# Stochastic Dynamics of a Trapped Bose-Einstein Condensate

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arXiv:cond-mat/0107432 v1 20 Jul 2001

## Abstract

We present a variational solution of the Langevin field equation describing the nonequilibrium dynamics of a harmonically trapped Bose-Einstein condensate. If the thermal cloud remains in equilibrium at all times, we find that the equation of motions for the parameters in our variational *ansatz* are equivalent to the Langevin equations describing the motion of a massive Brownian particle in an external potential. Moreover, these equations are coupled to a stochastic rate equation for the number of atoms in the condensate. As applications of our approach, we have calculated the collisional damping rates and frequencies of the low-lying collective excitations of a condensate with repulsive interactions, and have obtained a description of the growth and subsequent collapse of a condensate with attractive interactions. We have found a good agreement with the available experimental results in both cases.

PACS number(s): 03.75.Fi, 67.40.-w, 32.80.Pj

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## I. INTRODUCTION

The experimental realization of Bose-Einstein condensation in dilute atomic gases [1–3], has led to a large increase in the amount of both experimental and theoretical research on these quantum systems. Various theoretical predictions regarding equilibrium and nonequilibrium properties of degenerate Bose gases, can now be directly compared with experimental data. Regarding the zero-temperature behaviour of a Bose-Einstein condensate, a great deal of the physics is well described by the Gross-Pitaevskii equation, i.e., a mean-field equation for the macroscopic wave function of the condensate. It has led to very good agreement with experimental results on, for example, the condensate collective mode frequencies and the density profile of the condensate at zero temperature [4]. To understand the nonzero temperature behaviour of Bose-condensed gases, several proposals have been made to generalize the Gross-Pitaevskii equation and to include the effects of the thermal cloud on the condensate. At the mean-field level this is achieved by introducing in the Gross-Pitaevskii equation real and imaginary terms, that describe the coherent and incoherent effects of collisions between condensate and thermal atoms, and that in particular cause evaporation or growth of the condensate [5–9]. However, at nonzero temperatures also fluctuations can play an important role. An example of this is the reversible formation of a condensate, as experimentally achieved by Stamper-Kurn *et al.* [10]. Since the system is several times in the critical region, fluctuations of the order parameter around its mean-field value are of utmost importance to describe this experiment [11]. In addition, both quantum and thermal fluctuations are important to understand the stochastic nature of the collapse observed in $^7$Li [12–17] and the phenomena of phase 'diffusion' [18], in which case these fluctuations disturb the global phase of the condensate. Finally, from a fundamental point of view, a consistent description of a partially Bose-Einstein condensed gas requires that the fluctuation-dissipation theorem is obeyed, since this ensures relaxation of the system towards its correct physical equilibrium. Therefore, if dissipation is to be included in the generalized Gross-Pitaevskii equation, fluctuations must also be included.

Gardiner and Zoller have included such fluctuations in the description of a Bose-condensed system by deriving with second-order perturbation theory a master equation for the one-body density matrix [19], a procedure well-known from quantum optics. However, in this paper we will use the nonperturbative formulation developed previously by one of us [7,20]. Using field-theoretical techniques Stoof derived a Fokker-Planck equation describing the full nonequilibrium probability distribution of the order parameter. An equivalent formulation of this theory can be given in terms of a dissipative nonlinear Schrödinger equation with noise. Although in principle we can turn to numerical methods for the solution of the Fokker-Planck equation, or its corresponding Langevin equation [11], we find it more convenient here to proceed analytically, by means of a variational method. Variational approximations have previously provided a useful way to make analytical progress, and capture as many of the physics as possible. In particular, when applied to the zero-temperature Gross-Pitaevskii equation, a gaussian variational approximation has led to good results on the collective modes of the condensate [21–23], and on the description of the macroscopic tunneling of a condensate with attractive interactions [13,15,24,25]. It is the aim of this paper to apply a similar variational method also to the dissipative nonlinear Schrödinger equation with noise appropriate for nonzero temperatures. We achieve this by assuming

that the thermal cloud is in equilibrium at all times, and therefore acts as a 'heat bath' on the condensate. The stochastic nonlinear Schrödinger equation then obeys an equilibrium version of the fluctuation-dissipation theorem, which ensures that the condensate relaxes to the physically correct equilibrium. With this assumption, we are then able to derive Langevin equations for the variational parameters in our gaussian *ansatz*, which turn out to be equivalent to the equations of motion for a Brownian particle in a potential. These equations are coupled to a stochastic rate equation for the number of atoms in the condensate. Using these equations of motion, we are then able to describe collisional damping of the condensate collective modes at nonzero temperatures, and the condensate growth and stochastic initiation of the collapse, as recently observed in $^7$Li [16,17].

The rest of this paper is organized as follows. To make this paper selfcontained, we review in Sec. II the techniques of path integrals, and their application to stochastic differential equations. We use the method of functional integration throughout this paper. In Sec. III we review the Fokker-Planck equation describing the nonequilibrium dynamics of a Bose-Einstein condensed gas, and discuss the equilibrium solution of this Fokker-Planck equation. The most important result of this section is the Langevin field equation for the order parameter, that obeys the fluctuation-dissipation theorem. This Langevin field equation takes the form of a dissipative nonlinear Schrödinger equation with noise. We also derive the stochastic equations of motion for the density and phase of the condensate, and a damped wave equation describing the propagation of sound waves in a homogeneous Bose gas, at nonzero temperatures. In Sec. IV we present the variational approximation to our nonlinear dissipative Schödinger equation with noise, and also derive stochastic equations of motion for the variational parameters, which are the central result of this paper. To the best of our knowledge, a variational method for stochastic field equations, such as the stochastic nonlinear Schrödinger equation under consideration here, has not been developed previously. In Sec. V we apply our equations to calculate the temperature dependence of the damping and frequencies of the collective modes of a condensate, and to obtain a description of a growth-collapse curve of a condensate with attractive interactions. We end in Sec. VI with our conclusions.

## II. PATH INTEGRALS AND STOCHASTIC DIFFERENTIAL EQUATIONS

In this section, we discuss the application of path integrals to stochastic differential equations. First, we consider Brownian motion in the overdamped limit, which means that the acceleration of the particle can be neglected with respect its damping rate. In the second part we consider underdamped Brownian motion, in which the nonzero mass of the particle plays a crucial role.

### A. Overdamped Brownian motion

Consider the one-dimensional Langevin equation [26,27]

$$\dot{q}(t) = f(q(t)) + \eta(t). \tag{1}$$

In this equation, $q(t)$ denotes the position of the particle, and $\eta(t)$ is a gaussian noise term, which has a time-correlation function given by

$$\langle \eta(t')\eta(t)\rangle = \sigma\delta(t'-t). \tag{2}$$

Here, the brackets denote averaging over different realizations of the noise. The parameter $\sigma$ is positive and real, and is often referred to as the 'strength' of the noise. For a given initial condition $q(t_0) = q_0$, the Langevin equation in Eq. (1) generates a probability distribution $P[q,t;q_0,t_0]$. By definition, $P[q,t;q_0,t_0]dq$ denotes the probability that a solution $q(t)$ of Eq. (1) starting at $q_0$ at time $t_0$, has a value between $q$ and $q+dq$ at time $t$. From this definition, we can immediately write down an expression for this probability distribution, i.e.,

$$P[q,t;q_0,t_0] = \langle \delta(q(t)-q)\rangle. \tag{3}$$

Here $q(t)$ denotes again a solution of the Langevin equation for a particular realization of the noise, starting at $q(t_0) = q_0$. The brackets denote averaging over different realizations of the noise, as in Eq. (2).

We now want to derive a path-integral expression [28,29] for the probability distribution $P[q,t;q_0,t_0]$. To achieve this, we first divide the time interval $t-t_0$ into $N$ pieces, each of length $\Delta = (t-t_0)/N$. Using the notation $q(t_n) \equiv q_n$, and $\eta(t_n) \equiv \eta_n$, we discretize the Langevin equation in Eq. (1) as follows

$$\frac{1}{\Delta}(q_{n+1}-q_n) = f(q_n) + \eta_n. \tag{4}$$

The time-correlation of the noise is given by

$$\langle \eta_i \eta_j \rangle = \frac{\sigma}{\Delta}\delta_{ij}, \tag{5}$$

which reduces to Eq. (2) in the limit $\Delta \to 0$. Making use of the fact that the noise has a gaussian distribution, we now write the probability distribution for a particular realization of the noise as

$$P(\{\eta_n\}) = \left(\frac{2\pi\sigma}{\Delta}\right)^{N/2} \exp\left\{-\frac{\Delta}{2\sigma}\sum_{n=0}^{N-1}\eta_n^2\right\}. \tag{6}$$

Using this probability distribution, we are able to calculate averages over the noise as gaussian integrals. The two-point function of the noise is, for example, given by

$$\langle \eta_i \eta_j \rangle \equiv \int \prod_{n=0}^{N-1} d\eta_n \eta_i \eta_j P(\{\eta_n\}), \tag{7}$$

which reproduces the correct correlations, given in Eq. (5). We now first calculate the probability distribution $P[q_1,t_1;q_0,t_0]$, which is the probability distribution that a solution of the Langevin equation in Eq. (1) reaches the value $q_1$ at time $t_1 \equiv t_0 + \Delta$. Since we can easily solve the discrete Langevin equation in Eq. (4) explicitly for one time step, we find from the definition in Eq. (3) that

$$P[q_1,t_1;q_0,t_0] = \int d\eta_0 \left(\frac{2\pi\sigma}{\Delta}\right)^{1/2} \exp\left\{-\frac{\Delta}{2\sigma}\eta_0^2\right\} \delta\left(q_0 + \Delta(f(q_0)+\eta_0) - q_1\right). \tag{8}$$

Secondly, we integrate out the noise $\eta_0$ to obtain the result

$$P[q_1,t_1;q_0,t_0] = \frac{1}{\Delta}\exp\left\{-\frac{\Delta}{2\sigma}\left(\frac{q_1-q_0}{\Delta} - f(q_0)\right)^2\right\}. \tag{9}$$

We then use the this expression at each time step, and 'tie' them together using

$$P[q_{i+1},t_{i+1};q_{i-1},t_{i-1}] = \int dq_i P[q_{i+1},t_{i+1};q_i,t_i]P[q_i,t_i;q_{i-1},t_{i-1}], \tag{10}$$

which follows from the fact that the total probability is conserved. The result for $P[q_N,t_N;q_0,t_0]$ then becomes, after a combination of Eqs. (9) and (10) at each intermediate time step,

$$P[q_N,t_N;q_0,t_0] = \Delta^{-(N-2)}\int\left(\prod_{n=1}^{N-1}dq_n\right)\exp\left\{-\frac{\Delta}{2\sigma}\sum_{i=0}^{N-1}\left(\frac{q_{i+1}-q_i}{\Delta} - f(q_i)\right)^2\right\}. \tag{11}$$

Note that this expression explicitly shows that the integration is only over intermediate coordinates, and that the boundary values $q_N$ and $q_0$ are fixed. We now take the limit $N\to\infty$ and $\Delta\to 0$, while keeping $t_N - t_0$ fixed. If we absorb the prefactor in Eq. (11) in the integral measure, we get, after putting $q_N = q$ and $t_N = t$, the result

$$P[q,t;q_0,t_0] = \int_{q(t_0)=q_0}^{q(t)=q} d[q]e^{iS[q]/\hbar}. \tag{12}$$

In the exponent of the integrant we extracted a factor $i/\hbar$, where $\hbar$ is Planck's constant, for reasons that become clear shortly. The integral measure of the functional integral in Eq. (12) now denotes integration over all paths $q(t)$ with boundary conditions $q(t_0) = q_0$, and $q(t) = q$. Each of these paths gives a weighted contribution to the probability distribution, with a weight factor proportional to $e^{iS[q]/\hbar}$. The action $S[q]$ is given by

$$S[q] = i\hbar\int_{t_0}^{t} dt'\frac{1}{2\sigma}\left(\dot{q}(t') - f(q(t'))\right)^2. \tag{13}$$

From the path-integral expression in Eq. (12) we can now easily derive a partial differential equation for $P[q,t;q_0,t_0]$. This equation will turn out to be the Fokker-Planck equation [26,27] corresponding to the Langevin equation of Eq. (1). To derive the Fokker-Planck equation, we make a connection with quantum mechanics.

From ordinary quantum mechanics for a point particle with mass $m$ in a potential $V(q)$, we know that a matrix element of the evolution operator can be represented as a path integral [28,29], similar to Eq. (12). Denoting that matrix element by $W(q,t;q_0,t_0) = \langle q|e^{-i(t-t_0)\hat{H}(\hat{p},\hat{q})/\hbar}|q_0\rangle$, where $\hat{H}(\hat{p},\hat{q}) = \hat{p}^2/2m + V(\hat{q})$ is the Hamilton operator, this path integral is given by

$$W(q,t;q_0,t_0) = \int_{q(t_0)=q_0}^{q(t)=q} d[q]e^{iS^{\rm cl}[q]/\hbar}, \tag{14}$$

with $S^{\rm cl}[q]$ the classical action for the particle

$$S^{\rm cl}[q] = \int_{t_0}^{t} dt'\left(\frac{1}{2}m\dot{q}^2(t') - V(q(t'))\right). \tag{15}$$

On the other hand, the evolution operator also obeys the time-dependent Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}W(q,t;q_0,t_0) = H(p_q,q)W(q,t;q_0,t_0). \tag{16}$$

Here, $H(p_q,q)$ is the hamiltonian corresponding to the classical action in Eq. (15), in the position representation. It is essential that $H(p_q,q)$ is normal ordered, i.e., that the momentum operators are positioned left to the position operators, for the path-integral expression in Eq. (14) to be valid. We can now derive the Fokker-Planck equation by noting that Eq. (12) can be interpreted as the path-integral expression for the time evolution operator with a 'classical' action given by

$$S[q] \equiv \int_{t_0}^{t} dt' L(t'), \tag{17}$$

where $L(t)$ is by definition the lagrangian. The desired Fokker-Planck equation is, therefore, exactly the Schrödinger equation corresponding to the action $S[q]$. To derive the latter, we have to quantize this action, which can be done in a straightforward manner. The momentum conjugate to the position variable $q$ is given by

$$p_q \equiv \frac{\partial L}{\partial \dot{q}} = \frac{i\hbar}{\sigma}(\dot{q} - f(q)). \tag{18}$$

The hamiltonian thus becomes

$$H(p_q,q) \equiv p_q\dot{q} - L(q,\dot{q}) = \frac{\sigma}{2i\hbar}p_q^2 + p_q f(q). \tag{19}$$

Note that we have positioned the momentum operators left to the position operators, in order to obtain the correct correspondence with Eq. (12). Next, we impose canonical commutation relations $[q,p_q] = i\hbar$, so we write in the position representation $p_q = -i\hbar\partial/\partial q$. The 'Schrödinger equation' for $P[q,t;q_0,t_0]$ is then given by

$$i\hbar\frac{\partial P[q,t;q_0,t_0]}{\partial t} = H(p_q,q)P[q,t;q_0,t_0], \tag{20}$$

which can be explicitly rewritten as

$$\frac{\partial P[q,t]}{\partial t} = \frac{\partial}{\partial q}\left[\frac{\sigma}{2}\frac{\partial}{\partial q} - f(q)\right]P[q,t]. \tag{21}$$

This is indeed the well-known Fokker-Planck equation corresponding to the Langevin equation in Eq. (1). It has a 'streaming' term linear in the derivative with respect to $q$, which represents the reversible part of the Langevin equation. The 'diffusion' term, quadratic in the derivatives, represents the irreversible stochastic behaviour. All Fokker-Planck equations that correspond to a Langevin equation with gaussian noise have this general structure. Note that in writing down the Fokker-Planck equation, we dropped the dependence of the probability distribution on the initial conditions. This is because the form of the Fokker-Planck equation is independent of these initial conditions. Although the hamiltonian in Eq. (19)

is not hermitian, the Fokker-Planck equation in Eq. (21) conserves the normalization of the probability distribution, as can be seen from

$$\frac{d}{dt}\int dq P[q,t] = \int dq \frac{\partial}{\partial q}\left[\frac{\sigma}{2}\frac{\partial}{\partial q} - f(q)\right] P[q,t] = \left[\frac{\sigma}{2}\frac{\partial}{\partial q} - f(q)\right] P[q,t]\Big|_{-\infty}^{+\infty} = 0, \tag{22}$$

where we made use of the fact that a normalized probability distribution has to vanish at the boundaries. The equilibrium solution of the Fokker-Planck equation is given by

$$P[q, t \to \infty] \propto \exp\left\{\frac{2}{\sigma}\int^q dq' f(q')\right\}. \tag{23}$$

Since we have used units such that the mass of the Brownian particle and the friction coefficient are equal to one in the Langevin equation, we see that, in order for the probability distribution to relax to the correct Boltzmann distribution, we have to take the strength of the noise equal to $\sigma = 2k_{\rm B}T$, where $k_{\rm B}$ is Boltzmann's constant, and $T$ is the temperature. This is the fluctuation-dissipation theorem, which causes the probability distribution to relax to the correct physical equilibrium.

At this point we would also like to make clear that in writing down the time sliced version of the Langevin equation, we have made the choice to interpret the noise term as a so-called Ito process, as opposed to a Stratonovich process. The difference between Ito and Stratonovich calculus emerges when one deals with multiplicative noise. For example, let us consider the Langevin equation

$$\dot{q}(t) = f(q(t)) + g(q(t))\eta(t), \tag{24}$$

where $\eta(t)$ is a gaussian noise term with correlations given by Eq. (2). If one interprets the noise in Eq. (24) as an Ito process, the discretization reads

$$\frac{1}{\Delta}(q_{n+1} - q_n) = f(q_n) + g(q_n)\eta_n. \tag{25}$$

The corresponding Fokker-Planck equation then becomes

$$\frac{\partial P^{\rm I}[q,t]}{\partial t} = \frac{\partial}{\partial q}\left[\frac{\sigma}{2}\frac{\partial}{\partial q}g^2(q) - f(q)\right] P^{\rm I}[q,t]. \tag{26}$$

However, the time sliced version of Eq. (24) is in the Stratonovich calculus given by [30]

$$\frac{1}{\Delta}(q_{n+1} - q_n) = \frac{1}{2}[f(q_n) + f(q_{n+1})] + \frac{1}{2}[g(q_n) + g(q_{n+1})]\eta_n. \tag{27}$$

In this case, the Fokker-Planck equation reads

$$\frac{\partial P^{\rm S}[q,t]}{\partial t} = \frac{\partial}{\partial q}\left[\frac{\sigma}{2}g^2(q)\frac{\partial}{\partial q} - f(q)\right] P^{\rm S}[q,t]. \tag{28}$$

The Stratonovich interpretation therefore leads to an additional noise-induced drift term in the equation for the average of $q(t)$, as can be seen from

$$\frac{d\langle q\rangle^{\rm S}(t)}{dt} = \langle f(q)\rangle^{\rm S}(t) + \sigma\langle g(q)g'(q)\rangle^{\rm S}(t), \tag{29}$$

where $g'(q) = \partial g/\partial q$. This result follows straightforward from the Fokker-Planck equation in Eq. (28), with the use of partial integration. Note that the second term on the right-hand side of the last equation, which is the so-called spurious or noise-induced drift term, is absent in the case of an Ito process. In physics, a Stratonovich process arises naturally when the delta function in the time correlation of the noise is the result of a limiting procedure in which the correlation time becomes equal to zero.

With these important remarks we conclude the discussion of the one-dimensional Langevin equation and its corresponding Fokker-Planck equation. As a further illustration of the path-integral methods, we now discuss the case of the Brownian motion of a massive particle in a potential.

### B. Underdamped Brownian motion

Let us consider the Langevin equation for the Brownian motion in one dimension of a particle with mass $m$ in a potential $V(q)$,

$$\ddot{q}(t) + \gamma\dot{q}(t) = -\frac{1}{m}\frac{\partial V}{\partial q}(q(t)) + \eta(t). \tag{30}$$

In this equation $\eta(t)$ is a fluctuating force per unit mass, with a gaussian probability distribution. The parameter $\gamma > 0$ is a friction constant. The time-correlation of the noise is given by

$$\langle \eta(t')\eta(t)\rangle = \frac{2\gamma}{m\beta}\delta(t'-t). \tag{31}$$

Here $\beta = 1/k_{\rm B}T$ is the inverse temperature. Note that the strength of the fluctuations is related to the amount of dissipation $\gamma$ through Eq. (31). As mentioned previously, this is the fluctuation-dissipation theorem, which ensures that the probability distribution for the position and velocity of the Brownian particle relaxes to the Boltzmann distribution, as we will see in detail lateron. We now want to derive the Fokker-Planck equation associated with this Langevin equation of motion. To do so, we first write it as a set of two first-order differential equations

$$\begin{aligned}
&\dot{q}(t) = v(t);\\
&\dot{v}(t) = -\gamma v(t) - \frac{1}{m}\frac{\partial V}{\partial q}(q(t)) + \eta(t).
\end{aligned} \tag{32}$$

We are interested in the probability distribution $P[q, v, t; q_0, v_0, t_0]$, which is defined as the probability density that a particle with velocity $v_0$ and position $q_0$ at an initial time $t_0$, has a velocity $v$ and a position $q$ at time $t$. This probability distribution is thus given by

$$P[q, v, t; q_0, v_0, t_0] = \langle \delta(q(t) - q)\delta(v(t) - v)\rangle, \tag{33}$$

where $(q(t), v(t))$ is a solution of Eq. (32), with initial conditions $(q(t_0), v(t_0)) = (q_0, v_0)$. Following the same time-slicing procedure as in the previous section, we can derive a path-integral expression for this probability distribution. This path integral is in first instance given by

$$P[q,v,t;q_0,v_0,t_0] = \int_{q(t_0)=q_0}^{q(t)=q} d[q] \int_{v(t_0)=v_0}^{v(t)=v} d[v]\ \delta[v(t') - \dot{q}(t')] \times \exp\left\{ -\frac{m\beta}{4\gamma} \int_{t_0}^{t} dt' \left( \dot{v}(t') + \gamma v(t') + \frac{1}{m}\frac{\partial V}{\partial q}(q(t')) \right)^2 \right\}. \tag{34}$$

We next represent the delta functional by a Fourier path integral over an auxiliary coordinate $p_q$. As the notation suggests, this turns out to be the momentum conjugate to $q$. The path-integral expression for $P[q,v,t;q_0,v_0,t_0]$ then becomes

$$P[q,v,t;q_0,v_0,t_0] = \int_{q(t_0)=q_0}^{q(t)=q} d[q] \int d[p_q] \int_{v(t_0)=v_0}^{v(t)=v} d[v] \times \exp\left\{ \frac{i}{\hbar} \int_{t_0}^{t} dt' \left( p_q(t')(\dot{q}(t') - v(t')) + \frac{i\hbar m\beta}{4\gamma} \left( \dot{v}(t') + \gamma v(t') + \frac{1}{m}\frac{\partial V}{\partial q}(q(t')) \right)^2 \right) \right\}. \tag{35}$$

In this expression, we again extracted a factor of $i/\hbar$ outside the time integral in the exponent to make the connection with quantum mechanics more obvious, and we also absorbed some normalization factors in the path-integral measure. We then introduce the momentum conjugate to $v$, denoted by $p_v$, by multiplying the integrant in Eq. (35) by a factor one, written as the gaussian functional integral over $p_v$ given by

$$1 = \int d[p_v]\ \exp\left\{ \frac{i}{\hbar} \int_{t_0}^{t} dt' \frac{i\gamma}{\hbar m\beta} \left( p_v(t') - \frac{i\hbar m\beta}{2\gamma} \left( \dot{v}(t') + \gamma v(t') + \frac{1}{m}\frac{\partial V}{\partial q}(q(t')) \right) \right)^2 \right\}. \tag{36}$$

This procedure is generally known as a Hubbard-Stratonovich transformation [31]. After this procedure, the result for the probability distribution $P[q,v,t;q_0,v_0,t_0]$ reads

$$P[q,v,t;q_0,v_0,t_0] = \int_{q(t_0)=q_0}^{q(t)=q} d[q] \int d[p_q] \int_{v(t_0)=v_0}^{v(t)=v} d[v] \int d[p_v] \times \exp\left\{ \frac{i}{\hbar} \int_{t_0}^{t} dt' \left( p_q(t')q(t') + p_v(t')v(t') - H(p_q,q;p_v,v) \right) \right\}, \tag{37}$$

with a hamiltonian given by

$$H(p_q,q;p_v,v) = p_q v - \frac{i\gamma}{\hbar m\beta} p_v^2 - p_v \left( \gamma v + \frac{1}{m}\frac{\partial V}{\partial q} \right). \tag{38}$$

At this point it might be somewhat confusing that the momentum conjugate to $q$ is not simply proportional to $v$. We are, however, not quantizing a classical system, but instead trying to derive a path integral for the probability distribution generated by a classical stochastic equation of motion. The connection with quantum mechanics lies in the fact that we can identify this probability distribution with a quantum mechanical amplitude for some

quantum system. This does not mean that the Brownian particle has wave-like properties. Nevertheless, Eq. (37) is precisely the canonical path-integral representation for a matrix element of the evolution operator. Therefore, the probability distribution $P[q, v, t; q_0, v_0, t_0]$ obeys the time-dependent Schrödinger equation with the hamiltonian $H(p_q, q; p_v, v)$ given by Eq. (38), in the position representation. It is again essential that we use normal ordering. We can thus quantize this hamiltonian, by putting $[q, p_q] = [v, p_v] = i\hbar$, with all other commutators equal to zero. So we have in the position representation $p_q = -i\hbar\partial/\partial q$ and $p_v = -i\hbar\partial/\partial v$. The Schrödinger equation now results in

$$\frac{\partial P[q,v,t]}{\partial t} = \left[-\frac{\partial}{\partial q}v + \frac{\partial}{\partial v}\left(\gamma v + \frac{1}{m}\frac{\partial V}{\partial q}\right) + \frac{\gamma}{m\beta}\frac{\partial^2}{\partial v^2}\right] P[q,v,t], \tag{39}$$

which is indeed the Fokker-Planck equation associated with the Brownian motion of a particle with mass $m$ in a potential $V(q)$, and known as the Kramers-Klein equation [32]. It can be shown that the stationary solution of Eq. (39) is given by the Boltzmann distribution

$$P[q, v, t \to \infty] \propto \exp\left\{-\beta\left(\frac{1}{2}mv^2 + V(q)\right)\right\}, \tag{40}$$

which may be checked by insertion. It is important to realize that the fluctuation-dissipation theorem in Eq. (31) is again essential for the probability distribution to relax to the correct equilibrium distribution. It embodies the fact that dissipation and thermal fluctuations cooperate to achieve thermal equilibrium. This concludes our brief review of the connection between functional integration and stochastic differential equations. In the next section we use these techniques in the treatment of the nonequilibrium dynamics of a Bose-Einstein condensate.

## III. NONEQUILIBRIUM DYNAMICS

In this section, we present the Fokker-Planck equation describing the nonequilibrium dynamics of a Bose-Einstein condensed gas, and its corresponding Langevin field equation. The so-called hydrodynamic formulation will also be discussed. Since the Langevin field equation generalizes the Gross-Pitaevskii equation to nonzero temperature, we start our discussion by recalling this well-known equation.

### A. Stochastic nonlinear Schrödinger equation

The dynamics of a trapped Bose-Einstein condensate is at sufficiently low temperatures very well described by the time-dependent Gross-Pitaevskii equation

$$i\hbar\frac{\partial\Psi(\mathbf{x},t)}{\partial t} = \left\{-\frac{\hbar^2\nabla^2}{2m} + V^{\rm ext}(\mathbf{x}) + T^{\rm 2B}|\Psi(\mathbf{x},t)|^2\right\}\Psi(\mathbf{x},t), \tag{41}$$

where $\hbar$ is Planck's constant, $m$ is the mass of a single atom, $V^{\rm ext}(\mathbf{x})$ is the external trapping potential and $T^{\rm 2B} = 4\pi a\hbar^2/m$ is the two-body transition matrix, with $a$ the s-wave scattering length. The Gross-Pitaevskii equation arises as the equation of motion for the superfluid

order parameter, which is the expectation value of the Bose field operator $\hat{\psi}(\mathbf{x},t)$, that annihilates an atom at position $\mathbf{x}$ and at time $t$. The Gross-Pitaevskii equation is also referred to as the non-linear Schrödinger equation for the macroscopic condensate wave function $\Psi(\mathbf{x},t)$, since the condensate density is given by

$$n_c(\mathbf{x},t) = |\Psi(\mathbf{x},t)|^2. \tag{42}$$

The time-dependent Gross-Pitaevskii equation has stationary solutions of the form $\Psi(\mathbf{x},t) = \Psi(\mathbf{x})e^{i\mu t/\hbar}$, where the parameter $\mu$ is the chemical potential that fixes the number of atoms in the condensate and $\Psi(\mathbf{x})$ now obeys the time-independent Gross-Pitaevskii equation,

$$\left\{-\frac{\hbar^2\nabla^2}{2m} + V^{\rm ext}(\mathbf{x}) - \mu + T^{\rm 2B}|\Psi(\mathbf{x})|^2\right\}\Psi(\mathbf{x}) = 0. \tag{43}$$

The time-dependent Gross-Pitaevskii equation is a semiclassical mean-field equation, describing the average dynamics of the condensate only. It contains no description of the relaxation of the condensate towards equilibrium, and neither does it contain condensate growth from the thermal cloud, or condensate evaporation, at nonzero temperatures. Moreover, it completely neglects fluctuations of the order parameter around its mean value in the description. Therefore, we would like to modify the Gross-Pitaevskii equation such that it contains fluctuations due to incoherent collisions between condensate and noncondensate atoms, as well as condensate growth and evaporation. In order to do so consistently, we have to consider the full probability distribution for the order parameter, which can be found by means of the many-body T-matrix approximation to a field-theoretic formulation of the Keldysh theory [7,20]. It is given as a functional integral by

$$P[\phi,\phi^*;t] = \int_{\phi(\mathbf{x},t)=\phi(\mathbf{x})}^{\phi^*(\mathbf{x},t)=\phi^*(\mathbf{x})} d[\phi^*]d[\phi]\exp\left\{\frac{i}{\hbar}S^{\rm eff}[\phi^*,\phi]\right\}, \tag{44}$$

with an effective action

$$\begin{aligned} S^{\rm eff}[\phi^*,\phi] \quad &= \int_{t_0}^{t} dt' \int d\mathbf{x}\frac{2}{\hbar\Sigma^{\rm K}(\mathbf{x},t')} \\ &\times\left|\left(i\hbar\frac{\partial}{\partial t'} + \frac{\hbar^2\nabla^2}{2m} - V^{\rm ext}(\mathbf{x}) + iR(\mathbf{x},t') + \mu(t') + T^{\rm 2B}|\phi(\mathbf{x},t')|^2\right)\phi(\mathbf{x},t')\right|^2. \end{aligned} \tag{45}$$

In this effective action, the imaginary term $iR(\mathbf{x},t)$ describes the exchange of atoms between the condensate and the thermal cloud. Since, at this point, we also want to be able to describe a thermal cloud that is not in thermal equilibrium, we have to allow for a time dependent chemical potential. Before we discuss the physical content of the expressions in Eqs. (44) and (45) further, let us first derive the Fokker-Planck equation determining the time-dependence of $P[\phi,\phi^*;t]$. To do so, we note that the expressions in Eqs. (44) and (45) are very similar to the path-integral expressions we encountered in the previous section for the probability distribution generated by a stochastic differential equation. The only difference between Eqs. (12) and (44) is that the functional integration is now over all complex fields $\phi^*(\mathbf{x},t)$ and $\phi(\mathbf{x},t)$, instead of real functions $q(t)$. Also note that we did not specify the initial conditions at the time $t_0$. This is because we are only interested in

the universal long-time dynamics of the gas, which are independent of the specific form of the initial conditions. Moreover, as we have seen in the previous section, the form of the Fokker-Planck equation is in fact independent of these initial conditions. From Eq. (44) we can derive the Fokker-Planck equation by quantizing the effective action in Eq. (45), just as in the previous section. It is ultimately given by

$$
\begin{aligned}
&i\hbar\frac{\partial}{\partial t}P[\phi^*,\phi;t] = \\
&-\int d\mathbf{x}\frac{\delta}{\delta\phi(\mathbf{x})}\left(-\frac{\hbar^2\nabla^2}{2m}+V^{\rm ext}(\mathbf{x})-\mu(t)-iR(\mathbf{x},t)+T^{\rm 2B}|\phi(\mathbf{x})|^2\right)\phi(\mathbf{x})P[\phi^*,\phi;t] \\
&+\int d\mathbf{x}\frac{\delta}{\delta\phi^*(\mathbf{x})}\left(-\frac{\hbar^2\nabla^2}{2m}+V^{\rm ext}(\mathbf{x})-\mu(t)+iR(\mathbf{x},t)+T^{\rm 2B}|\phi(\mathbf{x})|^2\right)\phi^*(\mathbf{x})P[\phi^*,\phi;t] \\
&-\frac{1}{2}\int d\mathbf{x}\frac{\delta^2}{\delta\phi(\mathbf{x})\delta\phi^*(\mathbf{x})}\hbar\Sigma^{\rm K}(\mathbf{x},t)P[\phi^*,\phi;t]. \qquad (46)
\end{aligned}
$$

This Fokker-Planck equation describes the time evolution of the probability distribution of the condensate wave function at nonzero temperatures, in the presence of a thermal cloud.

The dissipation term $R(\mathbf{x},t)$ describes the exchange of atoms between the thermal cloud and the condensate, due to elastic collisions. In the Hartree-Fock approximation, which is sufficiently accurate for the nonzero temperatures of interest here, it is given by [20]

$$
\begin{aligned}
R(\mathbf{x},t) = 2\pi\left(T^{\rm 2B}\right)^2\int\frac{d\mathbf{k}_1}{(2\pi)^3}\int\frac{d\mathbf{k}_2}{(2\pi)^3}\int\frac{d\mathbf{k}_3}{(2\pi)^3}(2\pi)^3\delta(\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3) \\
\times\delta\left(\epsilon+\epsilon_1-\epsilon_2-\epsilon_3\right)\left[N_1(1+N_2)(1+N_3)-(1+N_1)N_2N_3\right], \qquad (47)
\end{aligned}
$$

In this expression, $N_i \equiv N(\mathbf{x},\mathbf{k}_i,t)$ is the Wigner distribution function of the thermal cloud, which can be determined by solving the corresponding quantum Boltzmann equation. We will not do this explicitly here, since lateron we assume that the noncondensed cloud is in thermal equilibrium. The energy of a thermal atom is given by

$$
\epsilon_i = \frac{\hbar^2\mathbf{k}_i^2}{2m}+V^{\rm ext}(\mathbf{x})+2T^{\rm 2B}|\langle\phi(\mathbf{x})\rangle(t)|^2. \qquad (48)
$$

Note that both in this expression, and in the Fokker-Planck equation in Eq. (46) we neglected the effect of the mean field of the thermal atoms, because it plays a minor role in the dynamics of the condensate. Note also that, for the average value of the order parameter calculated with the probability distribution in Eq. (44), we used the notation $\langle\phi(\mathbf{x})\rangle(t)$. The noisy order parameter field will be denoted by $\phi(\mathbf{x},t)$ and for stochastic averages of this quantity we will use the notation $\langle\phi(\mathbf{x},t)\rangle$. Since the Fokker-Planck equation and its corresponding Langevin equation are equivalent, we have of course that $\langle\phi(\mathbf{x})\rangle(t) = \langle\phi(\mathbf{x},t)\rangle$. The Keldysh self-energy $\hbar\Sigma^{\rm K}(\mathbf{x},t)$ in the Fokker-Planck equation describes the thermal fluctuations due to incoherent collisions between condensate and noncondensate atoms. It is given explicitly by [20]

$$
\begin{aligned}
\hbar\Sigma^{\rm K}(\mathbf{x},t) = -4\pi i\left(T^{\rm 2B}\right)^2\int\frac{d\mathbf{k}_1}{(2\pi)^3}\int\frac{d\mathbf{k}_2}{(2\pi)^3}\int\frac{d\mathbf{k}_3}{(2\pi)^3}(2\pi)^3\delta(\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3) \\
\times\delta\left(\epsilon+\epsilon_1-\epsilon_2-\epsilon_3\right)\left[N_1(1+N_2)(1+N_3)+(1+N_1)N_2N_3\right]. \qquad (49)
\end{aligned}
$$

Note that both the dissipation $R(\mathbf{x},t)$ and the Keldysh self-energy $\hbar\Sigma^{\rm K}(\mathbf{x},t)$, depend on the energy $\epsilon$ to take a condensate atom out of the gas at position $\mathbf{x}$ and time $t$, which has to be determined selfconsistently. This implies that $\epsilon$ is actually an operator in the configuration space of the order parameter, and given by [20]

$$\epsilon = -\frac{\hbar^2\nabla^2}{2m} + V^{\rm ext}(\mathbf{x}) + T^{\rm 2B}|\phi(\mathbf{x})|^2. \tag{50}$$

The fact that $\epsilon$ should be viewed as an operator will turn out to be crucial for the probability distribution of the order parameter to relax to the correct equilibrium distribution function.

Although our Fokker-Planck equation for the condensate, coupled to the appropriate quantum Boltzmann equation for the Wigner distribution function of the thermal cloud, describes in principle the full nonequilibrium dynamics of the Bose-condensed gas, its solution is very difficult even numerically. This is because of the fact that in the Fokker-Planck equation the dissipation $R(\mathbf{x},t)$ and the Keldysh self-energy also depend on the condensate wave function, through their dependence on $\epsilon$ given in Eq. (50), and through the mean-field effect of the condensate on the thermal atoms. As a result, writing down the corresponding Langevin equation results in a stochastic equation with multiplicative noise, and with a prefactor of the noise that has a complicated dependence on $\phi(\mathbf{x},t)$. We can, however, make progress by assuming that the thermal cloud is sufficiently close to equilibrium, which is, for example, justified for linear-response calculations around equilibrium, and also for condensate growth if the evaporative cooling is performed sufficiently slowly. From now on, we therefore assume that the thermal cloud can be described by a Bose distribution function

$$N(\epsilon_i) = \left[e^{\beta(\epsilon_i-\mu)} - 1\right]^{-1}, \tag{51}$$

with a chemical potential $\mu$ and an inverse temperature $\beta = 1/k_{\rm B}T$. The thermal cloud therefore now acts as a 'heat bath' on the condensate. Making the $\epsilon$-dependence explicit for a moment, we can relate the dissipation $R(\mathbf{x};\epsilon)$, and the Keldysh self-energy $\hbar\Sigma^{\rm K}(\mathbf{x};\epsilon)$ by means of

$$iR(\mathbf{x};\epsilon) = -\frac{1}{2}\hbar\Sigma^{\rm K}(\mathbf{x};\epsilon)[1+2N(\epsilon)]^{-1}, \tag{52}$$

which follows simply from the form of the Bose distribution function, together with the energy conserving delta function in Eqs. (47) and (49). This relation between the dissipation $R(\mathbf{x};\epsilon)$ and the Keldysh self-energy $\hbar\Sigma^{\rm K}(\mathbf{x};\epsilon)$ determining the strength of the fluctuations, is in fact the fluctuation-dissipation theorem. Just as in the case of the Brownian motion of a particle discussed in the previous section, it causes the system to relax to the correct equilibrium distribution, as we will see below. Since we are dealing with Bose condensation, the occupation numbers $N(\epsilon)$ are generally very large, and we have in a good approximation

$$[1+2N(\epsilon)]^{-1} \simeq \frac{1}{2}[\beta(\epsilon-\mu)]. \tag{53}$$

If we combine this result with Eq. (52), and substitute the operator in Eq. (50), we arrive at the approximation

$$iR(\mathbf{x},t) \simeq -\frac{\beta}{4}\hbar\Sigma^{\rm K}(\mathbf{x},t)\left[-\frac{\hbar^2\nabla^2}{2m} + V^{\rm ext}(\mathbf{x}) - \mu + T^{\rm 2B}|\phi(\mathbf{x})|^2\right], \tag{54}$$

where $\hbar\Sigma^{\rm K}(\mathbf{x},t) \equiv \hbar\Sigma^{\rm K}(\mathbf{x};\langle\mu_{\rm c}(\mathbf{x},t)\rangle)$, and the local chemical potential of the condensate $\langle\mu_{\rm c}(\mathbf{x},t)\rangle$ is given by

$$\langle\mu_{\rm c}(\mathbf{x},t)\rangle = \frac{\delta}{\delta|\langle\phi(\mathbf{x})\rangle(t)|^2}\int d\mathbf{x}\langle\phi^*(\mathbf{x})\rangle(t)\left(-\frac{\hbar^2\nabla^2}{2m}+V^{\rm ext}(\mathbf{x})+\frac{T^{\rm 2B}}{2}|\langle\phi(\mathbf{x})\rangle(t)|^2\right)\langle\phi(\mathbf{x})\rangle(t). \tag{55}$$

We now show that the above 'classical' approximation to the fluctuation-dissipation theorem indeed leads to the correct equilibrium. Let us therefore substitute Eq. (54) into the Fokker-Planck equation, which simplifies to

$$\begin{aligned}
&i\hbar\frac{\partial}{\partial t}P[\phi^*,\phi;t] = \\
&-\frac{\beta}{4}\int d\mathbf{x}\;\hbar\Sigma^{\rm K}(\mathbf{x},t)\frac{\delta}{\delta\phi(\mathbf{x})}\left(-\frac{\hbar^2\nabla^2}{2m}+V^{\rm ext}(\mathbf{x})-\mu+T^{\rm 2B}|\phi(\mathbf{x})|^2\right)\phi(\mathbf{x})P[\phi^*,\phi;t] \\
&-\frac{\beta}{4}\int d\mathbf{x}\;\hbar\Sigma^{\rm K}(\mathbf{x},t)\frac{\delta}{\delta\phi^*(\mathbf{x})}\left(-\frac{\hbar^2\nabla^2}{2m}+V^{\rm ext}(\mathbf{x})-\mu+T^{\rm 2B}|\phi(\mathbf{x})|^2\right)\phi^*(\mathbf{x})P[\phi^*,\phi;t] \\
&-\frac{1}{2}\int d\mathbf{x}\frac{\delta^2}{\delta\phi(\mathbf{x})\delta\phi^*(\mathbf{x})}\hbar\Sigma^{\rm K}(\mathbf{x},t)P[\phi^*,\phi;t].
\end{aligned} \tag{56}$$

The stationary solution of this Fokker-Planck equation is given by

$$P[\phi^*,\phi;t\to\infty] \propto \exp\left\{-\beta\int d\mathbf{x}\;\phi^*(\mathbf{x})\left(-\frac{\hbar^2\nabla^2}{2m}+V^{\rm ext}(\mathbf{x})-\mu+\frac{T^{\rm 2B}}{2}|\phi(\mathbf{x})|^2\right)\phi(\mathbf{x})\right\}, \tag{57}$$

as can be checked by substitution. To see that Eq. (57) is in fact the correct equilibrium distribution, we have to show that the macroscopic condensate wave function $\langle\phi(\mathbf{x})\rangle$ obeys the time-independent Gross-Pitaevskii equation. To see this, we first note that

$$\int d[\phi^*]d[\phi]\frac{\delta}{\delta\phi^*(\mathbf{x})}P[\phi^*,\phi;t] = 0 \tag{58}$$

for a general probability distribution which vanishes at the boundaries of the domain of integration. If we apply this to the equilibrium distribution $P[|\phi|;t\to\infty]$ we get, by applying the mean-field approximation $\langle|\phi(\mathbf{x})|^2\phi(\mathbf{x})\rangle \simeq |\langle\phi(\mathbf{x})\rangle|^2\langle\phi(\mathbf{x})\rangle$, the desired result

$$\left(-\frac{\hbar^2\nabla^2}{2m}+V^{\rm ext}(\mathbf{x})-\mu+T^{\rm 2B}|\langle\phi(\mathbf{x})\rangle|^2\right)\langle\phi(\mathbf{x})\rangle = 0, \tag{59}$$

which is precisely the time-independent Gross-Pitaevskii equation. Note that Eq. (54) together with the time-independent Gross-Pitaevskii equation implies that in equilibrium $\langle R(\mathbf{x},t)\rangle = 0$. This means that there is detailed balance between the condensate and the thermal cloud, and that there is, on average, no condensate growth or evaporation when the system has relaxed to equilibrium.

Using the results of the previous section, we now give a formulation of the nonequilibrium theory discussed above in terms of a Langevin field equation corresponding to the Fokker-Planck equation in Eq. (56). This Langevin field equation takes the form of a dissipative nonlinear Schrödinger equation with noise, given by

$$i\hbar\frac{\partial\phi(\mathbf{x},t)}{\partial t} = \left(1+\frac{\beta}{4}\hbar\Sigma^{\rm K}(\mathbf{x},t)\right) \times\left\{-\frac{\hbar^2\nabla^2}{2m}+V^{\rm ext}(\mathbf{x})-\mu+T^{\rm 2B}|\phi(\mathbf{x},t)|^2\right\}\phi(\mathbf{x},t)+\eta(\mathbf{x},t). \quad (60)$$

This Langevin equation quite generally generalizes the Gross-Pitaevskii equation to nonzero temperatures, and includes both dissipation and thermal fluctuations. The complex gaussian noise in the Langevin field equation has correlations [20,11]

$$\langle\eta^*(\mathbf{x},t)\eta(\mathbf{x}',t')\rangle = \frac{i\hbar^2}{2}\Sigma^{\rm K}(\mathbf{x},t)\delta(t-t')\delta(\mathbf{x}-\mathbf{x}'), \quad (61)$$

where the strength of the noise is determined by a Keldysh self-energy, given by

$$\hbar\Sigma^{\rm K}(\mathbf{x},t) = -4\pi i\left(T^{\rm 2B}\right)^2\int\frac{d\mathbf{k}_1}{(2\pi)^3}\int\frac{d\mathbf{k}_2}{(2\pi)^3}\int\frac{d\mathbf{k}_3}{(2\pi)^3}(2\pi)^3\delta(\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3) \times\delta\left(\langle\mu_{\rm c}(\mathbf{x},t)\rangle+\epsilon_1-\epsilon_2-\epsilon_3\right)\left[N_1(1+N_2)(1+N_3)+(1+N_1)N_2N_3\right]. \quad (62)$$

In this expression, $N_i$ is again the Bose-distribution function of the thermal cloud, evaluated at an energy of a thermal particle, which is in the Hartree-Fock approximation given by

$$\epsilon_i = \frac{\hbar^2\mathbf{k}_i^2}{2m}+V^{\rm ext}(\mathbf{x})+2T^{\rm 2B}|\langle\phi(\mathbf{x},t)\rangle|^2. \quad (63)$$

The average local chemical potential of the condensate atoms $\langle\mu_{\rm c}(\mathbf{x},t)\rangle$, is given by Eq. (55). Notice that in the latter equation, and in the above expression for the energy of a thermal particle, $\langle\phi(\mathbf{x},t)\rangle$ has to be determined self-consistently, since only then the probability distribution generated by the Langevin equation in Eq. (60) relaxes to the correct equilibrium.

The stochastic non-linear Schrödinger equation in Eq. (60), together with the expression for the Keldysh self-energy in Eq. (62), gives a nonequilibrium description of the condensate, that obeys the fluctuation-dissipation theorem. In Sec. IV we use a variational *ansatz* to solve this equation. However, we first derive the corresponding noisy hydrodynamic formulation.

### B. Stochastic hydrodynamics

The condensate is often described in terms of its density and its phase, by making the transformation $\phi=\sqrt{\rho}e^{i\theta}$. When applied to the Gross-Pitaevskii equation, this transformation results in the so-called Josephson equation for the phase, and a continuity equation for the density [4,9]. We now want to derive the generalization of these two equations to the case of our Langevin equation for the condensate. To do this, we first substitute the 'classical' approximation to the fluctuation-dissipation theorem into the effective action $S^{\rm eff}[\phi^*,\phi]$, which now reads

$$S^{\rm eff}[\phi^*,\phi] = \int_{t_0}^{t}dt'\int d\mathbf{x}\frac{2}{\hbar\Sigma^{\rm K}(\mathbf{x},t')}\left|\left(i\hbar\frac{\partial}{\partial t'}+\left\{1+\frac{\beta}{4}\hbar\Sigma^{\rm K}(\mathbf{x},t')\right\} \times\left[\frac{\hbar^2\nabla^2}{2m}-V^{\rm ext}(\mathbf{x})+\mu+T^{\rm 2B}|\phi(\mathbf{x},t')|^2\right]\right)\phi(\mathbf{x},t')\right|^2. \quad (64)$$

The reason for this substitution is that we have defined the fluctuation-dissipation theorem as an operator equation in the configuration space of the order parameter, and not in terms of its density and phase. We can now easily substitute $\phi = \sqrt{\rho}e^{i\theta}$ into this effective action. This substitution results in an effective action in terms of the density and the phase of the order parameter, i.e.,

$$\begin{aligned} S^{\rm eff}[\rho,\theta] = \int_{t_0}^{t} dt' \int d\mathbf{x} \frac{2}{\hbar\Sigma^{\rm K}(\mathbf{x},t')} \\ \left\{ \rho(\mathbf{x},t') \left( \hbar \frac{\partial\theta(\mathbf{x},t')}{\partial t'} - \frac{\beta}{4} i\hbar\Sigma^{\rm K}(\mathbf{x},t') \frac{\hbar\boldsymbol{\nabla}\cdot(\rho(\mathbf{x},t')\mathbf{v}_{\rm s}(\mathbf{x},t'))}{2\rho(\mathbf{x},t')} \right. \right. \\ \left. + \mu_{\rm c}(\mathbf{x},t') - \mu \right)^2 \\ + \frac{\hbar^2}{4\rho(\mathbf{x},t')} \left( \frac{\partial\rho(\mathbf{x},t')}{\partial t'} + \boldsymbol{\nabla}\cdot(\rho(\mathbf{x},t')\mathbf{v}_{\rm s}(\mathbf{x},t')) \right. \\ \left. \left. + \frac{\beta}{2} i\Sigma^{\rm K}(\mathbf{x},t') \left(\mu_{\rm c}(\mathbf{x},t') - \mu\right) \rho(\mathbf{x},t') \right)^2 \right\}, \end{aligned} \tag{65}$$

where we used that $\Sigma^{\rm K}(\mathbf{x},t)$ only has a negative imaginary part, as seen from Eq. (62), and thus $i\Sigma^{\rm K}(\mathbf{x},t)$ is a positive and real quantity. We defined the superfluid velocity $\mathbf{v}_{\rm s}(\mathbf{x},t)$, and the condensate chemical potential $\mu_{\rm c}(\mathbf{x},t)$ by means of [33],

$$\begin{aligned} \mu_{\rm c}(\mathbf{x},t) &= -\frac{\hbar^2\nabla^2\sqrt{\rho(\mathbf{x},t)}}{2m\sqrt{\rho(\mathbf{x},t)}} + V^{\rm ext}(\mathbf{x}) + T^{\rm 2B}\rho(\mathbf{x},t) + \frac{1}{2}m\mathbf{v}_{\rm s}^2(\mathbf{x},t); \\ \mathbf{v}_{\rm s}(\mathbf{x},t) &= \frac{\hbar}{m}\boldsymbol{\nabla}\theta(\mathbf{x},t), \end{aligned} \tag{66}$$

which coincides with the expression in Eq. (55). The effective action $S^{\rm eff}[\rho,\theta]$ yields two stochastic equations of motion. The equation for the phase of the condensate takes the form of a stochastic Josephson equation

$$\hbar\frac{\partial\theta(\mathbf{x},t)}{\partial t} - \frac{\beta}{4} i\hbar\Sigma^{\rm K}(\mathbf{x},t) \frac{\hbar^2\boldsymbol{\nabla}\cdot(\rho(\mathbf{x},t)\boldsymbol{\nabla}\theta(\mathbf{x},t))}{2m\rho(\mathbf{x},t)} = \mu - \mu_{\rm c}(\mathbf{x},t) + \frac{\nu(\mathbf{x},t)}{\sqrt{\rho(\mathbf{x},t)}}. \tag{67}$$

Here, the real gaussian noise $\nu(\mathbf{x},t)$ has correlations given by

$$\langle \nu(\mathbf{x},t)\nu(\mathbf{x}',t')\rangle = \frac{i\hbar^2}{4}\hbar\Sigma^{\rm K}(\mathbf{x},t)\delta(t-t')\delta(\mathbf{x}-\mathbf{x}'). \tag{68}$$

The stochastic Josephson equation has two modifications with respect to the ordinary Josephson equation. First, it has a spatial diffusion-like term proportional to $i\hbar\Sigma^{\rm K}(\mathbf{x},t)$. This term will cause the phase to undergo spatial diffusion due to collisions of thermal atoms with the condensate atoms, not to be confused with the phenomenon of phase 'diffusion', which corresponds to spreading of the global phase due to quantum fluctuations [18], and therefore relax to a state where the phase is position independent. So, in equilibrium we have $\langle \mathbf{v}_{\rm s}\rangle \equiv \hbar\langle\boldsymbol{\nabla}\theta\rangle/m = 0$, as expected. We will see lateron that this tendency

towards equilibrium will give rise to an increase in the sound velocity in the Bose condensate. Secondly, the Josephson equation has a noise term inversely proportional to the square root of the density. This noise represents the fluctuations in the phase of the condensate due to incoherent collisions of thermal atoms with condensate atoms, i.e., due to thermal fluctuations.

The equation of motion for the density is a stochastic continuity equation with a source term,

$$\frac{\partial \rho(\mathbf{x},t)}{\partial t} + \boldsymbol{\nabla}\cdot(\rho(\mathbf{x},t)\mathbf{v}_{\rm s}(\mathbf{x},t)) = -\frac{\beta}{2} i\Sigma^{\rm K}(\mathbf{x},t)\,(\mu_{\rm c}(\mathbf{x},t)-\mu)\,\rho(\mathbf{x},t) + 2\sqrt{\rho(\mathbf{x},t)}\xi(\mathbf{x},t), \tag{69}$$

with correlations of the gaussian noise $\xi(\mathbf{x},t)$ given by

$$\langle \xi(\mathbf{x},t)\xi(\mathbf{x}',t')\rangle = \frac{i\Sigma^{\rm K}(\mathbf{x},t)}{4}\delta(\mathbf{x}-\mathbf{x}')\delta(t-t'). \tag{70}$$

In appendix A it is explained that we have to interpret this noise as a Stratonovich process. This gives rise to additional drift terms in the equation of motion for the average of the density, because in Eq. (69) we are dealing with multiplicative noise. Note that from Eq. (69) it is explicitly seen that there is condensate growth if $\mu > \mu_{\rm c}$, i.e., if the chemical potential lies above the chemical potential of the condensate. If $\mu < \mu_{\rm c}$, there is condensate evaporation.

We will omit here the Fokker-Planck equation in terms of $\rho$ and $\theta$, but only discuss the equilibrium distribution generated by Eqs. (67) and (69). It is simply determined from $P[\phi^*,\phi;t\to\infty]$ in Eq. (57) by the substitution $\phi = \sqrt{\rho}e^{i\theta}$, since the jacobian of this transformation is equal to one. So we have

$$P[\rho,\theta;t\to\infty] \propto \exp\left\{-\beta\int d\mathbf{x}\rho(\mathbf{x})\left(-\frac{\hbar^2\nabla^2\sqrt{\rho(\mathbf{x})}}{2m\sqrt{\rho(\mathbf{x})}} + V^{\rm ext}(\mathbf{x}) + \frac{T^{\rm 2B}}{2}\rho(\mathbf{x}) + \frac{1}{2}m\mathbf{v}_{\rm s}^2(\mathbf{x}) - \mu\right)\right\}.$$

We see from this probability distribution that $\langle \mathbf{v}_{\rm s}\rangle = 0$, which should be the case in equilibrium. The average density profile is again determined by the time-independent Gross-Pitaevskii equation, as explained before.

To discuss the physical content of the stochastic continuity equation and the stochastic Josephson equation further, we now derive the wave equation describing the propagation of sound waves in a Bose Einstein condensate. For simplicity, we discuss here the homogeneous case, where $V^{\rm ext}(\mathbf{x}) = 0$. A treatment of the trapped case is presented in Sec.V A. We linearize the averages of Eqs. (69) and (67) around their equilibrium solutions $\langle\rho(\mathbf{x},t)\rangle = \rho_0$ and $\langle \mathbf{v}_{\rm s}(\mathbf{x},t)\rangle = 0$. Therefore, we write $\langle\rho(\mathbf{x},t)\rangle = \rho_0 + \delta\rho(\mathbf{x},t)$, and $\langle \mathbf{v}_s(\mathbf{x},t)\rangle = \delta\mathbf{v}_{\rm s}(\mathbf{x},t)$, and substitute this into the average of Eqs. (67) and(69). Linearization results in two coupled equations of motion for the deviations, i.e.,

$$\begin{aligned} m\frac{\partial \delta\mathbf{v}_s(\mathbf{x},t)}{\partial t} &= -T^{\rm 2B}\boldsymbol{\nabla}(\delta\rho(\mathbf{x},t)) + \frac{\beta i\hbar^2\Sigma^{\rm K}}{8}\nabla^2(\delta\mathbf{v}_{\rm s}(\mathbf{x},t)); \\ \frac{\partial\delta\rho(\mathbf{x},t)}{\partial t} + \rho_0\boldsymbol{\nabla}\cdot(\delta\mathbf{v}_{\rm s}(\mathbf{x},t)) &= -\frac{\beta}{2}i\Sigma^{\rm K}\left(2T^{\rm 2B}\rho_0 - \mu\right)\delta\rho(\mathbf{x},t). \end{aligned} \tag{71}$$

Note that we have made use of the fact that $\hbar\Sigma^{\rm K}$ is independent of the spatial coordinates for a homogeneous Bose gas, as can be seen from Eq. (62). Next, we combine these two equations to obtain a single damped wave equation for the propagation of sound waves in a homogeneous Bose gas,

$$\left(\frac{\partial^2}{\partial t^2} - c^2\nabla^2\right)\delta\rho(\mathbf{x},t) = -\frac{1}{\tau}\frac{\partial\delta\rho(\mathbf{x},t)}{\partial t}. \tag{72}$$

The relaxation time $\tau$ is defined to be inversely proportional to the damping rate of the waves. Physically, this damping arises because the excitation of a sound wave slightly disturbs the equilibrium situation where the average growth or evaporation of the condensate is equal to zero. Hence, there is no longer detailed balance between the condensate and the thermal cloud, and the collisions between the condensate and thermal atoms drive the condensate back to the equilibrium situation, where $\delta\rho(\mathbf{x},t) = 0$, and $\delta\mathbf{v}_{\rm s}(\mathbf{x},t) = 0$. The relaxation time $\tau$ is given by

$$\frac{1}{\tau} = \frac{\beta}{2} i\Sigma^{\rm K} T^{\rm 2B}\rho_0, \tag{73}$$

where we used the time-independent Gross-Pitaevskii equation which reduces in this case to $\mu = T^{\rm 2B}\rho_0$, to eliminate the chemical potential. The sound velocity $c$ in Eq. (72) is given by

$$c^2 = c_0^2\left[1 + \frac{1}{16}\left(\beta i\hbar\Sigma^{\rm K}\right)^2\right], \tag{74}$$

where $c_0 = (T^{\rm 2B}\rho_0/m)^{1/2}$ is the well-known zero temperature sound velocity, predicted by the Gross-Pitaevskii equation, and first obtained by Bogoliubov [34]. We see that our nonequilibrium treatment results in an increased sound velocity. This increase is a result from the term in the stochastic Josephson equation in Eq. (67) proportional to $i\hbar\Sigma^{\rm K}$. Physically, this term represents the fact that the phase of the condensate undergoes spatial diffusion due to collisions between condensate and thermal atoms, and therefore relaxes to a state where $\langle\mathbf{v}_{\rm s}\rangle = 0$. The spatial diffusion of the phase therefore increases the 'stiffness' of the condensate, and hence results in an increase of the sound velocity. Since the increase in the sound velocity is of order $\mathcal{O}(|\beta\hbar\Sigma^{\rm K}|^2)$, its effect is in general small below the critical temperature, as the collisionless limit is determined by $|\beta\hbar\Sigma^{\rm K}| \ll 1$ and experiments are usually in this limit. The damped wave equation in Eq. (72) should be compared to the result found by Williams and Griffin [35]. These authors use a dissipative non-linear Schödinger equation, with a damping term similar to Eq. (47), to arrive at a damped wave equation describing the propagation of sound in a trapped Bose-Einstein condensate in the presence of a static thermal cloud. Note, however, that although the microscopic expression used by Williams and Griffin is of the same form as in Eq. (47), the chemical potential of the condensate used by these authors in the calculation of $R(\mathbf{x},t)$ is not the operator given by Eq. (50). Instead, they use for the condensate energy in the energy-conserving delta function the expression $\langle\mu_{\rm c}(\mathbf{x},t)\rangle$, which in principle violates the fluctuation-dissipation theorem. As a consequence, these authors do not find an increase in the sound velocity at nonzero temperatures.

## IV. VARIATIONAL APPROXIMATION

Although the stochastic non-linear Schrödinger equation given in Eq. (60), or equivalently, the hydrodynamic formulation given in Eqs. (67) and (69), give a full nonequilibrium description of the condensate that can in principle be solved numerically [11], we find it more convenient to make analytical progress. Therefore, in the case of a harmonic trapping potential $V^{\rm ext}(\mathbf{x}) = \sum_j m\omega_j x_j^2/2$, we consider a gaussian variational *ansatz* for the condensate wave function

$$\phi(\mathbf{x},t) = \sqrt{N_{\rm c}(t)}e^{i\theta_0(t)}\prod_j \left(\frac{1}{\pi q_j^2(t)}\right)^{\frac{1}{4}} \exp\left\{-\frac{x_j^2}{2q_j^2(t)}\left(1-\frac{im}{\hbar}q_j(t)\dot{q}_j(t)\right)\right\}. \tag{75}$$

Here, the variational parameters $q_j(t)$ denote the gaussian widths of the condensate in the three spatial directions. The wave function is normalized to the number of atoms in the condensate $N_{\rm c}(t)$. This *ansatz* is different from the ones used in previous work, in the sense that it also contains a global phase $\theta_0(t)$. This turns out to be crucial, since the number of particles $N_{\rm c}(t)$, which is the variable conjugate to the global phase, is not constant in our case. Therefore, one must also allow for fluctuations in the global phase $\theta_0(t)$ of the condensate. We expect this *ansatz* to give correct results when the number of particles is small, because the mean-field interaction of the condensate will then be small, and the condensate density profile will be close to the ideal gas solution. Moreover, it has also proven to give correct results for the frequencies of the collective modes of the condensate, even in the Thomas-Fermi regime, where the mean-field interaction, and thus the number of atoms in the condensate, is large [21]. Therefore, we expect also to obtain physically sensible results even in this case.

When the gaussian *ansatz* is applied to the Gross-Pitaevskii equation we find that the variational parameters $q_j(t)$ obey Newton's equations of motion [13,21,24]

$$\frac{1}{2}mN_{\rm c}(t)\ddot{q}_j(t) = -\frac{\partial V}{\partial q_j}(\mathbf{q}(t), N_{\rm c}(t)), \tag{76}$$

with a potential energy equal to

$$V(\mathbf{q}, N_{\rm c}) = \sum_j \left(\frac{N_{\rm c}\hbar^2}{4mq_j^2} + \frac{1}{4}mN_{\rm c}\omega_j^2 q_j^2\right) + \frac{a\hbar^2 N_{\rm c}^2}{\sqrt{2\pi}mq_xq_yq_z}. \tag{77}$$

From the action in Eq. (64), we want to derive similar equations of motion, extended to the nonequilibrium case. For simplicity, we first consider an ideal gas, i.e., we drop the mean-field interaction term $T^{\rm 2B}|\phi(\mathbf{x},t)|^2$. The condensate remains, however, in contact with the thermal cloud, that acts as a 'heat bath'. Secondly, we assume that the Keldysh self-energy is constant over the size of the condensate. Although this assumption is not justified in general, we can always approximately compensate for this, by calculating a position independent Keldysh self-energy $\hbar\Sigma^{\rm K}(t)$ by means of an appropriately averaged $\hbar\Sigma^{\rm K}(\mathbf{x},t)$ over the size of the condensate. We substitute our trial wave function into the effective action in Eq. (64), to obtain a probability distribution in terms of $N_{\rm c}, \theta_0$ and $\mathbf{q}$. It is given by

$$P[N_{\rm c},\theta_0,\mathbf{q};t]=\int d[N_{\rm c}]\;d[\theta_0]\;d[\mathbf{q}]\exp\left\{\frac{i}{\hbar}S^{\rm eff}[N_{\rm c},\theta_0,\mathbf{q}]\right\}, \tag{78}$$

with an effective action that reads

$$\begin{aligned}S^{\rm eff}[N_{\rm c},\theta_0;\mathbf{q}]=\int_{t_0}^{t}dt'\frac{2}{\hbar\Sigma^{\rm K}(t')}\Bigg\{&N_{\rm c}(t')\left(\hbar\frac{d\theta_0(t')}{dt'}+\mu_{\rm c}(t')-\sum_j\frac{1}{4}m\dot{q}_j^2(t')+\sum_j\frac{1}{4}mq_j(t')\ddot{q}_j(t')-\mu\right)^2\\
&+\frac{\hbar^2}{4N_{\rm c}(t')}\left(\frac{dN_{\rm c}(t')}{dt'}+\frac{\beta}{2}i\Sigma^{\rm K}(t')\left[\mu_{\rm c}(t')-\mu\right]N_{\rm c}(t')\right)^2\\
&+\sum_j\frac{q_j^2(t')}{2N_{\rm c}(t')}\left(\frac{1}{2}mN_{\rm c}(t')\ddot{q}_j(t')+\frac{N_{\rm c}(t')\beta}{4}i\hbar^2\Sigma^{\rm K}(t')\frac{\dot{q}_j(t')}{q_j^2(t')}+\frac{\partial V}{\partial q_j}(\mathbf{q}(t'),N_{\rm c}(t'))\right)^2\\
&+\mathcal{O}\left((\beta i\hbar\Sigma^{\rm K})^2\right)\Bigg\}.\end{aligned} \tag{79}$$

The potential in this effective action is defined by

$$V(\mathbf{q},N_{\rm c})=\sum_j\left(\frac{N_{\rm c}\hbar^2}{4mq_j^2}+\frac{1}{4}mN_{\rm c}\omega_j^2q_j^2\right), \tag{80}$$

which is precisely the potential given by Eq. (77), without the mean-field interaction term. The condensate chemical potential for the gaussian *ansatz* is given by

$$\mu_{\rm c}(t)=\frac{\partial V}{\partial N_{\rm c}}(\mathbf{q}(t),N_{\rm c}(t))+\sum_j\frac{1}{4}m\dot{q}_j^2(t), \tag{81}$$

as expected. We now assume the dimensionless parameter $\beta i\hbar\Sigma^{\rm K}$ to be small, and thus restrict ourselves to a temperature regime sufficiently far below the critical temperature, where $|\beta\hbar\Sigma^{\rm K}|\ll 1$, i.e., the collisionless regime. We can then to a good approximation neglect the terms quadratic in $\beta i\hbar\Sigma^{\rm K}$. The effective action in Eq. (79) thus becomes a sum of three squares, and we can extract equations of motion with gaussian noise terms, exactly as in Sec. II. Since the action is quadratic in $\theta_0(t)$, we can integrate over this global phase exactly, because it only requires a gaussian integral. However, before we perform this integration, we discuss the stochastic equation of motion for $\theta_0(t)$. With the techniques discussed in the Sec. II, we easily see that it is given by

$$\hbar\frac{d\theta_0(t)}{dt}=\mu-\mu_{\rm c}(t)+\sum_j\frac{1}{4}m\dot{q}_j^2(t)-\sum_j\frac{1}{4}mq_j(t)\ddot{q}_j(t)+\frac{\nu(t)}{\sqrt{N_{\rm c}(t)}}, \tag{82}$$

with time correlation of the noise

$$\langle\nu(t')\nu(t)\rangle=\frac{i\hbar^2\Sigma^{\rm K}(t)}{4}\delta(t'-t). \tag{83}$$

This stochastic equation again has the form of a Josephson equation with a noise term added, similar to Eq. (67). Note that the noise term in Eq. (82) is inversely proportional to the square root of the number of particles in the condensate. As a result, the Fokker-Planck

equation for the probability distribution of $\theta_0(t)$, associated with the Langevin equation in Eq. (82), will have a 'diffusion' term inversely proportional to the number of particles. This means that the global phase is only well determined if there is a infinite number of atoms in the condensate, otherwise the global phase undergoes phase diffusion, due to thermal fluctuations. This mechanism for phase diffusion is different than the phase 'diffusion' considered by Lewenstein and You [18], who considered phase spreading due to quantum fluctuations.

Having made these remarks, we perform the integration over $\theta_0(t)$, and are left with a probability distribution for $N_{\rm c}$ and $\mathbf{q}$. It is given by

$$P[N_{\rm c},\mathbf{q},\mathbf{v};t]=\int d[N_{\rm c}]\;d[\mathbf{q}]\;d[\mathbf{v}]\;\delta[\mathbf{v}(t')-\dot{\mathbf{q}}(t')]\;\exp\left\{\frac{i}{\hbar}S^{\rm eff}[N_{\rm c},\mathbf{q},\mathbf{v}]\right\}. \tag{84}$$

Here we introduced the velocity $\mathbf{v}(t)=\dot{\mathbf{q}}(t)$ by means of a delta functional. The resulting effective action reads

$$\begin{aligned}S^{\rm eff}[N_{\rm c},\mathbf{q},\mathbf{v}]=\int_{t_0}^{t}dt'\frac{2}{\hbar\Sigma^{\rm K}(t')}\Bigg\{&\frac{\hbar^2}{4N_{\rm c}(t')}\left(\frac{dN_{\rm c}(t')}{dt'}+\frac{\beta}{2}i\Sigma^{\rm K}(t')\left[\mu_{\rm c}(t')-\mu\right]N_{\rm c}(t')\right)^2\\&+\sum_j\frac{q_j^2(t')}{2N_{\rm c}(t')}\left(\frac{1}{2}mN_{\rm c}(t')\dot{v}_j(t')+\frac{N_{\rm c}(t')\beta}{4}i\hbar^2\Sigma^{\rm K}(t')\frac{v_j(t')}{q_j^2(t')}+\frac{\partial V}{\partial q_j}(\mathbf{q}(t'),N_{\rm c}(t'))\right)^2\Bigg\}\end{aligned} \tag{85}$$

Eqs. (84) and (85) are similar to the path-integral expressions we encountered in our discussion of the Brownian motion of a particle in a potential in Sec. II. Therefore we immediately conclude that the equations of motion for the variational parameters are given by

$$\frac{1}{2}mN_{\rm c}(t)\ddot{q}_j(t)+\frac{N_{\rm c}(t)\beta}{4}i\hbar^2\Sigma^{\rm K}(t)\frac{\dot{q}_j(t)}{q_j^2(t)}=-\frac{\partial V}{\partial q_j}(\mathbf{q}(t),N_{\rm c}(t))+\frac{\sqrt{2N_{\rm c}(t)}}{q_j(t)}\xi_j(t), \tag{86}$$

with the time correlations of the gaussian noise terms $\xi_j(t)$ given by

$$\langle\xi_j(t)\xi_k(t')\rangle=\frac{i\hbar^2\Sigma^{\rm K}(t)}{4}\delta_{jk}\delta(t-t'). \tag{87}$$

So we have found the important result that the variational parameters obey the equations of motion of a Brownian particle with mass $mN_{\rm c}/2$ in a potential $V(\mathbf{q},N_{\rm c})$. Physically, the variational description of the condensate with the Langevin equation in Eq. (86), as opposed to Eq. (76), has two important extra features. First, there is a damping term present, i.e., a term proportional to the velocity $\dot{q}_j(t)$. This damping term can, for example, be used to calculate the damping on the collective modes of the condensate. Since the damping term is proportional to $\hbar\Sigma^{\rm K}$, we conclude that it arises because of incoherent collisions between condensate and thermal atoms, which drive the condensate back to equilibrium, and let the phase of the condensate relax to a state where the phase is, on average, position independent. Secondly, since the Langevin equation also contains fluctuations, it can, for example, be used to describe the stochastic initiation of the collapse observed in $^7$Li [14–17]. Our description contains thermal fluctuations, that cause the condensate to overcome the macroscopic energy barrier and start the collapse. In the next section, we will present the result of calculations that we have done on the two above mentioned phenomena.

Since the potential in the Langevin equation in Eq. (86) depends on the number of condensate particles $N_{\rm c}$, we have to couple Eq. (86) to a rate equation for the number of

condensate atoms. This stochastic rate equation also follows directly from the effective action in Eq. (85) with the techniques discussed previously, and is given by

$$\frac{dN_{\rm c}(t)}{dt} = -\frac{\beta}{2} i\Sigma^{\rm K}(t)\left[\mu_{\rm c}(t) - \mu\right] N_{\rm c}(t) + 2\sqrt{N_{\rm c}(t)}\eta(t), \tag{88}$$

with correlations of the gaussian noise given by

$$\langle \eta(t')\eta(t)\rangle = \frac{i\Sigma^{\rm K}(t)}{4}\delta(t'-t). \tag{89}$$

As explained in appendix A, we have to treat the multiplicative noise in Eq. (88) again as a Stratonovich process, to achieve the correct equilibrium distribution. Physically, Eq. (88) describes the growth or evaporation of the condensate. The noise term in Eq. (88) represents the fluctuations in the number of particles.

To see that Eqs. (86) and (88) generate the correct equilibrium distribution, we now discuss the Fokker-Planck equation for $P[N_{\rm c}, \mathbf{q}, \mathbf{v}; t]$. However, let us first discuss the equilibrium solution we expect on basis of Eq. (57). A substitution of the gaussian *ansatz* in Eq. (75) into equation Eq. (57) results in

$$P[N_{\rm c}, \mathbf{q}, \mathbf{v}; t \to \infty] \propto \exp\left\{-\beta\left(\sum_j \frac{1}{4} m N_{\rm c} v_j^2 + V(\mathbf{q}, N_{\rm c}) - \mu N_{\rm c}\right)\right\}, \tag{90}$$

where $V(\mathbf{q}, N_{\rm c})$ is the potential given by Eq. (77). Although the Langevin equations for $q_j(t)$ and $N_{\rm c}(t)$ did, in first instance, not include the mean-field interactions, we argue that they also are correct for the interacting case. The reason for this is, that in this manner we are led to the correct equilibrium distribution as we show now. Let us therefore determine the Fokker-Planck equation for the probability distribution of $N_{\rm c}$, $\mathbf{q}$ and $\mathbf{v}$, generated by the stochastic equations in Eqs. (86) and (88) with the interacting potential in Eq. (77). It is given by

$$\begin{aligned}
&\frac{\partial P[N_{\rm c}, \mathbf{q}, \mathbf{v}; t]}{\partial t} = \\
&\sum_j \left[-\frac{\partial}{\partial q_j} v_j + \frac{\partial}{\partial v_j}\left(\frac{\beta i\hbar^2 \Sigma^{\rm K}(t)}{2m q_j^2} v_j + \frac{2}{m N_{\rm c}} \frac{\partial V}{\partial q_j}(\mathbf{q}, N_{\rm c})\right) + \frac{i\hbar^2 \Sigma^{\rm K}(t)}{m^2 N_{\rm c} q_j^2} \frac{\partial^2}{\partial v_j^2}\right] P[N_{\rm c}, \mathbf{q}, \mathbf{v}; t] + \\
&\left[\frac{\partial}{\partial N_{\rm c}}\left(\frac{\beta}{2} i\Sigma^{\rm K}(t)\left[\frac{\partial V}{\partial N_{\rm c}}(\mathbf{q}, N_{\rm c}) + \sum_j \frac{1}{4} m v_j^2 - \mu\right] N_{\rm c}\right) + \frac{i\Sigma^{\rm K}(t)}{2} \frac{\partial}{\partial N_{\rm c}} N_{\rm c} \frac{\partial}{\partial N_{\rm c}}\right] P[N_{\rm c}, \mathbf{q}, \mathbf{v}; t],
\end{aligned} \tag{91}$$

and insertion of the equilibrium distribution shows that it is indeed a stationary solution of this Fokker-Planck equation. Thus we conclude that the Langevin equations for $q_j(t)$ and $N_{\rm c}(t)$ give, with the potential $V(\mathbf{q}, N_{\rm c})$ given in Eq. (77), the correct description of the nonequilibrium dynamics of a Bose-Einstein condensate in the gaussian approximation. This description includes damping of the collective modes of the condensate, as well as condensate growth and evaporation. The essence of our method lies in the fluctuation-dissipation theorem, which ensures the relaxation towards the correct physical equilibrium distribution.

## V. APPLICATIONS

In this section we first apply the Langevin equations for the variational parameters, derived in the previous section, to the calculation of the damping and frequency of the collective modes of the condensate. As a second application, we also obtain a description of the initial growth of a condensate.

### A. Collective modes of the condensate

In this section we use the Langevin equations in Eq. (86) for the gaussian variational parameters, and the stochastic rate equation in Eq. (88) for the number of particles in the condensate, to obtain a description of the collective modes of the condensate. We calculate the frequency and damping of both the monopole and quadrupole mode in an isotropic trap, and compare with the theoretical results found by Williams and Griffin [35,36]. Since we are considering the case of a static thermal cloud, our results will be correct only for the modes where the thermal cloud does not play an important role, i.e., for the out-of-phase modes [22,23]. In the experiments of Jin *et al.* [37], this turns out to be the quadrupole mode. We calculate the frequency of the quadrupole mode for this experiment, by means of a fit to the experimental data for the damping.

The frequency and damping of the collective modes are, as measured in experiment, averaged quantities. Therefore, we first write down the equations of motion for the averages of the gaussian variational parameters and the number of particles in the condensate. The equations of motion for the average of the gaussian widths read

$$\frac{1}{2}mN_{\rm c}(t)\ddot{q}_j(t) + \frac{N_{\rm c}(t)\beta}{4}i\hbar^2\Sigma^{\rm K}\frac{\dot{q}_j(t)}{q_j^2(t)} = -\frac{\partial V}{\partial q_j}(\mathbf{q}(t), N_{\rm c}(t)). \tag{92}$$

For notational convenience we omit the brackets $\langle\cdots\rangle$ denoting the noise average of a stochastic quantity, and denoted the averages of the gaussian variational parameters simply by $q_j(t)$, where $j$ equals $x$, $y$ or $z$. The average equation in Eq. (92) is obtained from the Langevin equation for $q_j(t)$ by simply leaving out the noise term. This can be done because the noise in the Langevin equation does not induce a drift term for the average. This follows directly from the Fokker-Planck equation, with the use of partial integration. Since we want to describe a perturbation around a static equilibrium, the Keldysh self-energy will be time independent to a good approximation, and we thus drop its explicit dependence on time. For a description of the collective modes, we also have to consider variations in the average number of particles of the condensate, caused by the excitation of a mode. This means that we also have to consider the rate equation for the average number of particles

$$\frac{dN_{\rm c}(t)}{dt} = -\frac{\beta}{2}i\Sigma^{\rm K}\left[\mu_{\rm c}(t) - \mu\right]N_{\rm c}(t) + \frac{i\Sigma^{\rm K}}{2}. \tag{93}$$

In writing down this equation, we again left out the brackets $\langle\cdots\rangle$, which denote averaging over different realizations of the noise in the stochastic rate equation in Eq. (88). The last term on the right hand side of Eq. (93) is a so-called noise-induced, or spurious drift term. It arises because in Eq. (88) we are dealing with multiplicative Stratonovich noise. Without this

drift term, the equilibrium number of particles predicted by Eq. (93) would not be correct, as we will see lateron. Note that the average of the stochastic rate equation in Eq. (93) is very similar to the result obtained by Gardiner *et al.* [38]. However, their expression for the chemical potential of the condensate is different since they do not consider a gaussian *ansatz*, and they also have not made the 'classical' approximation to the fluctuation-dissipation theorem.

To obtain a description of the collective modes of the condensate, we have to linearize the equations in Eqs. (92) and (93) around their time-independent equilibrium solutions. Let us therefore put $q_j(t) = q_j^{(0)} + \delta q_j(t)$ and $N_{\rm c}(t) = N_0 + \delta N(t)$ and substitute this in Eqs. (92) and (93). Equating the zeroth-order terms after linearization results for the average rate equation in

$$\left[\beta\left(\frac{\partial V(\mathbf{q}^{(0)}, N_0)}{\partial N_{\rm c}} - \mu\right) - \frac{1}{N_0}\right] N_0 = 0. \tag{94}$$

From this equation the equilibrium number of particles can be calculated. It is however much more convenient to use the number of particles in the condensate as experimental input, and calculate the chemical potential $\mu$ such that Eq. (94) is satisfied. The equilibrium conditions for the gaussian variational parameters read

$$\frac{\partial V(\mathbf{q}^{(0)}, N_0)}{\partial q_j} = 0, \tag{95}$$

from which $\mathbf{q}^{(0)}$ can be calculated. For the noninteracting case, where $a = 0$ in the potential, the above equations result in $q_j = \sqrt{\hbar/m\omega_j}$ as expected. In this case $N_0$ is given by

$$N_0 = \left[\beta(\frac{\hbar}{2}(\omega_x + \omega_y + \omega_z) - \mu)\right]^{-1}, \tag{96}$$

which is the correct equilibrium ground state occupation number of a non-interacting Bose gas, within the 'classical' approximation. Note that without the noise-induced drift term in the rate equation for the average number of particles the correct equilibrium would not have been obtained.

The linearized equations of motion for the deviations are found by equating the first order contribution on the left and right-hand side of Eqs. (92) and (93) after linearization. The equation for the deviation in the equilibrium number $N_0$ of condensate atoms due to the excitation of a collective mode is given by

$$\delta\dot{N}(t) = -\Gamma\ \delta N(t) - \sum_j \alpha_j \delta q_j(t). \tag{97}$$

Here, the parameter $\Gamma$ is given by

$$\Gamma = -\frac{\beta}{2} i\Sigma^{\rm K}\left[\frac{1}{\beta N_0} + N_0 \frac{\partial^2 V(\mathbf{q}^{(0)}, N_0)}{\partial N_{\rm c}^2}\right], \tag{98}$$

where we eliminated the chemical potential $\mu$ by using Eq. (94). Physically, $\Gamma$ describes the lack of detailed balance between the thermal cloud and the condensate due to an excitation

of a collective mode of the condensate. In general, this lack of detailed balance will cause damping, and will alter the frequency with respect to the undamped case. The parameters $\alpha_j$ are given by

$$\alpha_j = \frac{\beta}{2} i\Sigma^{\rm K} \frac{\partial^2 V(\mathbf{q}^{(0)}, N_0)}{\partial q_j \partial N_{\rm c}}, \tag{99}$$

and represent the response of the fluctuations in the number of particles due to a deformation of the condensate in the $j$-th direction, and vice-versa.

The linearized equations of motion for the deviation of the gaussian variational parameters take the form of damped harmonic equations

$$\delta\ddot{q}_j(t) + \Gamma_j \delta\dot{q}_j(t) = -\alpha_j/\alpha \; \delta N(t) - \Omega_j^2 \delta q_j(t) - \sum_{k\neq j} \Omega_{jk}^2 \delta q_k(t). \tag{100}$$

The damping rates $\Gamma_j$ are given by

$$\Gamma_j = \frac{\beta}{2m(q_j^{(0)})^2} i\hbar^2 \Sigma^{\rm K}, \tag{101}$$

and the frequencies $\Omega_j$ and $\Omega_{jk}$ read

$$\Omega_j^2 = \frac{2}{mN_0} \frac{\partial^2 V(\mathbf{q}^{(0)}, N_0)}{\partial q_j^2}; \quad \Omega_{jk}^2 = \frac{2}{mN_0} \frac{\partial^2 V(\mathbf{q}^{(0)}, N_0)}{\partial q_j \partial q_k}. \tag{102}$$

In Eq. (100) we introduced the parameter $\alpha = mN_0\beta i\Sigma^{\rm K}/4$ for later convenience. Physically, the damping rates $\Gamma_j$ arise because of collisions between thermal atoms and condensate atoms. This causes damping of the collective modes on the condensate, and also alters the frequencies with respect to the results obtained without damping. It should be noted here, that all the parameters mentioned above can be calculated microscopically, by using the expression for the Keldysh self-energy given in Eq. (62).

To obtain the eigenmodes of Eqs. (97) and (100) we have to consider solutions of the form $(\delta N(t), \delta\mathbf{q}(t)) = (\delta N(0), \delta\mathbf{q}(0))e^{-i\omega t}$. For such solutions we can rewrite these equations as

$$\begin{pmatrix} \alpha(\Gamma - i\omega) & \alpha_x & \alpha_y & \alpha_z \\ \alpha_x & \Omega_x^2 - i\omega\Gamma_x - \omega^2 & \Omega_{xy}^2 & \Omega_{xz}^2 \\ \alpha_y & \Omega_{xy}^2 & \omega_y^2 - i\omega\Gamma_y - \omega^2 & \Omega_{yz}^2 \\ \alpha_z & \Omega_{xz}^2 & \Omega_{yz}^2 & \Omega_z^2 - i\omega\Gamma_z - \omega^2 \end{pmatrix} \begin{pmatrix} \delta N(0)/\alpha \\ \delta q_x(0) \\ \delta q_y(0) \\ \delta q_z(0) \end{pmatrix} = 0. \tag{103}$$

This matrix equation only has non-trivial solutions if the determinant of the above matrix is equal to zero. Solving this condition for $\omega$ results in general in complex frequencies $\omega = \omega_{\rm R} - i\omega_{\rm I}$. The real part $\omega_{\rm R}$ then gives the frequency of a collective mode, whereas the imaginary part $\omega_{\rm I}$ gives the damping of this mode.

*1. Isotropic trapping potential*

We consider now the case of an isotropic trapping potential $V^{\rm ext}(\mathbf{x}) = \frac{1}{2}m\omega_0^2\mathbf{x}^2$ for a discussion of the frequencies and damping rate of the low-lying collective modes of the condensate. Because of the spherical symmetry of the condensate in equilibrium, the number of parameters reduces and the eigenvalue equation simplifies significantly. For the frequencies and damping rates we have:

$$\begin{aligned} \Omega_x = \Omega_y = \Omega_z &\equiv \Omega_r; \\ \Omega_{xy} = \Omega_{xz} = \Omega_{yz} &\equiv \Omega_{rr}; \\ \Gamma_x = \Gamma_y = \Gamma_z &\equiv \Gamma_r. \end{aligned} \tag{104}$$

For the parameters $\alpha_j$ we have the same simplification, and we denote these parameters by $\alpha_r$. With these simplifications, the eigenmodes can be calculated analytically. One mode is doubly degenerate, with eigenvectors $(0,1,-1,0)$ and $(0,1,0,-1)$, and is thus the quadrupole mode. From the form of the eigenvectors of the quadrupole mode, we are led to the important conclusion that the number of particles in the condensate is constant for this mode. Physically this can be understood from the fact that the motion of the condensate is 'volume-preserving' in this case: as one direction shrinks the other one expands and vice versa. This motion does not lead to a change in the chemical potential of the condensate, at least to a linear approximation, and therefore does not affect the average number of atoms in the condensate. The complex frequency of the quadrupole mode is given by

$$\omega_{\rm quad} = \sqrt{\Omega_r^2 - \Omega_{rr}^2 - \left(\frac{\Gamma_r}{2}\right)^2} - \frac{1}{2}i\Gamma_r. \tag{105}$$

Note that from this expression it is clearly seen that the damping also affects the frequencies of the collective modes [23].

The frequency and eigenvector of the monopole mode can also be calculated analytically for the isotropic case. However, because of the rather formidable expressions involved, we omit them here. For the monopole mode the number of atoms in the condensate is not constant, but oscillates out of phase with the spatial degrees of freedom of the condensate. This can be understood from the fact that the monopole motion leads to a global increase in the density of the condensate, and therefore affects the detailed balance of the condensate with the thermal cloud. In the case where we ignore the fluctuations of the number of atoms in the condensate, and take $\alpha = \alpha_r = \Gamma = 0$, the expression for the complex frequency of the monopole mode is given by

$$\omega_{\rm mono} = \sqrt{\Omega_r^2 + 2\Omega_{rr}^2 - \left(\frac{\Gamma_r}{2}\right)^2} - \frac{1}{2}i\Gamma_r. \tag{106}$$

Comparing the results in Eqs. (105) and (106) we see that the damping rate of both modes is equal in first order in $\Gamma_r$, at least within our variational approximation.

We now turn to an explicit calculation of the frequencies and damping rates of the quadrupole and monopole mode in an isotropic trap. We have used the same parameters as Williams and Griffin [35,36], and thus have taken the trapping potential frequency equal to

$\omega_0/2\pi = 10$ Hz. The calculations are performed for $^{87}$Rb, which has a scattering length of $a = 5.7$ nm. We take the total number of atoms equal to $N_{\rm total} = 2 \times 10^6$. Since the number of atoms in the condensate is large at most temperatures below the critical temperature, we are mainly in the Thomas-Fermi regime, and can neglect the kinetic energy of the condensate atoms with respect to their mean-field interaction. Therefore, we have used a Thomas-Fermi profile for the condensate to calculate the collision integral in the expression for the Keldysh self-energy in Eq. (62). As a result the Keldysh self-energy turns out not to be constant over the size of the condensate in this limit, contrary to our assumption in the derivation of the stochastic equations for the variational parameters. To compensate for this effect, we have calculated $\hbar\Sigma^{\rm K}$ by taking a volume average of $\hbar\Sigma^{\rm K}(\mathbf{x})$ over the size of the condensate. We report our results for the number of condensate atoms, the frequencies and the damping rates as a function of the reduced temperature $T/T_{\rm BEC}$, where $T_{\rm BEC}$ is the critical temperature for an ideal Bose gas.

The procedure for calculating the number of atoms in the condensate as a function of temperature is as follows. For a given number of condensate atoms $N_0$ we first calculate the average condensate density profile and the chemical potential from the time-independent Gross-Pitaevskii equation. In the Thomas-Fermi limit the average condensate density profile is given by

$$|\langle\phi(\mathbf{x})\rangle|^2 = \frac{1}{T^{\rm 2B}}\left(\mu - V^{\rm ext}(\mathbf{x})\right), \tag{107}$$

with a condensate chemical potential

$$\mu = \frac{\hbar\omega_0}{2}\left(15 N_0 a/l\right)^{2/5}, \tag{108}$$

where $l = \sqrt{\hbar/m\omega_0}$ is the harmonic oscillator length. Clearly, the condensate density can not be negative, so Eq. (107) is only valid if the condensate density is positive, otherwise it should be taken equal to zero. At $|\mathbf{x}| = l\sqrt{\mu/\hbar\omega_0} \equiv R_{\rm TF}$ the condensate density is equal to zero. Next we calculate the number of atoms in the thermal cloud with the value of the chemical potential determined by Eq. (108) using

$$N_{\rm thermal} = \int d\mathbf{x} \int \frac{d\mathbf{k}}{(2\pi)^3} N(\epsilon(\mathbf{x},\mathbf{k})), \tag{109}$$

with $N(\epsilon(\mathbf{x},\mathbf{k}))$ given by Eqs. (51) and (63). We repeat these steps for a variable number of condensate atoms until $N_0 + N_{\rm thermal} = N_{\rm total}$. The result of the calculation is shown in Fig. 1, together with the result for an ideal Bose gas in the thermodynamic limit. Using this result for $N_0$ as a function of the temperature $T$, we have calculated the Keldysh self-energy as a function of temperature using the expression in Eq. (62) [39]. In Fig. 2 the function $|\beta\hbar\Sigma^{\rm K}(\mathbf{x})|$ is shown, for $T = 0.5 T_{\rm BEC}$. It is clear from this figure that the Keldysh self-energy is not constant over the size of the condensate, and even diverges in the Thomas-Fermi approximation at $|\mathbf{x}| = R_{\rm TF}$. The equilibrium values of the variational parameters were calculated using Eq. (95). Subsequently, the various parameters were calculated from Eqs. (98), (99), (101) and (102). The complex frequency of the quadrupole was calculated from Eq. (105), and the complex frequency of the monopole mode was calculated from the corresponding analytical expression, which we have omitted here.

The results for the frequencies are presented in Fig. 3. The dashed lines are the $T = 0$ frequencies obtained by Stringari [40], in the Thomas-Fermi limit. Since the calculations are in the collisionless limit, where $|\beta\hbar\Sigma^{\rm K}| \ll 1$, the results are essentially $T = 0$ results for a variable number of condensate atoms. In the Thomas-Fermi limit these frequencies are independent of the number of atoms in the condensate, but below $N_0 \simeq 10^4$ condensate atoms, where the Thomas-Fermi approximation starts to break down, the frequencies deviate from the $T = 0$ results, as is seen in Fig. 3. Although for the monopole mode the full expression for the frequency and damping involves the parameters $\Gamma$, $\alpha_r$ and $\alpha$, which are related to the fluctuations in the number of condensate atoms due to the excitation of a collective mode, this hardly affects the results for the frequencies and damping of this mode. We therefore conclude that the fluctuations in the number of condensate atoms during the excitation of a collective mode hardly affect the damping and frequency of this mode, and that the expression in Eq. (106) is valid as long as $|\beta\hbar\Sigma^{\rm K}| \ll 1$.

The results for the damping rate are shown in Fig. 4, together with the result for $|\beta\hbar\Sigma^{\rm K}|$. Within our variational approximation, the damping rates for both the quadrupole and the monopole modes are found to be the same in the collisionless regime considered here. As clearly seen from Fig. 4, the damping rate increases with increasing temperature. This is because the density of the thermal cloud becomes larger with increasing temperature, and there are therefore more collisions between condensate and thermal atoms, which cause the damping. Williams and Griffin [35] have also calculated the damping of the monopole mode for a condensate in a spherical trap in presence of a static thermal cloud. These authors have generalized the wave equation derived by Stringari [40] to nonzero temperatures, similar to our result in Eq. (72), albeit that their work does not obey the fluctuation-dissipation theorem as mentioned previously. They have calculated the damping of the monopole mode in perturbation theory, considering the damping as a perturbation parameter. Our results for the damping of the monopole mode have the same order of magnitude as their results. There are however some qualitative differences. We find that the damping rate increases very slowly with temperature for a large temperature regime $T < 0.95T_{\rm BEC}$, and then increases dramatically as the temperature approaches the critical temperature. Williams and Griffin find that the damping rate increases much more gradually with increasing temperature. These differences are probably due to the fact that these authors take into account that the collision integral in Eq. (62) has a position dependence. In Ref. [36] Williams and Griffin improve upon their Thomas-Fermi calculation for the damping rate by using the Bogoliubov-deGennes equations that follow from linearizing the Gross-Pitaevskii equation with an imaginary term. In this case, their results for the damping also show a dramatic increase as the temperature reaches the critical temperature and the Thomas-Fermi approximation breaks down. In this latter work it is also found that the damping for the quadrupole mode and the monopole mode are slightly different. The fact that we find that the damping for both the monopole mode and the quadrupole mode are equal to first order in $\Gamma_r$ is a result of neglect of the spatial dependence of the Keldysh self-energy. Williams and Griffin also find that there is no first-order correction in the damping to the real part of the frequencies, a conclusion consistent with Eqs. (105) and (106).

Summarizing, we have calculated the damping and frequencies for the quadrupole mode and the monopole mode of a condensate in a spherical trap. Our results differ slightly both qualitatively and quantitatively from the theoretical results found by Williams and Griffin

[35,36]. These differences are probably mostly due to the fact that the calculations are performed for a large number of atoms in the trap, which implies that the collision integral involved in the calculation has a significant position dependence, which our variational approach does not properly account for. However, for a smaller amount of atoms in the trap, we believe that our method should give accurate results, and goes in principle beyond the perturbation theory considered in Refs. [35,36].

### *2. Anisotropic trapping potential*

We now calculate the frequency of the $m = 2$ quadrupole mode, where $m$ is the azimuthal quantum number of the angular momentum, for the experimental parameters of Jin *et al.* [37]. In this experiment, one loads $^{87}$Rb atoms into an anisotropic trap, with radial frequency $\omega_r/2\pi = 129$ Hz, and axial frequency $\omega_z/2\pi = 365$ Hz. Although the equilibrium shape of the condensate is now anisotropic, the expression for the frequency of the quadrupole mode found in the isotropic case in Eq. (105), turns out to be also correct for the $m = 2$ quadrupole mode. The parameters $\Omega_r$, $\Omega_{rr}$, and $\Gamma_r$, are now given by

$$\begin{aligned} \Omega_r &\equiv \Omega_x = \Omega_y; \\ \Omega_{rr} &\equiv \Omega_{xy} = \Omega_{yx}; \\ \Gamma_r &\equiv \Omega_x = \Omega_y, \end{aligned} \tag{110}$$

which follow from the axial symmetry of the condensate in equilibrium. It follows from the expression for the complex frequency of the quadrupole mode in Eq. (105) that the complex frequencies lie on a circle of radius $|\omega_{quad}| = \sqrt{\Omega_r^2 - \Omega_{rr}^2} \simeq \sqrt{2}\omega_r$. To test the validity of our expression for the frequency of the quadrupole mode, we have plotted the experimental data points taken from Ref. [37] in the complex $\omega$-plane. In Fig. 5 the result is shown, together with a circle of radius $\sqrt{2}\omega_r$. The good quantitative agreement with experiment, clearly visible in Fig. 5, implies that the expression in Eq. (105) for the frequency and damping of the quadrupole mode is correct, even in the hydrodynamic regime, where $|\beta\hbar\Sigma^{\rm K}| \gg 1$. This may in first instance come as a surprise, since our variational approximation to the stochastic non-linear Schrödinger Eq. (60) was derived in the collisionless regime, where $|\beta\hbar\Sigma^{\rm K}| \ll 1$. Apparently, the relation $|\omega_{\rm quad}| \simeq \sqrt{2}\omega_r$ is also valid in the hydrodynamic regime. This can be understood from the fact that this relation for the complex frequency is quite general for a damped harmonic oscillator and we expect on general grounds that the quadrupole mode of the condensate can be described in this way, both in the collisionless and in the hydrodynamic regime.

To determine $|\beta\hbar\Sigma^{\rm K}|$ as a function of temperature, we have fitted the imaginary part of Eq. (105) to the experimental data for the damping presented in Ref. [37]. From this fit, we have calculated the dimensionless parameter $|\beta\hbar\Sigma^{\rm K}|$, using Eqs. (101), (102), and (105). The results of this fit are presented in Fig. 6. The value for $|\beta\hbar\Sigma^{\rm K}|$ found in this manner is then used to calculate the real part of Eq. (105), i.e., the frequency of the collective mode. In both the fitting of the damping, and the calculation of the frequency, we used the fit presented in Ref. [23] to determine the experimental values for the number of condensate atoms as a function of the temperature. The result for the frequency is presented in Fig. 7, together with the frequency calculated from Eq. (105), with $\Gamma_r = 0$, i.e., the zero-temperature frequency

for a variable number of condensate atoms. The experimental points are also shown. In Fig. 7, a good quantitative agreement with the experimental results is found. Fig. 7 also shows clearly that at nonzero temperatures the damping seriously affects the frequency of the quadrupole mode, as also found by Al Khawaja and Stoof [23]. However, a microscopic calculation of $\hbar\Sigma^{\rm K}$ in the Thomas-Fermi limit for the experimental conditions of interest, by means of a volume average of Eq. (62) over the size of the condensate, turns out to give values for $|\beta\hbar\Sigma^{\rm K}|$ which are approximately one order of magnitude too small to explain the experimental data for the quadrupole mode. There are several possible reasons for this discrepancy. One possible reason is that the calculation of $\hbar\Sigma^{\rm K}$ as an average over the size of the condensate of $\hbar\Sigma^{\rm K}(\mathbf{x})$ is not a good approximation. Since $\hbar\Sigma(\mathbf{x})$ becomes large compared to $\hbar\Sigma^{\rm K}(\mathbf{0})$ at the edges of the condensate, where the depletion of the thermal cloud due to the condensate's mean field is relatively small, the position dependence of $\hbar\Sigma^{\rm K}(\mathbf{x})$ is of importance, in particular since the density fluctuations occur for the quadrupole mode precisely near the edges of the condensate. This is clearly shown in Fig. 2, which shows that the Keldysh self-energy even diverges at $|\mathbf{x}| = R_{\rm TF}$ in the Thomas-Fermi limit. With respect to this remark, we refer to future work concerning the full numerical solution of the average of the Langevin equation in Eq. (60), to investigate the importance of the position dependence of $\hbar\Sigma^{\rm K}(\mathbf{x})$, and a comparison of this numerically exact approach to the variational method developed here. Another possible reason for the discrepancy with the experimental results, is the presence of other sources of damping, such as Landau and Beliaev damping, which have not been included in our calculations.

### B. Condensate growth and collapse

Although so far we have focused on repulsive interactions, and thus a positive scattering length $a$, we consider in this section the case where the scattering length is negative. In this case the condensate energy in the gaussian approximation becomes

$$V(\mathbf{q}, N_{\rm c}) = \sum_j \left( \frac{N_{\rm c}\hbar^2}{4mq_j^2} + \frac{1}{4} m N_{\rm c} \omega_j^2 q_j^2 \right) - \frac{|a|\hbar^2 N_{\rm c}^2}{\sqrt{2\pi} m q_x q_y q_z}. \tag{111}$$

From this potential it can easily be seen that there is only a metastable condensate possible if the number of atoms in the condensate is smaller that a certain critical value. This is illustrated in Fig. 8, where we show the potential in Eq. (111), for several values of the number of atoms in an isotropic condensate. For an isotropic trap, the maximum condensate number $N_{\rm max}$ turns out to be given by the condition

$$N_{\rm max} \frac{|a|}{l} < \frac{2\sqrt{2\pi}}{5^{5/4}} \simeq 0.67. \tag{112}$$

If the number of atoms in the condensate is above this value, the potential in Eq. (111) has no (meta)stable minima. If the number of atoms is smaller than $N_{\rm max}$, the potential has a metastable minimum, and the condensate can start to collapse by overcoming the metastable energy barrier by either macroscopic quantum tunneling or thermal fluctuations [13]. The stability condition for the condensate, found by a full numerical solution of the

Gross-Pitaevskii, turns out to be $N_c a/l < 0.58$ [41], and thus we see that the gaussian approximation is only 16 % off.

The first experiments on Bose-Einstein condensation in a gas with attractive interactions where performed with $^7$Li [14], which has a negative scattering length of $a \simeq -1.45$ nm. In these experiments, the gas is evaporatively cooled below the critical temperature, which causes the condensate to undergo several growth and collapse cycles before relaxing to a metastable equilibrium [15]. Because of the stochastic initiation of the collapse, one could not make a sequence of destructive measurements. However, a statistical analysis revealed that during a collapse the number of condensate atoms is not reduced to zero, but that the collapse stops, presumably because of elastic and inelastic collisional loss processes, when the number of atoms in the condensate is about 200 [16]. In a recent experiment, one was able to make a sequence of destructive measurements by (partially) dumping the condensate by a two-photon pulse [17], and thus start each measurement with approximately the same initial number of condensate atoms.

Using the Langevin equations for $q_j(t)$, and the stochastic rate equation for the number of atoms in the condensate, we are able to describe this experiment. To do so in the easiest way, we want to model the thermal cloud by an equilibrium Bose distribution of a noninteracting gas. However, numerical solutions of the quantum Boltzmann equation for these experiments have shown that the thermal cloud is not equilibrium, but can be well modeled by a distribution function given by [16]

$$f(\epsilon) = \frac{\exp[\beta(\mu' - \mu)]}{\exp[\beta(\epsilon - \mu)] - 1} \equiv \frac{A}{\exp[\beta(\epsilon - \mu)] - 1}. \tag{113}$$

At high energies, $f$ has the form of a Boltzmann distribution with chemical potential $\mu'$ and temperature $1/k_{\rm B}\beta$. At low energies $f$ has the form $f(\epsilon) = A/\beta(\epsilon - \mu)$, which is precisely the low-energy tail of a Bose-distribution with effective temperature $A/k_{\rm B}\beta$. Therefore, we conclude that Eq. (54) is too a good approximation still valid for the distribution function given by Eq. (113), if we make the replacement $\beta \to \beta/A$ in the operator on the right hand side of Eq. (54). In this manner we obey the fluctuation-dissipation theorem, and have also accounted for the fact that the distribution function of the thermal cloud is a nonequilibrium distribution function. We expect that this approximation will give quantitatively correct results for the condensate growth rate, since it is particularly good for the low-lying energy levels, which dominate the condensate growth.

Before comparing to the experimental data reported in Ref. [17], we discuss some aspects of the numerical solutions of the stochastic rate equation coupled to the Langevin equation, for a condensate which has initially no atoms. In the experiment performed by Gerton *et al.* this would correspond to the situation where the condensate is dumped completely. The stochastic rate equation in Eq. (88) is well suited for this purpose, since it also contains fluctuations, which initiate the growth in this case. Without these fluctuations, the growth rate of the condensate would never become nonzero. These initial conditions for condensate growth are different from the experiments conducted by Miesner *et al.* [42], in which the condensate growth is observed after evaporatively cooling the gas. In this case, the ground state already has a rather large nonzero occupation number above the critical temperature, which causes a growth process dominated by Bose-stimulation. Therefore, for a theoretical description of the condensate growth it is not so essential to include fluctuations in this case

[43,44]. We perform our simulations for the experimental conditions reported in Ref. [17]. The trap frequencies are given by $\omega_r/2\pi = 151$ Hz and $\omega_z/2\pi = 131.5$ Hz. We consider a thermal cloud with 70000 atoms at a temperature $T = 170$ nK. The parameter $A$ is taken equal to 4. These values correspond to typical experimental conditions [45]. The Keldysh self-energy $\hbar\Sigma^{\rm K}$ is calculated using Eq. (62), with the nonequilibrium distribution function $f(\epsilon)$ given by Eq. (113). The mean-field effects of the condensate on the thermal cloud are neglected, an approximation which will certainly be valid in the initial stage of the condensate growth, when the condensate is small. Moreover, we take the chemical potential of the condensate $\langle\mu_{\rm c}(\mathbf{x},t)\rangle$ in Eq. (62) equal to zero. Since the density of thermal atom will be the largest in the center of the trap, and the condensate is small in this case, we do not perform an average to calculate $\hbar\Sigma^{\rm K}$, but simply take $\hbar\Sigma^{\rm K} = \hbar\Sigma^{\rm K}(\mathbf{0})$.

We solve the Langevin equations coupled to the stochastic rate equation, using standard numerical techniques for stochastic differential equations [46]. Since the initial number of condensate atoms is equal to zero, we put at time $t = 0$ the values of the gaussian variational parameters equal to $q_j = \sqrt{\hbar/m\omega_j}$, which is their equilibrium value in the limit where the number of condensate atoms approaches zero. A slight subtlety arises in the use of the stochastic rate equation. One has to realize that the chemical potential of the thermal cloud in this equation is measured with respect to the energy of the lowest excited level, since we want the distribution function to describe the non-condensed atoms only. This means that we should use in the rate equation the chemical potential found in matching the distribution function in Eq. (113) to the number of thermal atoms, and add the energy of the lowest excited level to it. This is immediately clear when we write the number of atoms in the thermal cloud as a sum over occupation numbers, instead of an integral over energy. Fig. 9 shows the results of our simulations. In Fig. 9 (a), we plot the number of condensate atoms as a function of time, for the solutions of the Langevin equations and the stochastic rate equation for three different realizations of the noise. We assume that during the growth and subsequent collapse of the condensate, the distribution function of the thermal cloud is not affected. The maximum number of atoms in the condensate is for the parameters under consideration here equal to $N_{\rm max} \simeq 1470$ atoms, within the gaussian approximation. This means that during one growth-collapse cycle the number of atoms in the thermal cloud is reduced by only approximately 2 %, and therefore this approximation seems valid for the description of one growth-collapse curve. Once a collapse is initiated by the noise in the Eqs. (86) and (88), we model the collapse by putting the number of condensate atoms instantaneously equal to a gaussian random number with a mean value of 200 and a deviation of 40, which corresponds to the 20 % systematic uncertainty reported in [16], and the variational parameters $q_j(t)$ equal to their corresponding equilibrium values, given by Eq. (95). The collapse occurs on a time-scale $\mathcal{O}(1/\omega)$, which is much faster than the time-scale on which the condensate grows. Since we are interested in the growth process here, and not in the loss process which stops the collapse, this appears a reasonable way to model the collapse. Fig. 9 (a) clearly shows that when the number of condensate atoms approaches the maximum number $N_{\rm max}$, the condensate tends to collapse. We found that the collapse is initiated stochastically by the fluctuations in the number of atoms in the condensate, which cause density fluctuations that cause the condensate to overcome the macroscopic energy barrier and start the collapse. Since our description only includes thermal fluctuations, we may ask if macroscopic quantum tunneling might be of importance. However, previous

work has shown that decay by thermal fluctuations is the main decay mechanism for the temperatures of interest [15]. In Fig. 9 (b) we plot the number of atoms in the condensate as a function of time, averaged over different realizations of the noise. The red line is an average over 5 different realizations of the noise. A growth-collapse signature is still visible in this curve, although the stochastic growth process and initiation of the collapse has led to a dephasing of the moment of collapse. The green line shows an average over 10 realizations of the noise. Although the initial growth is clearly visible in this curve, the collapse can hardly be seen from this average, since the noise has led to an almost complete dephasing, and the collapse is 'averaged out'. Finally, the blue curve shows an average over 1000 realizations of the noise. No signature of the collapse is visible in this curve, because the averaging leads to a complete dephasing of the moment of the collapse.

We now discuss the simulation of the experiments performed by Gerton *et al.* [17]. To make a comparison with experiment, one has to realize that each data point is obtained as an average over 5 or 10 individual runs. Since the condensate number is probed by means of a destructive measurement, each experimental curve should not be viewed as an average of curves. Instead, each point is an average over different experimental runs, and the time-correlation between different experimental points is only caused by the initial conditions, which are approximately the same for each experimental run. To simulate this experiment by means of a numerical solution of the Langevin equations in Eqs. (86) and (88), we therefore have to let the numerical solutions evolve up to a certain point in time, and then make a numerical measurement. We then average over 5 or 10 measurements to obtain a data-point and its uncertainty, and repeat this procedure at a different measurement time. In this way, we are certain that each individual solution of our stochastic equations, is done for a different realization of the noise. Note that this procedure is very reminiscent of the method of Monte Carlo simulation.

We have done simulations for the three different experimental situations presented in Ref. [17]. The results of our simulations are presented in Fig. 10, as red triangles. The experimental data-points are also shown, and denoted by black circles. The Keldysh self-energy was calculated as in the simulations described above, using the averages of the full experimental data on the number of atoms in the thermal cloud and their temperature [45]. For the parameter $A$ of the nonequilibrium distribution function in Eq. (113) we use the average of the fits obtained by Gerton *et al.* [45]. For Fig. 10 (a) and (c) this corresponds to a thermal cloud of approximately 65000 atoms at a temperature of $T = 170$ nK. The parameter of the nonequilibrium distribution is equal to $A = 4$ for these simulations. For Fig. 10 (b) the thermal cloud contains approximately 100000 atoms at a temperature of $T = 200$ nK. The parameter $A = 2$ in this case. For Fig. 10 (a) and (b) the averages are taken over 5 runs for each data-point, whereas for (c) 10 runs are used. The error bars denote the uncertainty in the average. In Fig. 10 (a) and (b) the initial number of atoms was take equal to $N_{\rm c}(0) = 100$, and $N_{\rm c}(0) = 438$ respectively. For Fig. 10 (c) we have taken $N_{\rm c}(0) = 0$, since the condensate was in this case dumped completely to within the experimental resolution [17].

The results of our simulations presented in Fig. 10 shows good agreement with experiment for the initial stage of the growth, where the condensate is small. In particular, Fig. 10 (a) show good agreement in the initial stage where $N_{\rm c} < 400$ atoms, whereas Fig. 10 (c) shows good agreement in the regime where $N_{\rm c} < 600$ atoms. This is to be expected, since the

gaussian *ansatz* is a very good approximation for a small number of atoms in the condensate, whereas it becomes worse for a larger number of atoms in the condensate. The fact that the error bars of the experimental data points have the same order of magnitude as the error bars on our simulations, indicates that the fluctuations, i.e., the noise in our stochastic equation, have indeed the correct order of magnitude. As mentioned in the discussion of the individual solutions of our stochastic equations, we find that the collapse is initiated by fluctuations in the number of atoms, and that these fluctuations thus lead to a dephasing of the moment of the collapse. In principle also the fluctuations in the initial number of atoms in the condensate lead to dephasing. However, the uncertainty in $N_{\rm c}(0)$ is small compared to the uncertainty in $N_{\rm c}(t)$ at later times, and we therefore conclude that fluctuations in the initial conditions for the condensate are presumably less important for an understanding of the dephasing of the moment of the collapse. Moreover, there are also fluctuations in the properties of the thermal cloud for each individual experimental run, which we have not taken into account. With respect to this point we also note that our method does not display the saturation in the growth rate, observed in numerical solutions of the quantum Boltzmann equation [17]. This effect is also observable in the experimental data in Fig. 10 (c), where the growth is exponential in the first stage, but then turns linear. This saturation in the growth rate is caused by the fact that the condensate mostly grows from the low-lying excited states, which in turn have to be feeded by collisions in higher energy states, which are not Bose enhanced. Since in our simulations the thermal cloud is taken to be static, our simulations do not display this saturation effect. In conclusion, we like to point out that, to make a sensible quantitative comparison to the experimental results in the whole time-domain, we have to compare the converged averages of both the experimental runs and the theoretical simulations. This is because of the fact that the fluctuations are so large, that each individual growth curve can differ substantially, as seen from Fig. 9 (a). In turn, this leads to averages that can differ qualitatively, depending on the number of runs one averages over, as also clearly seen in Fig. 9 (b).

## VI. CONCLUSIONS

We have presented a Fokker-Planck equation that describes the nonequilibrium dynamics of an atomic Bose-Einstein condensate. We have discussed an approximation to this Fokker-Planck equation, which assumes that the thermal cloud is close to equilibrium. Its corresponding Langevin equation has the form of a stochastic nonlinear Schrödinger equation with complex gaussian noise. Both the Fokker-Planck equation, and the Langevin equation obey the fluctuation-dissipation theorem, which ensures that the condensate relaxes to the correct equilibrium. We have also presented the hydrodynamic formulation corresponding to this stochastic non-linear Schrödinger equation, in which the condensate is described in terms of its density and its phase. These turn out to obey a stochastic continuity equation and a stochastic Josephson equation, respectively. To make analytical progress, we have then extended the variational calculus, commonly applied to the Gross-Pitaevskii equation, also to the case of the stochastic non-linear Schrödinger equation. The equations of motion for the variational parameters turn out to be identical to the Langevin equations describing the Brownian motion of a particle in a potential. These equations are then coupled to a stochastic rate equation for the number of atoms in the condensate. We have applied these

equations to calculate the damping and frequencies of the collective modes of the condensate, and to obtain a description of the growth-collapse curve of a condensate with attractive interactions. However, there are much more applications possible with the variational method presented here. With a slight extension, it can also be used to calculate the frequency and damping of the scissor modes of the condensate [47,48], at nonzero temperatures. Moreover, applying the method to a Thomas-Fermi density profile, we can obtain a simple description of the growth of a condensate with repulsive interactions. The treatment of the dissipative dynamics of vortices and other topological excitations such as skyrmions [49], is also feasible within this variational method.

The only quantity that characterizes the thermal cloud is, in our approach, the Keldysh self-energy $\hbar\Sigma^{\rm K}(\mathbf{x},t)$. The parameter that enters the equations of motions for the variational parameters, turns out to be some spatial average of this quantity. In our calculations presented here, we have taken an average over the size of the condensate, when calculating the frequency and damping of the collective modes. Future work will include a numerical solution of the stochastic non-linear Schrödinger equation, to investigate the importance of the spatial dependence of the Keldysh self-energy, which we have not taken into account here. Nevertheless, we believe that the variational method presented here, provides a satisfying picture of the nonequilibrium dynamics of a Bose-Einstein condensate at nonzero temperatures. Moreover, as we have shown, calculations done within this variational approximation, lead already to a good agreement with experiments on collective modes and on condensate growth.

## ACKNOWLEDGEMENTS

It is a great pleasure to thank Ionut Prodan and Randy Hulet for kindly providing us with their raw experimental data. We also thank Usama Al Khawaja for useful remarks, and for providing us with some numerical results.

## APPENDIX A: AMPLITUDE AND PHASE VARIABLES

The condensate is often described in terms of density and phase variables, by making a canonical transformation $\phi = \sqrt{\rho}e^{i\theta}$. In this appendix we discuss the derivation of the Langevin equations of motion for $\rho(\mathbf{x},t)$ and $\theta(\mathbf{x},t)$. For simplicity, we first discuss the single-mode version of the probability distribution in Eq. (44) in the non-interacting case. So we consider a probability distribution for a single-mode complex order parameter, which reads

$$P[\phi^*,\phi;t] = \int_{\phi(t)=\phi}^{\phi^*(t)=\phi^*} d[\phi^*]d[\phi]\exp\left\{\frac{i}{\hbar}\left(\int_{t_0}^{t}dt'\frac{2}{\hbar\Sigma^{\rm K}}\left|\left(i\hbar\frac{\partial}{\partial t'}+\mu-\mu_{\rm c}+iR\right)\phi(t')\right|^2\right)\right\}. \tag{A1}$$

Physically, this probability distribution describes the nonequilibrium dynamics of a non-interacting Bose-Einstein condensate, in contact with a thermal cloud characterized by a Keldysh self-energy $\hbar\Sigma^{\rm K}$, with inverse temperature $\beta$ and a chemical potential $\mu$. The

energy per particle in the single-mode system is equal to $\mu_{\rm c}$. The dissipation $R$ is again related to the Keldysh self-energy by the fluctuation-dissipation theorem in Eq. (54), which in this simple case reads

$$iR = -\frac{\beta}{4}\hbar\Sigma^{\rm K}[\mu_{\rm c} - \mu]. \tag{A2}$$

From the previous sections we know that the probability distribution $P[\phi^*, \phi; t]$ is generated by the Langevin equation

$$i\hbar\frac{\partial\phi(t)}{\partial t} = (\mu_{\rm c} - \mu - iR)\,\phi(t) + \eta(t), \tag{A3}$$

where the complex noise has a time correlation function given by

$$\langle\eta^*(t')\eta(t)\rangle = \frac{i\hbar^2}{2}\Sigma^{\rm K}\delta(t'-t). \tag{A4}$$

As explained in the first section, the noise $\eta(t)$ has to be interpreted as an Ito process. We can again derive the Fokker-Planck equation for $P[\phi^*, \phi; t]$ by noting that it is in fact the Schrödinger equation in the position representation. We will do this in some detail once more, to make clear the different steps of the derivation. The first step is to determine the momenta conjugate to the coordinates $\phi$ and $\phi^*$. Since we have a lagrangian equal to

$$L[\phi^*, \phi] = \frac{2}{\hbar\Sigma^{\rm K}}\left|\left(i\hbar\frac{\partial}{\partial t'} + \mu - \mu_{\rm c} + iR\right)\phi(t')\right|^2, \tag{A5}$$

we can define the momentum conjugate to $\phi$ in the usual way

$$p_\phi = \frac{\partial L}{\partial\dot{\phi}} = \frac{2i}{\hbar\Sigma^{\rm K}}\left(-i\hbar\frac{\partial}{\partial t} - \mu_{\rm c} - iR + \mu\right), \tag{A6}$$

with the complex conjugate expression for $p_{\phi^*}$. The second step is to derive the hamiltonian. Although it has in principle ordering problems, we overcome these by noting that in the path-integral formulation of quantum mechanics we are always dealing with a normal ordered hamiltonian. The normal ordered hamiltonian, i.e., with the momentum operators positioned left to the coordinate operators, is now given by

$$H[p_\phi, \phi; p_{\phi^*}, \phi^*] = p_\phi\dot{\phi} + p_{\phi^*}\dot{\phi}^* - L[\phi^*, \phi]. \tag{A7}$$

The last step towards the Fokker-Planck equation is to quantize the hamiltonian, and to write down the Schrödinger equation in the position representation. So we have $p_\phi = -i\hbar\partial/\partial\phi$, and similarly $p_{\phi^*} = -i\hbar\partial/\partial\phi^*$. The Fokker-Planck equation becomes

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}P[\phi^*, \phi; t] = &-\frac{\partial}{\partial\phi}(\mu_{\rm c} - \mu - iR)\phi P[\phi^*, \phi; t] \\ &\frac{\partial}{\partial\phi^*}(\mu_{\rm c} - \mu - iR)\phi^* P[\phi^*, \phi; t] \\ &-\frac{1}{2}\frac{\partial^2}{\partial\phi^*\partial\phi}\hbar\Sigma^{\rm K}P[\phi^*, \phi; t]. \end{aligned} \tag{A8}$$

With the use of the fluctuation-dissipation theorem in Eq. (A2) we then show that it has as a stationary solution

$$P[|\phi|, t \to \infty] \propto \exp\left\{-\beta\phi^*(\mu_c - \mu)\phi\right\}, \tag{A9}$$

which only depends on the amplitude of $\phi$ and $\phi^*$.

We now repeat the above discussion in terms of amplitude and phase variables, defined by $\sqrt{N}e^{i\theta}$. Let us first discuss the equilibrium properties we expect in terms of the number of particles $N$ and the phase $\theta$. Since the transformation to $N$ and $\theta$ has a jacobian equal to zero, we can just substitute it into Eq. (A9), to obtain the equilibrium distribution in terms of the number of particles. It is given by

$$P[N; t \to \infty] \propto \exp\left\{-\beta(\mu_c - \mu)N\right\}. \tag{A10}$$

We can easily check that the average number of particles in the single mode is in equilibrium given

$$\langle N \rangle \equiv \frac{\int_0^\infty dN\ N P[N; t \to \infty]}{\int_0^\infty dN\ P[N; t \to \infty]} = [\beta(\mu_c - \mu)]^{-1}. \tag{A11}$$

This is precisely the Bose distribution for $\beta(\mu_c - \mu) \ll 1$. If we do not apply the 'classical' approximation to the fluctuation-dissipation theorem, as in Eq. (A2), but use the exact relation

$$iR = -\frac{1}{2}\hbar\Sigma^K\left(1 + 2N(\mu_c)\right)^{-1}, \tag{A12}$$

instead, we find the Bose distribution as the equilibrium number of particles, as expected. For the description of a single-mode Bose-Einstein condensate Eq. (A2) is in general a good approximation, since $\mu$ is very close to $\mu_c$ below the critical temperature. Let us now try to derive the Fokker-Planck equation for $N$ and $\theta$. We can do this by substitution of $\phi = \sqrt{N}e^{i\theta}$ into the action in the exponent of Eq. (A1). Since the jacobian of the transformation is equal to one, we simply have

$$P[N, \theta; t] = \int_{\theta(t)=\theta}^{N(t)=N} d[N]d[\theta] \exp\left\{\frac{i}{\hbar}S[N, \theta]\right\}, \tag{A13}$$

with an action equal to

$$S[N, \theta] = \int_{t_0}^{t} dt' \left( \frac{2N(t')}{\hbar\Sigma^K}\left(\hbar\dot{\theta}(t') + \mu_c - \mu\right)^2 + \frac{\hbar}{2\Sigma^K N(t')}\left(\dot{N}(t') + \frac{2R}{\hbar}N(t')\right)^2 \right). \tag{A14}$$

Naively, we could derive the Fokker-Planck equation from the above path-integral expression by going through the same steps as before. If we again apply normal ordering to the hamiltonian with respect to $N$ and $\theta$ and their conjugate momenta, the Fokker-Planck equation reads

$$\begin{aligned}\frac{\partial P[N, \theta; t]}{\partial t} &= \left(\frac{i\Sigma^K}{2}\frac{\partial^2}{\partial N^2}N + \frac{2R}{\hbar}\frac{\partial}{\partial N}N\right) P[N, \theta; t] \\ &+ \left(\frac{i\Sigma^K}{8N}\frac{\partial^2}{\partial\theta^2} + \frac{1}{\hbar}\frac{\partial}{\partial\theta}(\mu_c - \mu)\right) P[N, \theta; t].\end{aligned} \tag{A15}$$

This Fokker-Planck equation is however incorrect, since it is easily seen that the equilibrium distribution in Eq. (A10) is not a solution of this Fokker-Planck equation. The fact that the Fokker-Planck equation in Eq. (A15) turns out to be incorrect has to do with the fact that we have normal ordered the hamiltonian in terms of the variables $N$ and $\theta$. Although normal ordering of the hamiltonian $H[p_\phi, \phi; p_{\phi^*}, \phi^*]$ did give the correct results, this, however, does not imply that we also have to normal order the hamiltonian in terms of $N$ and $\theta$. Let us therefore proceed more carefully, and rewrite the Fokker-Planck equation in Eq. (A8) for $P[\phi^*, \phi; t]$ in terms of $N$ and $\theta$. With the use of the chain rule for differentiation it is easy to show that, for a general function $f$

$$\frac{\partial f}{\partial \phi^*} = \sqrt{N} e^{i\theta} \frac{\partial f}{\partial N} + \frac{i}{2\sqrt{N}} e^{i\theta} \frac{\partial f}{\partial \theta}, \tag{A16}$$

with the complex conjugate expression for $\partial f / \partial \phi$. Substitution of this result in the Fokker-Planck equation in Eq. (A8) yields the Fokker-Planck equation for $P[N, \theta; t]$. It is given by

$$\begin{aligned} \frac{\partial P[N,\theta;t]}{\partial t} = & \left( \frac{i\Sigma^{\rm K}}{2} \frac{\partial}{\partial N} N \frac{\partial}{\partial N} + \frac{2R}{\hbar} \frac{\partial}{\partial N} N \right) P[N,\theta;t] \\ & + \left( \frac{i\Sigma^{\rm K}}{8N} \frac{\partial^2}{\partial \theta^2} + \frac{1}{\hbar} \frac{\partial}{\partial \theta} (\mu_{\rm c} - \mu) \right) P[N,\theta;t]. \end{aligned} \tag{A17}$$

Comparison of the Fokker-Planck equations in Eqs. (A15) and (A17) shows that in Eq. (A15) we have misinterpreted the noise on $N(t)$ as an Ito process, whereas in the correct Fokker-Planck equation in Eq. (A17) we are clearly dealing with a Stratonovich process. Note that the same conclusion can also be reached by determining the equation of motion of $\langle N \rangle(t)$ from a variation of the action $S[N, \theta]$.

From the action in Eq. (A14) we can read of the Langevin equations for $N$ and $\theta$. The Langevin equation for $N(t)$ is given by

$$\begin{aligned} \dot{N}(t) &= -\frac{2R}{\hbar} N(t) + 2\sqrt{N(t)}\eta(t'); \\ \langle \eta(t')\eta(t) \rangle &= \frac{i\Sigma^{\rm K}}{4} \delta(t'-t), \end{aligned} \tag{A18}$$

and the Langevin equation for $\theta$ reads

$$\begin{aligned} \hbar \dot{\theta}(t) &= \mu - \mu_{\rm c} + \frac{\nu(t)}{\sqrt{N(t)}}; \\ \langle \nu(t')\nu(t) \rangle &= \frac{i\hbar^2 \Sigma^{\rm K}}{4} \delta(t'-t). \end{aligned} \tag{A19}$$

From the above discussion we thus conclude that we have to interpret the noise in the Langevin equation in Eq. (A18) for the number of particles $N(t)$ as a Stratonovich process, to achieve the correct equilibrium distribution. This is the main conclusion of this appendix. It is straightforward to show that the above discussion generalizes to the case of a multi-mode description of the condensate in terms of the complex field $\phi(\mathbf{x}, t)$. When we make the transformation to density and phase variables by setting $\phi = \sqrt{\rho} e^{i\theta}$, we again have to be careful, and interpret the multiplicative noise that enters the equation of motion for the density $\rho(\mathbf{x}, t)$ as a Stratonovich process.

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## FIGURES

FIG. 1. The solid line gives the condensate fraction for a $^{87}$Rb gas of $2 \times 10^6$ atoms in an isotropic trap with $\omega_0/2\pi = 10$ Hz. The dashed line corresponds to the ideal Bose gas in the thermodynamic limit.

FIG. 2. The dimensionless quantity $|\beta\hbar\Sigma^{\rm K}(\mathbf{x})|$. At $|\mathbf{x}| = R_{\rm TF}$ there is a divergence, when the collision integral in Eq. (62) is calculated in the Thomas-Fermi limit. This is indicated by the dashed line. The calculation is performed for $T = 0.5T_{\rm BEC}$ with the same parameters as in Fig. 1

FIG. 3. Frequencies for both the monopole and quadrupole mode as a function of the temperature. The dashed lines indicate the zero-temperature results found by Stringari [40], i.e., $\omega_{\rm quad} = \sqrt{2}\omega_0$ and $\omega_{\rm mono} = \sqrt{5}\omega_0$. The plot starts at $T = 0.8T_{\rm BEC}$ since for smaller temperatures the deviation from these values is neglegible. The parameters are the same as in Fig. 1.

FIG. 4. The damping rate for both the quadrupole and monopole mode for a condensate in an isotropic trap, which is the same for small damping in our variational approximation. The inset shows the dimensionless parameter $|\beta\hbar\Sigma^{\rm K}|$. The parameters are the same as in Fig. 1.

FIG. 5. The complex $\omega$-plane. In our expression for the frequency and damping rate of the $m = 2$ quadrupole mode in Eq. (105) the complex frequencies lie on a circle of radius $\sqrt{2}\omega_r$. The experimental points taken from Ref. [37] are also shown.

FIG. 6. Fit to the damping rate of the quadrupole mode, as measured by Jin *et al.* [37]. The solid line shows the fit and the experimental points are taken from Ref. [37]. The inset shows the value of $|\beta\hbar\Sigma^{\rm K}|$, as calculated from this fit with Eqs. (101), (102) and (105).

FIG. 7. The frequency of the quadrupole mode as a function of temperature, calculated with Eq. (105), by using the fit shown in Fig. 6. The dashed line show the $T = 0$ frequency for a variable number of condensate atoms. The experimental points taken from Jin *et al.* [37] are also shown.

FIG. 8. The condensate energy as a function of the condensate width in the gaussian approximation for three different values of the condensate atom number. If the number of atoms in the condensate is larger than $N_{\rm max}$ the condensate is unstable, otherwise a metastable condensate is possible.

FIG. 9. (a) Growth-collapse curves of a $^7$Li condensate, and (b) their averages. The colored lines in (a) display the number of condensate atoms for solutions of the Langevin equation for $q_j(t)$, coupled to the stochastic rate equation for $N_{\rm c}(t)$, for three different realizations of the noise. In (b) the red line correspond to an average over 5 realizations, the green line to 10, and the blue line to an average over 1000 different realizations of the noise. The simulations are done for a thermal cloud of 70000 atoms with $T = 170$ nK, and $A = 4$.

FIG. 10. Simulations of the experiment performed by Gerton *et al.* [17]. The results of the simulations are denoted by red triangles, the experimental data is shown as black circles. In (a) and (b) each data-point of the simulations is an average over 5 runs, as in the experiment. For (c) 10 runs per point were done. The error bars in both the experimental data, and the data obtained by the simulations, denote the uncertainty in the mean.