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A NOTE ON ELLIPTIC FUNCTIONS AND APPROXIMATION
BY ALGEBRAIC NUMBERS OF BOUNDED DEGREE

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Résumé : Soit p une fonction elliptique de Weierstrass d'invariants g_2 et g_3 algébriques. Par un contre-exemple, on montre que pour l'obtention d'une minoration pour l'approximation simultanée de $p(a)$, b et $p(ab)$ par des nombres algébriques de degré borné, une hypothèse supplémentaire sur les nombres β qui approximent b est nécessaire.

Summary : Let p be a Weierstrass elliptic function with algebraic invariants g_2 and g_3 . By a counterexample it is shown that lower bounds for the simultaneous approximation of $p(a)$, b and $p(ab)$ by algebraic numbers of bounded degree cannot be given without an added hypothesis on the numbers β approximating b .

Let p be a Weierstrass elliptic function with algebraic invariants g_2, g_3 ; for $a, b \in \mathbb{C}$ such that a and ab are not poles of p , we consider the simultaneous approximation of $p(a)$, b and $p(ab)$ by algebraic numbers. It was shown in [2], Theorem 2, that lower bounds for the approximation errors in terms of the heights and degrees of these algebraic numbers can only be given if the numbers β used to approximate b do not lie in the field \mathbb{K} of complex multiplication of p . (As this condition is equivalent to the algebraic independence of $p(z)$ and $p(\beta z)$ as functions of z , the result proves the conjecture on admissible sets in Appendix 2 of [3]).

Now consider simultaneous approximation of the same numbers by algebraic numbers of bounded degree. The sequences of algebraic numbers constructed in [2] have rapidly rising

degrees, so they do not provide a relevant counterexample. It is the purpose of this note to show how the original example should be modified for the new problem.

Let $\Omega = \mathbb{Z} \omega_1 + \mathbb{Z} \omega_2$ denote the period lattice of p , and IF the field $\mathbb{Q}(g_2, g_3)$. For every $d \in \mathbb{N}$, the set of $z \in \mathbb{C} \setminus \Omega$ such that $p(z)$ is algebraic of degree at most d is denoted by A_d . Let B be an open set in \mathbb{C} such that its closure \bar{B} is contained in the interior of the fundamental parallelogram $[0,1]\omega_1 + [0,1]\omega_2$.

LEMMA 1. *For every $d \geq 2$, the set A_d is dense in \mathbb{C} .*

Proof. Let $\mathcal{O} \subset \mathbb{C}$ be an arbitrary open set. Take $a \in \mathcal{O} \setminus \Omega$ with $p'(a) \neq 0$. According to [1], Chapter 4, Theorem 11, Corollary 2, there exist open sets U, V with $a \in U \subset \mathcal{O}, p(a) \in V$, such that p induces a bijection from U onto V . As $\{z \in \bar{\mathbb{Q}} \mid \deg z \leq d\}$ is dense in \mathbb{C} , we can find $z \in V \cap \bar{\mathbb{Q}}$ with $\deg z \leq d$. For the unique $u \in U$ with $p(u) = z$, we have $u \in \mathcal{O} \cap A_d$. ■

LEMMA 2. *Assume $d \geq 2[\text{IF} : \mathbb{Q}]$. Then, for every $g : \mathbb{N} \rightarrow \mathbb{R}$, there exist sequences $(u_n)_{n=1}^\infty, (\beta_n)_{n=1}^\infty, (v_n)_{n=1}^\infty, (\epsilon_n)_{n=1}^\infty$, such that for all $n \in \mathbb{N}$ the following statements are true :*

- (1) $u_n \in A_d \cap B, \beta_n \in [0,1] \cap \bar{\mathbb{Q}}, v_n \in A_d, v_n = \beta_n u_n, \epsilon_n \in]0,1[$;
- (2) $\epsilon_{n+1} < \exp(-n |g(H_n)|)$, where $H_n := \max(H(p(u_n)), H(\beta_n), H(p(v_n)))$;
- (3) $\epsilon_{n+1} < \epsilon_n^2, \epsilon_{n+1} < \frac{1}{4} \deg^{-4} \beta_n$;
- (4) $0 < |\beta_n - \beta_{n+1}| < \epsilon_{n+1}, |u_n - u_{n+1}| < \epsilon_{n+1}$.

Proof. Take $u_1 \in A_d \cap B$ (the existence of such an u_1 follows from Lemma 1). Define $v_1 := u_1$, $\beta_1 := 1$, $\epsilon_1 := \frac{1}{2}$. Then (1) is true for $n = 1$. Now suppose $u_1, \dots, u_N, \beta_1, \dots, \beta_N, v_1, \dots, v_N, \epsilon_1, \dots, \epsilon_N$ have been chosen in such a way that (1) holds for $n = 1, \dots, N$ and (2), (3), (4) hold for $n = 1, \dots, N-1$, and proceed by induction.

Choose $\epsilon_{N+1} \in]0,1[$ so small that (2) and (3) hold for $n = N$. Take $r > \epsilon_{N+1}^{-1}$ and consider the function $f : \mathbb{C} \rightarrow \mathbb{C}$ defined by $f(z) := rz$. As f is a continuous bijection, there exists an open set $U \subset \mathbb{C}$ with $fU \subset B \cap B(u_N, \epsilon_{N+1})$. Take $w \in U$ such that $p(w) \in \bar{\mathbb{Q}}$ with $\deg p(w) \leq 2$ (the existence of w again follows from Lemma 1). Define $u_{N+1} := rw$. By Lemma 6.1 of [4], $p(u_{N+1}) \in \bar{\mathbb{Q}}$ and

$$\deg p(u_{N+1}) \leq [\text{IF}(p(w)) : \mathbb{Q}] \leq 2[\text{IF} : \mathbb{Q}] \leq d,$$

so $u_{N+1} \in A_d$. Furthermore the definition of U gives $u_{N+1} \in B$ and $|u_N - u_{N+1}| < \epsilon_{N+1}$. Take

$s \in \mathbb{N}$ with $0 \leq s \leq r$ and $0 < |\beta_N - \frac{s}{r}| < \epsilon_{N+1}$; define $\beta_{N+1} := \frac{s}{r}$; then $\beta_{N+1} \in [0, 1] \cap \mathbb{Q}$ and (4) holds for $n = N$. Define $v_{N+1} := \beta_{N+1} u_{N+1} = sw$; then as above we find that $v_{N+1} \in A_d$ and (1) holds for $n = N + 1$. ■

THEOREM. Assume $d \geq 2$ [IF : \mathbb{Q}]. Then, for every $g : \mathbb{N} \rightarrow \mathbb{R}$, there exist $a \in \mathbb{C}$, $b \in \mathbb{C} \setminus \mathbb{K}$, such that a and ab are not poles of p and such that for every $C \in \mathbb{R}$ there exist infinitely many tuples $(u, \beta, v) \in \mathbb{C}^3$ satisfying $u, v \in A_d$, $\beta \in \mathbb{Q}$ and

$$\max(|p(a) - p(u)|, |b - \beta|, |p(ab) - p(v)|) < \exp(-Cg(H))$$

while $\max(H(p(u)), H(\beta), H(p(v))) \leq H$.

Proof. According to Lemma 3 of [2], the sequences $(u_n)_{n=1}^\infty$ and $(\beta_n)_{n=1}^\infty$ constructed in Lemma 2 above are Cauchy sequences and their limits a, b satisfy

$$(5) \quad \max(|a - u_n|, |b - \beta_n|) \leq \epsilon_{n+1}^{1/2}$$

for almost all n . Thus $a \in \overline{B}$ and therefore a cannot be a pole of p . Formula (4) implies the existence of arbitrarily large n for which $\beta_n \neq b$; as by (3) and (5), every β_n is a convergent of the continued fraction expansion of b and $\lim \beta_n = b$, it follows that b has infinitely many convergents. Thus $b \in \mathbb{R} \setminus \mathbb{Q}$ and therefore $b \notin \mathbb{K}$. On particular, $b \neq 0$; hence ab cannot be a pole of p either.

By the continuity of p in ab , (5) implies

$$(6) \quad \max(|p(a) - p(u_n)|, |b - \beta_n|, |p(ab) - p(v_n)|) \leq c\epsilon_{n+1}^{1/2}$$

for almost all n , where c does not depend on n . In the notation of (2), the right hand member of (6) satisfies

$$c\epsilon_{n+1}^{1/2} < c \exp(-\frac{1}{2} n |g(H_n)|) < \exp(-Cg(H_n))$$

if n is sufficiently large in terms of C and c . ■

REFERENCES

- [1] L.V. AHLFORS. «*Complex analysis*». 2nd edition. Mac Graw-Hill Book Co., New-York, 1966.
- [2] A. BIJLSMA. «*An elliptic analogue of the Franklin-Schneider theorem*». Ann. Fac. Sci. Toulouse (5) 2 (1980), 101-116.
- [3] W.D. BROWNAWELL & D.W. MASSER. «*Multiplicity estimates for analytic functions*». I.J. Reine Angew. Math. 314 (1980), 200-216.
- [4] D.W. MASSER. «*Elliptic functions and transcendence*». Lecture Notes in Mathematics 437. Springer-Verlag, Berlin, 1975.

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