{A}$ must factor the higher Hamiltonian structure for $\mathfrak{B}$. Hence, we have $B_{k'} = \square \circ A_1 \circ \square^*$, where $k' = k'\bigl(\square, (\ell_w^{(s)})^*\bigr) \geq 2$.

Example 2. We consider Lagrangian two-dimensional Toda system (1). The density L of its Lagrangian is

$$L = -\frac{1}{2}\bigl((2u_x - v_x)u_y + (2v_x - u_x)v_y\bigr) - e^{2u-v} - e^{2v-u}.$$

Therefore, we define the momenta as $\mathfrak{m}^1 := 2u_x - v_x$ and $\mathfrak{m}^2 := 2v_x - u_x$, and we then express the integrals in terms of them: $w = w[\mathfrak{m}]$. All symmetries of (1) are of the form (up to $x \leftrightarrow y$) $\varphi = \square\bigl(\boldsymbol{\phi}(x, [w^1], [w^2])\bigr)$, where $\boldsymbol{\phi} = {}^{\mathrm{t}}(\phi_1, \phi_2)$ is a pair of arbitrary functions and the $(2{\times}2)$-matrix total differential operator is

$$\square = \ell_{w^1,w^2}^{(\mathfrak{m}^1,\mathfrak{m}^2)} = \begin{pmatrix} u_x + D_x & -\frac{2}{3}D_x^2 - u_x D_x - \frac{1}{3}u_x^2 - \frac{2}{3}u_x v_x + \frac{2}{3}v_x^2 + \frac{1}{3}u_{xx} - \frac{2}{3}v_{xx} \\ v_x + D_x & -\frac{1}{3}D_x^2 + \frac{2}{3}u_{xx} - \frac{1}{3}v_{xx} - \frac{2}{3}u_x^2 + \frac{2}{3}u_x v_x + \frac{1}{3}v_x^2 \end{pmatrix}.$$

The elements of the resulting Hamiltonian operator $\hat{A}_2 = \|A_{ij}, 1 \leq i, j \leq 2\|$ are [13]

$$\begin{aligned}
A_{11} &= 2D_x^3 + 2w^1 D_x + w_x^1,\\
A_{12} &= -D_x^4 - w^1 D_x^2 + (3w^2 - 2w_x^1)D_x + (2w_x^2 - w_{xx}^1),\\
A_{21} &= D_x^4 + w^1 D_x^2 + 3w^2 D_x + w_x^2,\\
A_{22} &= -\frac{2}{3}D_x^5 - \frac{4}{3}w^1 D_x^3 - 2w_x^1 D_x^2 + \left(2w_x^2 - 2w_{xx}^1 - \frac{2}{3}(w^1)^2\right)D_x +\\
&\quad + \frac{1}{3}(3w_{xx}^2 - 2w_{xxx}^1 - 2w^1 w_x^1).
\end{aligned}$$

[1] In most cases, this is one of the higher structures for $\mathfrak{A}$, which is indicated by the subscript $k = k(\square, \mathfrak{m}) \geq 2$. The choice of the "junior" operator $\hat{A}_1$ for $\mathfrak{A}$ is discussed below.

The shift $w^2 \mapsto w^2 + \lambda$ of the second integral yields the "junior" Hamiltonian operator[2]

$$\hat{A}_1^{(2)} := \frac{d}{d\lambda}\Big|_{\lambda=0}(\hat{A}_2) = \begin{pmatrix} 0 & 3D_x \\ 3D_x & 0 \end{pmatrix},$$

which is compatible with $\hat{A}_2$.

The pair $(\hat{A}_1^{(2)}, \hat{A}_2)$ is the bi-Hamiltonian structure for the Boussinesq equation

$$w_t^1 = 2w_x^2 - w_{xx}^1, \qquad w_t^2 = -\frac{2}{3}w_{xxx}^1 - \frac{2}{3}w^1 w_x^1 + w_{xx}^2.$$

The symmetry $w_x = (\hat{A}_2 \circ \delta/\delta w)\bigl(\int w^1\,dx\bigr)$ starts the second sequence of Hamiltonian flows in the Boussinesq hierarchy $\mathfrak{A}$. The modified Boussinesq hierarchy $\mathfrak{B}$ inherits both sequences of Hamiltonians from $\mathfrak{A}$ by the Miura substitution $w = w[\mathfrak{m}]$, where we take $\mathfrak{m} = \mathfrak{m}[u]$ into account. Namely, for any Hamiltonian $\mathcal{H}[w]$, the flows $u_\tau = \delta\mathcal{H}[\mathfrak{m}]/\delta\mathfrak{m}$ and $\mathfrak{m}_\tau = -\delta\mathcal{H}\bigl[\mathfrak{m}[u]\bigr]/\delta u$ belong to $\mathfrak{B}$. The velocities u_τ constitute the commutative subalgebra of Noether symmetries for two-dimensional Toda chain (1).

3. Commutation relations in sym $\mathcal{E}_{\mathrm{L}}$

In this section, we prove the commutation closure for the images of operators (3) using the well-known analogous property of auxiliary Hamiltonian operators (7). At the same time, we describe the relations in the symmetry algebra generated by (5) for $\mathcal{E}_{\mathrm{L}}$.

First, we consider an arbitrary linear total differential operator A whose arguments $\boldsymbol{\phi}(x,[w]) = {}^{\mathrm{t}}(\phi_1,\ldots,\phi_r)$ are the variational covectors for the infinite jet bundle over ξ. We assume that the image of A in the Lie algebra of evolutionary vector fields ∂_φ is closed under commutation: $[\mathrm{im}\,A, \mathrm{im}\,A] \subseteq \mathrm{im}\,A$. By the Leibnitz rule, two sets of summands appear in the bracket of the fields $A(\phi')$ and $A(\phi'')$ that belong to the image of A:

$$[A(\phi'), A(\phi'')] = A\bigl(\partial_{A(\phi')}(\phi'') - \partial_{A(\phi'')}(\phi')\bigr) + \bigl(\partial_{A(\phi')}(A)(\phi'') - \partial_{A(\phi'')}(A)(\phi')\bigr).$$

In the first summand, we use the permutability of evolutionary derivations and total derivatives. The second summand hits the image of A by construction.

The commutator $[\,\cdot\,,\,\cdot\,]|_{\mathrm{im}\,A}$ induces a Lie algebra structure $[\,\cdot\,,\,\cdot\,]_A$ in the quotient $\Omega(\xi_\pi)$ of the domain of A by its kernel:

$$[A(\phi'), A(\phi'')] = A([\phi', \phi'']_A), \quad \phi', \phi'' \in \Omega(\xi_\pi). \tag{9a}$$

This bracket, which is defined up to $\ker A$, equals

$$[\phi', \phi'']_A = \partial_{A(\phi')}(\phi'') - \partial_{A(\phi'')}(\phi') + \{\{\phi', \phi''\}\}_A. \tag{9b}$$

It contains the two standard summands and the skew-symmetric bilinear bracket $\{\{\,\cdot\,,\,\cdot\,\}\}_A$.

Lemma 2 [13], [14]**.** *The image of a Hamiltonian operator $\hat{A} = \|\sum_{\boldsymbol{\tau}} A^{ij}_{\boldsymbol{\tau}}(x,[w]) \cdot D_{\boldsymbol{\tau}}\|$ is closed under commutation. The kth component $(1 \leq k \leq r)$ of the resulting bracket $\{\{\,\cdot\,,\,\cdot\,\}\}_{\hat{A}}$ on the domain of $\hat{A}$ is calculated by the formula*

$$\{\{\phi', \phi''\}\}^k_{\hat{A}} = \sum_{|\boldsymbol{\sigma}|\geq 0}\sum_{i=1}^{r}(-1)^{|\boldsymbol{\sigma}|}\biggl(D_{\boldsymbol{\sigma}} \circ \biggl[\sum_{|\boldsymbol{\tau}|\geq 0}\sum_{j=1}^{r} D_{\boldsymbol{\tau}}(\phi'_j)\cdot\frac{\partial A^{ij}_{\boldsymbol{\tau}}}{\partial w^k_{\boldsymbol{\sigma}}}\biggr]\biggr)(\phi''_i). \tag{10}$$

The coefficients of the terms, which are bilinear in ϕ' and ϕ'', in the bracket $\{\{\,\cdot\,,\,\cdot\,\}\}_{\hat{A}}$ are differential functions of the variables w.

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[2]The analogous operator $\hat{A}_1^{(1)} = \frac{d}{d\mu}\big|_{\mu=0}(\hat{A}_k)$, where $w^1 \mapsto w^1 + \mu$, is not Hamiltonian.

We now pass from Hamiltonian operators (7) to the operators $\square$ that have the same domain as $\hat{A}_k$ but take values in a different Lie algebra. Our main result is as follows.

Theorem 2. *Let the following conditions*[3] *be satisfied on an open dense subset of the Lagrangian Liouville-type system* $\mathcal{E}_{\mathrm{L}} = \{\boldsymbol{u}_{xy} = f(\boldsymbol{u}; x, y)\}$ (*see Proposition* 1):

a. *the constant symmetric real matrix κ in the kinetic term of the Lagrangian density $\mathcal{L}$ is invertible,*

b. *the linearization* $\ell_f^{(u)} = \|\partial f^i/\partial u^j\| \equiv f'(\boldsymbol{u}; x, y)$ *of the right-hand side in $\mathcal{E}_{\mathrm{L}}$ is an invertible matrix,*

c. *there are as many integrals* $w^i(x, [\mathfrak{m}]) \in \ker D_y|_{\mathcal{E}_{\mathrm{L}}}$ *as there are unknowns u^j,*

d. *the integrals w are minimal, i.e.,* $\Phi \in \ker D_y|_{\mathcal{E}_{\mathrm{L}}}$ *implies* $\Phi = (x, [w])$,

e. *the integrals w are differential-functionally independent,*[4] *in the sense that* $\Phi\bigl(x, \bigl[w[\mathfrak{m}]\bigr]\bigr) = 0$ *implies* $\Phi \equiv 0$, *and*

f. *the $r{\times}m$ matrix* $\Lambda = \|\partial w^i/\partial \mathfrak{m}^j_{d(i)}\|$, *where* $d(i) := \operatorname{ord}_x w^i$ *is the differential order of the ith integral* $w^i[\mathfrak{m}]$, *is invertible* (*i.e., its rank is maximal*).

Then the following statements hold:

1. *the image of operator* (3) *is closed under commutation of the symmetries* $\varphi = \square\bigl(\boldsymbol{\phi}(x, [w])\bigr) \in \operatorname{sym} \mathcal{E}_{\mathrm{L}}$,

2. *the bracket* $\{\{\,\cdot\,,\,\cdot\,\}\}_\square$ *resulting on the domain of the operator $\square$ satisfies the equality*

$$\{\{\phi', \phi''\}\}_\square = \{\{\phi', \phi''\}\}_{\hat{A}_k}, \quad \phi', \phi'' \in \Omega(\xi_\pi), \tag{11}$$

   and

3. *the coefficients of the bilinear terms in the bracket* $\{\{\,\cdot\,,\,\cdot\,\}\}_\square$ *are differential functions of the integrals* w.

Remark 3. The first hypothesis of the theorem implies that the system $\mathcal{E}_{\mathrm{L}}$ is determined, normal, and ℓ-normal (see Sec. 4). We emphasize that $\mathcal{E}_{\mathrm{L}}$ by assumption is the entire system of equations imposed on the sections $u = s(x, y) \in \Gamma(\pi)$.

Our second statement means that the ambiguity (up to $\ker \hat{A}_k$) in the choice of a representative from the equivalence class $\{\{\phi', \phi''\}\}_{\hat{A}_k}$ in the right-hand side of (11) amounts to the choice of an element from $\ker \square \subseteq \ker \hat{A}_k$. We prove the equality of the kernels for all $\boldsymbol{\phi}([w]) \in \Omega(\xi_\pi)$. This implies that commutation relations in $\operatorname{sym} \mathcal{E}_{\mathrm{L}}$, which are determined by the Lie algebra structure (9b) on $\Omega(\xi_\pi)$, are obtained explicitly via (10) for $\hat{A}_k$.

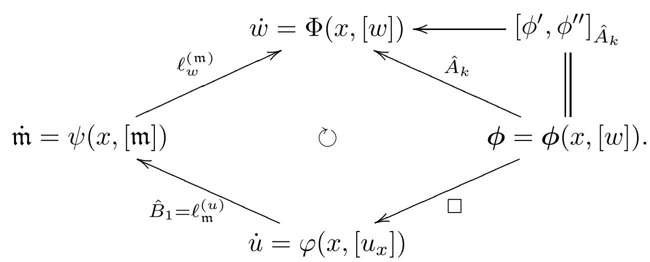
Being a corollary of Lemma 2 and the first two statements in Theorem 2, the third statement is simultaneously a special case of Proposition 2 (see below).

Proof of Theorem 2. We first note that symmetries (5) are independent of u and of $u_y, u_{yy}, \ldots$. Hence, this also holds for the commutator $\varphi = [\varphi', \varphi''] \in \operatorname{sym} \mathcal{E}_{\mathrm{L}}$ of two such symmetries $\varphi' = \square\bigl(\boldsymbol{\phi}'(x, [w])\bigr)$ and $\varphi'' = \square\bigl(\boldsymbol{\phi}''(x, [w])\bigr)$ because the Lie bracket is a local bidifferential operator.

[3] The list can be nonminimal such that it is easier to verify each requirement.

[4] The nonexistence of a nontrivial Φ and hence of its nonzero linearization $\ell_\Phi^{(\mathfrak{m})} = \ell_\Phi^{(w)} \circ \ell_w^{(\mathfrak{m})}$ is equivalent to the implication $(\nabla \circ \ell_w^{(\mathfrak{m})} = 0$ with $\nabla = \ell_{(\cdot)}^{(\mathfrak{m})}) \Rightarrow \nabla = 0$. In this form, the nondegeneracy requirement is dual to (12) (see below).

Factorization (7) and Lemma 2 can be represented by the diagram



The commutator $\varphi = [\varphi',\varphi'']$ determines the velocity $\Phi(x,[w])$ of the integrals that equals[5] $\partial_{[\varphi',\varphi'']}(w) = [\hat A_k(\phi'),\hat A_k(\phi'')]$. Because the image of the Hamiltonian operator $\hat A_k$ is closed under commutation, we obtain the equivalence class $\boldsymbol\phi(x,[w]) = [\phi',\phi'']_{\hat A_k}$ of sections such that $\Phi = \hat A_k(\boldsymbol\phi) = \partial_{\square(\boldsymbol\phi)}(w)$. By the construction of $\hat A_k$, the commutator of φ' and φ'' belongs to the set of symmetries $\square(\boldsymbol\phi)$. This proves the first statement in the theorem.

But there may be many such $\varphi = \square(\boldsymbol\phi)$ that induce the same velocity $\dot w = \Phi$. Because $\ker\square \subseteq \ker \hat A_k$, the equivalence class $[\phi',\phi'']_{\hat A_k}$ in principle may contain elements that do not belong to $[\phi',\phi'']_{\square}$.

We demonstrate that all representatives of the equivalence class $[\phi',\phi'']_{\hat A_k}$ determine the same symmetry φ of $\mathcal E_{\mathrm L}$. Therefore, this $\varphi = \square(\boldsymbol\phi)$ is the commutator $[\square(\boldsymbol\phi'),\square(\boldsymbol\phi'')]$ of the two symmetries because the image of $\boldsymbol\phi$ under $\square$ must contain it. It suffices to prove that the trivial solution φ of the linear homogeneous equation $\ell_w^{(u)}(\varphi) = (\ell_{w[\mathfrak m]}^{(\mathfrak m)} \circ \ell_{\mathfrak m}^{(u)})(\varphi) = 0$ is unique.

There are three ways to obtain the zero velocity $\Phi(x,[w]) \equiv 0$ of the integrals $w\bigl[\mathfrak m[u]\bigr]$ along $\varphi \in \operatorname{sym}\mathcal E_{\mathrm L}$. The first possibility is if the integrals were differentially dependent, which is excluded by the assumption in the theorem. The second possibility is if the intermediate equation $\kappa\cdot D_x(\varphi) = 0$, $\det\kappa \neq 0$, can have nontrivial solutions only if some shifts $\varphi = \mathrm{const}$ are symmetries of $\mathcal E_{\mathrm L}$. But the determining equation

$$\bigl(D_xD_y - f'(\boldsymbol u; x, y)\bigr)(\mathrm{const}) \doteq 0$$

on $\mathcal E_{\mathrm L}$ then implies linear dependence between the differentials of its right-hand sides. This contradicts another initial assumption.

Therefore, the proof of the second statement reduces to the uniqueness problem for the zero solution $\psi = 0$ of the linear homogeneous equation $\ell_{w[\mathfrak m]}^{(\mathfrak m)}(\psi) = 0$. Because $\mathcal E_{\mathrm L}$ is the only system[6] of differential relations imposed upon the sections $u = s(x,y) \in \Gamma(\pi)$, the zero in the right-hand side of $\ell_w^{(\mathfrak m)}(\psi) = 0$ is achieved identically with respect to $[\mathfrak m]$ (otherwise, it would overdetermine $\mathcal E_{\mathrm L}$).

There remain two situations where a velocity ψ of the momenta $\mathfrak m = -\kappa u_x/2$ makes $\dot w = \Phi$ zero. One reason is obvious: it is using the equation $\mathcal E_{\mathrm L} = \{\boldsymbol u_{xy} = f\}$ itself. Indeed, if the "time" along the flow $\dot{\mathfrak m} = \psi$ is the variable y, i.e., $\mathfrak m_y = -\delta H_{\mathrm L}/\delta u$, then we have $\partial_{\mathfrak m_y}^{(\mathfrak m)}(w) = D_y\bigl(w[\mathfrak m]\bigr) \doteq 0$ on $\mathcal E_{\mathrm L}$. But the presence of $\boldsymbol u$ among the arguments of f excludes such solutions ψ from further consideration.

Without restricting the generality, we assume that the integral $w^r[\mathfrak m]$ has the highest differential order: $d(r) \geq d(i)$ for all $i < r$. We calculate the velocities of the nonminimal integrals $(w')^i := D_x^{d(r)-d(i)}(w^i)$. Using the permutability $[D_x, \partial_\psi^{(\mathfrak m)}] = 0$ of evolutionary derivations with total derivatives, we deduce $\partial_\psi^{(\mathfrak m)}(w') = 0$ from the identities $\partial_\psi^{(\mathfrak m)}(w^1) = \cdots = \partial_\psi^{(\mathfrak m)}(w^r) = 0$. It is readily seen that the linearization of the new integrals has the form $\ell_{w'}^{(\mathfrak m)} = \Lambda\cdot D_x^{d(r)} + O\bigl(d(r)-1\bigr)$, where the matrix Λ is invertible

[5]The ℓ-normality of $\mathcal E_{\mathrm L}$ implies that φ is its symmetry whenever the velocity $\partial_\varphi(w)$ of the minimal integrals is in $\ker D_y|_{\mathcal E_{\mathrm L}}$ (see (16) below).

[6]The hyperbolic system $\mathcal E_{\mathrm L}$ is formally integrable [12]: its infinite prolongation $\mathcal E_{\mathrm L}^\infty$ exists, and there is an epimorphism $\mathcal E_{\mathrm L}^\infty \to \mathcal E_{\mathrm L}$.

by our initial assumption. Multiplying the new equation $\ell_{w'}^{(\mathfrak{m})}(\psi)=0$ by Λ^{-1}, we find by induction that $\psi(x,[\mathfrak{m}])$ is independent of $[\mathfrak{m}]$ and we therefore have $\psi=\psi(x)$. Consequently, the admissible sections φ that solve the intermediate equation $\psi(x)=-\kappa D_x(\varphi)/2$ also depend only on x: $\varphi=\varphi(x)$. But such sections, whenever nonzero, cannot be symmetries[7] of the hyperbolic system $u_{xy}-f(\boldsymbol{u};x,y)=0$, as a result of the nondegeneracy because $\det f'(\boldsymbol{u})\neq 0$. In this notation, the "symmetry" $\varphi(x)$ must satisfy the determining equation $\bigl(D_x\circ D_y-f'(\boldsymbol{u})\bigr)\varphi(x)\doteq 0$ on $\mathcal{E}_{\mathrm{L}}$. The first summand vanishes because $D_y(x)\equiv 0$. We thus obtain $f'(u)\cdot\varphi(x)=0$, where the linearization matrix $f'(\boldsymbol{u})$ is invertible. Hence, $\varphi(x)\equiv 0$. This completes the proof.

Remark 4. In the presented proof, we obtained the linear ODE $\ell_w^{(\mathfrak{m})}\bigl(\psi(x)\bigr)=0$, which holds simultaneously for all sections $s\in\Gamma(\pi)$, although nonzero solutions $\psi(x)$ do not contribute to the symmetry algebra. The presence of nontrivial solutions is possible, first, if there is a total differential operator ∇ such that $\ell_w^{(\mathfrak{m})}\circ\nabla=0$ (for instance, the identity $\bigl(\begin{smallmatrix}D_x & 1\\ 0 & 0\end{smallmatrix}\bigr)\bigl(\begin{smallmatrix}1\\ -D_x\end{smallmatrix}\bigr)\bigl(h(x)\bigr)\equiv 0$ holds for all $h(x)$). To avoid this, we must require that the linearization of the integrals be *nondegenerate*:

$$\ell_{w[\mathfrak{m}]}^{(\mathfrak{m})}\circ\nabla=0\Longrightarrow\nabla=0. \tag{12}$$

In the adjoint form, $\nabla^*\circ\square=0\Rightarrow\nabla^*=0$, Eq. (12) expresses the absence of linear differential relations[8] between components of the symmetries $\varphi=\square\bigl(\boldsymbol{\phi}(x,[w])\bigr)$. This property is dual to the nondegeneracy $\nabla\circ\ell_w^{(\mathfrak{m})}=0\Rightarrow\nabla=0$ that originates from the differential-functional independence $\Phi(x,[w])=0\Rightarrow\Phi\equiv 0$ via $\nabla=\ell_\Phi^w$ (see footnote 4).

Finally, let the section $s(x,y)\in\operatorname{Sol}\mathcal{E}_{\mathrm{L}}$ be a solution of the Darboux-integrable Liouville-type system $\mathcal{E}_{\mathrm{L}}$. Taking the restriction $\mathcal{L}^s=\ell_w^{(\mathfrak{m})}\big|_{j_\infty(s)}$ of the linearization operator to the jet of s, we obtain the ODE $\mathcal{L}^s\bigl(\psi(x)\bigr)=0$. For each s, the linear space $\mathcal{O}(s)$ of its solutions is finite dimensional. (For example, its dimension is equal to the sum of the exponents of a semisimple complex Lie algebra if $\mathcal{E}_{\mathrm{L}}$ is the associated two-dimensional Toda chain.) Therefore, the requirement

$$\bigcap_{s\in\operatorname{Sol}\mathcal{E}_{\mathrm{L}}\subset\Gamma(\pi)}\mathcal{O}(s)=\{0\}$$

in combination with (12) allows eliminating the excessive freedom in the choice of solutions $\psi(x)$ of the equation $\ell_w^{(\mathfrak{m})}(\psi)=0$.

Theorem 2 was illustrated for semisimple complex Lie algebras of rank two in [16], where the Hamiltonian operators $\hat{A}_1$ and $\hat{A}_k$ were constructed for the corresponding Drinfel'd–Sokolov hierarchies [11] and the commutation relations in $\operatorname{sym}\mathcal{E}_{\mathrm{L}}$ were calculated for the two-dimensional Toda chains $\boldsymbol{u}_{xy}=e^{K\boldsymbol{u}}$.

Example 3 (The modified Kaup–Boussinesq equation)**.** We consider a Lagrangian extension of the scalar Liouville equation [15],

$$A_{xy}=-\frac{1}{8}Ae^{-B/4},\qquad B_{xy}=\frac{1}{2}e^{-B/4}. \tag{13}$$

We let $a=B_x/2$ and $b=A_x/2$ denote the momenta. The minimal integrals of system (13) are $w_1=-a^2/4-a_x$ and $w_2=ab+2b_x$, and hence $D_y(w_i)\doteq 0$, $i=1,2$, by virtue of (13). Therefore, the operator

$$\square=\bigl(\ell_{w_1,w_2}^{(a,b)}\bigr)^*=\begin{pmatrix}-\frac{1}{4}B_x+D_x & \frac{1}{2}A_x\\ 0 & \frac{1}{2}B_x-2D_x\end{pmatrix}$$

[7] This is an immediate, pointwise generalization of the fact (see above) that $\varphi=\mathrm{const}\neq 0$ is not a symmetry of $\mathcal{E}_{\mathrm{L}}$.

[8] Hence, nondegeneracy (12) is analogous to the notion of ℓ-normal differential equations in the analysis of their formal integrability (see Sec. 4).

determines Noether symmetries (see (4)) of (13). The bracket $\{\{\,\cdot\,,\,\cdot\,\}\}_{\square}$ induced in the domain of $\square$ is

$$\{\{\boldsymbol{\psi},\boldsymbol{\chi}\}\}_{\square}=\frac{1}{2}\begin{pmatrix}\psi^2_x\chi^1-\psi^1\chi^2_x+\psi^1_x\chi^2-\psi^2\chi^1_x\\ \psi^2_x\chi^2-\psi^2\chi^2_x\end{pmatrix},$$

where $\boldsymbol{\psi}={}^{\mathrm{t}}(\psi^1,\psi^2)$ and $\boldsymbol{\chi}={}^{\mathrm{t}}(\chi^1,\chi^2)$.

We consider a symmetry of (13),

$$A_t=\frac{1}{2}A_xA_{xx}+\frac{1}{2}\left(\frac{1}{4}A_x^2-1\right)B_x,\qquad B_t=-2A_{xxx}+\frac{1}{8}A_xB_x^2-\frac{1}{2}A_xB_{xx}. \tag{14}$$

We choose an equivalent pair of integrals $u=w_2$ and $v=w_1+w_2^2/4$. The evolution of u and v along (14) is

$$u_t=uu_x+v_x,\qquad v_t=(uv)_x+u_{xxx}. \tag{15}$$

This is the Kaup–Boussinesq system, and (14) is the potential twice-modified Kaup–Boussinesq equation (see [22]). The right-hand side of integrable system (14) belongs to the image of the adjoint linearization $\widetilde{\square}=(\ell^{(a,b)}_{(u,v)})^*$. The operator $\widetilde{\square}$ factors the third Hamiltonian structure $\hat{A}_3^{\mathrm{KB}}=\widetilde{\square}^*\circ(\ell^{(A,B)}_{(a,b)})^*\circ\widetilde{\square}$ for (15); we have $k=3$ and

$$\hat{A}_3^{\mathrm{KB}}=\begin{pmatrix}uD_x+\frac{1}{2}u_x & D_x^3+\left(\frac{1}{4}u^2+v\right)D_x+\frac{1}{4}(u^2+2v)_x\\ D_x^3+\left(\frac{1}{4}u^2+v\right)D_x+\frac{1}{2}v_x & uD_x^3+\frac{3}{2}u_xD_x^2+\left(\frac{3}{2}u_{xx}+uv\right)D_x+\frac{1}{2}u_{xxx}+\frac{1}{2}\,(uv)_x\end{pmatrix}.$$

By Theorem 2, the bracket $\{\{\,\cdot\,,\,\cdot\,\}\}_{\widetilde{\square}}$ is equal to $\{\{\,\cdot\,,\,\cdot\,\}\}_{\hat{A}_3^{\mathrm{KB}}}$, given by formula (10). We obtain

$$\{\{\boldsymbol{\psi},\boldsymbol{\chi}\}\}_{\widetilde{\square}}=\{\{\boldsymbol{\psi},\boldsymbol{\chi}\}\}_{\hat{A}_3^{\mathrm{KB}}}=\begin{pmatrix}\boldsymbol{\psi}\cdot\nabla_1(\boldsymbol{\chi})-\nabla_1(\boldsymbol{\psi})\cdot\boldsymbol{\chi}\\ \boldsymbol{\psi}\cdot\nabla_2(\boldsymbol{\chi})-\nabla_2(\boldsymbol{\psi})\cdot\boldsymbol{\chi}\end{pmatrix},$$

where

$$\nabla_1=-\frac{1}{2}\begin{pmatrix}D_x & 0\\ uD_x & D_x^3+vD_x\end{pmatrix},\qquad \nabla_2=-\frac{1}{2}\begin{pmatrix}0 & D_x\\ D_x & uD_x\end{pmatrix}.$$

The operator $\hat{A}_1=\left(\begin{smallmatrix}0 & D_x\\ D_x & 0\end{smallmatrix}\right)$ is the first Hamiltonian structure for (15); its inverse $A_1=\hat{A}_1^{-1}$ factors the second Hamiltonian structure $B_2=\widetilde{\square}\circ A_1\circ\widetilde{\square}$ for (14).

4. Non-Lagrangian Liouville-type systems

Let $\mathcal{E}=\{F=0\}$ be a Liouville-type system but not necessarily Euler–Lagrange. We assume that a column $w\in\ker D_y|_{\mathcal{E}}$ is composed of minimal integrals for $\mathcal{E}$ and hence $D_y(w)=\nabla(F)$ for some operator ∇. From the structure of the Liouville-type systems $\mathcal{E}$, there are no differential relations (*syzygies*) between the hyperbolic equations $\{F^i=0\}$ in them: $\Delta(F)=0$ implies $\Delta=0$. For the same reason, the systems $\mathcal{E}$ are ℓ-normal [12], [14]: $\Delta\circ\ell_F\doteq 0$ on $\mathcal{E}$ also requires $\Delta=0$. Consequently, an evolutionary vector field ∂_φ is a symmetry of a Liouville-type system $\mathcal{E}$ if and only if it preserves the integrals,

$$D_y\big(\partial_\varphi(w)\big)=\partial_\varphi(\nabla)(F)+\nabla\big(\partial_\varphi(F)\big)\doteq\nabla\big(\ell_F(\varphi)\big)\quad\text{on }\mathcal{E}. \tag{16}$$

We consider the operator equation

$$D_y\circ\ell_w^{(u)}\doteq\nabla\circ\ell_F\quad\text{on }\mathcal{E}.$$

If, hypothetically, a total differential operator $\square$ such that

$$\ell_w^{(u)} \circ \square \in \mathcal{C}\,\mathrm{Diff}(\ker D_y|_{\mathcal{E}} \to \ker D_y|_{\mathcal{E}}) \tag{17}$$

were constructed, then it would assign symmetries $\varphi = \square(\boldsymbol{\phi})$ of the Liouville-type system $\mathcal{E}$ to arbitrary r-tuples $\boldsymbol{\phi}(x, [w])$ of the integrals (see (5)).

In [10], there is an algorithm for constructing operators $\square$, solutions of the equation in total derivatives

$$\ell_w^{(u)} \circ \square = \mathbf{1}_{m\times m} \cdot D_x^k \quad \mod \mathcal{C}\,\mathrm{Diff}_{<k}(\ker D_y|_{\mathcal{E}} \to \ker D_y|_{\mathcal{E}}). \tag{18}$$

Most remarkably, truncation of the sequence of coefficients of derivative powers in $\square$ from below is a consequence of the presence of a complete set of the integrals $\overline{w} \in \ker D_x|_{\mathcal{E}}$ for $\mathcal{E}$. But the minimal integrals w must be "spoiled" by differentiating with respect to x a suitable number of times. Consequently, instead of the Hamiltonian operator $\hat{A}_k = \ell_w^{(u)} \circ \square$ (see (7)), we obtain the right-hand side of (18). Moreover, the images of the operators constructed in [10] do not always span the entire x components of the Lie algebras $\operatorname{sym}\mathcal{E}$, and the images are generally not closed under commutation. Further, the transformation rules in the domains of $\square$ under reparameterizations $\widetilde{w}[w]$ of the integrals remain unspecified for non-Lagrangian Liouville-type systems.

Proposition 2. *If the image of a solution $\square$ of operator equation* (17) *for a Liouville-type system $\mathcal{E}$ is closed under commutation, then all coefficients of the bracket $\{\{\cdot,\cdot\}\}_\square$ on its domain* (*see* (9)) *belong to* $\ker D_y|_{\mathcal{E}}$.

Proof. By assumption, we have $(D_y \circ \ell_w^{(u)} \circ \square)([\phi', \phi'']_\square) \doteq 0$ for all $\phi', \phi''(x, [w])$. We write this equality in more detail,

$$0 \doteq (D_y \circ \underline{\ell_w^{(u)} \circ \square})\big(\partial_{\square(\phi')}(\phi'') - \partial_{\square(\phi'')}(\phi') + \{\{\phi', \phi''\}\}_\square\big) \doteq (\ell_w^{(u)} \circ \square)\big(D_y(\{\{\phi', \phi''\}\}_\square)\big)$$

because the underlined composition satisfies (17). Clearly, $D_y(\phi')$ and $D_y(\phi'')$ vanish on $\mathcal{E}$ for arbitrary ϕ' and ϕ''. For the same reason, not only the whole bracket $\{\{\phi', \phi''\}\}_\square$ but also each particular coefficient of the bilinear terms in it is in $\ker D_y|_{\mathcal{E}}$.

Example 4. We consider the parametric extension of the scalar Liouville equation,

$$\mathcal{E}(\varepsilon) = \big\{u_{xy} = e^{2u}\sqrt{1 + 4\varepsilon^2 u_x^2}\,\big\}, \quad \varepsilon \in \mathbb{R}. \tag{19}$$

This equation is ambient with respect to the hierarchy of Gardner's deformation of the potential modified KdV equation (see [15]). The contraction $\mathcal{U} = \mathcal{U}\big(\varepsilon, [u(\varepsilon)]\big)$ from (19) to the nonextended equation $\mathcal{U}_{xy} = e^{2\mathcal{U}}$ is $\mathcal{U} = u + \operatorname{arcsinh}(2\varepsilon u_x)/2$; we obtain the third-order integral for (19) if we substitute it in the integral $w = \mathcal{U}_x^2 - \mathcal{U}_{xx}$ at $\varepsilon = 0$. But the regularized (at $\varepsilon = 0$) integral of lower (second) order for (19) is

$$w = \frac{1 - \sqrt{1 + 4\varepsilon^2 u_x^2}}{2\varepsilon^2} + \frac{u_{xx}}{\sqrt{1 + 4\varepsilon^2 u_x^2}}. \tag{20}$$

The second integral for (19) is $\overline{w} = u_{yy} - u_y^2 - \varepsilon^2 e^{4u} \in \ker D_x|_{\mathcal{E}(\varepsilon)}$. The operators

$$\begin{aligned} \overline{\square} &= u_y + \frac{1}{2}D_y, \\ \square &= \frac{1}{2}(1 + 4\varepsilon^2 u_x^2 - 2\varepsilon^2 u_{xx})D_x + u_x + 4\varepsilon^2 u_x^3 - 2\varepsilon^2 u_{xxx} + \frac{12\varepsilon^4 u_x u_{xx}^2}{1 + 4\varepsilon^2 u_x^2} \end{aligned} \tag{21}$$

assign the symmetries $\varphi = \square\big(\phi(x,[w])\big)$ and $\bar\varphi = \overline{\square}\big(\bar\phi(y,[\overline{w}])\big)$ of (19) to its integrals.

The image of each of the operators $\square$ and $\overline{\square}$ is a Lie subalgebra in $\operatorname{sym}\mathcal{E}(\varepsilon)$. The bracket $\{\{p,q\}\}_{\overline{\square}} = p_y q - p q_y$ for $\overline{\square}$ is familiar [3], [9]. The bracket induced in the domain of $\square$ has the following form: for any arguments p and q, we have

$$\begin{aligned}
\{\{p,q\}\}_{\square} = {}&\varepsilon^2(p_{xx}q_x - p_x q_{xx}) - 2\varepsilon^2(p_{xxx}q - p q_{xxx}) - \\
&- 12\varepsilon^4(8\varepsilon^2 u_x^3 u_{xx} - 4\varepsilon^2 u_x^2 u_{xxx} + 4\varepsilon^2 u_x u_{xx}^2 + 2u_x u_{xx} - u_{xxx}) \times \\
&\times (1 + 8\varepsilon^2 u_x^2 + 16\varepsilon^4 u_x^4 - 2\varepsilon^2 u_{xx} - 8\varepsilon^4 u_x^2 u_{xx})^{-1}(p_{xx}q - p q_{xx}) + \\
&+ (\underline{1} + 288\varepsilon^4 u_x^4 - 288\varepsilon^4 u_x^2 u_{xx} + 28\varepsilon^2 u_x^2 - 16\varepsilon^2 u_{xx} - 288\varepsilon^6 u_x u_{xx} u_{xxx} - \\
&- 96\varepsilon^6 u_{xx}^3 + 3072\varepsilon^{10} u_x^{10} + 24\varepsilon^6 u_{xxx}^2 + 24\varepsilon^4 u_{4x} + 1408\varepsilon^6 u_x^6 + 3328\varepsilon^8 u_x^8 - \\
&- 768\varepsilon^{10} u_{4x} u_{xx} u_x^4 - 384\varepsilon^8 u_{4x} u_x^2 u_{xx} - 2304\varepsilon^8 u_x^3 u_{xx} u_{xxx} + 384\varepsilon^8 u_{xx}^2 u_x u_{xxx} - \\
&- 4608\varepsilon^{10} u_x^5 u_{xx} u_{xxx} + 16\varepsilon^4 u_{xx}^2 - 5632\varepsilon^8 u_x^6 u_{xx} - 1920\varepsilon^6 u_{xx} u_x^4 + 3328\varepsilon^8 u_x^4 u_{xx}^2 + \\
&+ 512\varepsilon^6 u_{xx}^2 u_x^2 + 384\varepsilon^{10} u_x^4 u_{xxx}^2 - 960\varepsilon^{10} u_{xx}^4 u_x^2 - 48\varepsilon^4 u_x u_{xxx} - 3072\varepsilon^{10} u_x^7 u_{xxx} + \\
&+ 3072\varepsilon^{10} u_{xx}^3 u_x^4 - 2304\varepsilon^8 u_x^5 u_{xxx} - 576\varepsilon^6 u_x^3 u_{xxx} + 288\varepsilon^6 u_{4x} u_x^2 + 384\varepsilon^8 u_x^2 u_{xx}^3 + \\
&+ 6144\varepsilon^{10} u_{xx}^2 u_x^6 - 6144\varepsilon^{10} u_{xx} u_x^8 + 1152\varepsilon^8 u_{4x} u_x^4 + 1536\varepsilon^{10} u_{4x} u_x^6 + 192\varepsilon^8 u_{xxx}^2 u_x^2 + \\
&+ 240\varepsilon^8 u_{xx}^4 + 1536\varepsilon^{10} u_{xx}^2 u_x^3 u_{xxx} - 48\varepsilon^6 u_{4x} u_{xx}) \times \\
&\times (\underline{1} + 96\varepsilon^4 u_x^4 + 256\varepsilon^6 u_x^6 + 256\varepsilon^8 u_x^8 + 4\varepsilon^4 u_{xx}^2 - 48\varepsilon^4 u_x^2 u_{xx} + 32\varepsilon^6 u_{xx}^2 u_x^2 - \\
&- 4\varepsilon^2 u_{xx} - 256\varepsilon^8 u_x^6 u_{xx} + 64\varepsilon^8 u_x^4 u_{xx}^2 - 192\varepsilon^6 u_{xx} u_x^4 + 16\varepsilon^2 u_x^2)^{-1}(p_x q - p q_x).
\end{aligned}$$

The two underlined units correspond to the bracket $p_x q - p q_x$ on the domain of the operator $\square = \mathcal{U}_x + D_x/2$, which gives symmetries of the Liouville equation $\mathcal{U}_{xy} = e^{2\mathcal{U}}$ at $\varepsilon = 0$. In agreement with Proposition 2, the nonconstant coefficients of the bilinear terms $p_{xx}q - p q_{xx}$ and $p_x q - p q_x$ in $\{\{p,q\}\}_{\square}$ belong to $\ker D_y|_{\mathcal{E}(\varepsilon)}$. It is remarkable that because the entire construction (19)–(21) contains formal power series $u(\varepsilon)$ in ε, these are the two rational functions. An attempt to express them in terms of w and its derivatives with respect to x leads to formal series with unbounded growth of the differential orders of its coefficients.

5. Discussion

The matrix operators $\square = (\square^{i,j},\ 1 \le i \le m,\ 1 \le j \le r)$ given by (3) are generalizations of tensors of type $(2,0)$ (in the geometry of infinite jet bundles). We define the operators by using the two unrelated groups of differential reparameterizations for the respective coordinates in the domains and images. Furthermore, the operators $\square$ for the Liouville-type systems $\mathcal{E}_{\mathrm{L}}$ generalize the theory of Hamiltonian structures as follows: they map variational covectors for one equation (we recall that $\operatorname{sym}\mathcal{E}_{\varnothing} \supset \mathfrak{A}$) to symmetries of the other system $\mathcal{E}_{\mathrm{L}}$ (such that $\operatorname{sym}\mathcal{E}_{\mathrm{L}} \supset \mathfrak{B}$).

Unlike in [8], [10], we do not attempt to solve Eq. (17) for $\square$. On the contrary, we defined operators (3) by a geometric reasoning. Hence, we first obtained the Hamiltonian operators $\hat{A}_k = \ell_w^{(u)} \circ \square$ for the KdV-type hierarchies on Lagrangian systems of Liouville type [9], [11] and then proved that the images of the operators $\square$ are involutive (see [23]). In other words, we described an algorithm aimed at constructing completely integrable equations.

Formulas (3) and (7) prescribe the differential order of $\hat{A}_k$. Estimates for the orders of the integrals w for the two-dimensional Toda chains associated with semisimple complex Lie algebras $\mathfrak{g}$ are known from [2] (see Example 1) and are contained in [4], [16] in a different formulation. The upper bound, that the numbers $\operatorname{ord}_x w^i - 1$ are not greater than the exponents for $\mathfrak{g}$, is proved by using Schur polynomials to verify that Serre's relations $(\operatorname{ad} Y_i)^{-K^i_j+1}(Y_j) = 0$, $i \neq j$, hold for the generators

$$Y_i = \sum_{k\geq 0} \exp\biggl(-\sum_{j=1}^{m} K^i_j u^j\biggr) \cdot D_x^k\biggl(\exp\biggl(\sum_{j'=1}^{m} K^i_{j'} u^{j'}\biggr)\biggr) \cdot \frac{\partial}{\partial u^i_{k+1}}$$

of the characteristic Lie algebras (see [1], [2], [7], and also [16]) and by using the Frobenius theorem. The fact that the vector fields Y_i coincide with the Chevalley generators $\mathfrak{f}_i$ of the semisimple Lie algebra $\mathfrak{g}$ is important here. The same estimate from below follows from the absence of relations other than Serre's for the generators Y_i. This was established in [2] for the root systems A_n and D_n by explicitly listing the linearly independent nonzero iterated commutators.

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