Summary

Solving optimal stopping problems driven by Lévy processes has been a challenging task and has found many applications in modern theory of mathematical finance. For example situations in which optimal stopping problems typically arise include the problems of finding the arbitrage-free price of the *American put (call) option* and determining an optimal *bankruptcy level* in the problem of endogenous default.

The main concern in pricing the American put (call) option lies in finding the critical value of the stock price process below (above) which the option is exercised. In the case of endogenous default, the problem mainly deals with finding an optimal bankruptcy level of a firm which keeps a constant profile of debt and chooses its default level endogenously, to maximize the equity value. In the context of the theory of optimal stopping, the arbitrage-free price of the American put (call) option and the equity value of the defaultable firm correspond to the value function of an optimal stopping problem while the critical value of the stock price process and the optimal bankruptcy level correspond to the optimal stopping boundary.

In general, optimal stopping problems are two-dimensional in the sense that they consist of finding the value function and the optimal boundary simultaneously; that is to say that the value function can be seen as a function of an unknown stopping boundary. Thus, from an analytical point of view, solving the problem is difficult.

A major technique that has been widely used in the theory of optimal stopping problems driven by diffusion processes is the *free boundary* formulation for the value function and the optimal boundary. The free boundary formulation consists primarily of a partial differential equation and (among other boundary conditions) the *continuous* and *smooth pasting* conditions used to determine the unknown boundary and specify the value function. The first condition requires the value function to be continuous at the boundary while the second condition imposes a C^1 smoothness of the value function at the boundary. Depending on the nature of the problem and the sample paths of the Lévy process, the smooth pasting condition may break down. As will be shown in this thesis, this phenomenon can happen to be the case when the Lévy process has paths of bounded variation. As a result, for this type of Lévy processes, the continuous pasting condition appears to be the only criterion for choosing the boundary. Thus, a better understanding of the appropriate choice of pasting conditions to determine the boundary can play an important role in the theory.

Much of this thesis is concerned with solving optimal stopping problems driven by Lévy processes in a general setting. The aim is to propose a framework by which semi-explicit solutions can be obtained. Using such solutions, we give sufficient and necessary conditions for the continuous and smooth pasting conditions to occur in the considered problem. In this thesis we give examples of different cases.

For finite expiration date, we focus on the American put option problem where the evolution of the stock price is driven by a bounded variation Lévy process. The problem is solved by using a change of variable formula with local time on curves for bounded variation Lévy processes. Combining this with Itô-Doob-Meyer decomposition of the value process of the American put option problem into martingale and potential processes, we show that the optimal stopping boundary can be characterized as a solution to a nonlinear integral equation. Taking account of the continuous pasting condition, we show using the change of variable formula that such integral equation admits, under some conditions, a unique solution for the optimal boundary. By the uniqueness of such solution, we show that the value function of the American put option problem and the optimal stopping boundary represent the unique pair solution to a free boundary problem of parabolic integro-differential type.

In the case of infinite maturity, we give an optimal solution to a perpetual optimal stopping problem for a general class of payoff functions under Lévy processes. The solution is obtained by reducing the stopping problem into an averaging problem. Using solution to the latter problem, we obtain using the *Wiener-Hopf* factorization a *fluctuation identity* of Lévy processes. This fluctuation identity relates the solution of the averaging problem with the expected value of discounted payoff function up to a first passage time. Based on the identity, we show that if the solution to the averaging problem has a certain monotonicity property then an optimal solution to the stopping problem can be described in terms of such a monotone function, and the boundary is given by a level at which the function changes its sign. Using such solution, we are able to show that the smooth pasting condition is satisfied if and only if the optimal stopping boundary is regular for interior of the stopping region for the Lévy process. A number of problems are studied in detail, in particular for polynomial payoff and the arbitrage-free pricing of the American put and call options.

For the problem of endogenous default, we show that within a particular class of models, the issue of choosing an optimal bankruptcy level can be dealt with analytically and numerically when the underlying source of randomness for the value of the firm's asset is replaced by a general *Lévy process with no positive jumps*. By working with the latter process, we bring to light a new phenomenon, namely that, depending on the nature of the small jumps, the optimal bankruptcy level may be determined by a continuous pasting condition as opposed to the usual smooth pasting condition. Moreover, we are able to prove the optimality of the bankruptcy level according to the appropriate choice of pasting conditions.

Most of the main results presented in this thesis are verified by means of numerical examples for Lévy processes having one-sided jumps.