Chapter 7

Evaluating Scale Functions of Spectrally Negative Lévy Processes[1]

Abstract

In this chapter we discuss a robust numerical method to compute the scale function $\{W^{(q)}(x) : q \geq 0, x \in \mathbb{R}_+\}$ of a general spectrally negative Lévy process $(X, \mathbb{P})$. The method is based on the Esscher transform of measure $\mathbb{P}^\nu$ under which X is taken and the scale function is determined. This change of measure makes it possible for the scale function to be bounded and hence makes numerical computation easier, fast and stable. Working under the measure $\mathbb{P}^\nu$ and using the method of Abate and Whitt [1] and Choudhury et al. [31], we give a fast stable numerical algorithm for the computation of $W^{(q)}(x)$ for $q \geq 0$.

7.1 Introduction

In literature, we have seen that many fluctuation identities associated with the problem of first-exit from positive half-line or finite interval of a (reflected) spectrally negative Lévy process $(X, \mathbb{P})$ can be written in terms of the so called q-scale function $\{W^{(q)}(x) : q \geq 0, x \in \mathbb{R}_+\}$. For literature review, we refer to Bertoin [14], [15], Avram et al. [7], Lambert [75], Rogers [107], and the literature therein. In connection with pricing American put and Russian options driven by spectrally negative Lévy processes, the rational price of these options appears to be some functional of this function, see for instance Avram et al. [7]. In mathematical insurance, this function appears in the problem of finding optimal dividend payments, see for instance Avram et al. [8]. In credit risk theory, the scale function plays an important role in determining an endogenous bankruptcy level V_B as well as in assessing the optimality of V_B, see for instance Kyprianou and Surya [73] (see also Chapter 5 for more details). Working under a completely general spectrally negative Lévy process, it was shown recently

[1]Submitted for publication to *Journal of Applied Probability*.

in Kyprianou and Surya [73] that not only the analytical treatment of the optimal bankruptcy level is possible, but also the smooth pasting condition used by Leland and Toft [76] and Hilberink and Rogers [58] for optimality criterion for choosing V_B can be verified both analytically and numerically. It is also worth noting that the scale function appears to be an important factor in Queuing theory (see for instance Dube et al. [36]) and the theory of continuous state branching processes (we refer to Chapter 10 in Kyprianou [69] for details).

For some spectrally negative Lévy processes, the scale function is available in explicit form. Typical examples are standard Brownian motion, α-stable processes with $\alpha \in (0, 2)$, jump diffusion process with exponential negative jumps and Compound Poisson processes. See for instance Bertoin [14], [15] and Kyprianou [69] for more details. Due to complexity of the Laplace exponent of the Lévy process, the scale function is not available in explicit form in general. An example for this is a spectrally negative tempered stable process with index less than two. Thus, numerical inversion of Laplace transform can be used to compute the scale function numerically.

Quite recently, a fast and stable numerical inversion for the scale function $W^{(0)}(x)$ of a spectrally negative Lévy process containing Brownian motion whose Laplace exponent satisfying $\kappa'(0+) > 0$ was proposed by Rogers [107]. We will come back to this point later on Section 3. For the case $q \geq 0$ and X contains no Brownian motion, the issue of how to evaluate the q-scale function $W^{(q)}(x)$ was not addressed. As will be shown later, there is a problem on the computation. The problem is that for each $q > 0$ the q-scale function $W^{(q)}(x)$ is exponentially unbounded at infinity under the measure $\mathbb{P}$ and hence numerical inversion for producing $W^{(q)}(x)$ could be unstable. We try to overcome this problem by a change of measure using the Esscher transform $\mathbb{P}^\nu$ under which the scale function could be bounded. Working under the measure $\mathbb{P}^\nu$ and using the method of Abate and Whitt [1] and Choudhury et al. [31], we give a fast stable numerical algorithm for the computation of $W^{(q)}(x)$ for $q \geq 0$.

The organization of this chapter is as follows. In Section 2, we briefly discuss spectrally negative Lévy processes and its exponential change of measure $\mathbb{P}^\nu$. Section 3 defines the q-scale function of a general spectrally negative Lévy process. In Section 4 we discuss numerical methods for the computation of the q-scale function. Numerical examples of the computation are given in Section 5. Finally, we provide the MATLAB program code for the computation in Section 6.

7.2 Spectrally negative Lévy processes

In this chapter, we consider a Lévy process X having the canonical decomposition

$$X_t \;=\; \mu t + \sigma B_t + J_t^{(-)},$$

where $B = \{B_t, t \geq 0\}$ is a standard Brownian motion and $J^{(-)} = \{J_t^{(-)}, t \geq 0\}$ is a non-Gaussian Lévy process, having no positive jumps, independent of B. This class

of Lévy processes has a great interest from theoretical point of view, because they are processes for which fluctuation theory takes the nicest form and can be developed explicitly to its fuller extent. The degenerate case when X is either the negative of a subordinator or a deterministic drift has no interest and will not be discussed throughout. What we shall say here is based on (for the most part) Chapter VII in Bertoin [13].

The law of the Lévy process started at zero will be denoted by $\mathbb{P}$ (with the associated expectation operator $\mathbb{E}$). Since X has no positive jumps, the moment generating function $\theta \mapsto \mathbb{E}(e^{\theta X_t})$ exists for all $\theta \geq 0$ and is given by

$$\mathbb{E}(e^{\theta X_t}) \;=\; e^{t\kappa(\theta)},$$

where the function $\kappa : [0, \infty) \to (-\infty, \infty)$, also called as the Laplace exponent of X, is defined by

$$\kappa(\theta) \;=\; \mu\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{(-\infty,0)} \big(e^{\theta y} - 1 - \theta y \mathbf{1}_{\{y>-1\}}\big)\Pi(dy). \tag{7.2.1}$$

It is easily seen that κ is zero at the origin and is strictly convex with $\lim_{\theta\uparrow\infty} \kappa(\theta) = \infty$. Next we denote by $\Phi(\alpha)$ the largest solution of the equation

$$\kappa(p) \;=\; \alpha \quad \text{for all } \alpha \geq 0.$$

Note that due to the convexity of κ, there exists at most two roots for a given α and precisely one root when $\alpha > 0$. The asymptotic behaviour of X can be determined from the sign of $\kappa'(0+)$, the right-derivative of κ at zero. X drifts to $-\infty$, oscillates or drifts to $+\infty$ according to whether $\kappa'(0+)$ is negative, zero or positive. See for instance Kyprianou and Palmowski [70] for more details.

It is worth mentioning that under the *Esscher transform* $\mathbb{P}^\nu$ defined by

$$\frac{d\mathbb{P}^\nu}{d\mathbb{P}}\Big|_{\mathcal{F}_t} \;=\; e^{\nu X_t - \kappa(\nu)t} \quad \text{for all } \nu \geq 0, \tag{7.2.2}$$

the Lévy process $(X, \mathbb{P}^\nu)$ is still a spectrally negative Lévy process. The Laplace exponent of X under the measure $\mathbb{P}^\nu$ has changed to

$$\begin{aligned} \kappa_\nu(\theta) \;&= \frac{1}{t}\log \mathbb{E}^\nu\big(e^{\theta X_t}\big) \\ &= \frac{1}{t}\log \mathbb{E}\big(e^{(\theta+\nu)X_t - \kappa(\nu)t}\big) \\ &= \kappa(\theta+\nu) - \kappa(\nu). \end{aligned} \tag{7.2.3}$$

To each $\nu \geq 0$, we will denote by $\mathbb{P}^\nu_x$ the translation of $\mathbb{P}^\nu$ under which $X_0 = x$.

7.3 Scale functions

This section discusses the so called scale functions. (See Bertoin [13], [14] and [15] for the origin of this function). This function features invariably all known identities

for Laplace transforms of first-exit and overshoots for spectrally negative (reflected) Lévy processes. See the aforementioned literature for more details.

Definition 7.3.1 Let $q \geq 0$ and define $\Phi_\nu(q)$ as the largest root of $\kappa_\nu(\theta) = q$.

Definition 7.3.2 (q-Scale function) For a given spectrally negative Lévy process with Laplace exponent (7.2.1), there exists for every $q \geq 0$ a right-continuous function $W^{(q)} : [0, \infty) \to [0, \infty)$, called the q-scale function, with Laplace transform given by

$$\int_0^\infty e^{-\lambda x} W^{(q)}(x) dx \;=\; \frac{1}{\kappa(\lambda) - q} \quad \text{for} \;\; \lambda > \Phi(q), \tag{7.3.1}$$

where $\Phi(q)$ was defined in the previous section. We shall write for short $W^{(0)} = W$. Furthermore, we refer to $W_\nu^{(q)}(x)$ the scale function under the measure $\mathbb{P}^\nu$.

We assume throughout the remaining of this chapter that the Lévy measure Π has no atoms. It turns out that the smoothness properties of the q-scale functions $W^{(q)}(x)$ are very closely related to the roughness of the underlying paths of the associated Lévy process. If X has paths of unbounded variation or bounded variation and the Lévy measure Π has no atoms, it is known that the q-scale function $W^{(q)}(x)$ is continuously differentiable, see for instance Lambert [75], Chan and Kyprianou [28], and the literature therein for more details.

Lemma 7.3.3 (Asymptotic behaviour) *Suppose that either ($q > 0$) or ($q = 0$ and $\kappa'(0+) > 0$). Then the scale function $\{W_{\Phi(q)}(x) : q \geq 0, x \in \mathbb{R}\}$ of X taken under the measure $\mathbb{P}^{\Phi(q)}$ is increasing and bounded from above by $1/\kappa'(\Phi(q))$. However, when X is taken under the measure $\mathbb{P}$, the q-scale function $W^{(q)}(x)$ is given by*

$$W^{(q)}(x) \;=\; e^{\Phi(q)x} W_{\Phi(q)}(x), \tag{7.3.2}$$

and hence has the asymptotic $W^{(q)}(x) \sim \frac{e^{\Phi(q)x}}{\kappa'(\Phi(q))}$ as $x \to \infty$.

Remark 7.3.4 In the case of $q = 0$ and $\kappa'(0+) = 0$, the scale function $W_{\Phi(q)}(x)$ is increasing and unbounded at infinity and hence the numerical computation may produce instability in the tail (as $x \to \infty$).

Proof From (7.3.1) and (7.2.3), it is clear to see for all $q \geq 0$ that

$$\int_0^\infty e^{-\lambda x} W_{\Phi(q)}(x) dx \;=\; \frac{1}{\kappa_{\Phi(q)}(\lambda)} \;=\; \frac{1}{\kappa(\lambda + \Phi(q)) - \kappa(\Phi(q))}, \tag{7.3.3}$$

for $\lambda > 0$. It is clear following (7.3.3) and using a standard Tauberian theorem that

$$\lim_{x \uparrow \infty} W_{\Phi(q)}(x) \;= \lim_{\lambda \downarrow 0} \frac{\lambda}{\kappa(\lambda + \Phi(q)) - \kappa(\Phi(q))} \;=\; \frac{1}{\kappa'(\Phi(q))} \;<\; \infty. \tag{7.3.4}$$

Taking into account of (7.3.4), we see using integration by parts that the Laplace-Stieltjes transform of the scale function $W_{\Phi(q)}(x)$ is given by

$$\int_0^\infty e^{-\lambda x} dW_{\Phi(q)}(x) \;=\; \frac{\lambda}{\kappa(\lambda+\Phi(q)) - \kappa(\Phi(q))},$$

where $dW_{\Phi(q)}$ denotes the Stieltjes measure associated to the scale function $W_{\Phi(q)}(x)$ which gives the mass $W_{\Phi(q)}(0)$ zero value. Since the Lévy measure Π has no atoms so that $W_{\Phi(q)}$ is continuously differentiable, it is clear that the scale function $W_{\Phi(q)}(x)$ is increasing and bounded from above by $\frac{1}{\kappa'(\Phi(q))}$.

The claim that the q-scale function $W^{(q)}(x)$ is given by the expression (7.3.2) can be verified by applying Laplace transform to the both sides of (7.3.2). As a result of (7.3.2), it is clear by Tauberian theorem that the q-scale function $W^{(q)}(x)$ has the asymptotic $W^{(q)}(x) \sim \frac{e^{\Phi(q)x}}{\kappa'(\Phi(q))}$ as $x \to \infty$. Thus, our claim is then proved. □

Remark 7.3.5 Following (7.3.1), it is straightforward to check that

$$W_\nu^{(q)}(x) \;=\; e^{-\nu x} W^{(q+\kappa(\nu))}(x) \quad \text{for all } \nu \geq 0 \text{ and } q \geq -\kappa(\nu). \tag{7.3.5}$$

To verify the relation (7.3.5), apply Laplace transform to the both sides of (7.3.5).

7.4 Evaluating scale functions

In this section we discuss numerical algorithms based on a univariate version of Rogers' method [107], and the methods of Abate and Whitt [1] and Choudhury et al [31] for the computation of the q-scale function $W^{(q)}(x)$. In order to apply these algorithms, we need the result of Lemma 7.3.3 for the computation.

7.4.1 A method based on Rogers' approach

In [107], Rogers gives a fast stable numerical algorithm (which is a variant of Abate and Whitt [1]) to invert the bivariate Laplace transform

$$\int_0^\infty \int_0^\infty e^{-(\lambda x+qt)} \mathbb{P}(\tau_{-x}^- \geq t) dt dx \;=\; \frac{\lambda - \Phi(q)}{(\kappa(\lambda) - q)\lambda\Phi(q)} \tag{7.4.1}$$

of the probability of first passage time $\tau_x^- = \inf\{t > 0 : X_t < -x\}$ below a level $-x \leq 0$ of a spectrally negative Lévy process having Gaussian component.

Compared to the problem (7.4.1), the inversion problem (7.3.1) is different on its own. Unlike the inversion problem (7.4.1) the problem (7.3.1) is univariate. However, the algorithm for inverting the Laplace transform (7.4.1) could be adapted to accommodate the inversion problem (7.3.1).

By identifying the fact that $\mathbb{P}(\tau^-_{-x} \geq t) = \mathbb{P}(-\underline{X}_t \leq x)$, we can now reduce the problem (7.4.1) into a univariate case as

$$\int_0^\infty e^{-\lambda x}\mathbb{P}(-\underline{X}_{\mathbf{e}_q} \leq x)dx \;=\; \frac{q}{\Phi(q)}\frac{1}{(\kappa(\lambda)-q)} - \frac{q}{\lambda(\kappa(\lambda)-q)}. \tag{7.4.2}$$

Note that the expression in (7.4.2) is based on the Wiener-Hopf factorization formula (2.3.3) (see Section 3 of Chapter 2 for details) for spectrally negative Lévy processes and is obtained by applying integration by part in (2.3.3). Hence, as a result of inverting the Laplace transform (7.4.2), the q-scale function $W^{(q)}(x)$ in (7.3.1) might be deduced from the relation (7.4.2).

In the special case of $q = 0$ and $\kappa'(0+) > 0$, we see that the result of inverting (7.4.2) coincides up to multiplicative constant with that of (7.3.1). In this case, a univariate version of Rogers' algorithm might be applied to get the scale function $W^{(0)}(x)$. Due to the presence of the second term on the right-hand side of the equation (7.4.2), it is not straightforward, however, how to get $W^{(q)}(x)$ from (7.4.2).

Suppose that the problem (7.4.2) could be solved numerically using the univariate version of Rogers' algorithm to produce the distribution function $\mathbb{P}(-\underline{X}_{\mathbf{e}_q} \leq x)$. The goal is to find an expression for the q-scale functions $\{W^{(q)}(x), q \geq 0\}$ in terms of the function $\mathbb{P}(-\underline{X}_{\mathbf{e}_q} \leq x)$. To solve this problem, we can use the Esscher transform $\mathbb{P}^{\Phi(q)}$ (7.2.2) to first compute the scale function $W_{\Phi(q)}(x)$ and then use the transformation (7.3.2) to obtain $W^{(q)}(x)$ for $q \geq 0$. (Note that $W_{\Phi(q)}(x)$ plays the role of the scale function $W^{(q)}(x)$ for $q = 0$ when X is taken under the measure $\mathbb{P}^{\Phi(q)}$.) Under this measure, we obtain after some calculations that

$$W_{\Phi(q)}(x) \;=\; \frac{\Phi(q)}{q}e^{-\Phi(q)x}\mathbb{P}(-\underline{X}_{\mathbf{e}_q} \leq x) + \frac{\Phi(q)^2}{q}\int_0^x e^{-\Phi(q)y}\mathbb{P}(-\underline{X}_{\mathbf{e}_q} \leq y)dy. \tag{7.4.3}$$

To see that this is the correct expression for the scale function $W_{\Phi(q)}(x)$, take Laplace transform on both sides of the above expression and use (7.4.2) to get (7.3.3), i.e.,

$$\int_0^\infty e^{-\lambda x}W_{\Phi(q)}(x)dx \;=\; \frac{1}{(\kappa(\lambda+\Phi(q))-q)} \quad \text{for } \lambda > 0 \text{ and } q \geq 0. \tag{7.4.4}$$

Hence, following (7.4.3) and (7.3.2) the q-scale function $W^{(q)}(x)$ is finally given by

$$W^{(q)}(x) \;=\; \frac{\Phi(q)}{q}\mathbb{P}(-\underline{X}_{\mathbf{e}_q} \leq x) + \frac{\Phi(q)^2}{q}\int_0^x e^{-\Phi(q)(y-x)}\mathbb{P}(-\underline{X}_{\mathbf{e}_q} \leq y)dy.$$

Note that the algorithm given in Rogers [107] can be used to handle the case where a spectrally negative Lévy process contains a Gaussian component ($\sigma > 0$). To deal with a more general class of spectrally negative Lévy processes, we need to modify the algorithm based on the method of Abate and Whitt [1] and Choudhury et al. [31].

7.4.2 A method based on Abate and Whitt [1] and Choudhury et al. [31]

In this section we discuss a robust numerical method based on Abate and Whitt [1] and Choudhury et al. [31] for inverting univariate Laplace transform which works with a good accuracy for bounded functions. In this thesis, we use this method to invert the Laplace transform (7.4.4) to produce the function $\{W_{\Phi(q)}(x), q \geq 0\}$ and then apply the transformation (7.3.2) to obtain the q-scale function $\{W^{(q)}(x), q \geq 0\}$.

To start with, let f be a real-valued function (not necessarily a probability density) defined on the positive real line. For such a function f, it is often convenient to work with the Laplace transform

$$\widehat{f}(\lambda) \,=\, \int_0^\infty e^{-\lambda t} f(x) dx, \tag{7.4.5}$$

where λ is a complex variable for which the integral exists. The standard inversion integral for the Laplace transform $\widehat{f}$ is the *Bromwich contour integral*, which can also be expressed as the integral of a real-valued function of a real variable by choosing a specific contour by any vertical line $\lambda = a_1$ such that $\widehat{f}(\lambda)$ has no singularities on or to the right of the vertical line. By applying this integral, we obtain

$$f(x) \,=\frac{1}{2\pi i} \int_{a_1 - i\infty}^{a_1 + i\infty} e^{\lambda x} \widehat{f}(\lambda) d\lambda \,=\, \frac{e^{a_1 x}}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda x} \widehat{f}(a_1 + i\lambda) d\lambda. \tag{7.4.6}$$

For some Laplace transforms $\widehat{f}$, analytic expressions for f are available explicitly, see for instance Obberhettinger [93]. When the transform can not be inverted analytically, the function f can be approximated by means of numerical approximation.

Several numerical inversion algorithms have been proposed by several authors. The fast and stable one is given by Abate and Whitt [1] and Choudhury et al. [31]. Following Abate and Whitt [1], we use the trapezoidal rule to approximate the integral in (7.4.6) and analyze the corresponding discretization error using the *Poisson summation formula*. The trapezoidal rule approximates the integral of a function g over the bounded interval $[c, d]$ by the integral of the piecewise linear function obtained by connecting the $n + 1$ evenly spaced points $g(c + kh)$, $0 \leq k \leq n$ where $h = (d - c)/n$. Using the trapezoidal rule, we have

$$\int_c^d g(x) dx \,\approx\, h\Big[\frac{g(c) + g(d)}{2} + \sum_{k=1}^{n-1} g(c + kh)\Big], \tag{7.4.7}$$

see page 51 in Davis and Rabinowitz [34]. The trapezoidal rule (7.4.7) also applies for the case $c = -\infty$ and $d = \infty$ with the following modification

$$\int_{-\infty}^{\infty} g(x) dx \,\approx\, h_1 \sum_{j=-\infty}^{\infty} g(j h_1), \tag{7.4.8}$$

where h_1 is a small positive constant. Applying (7.4.8) to (7.4.6) with step size $h_1 = \pi/x$, $x > 0$, and letting $a_1 = A_1/x$ at the same time, we get

$$f(x) \approx \frac{e^{A_1}}{2x} \sum_{j=-\infty}^{\infty} (-1)^j \widehat{f}((A_1 + ij\pi)/x). \tag{7.4.9}$$

The advantage of using this simple numerical procedure of trapezoidal rule is that it tends to work well for periodic and oscillating integrands since the errors tend to cancel out and the realized errors can be substantially less than from other alternative numerical procedure such as Simpson's rule. Moreover, the Poisson summation formula can be applied to obtain a convenient representation of the discretization error associated with the trapezoidal rule. Using the Poisson summation formula, the approximation (7.4.9) can be obtained for an integrable function g by

$$\sum_{j=-\infty}^{\infty} g(x + 2\pi j/h_2) = \frac{h_2}{2\pi} \sum_{j=-\infty}^{\infty} e^{-ijh_2 x} \phi(jh_2) \tag{7.4.10}$$

where h_2 is some positive constant and $\phi(u) = \int_{-\infty}^{\infty} e^{iux} g(x)dx$, the Fourier transform of g. In order to control the aliasing[2] error, we do *exponential damping*; that is, if f is our original function of interest, then we replace $g(x)$ by the function $f(x)e^{-a_1 x}\mathbf{1}_{[0,\infty)}(x)$. Then $\phi(\lambda) = \widehat{f}(a_1 - i\lambda)$, and the right-hand side of (7.4.10) can be expressed in terms of Laplace transform values:

$$\sum_{j=0}^{\infty} e^{-a_1(x+2\pi j/h_2)} f(x + 2\pi j/h_2) = \frac{h_2}{2\pi} \sum_{j=-\infty}^{\infty} e^{-ijh_2 x} \widehat{f}(a_1 - ijh_2). \tag{7.4.11}$$

In addition, if we let $h_2 = \pi/(xl_1)$, with $l_1 \geq 1$, and $a_1 = A_1/(2xl_1)$ in (7.4.11) we obtain

$$f(x) = \frac{e^{A_1/(2l_1)}}{2xl_1} \sum_{j=-\infty}^{\infty} e^{-ij\pi/l_1} \widehat{f}\Big(\frac{A_1}{2xl_1} - \frac{ij\pi}{l_1 x}\Big) - \overline{e}_\infty, \tag{7.4.12}$$

where the error $\overline{e}_\infty$ is given by

$$\overline{e}_\infty = \sum_{j=1}^{\infty} e^{-2jA_1} f\big((2j+1)x\big).$$

Comparing (7.4.11) with (7.4.9), we conclude that $\overline{e}$ is an explicit expression for the discretization error associated with the trapezoidal rule approximation. This discretization error can easily be bounded whenever f is bounded. For example if $|f(x)| \leq C$ for some $C > 0$ and all x ($C = 1$ if $f(x)$ is a probability distribution), then we have

$$|\overline{e}_\infty| \leq \frac{Ce^{-2A_1}}{(1 - e^{-2A_1})}. \tag{7.4.13}$$

[2]Aliasing means that the new function is constructed by adding a translated version of the original function

Therefore, an approximation of the function f (7.4.6) is given by

$$S_N(x) \;=\; \frac{e^{A_1/(2l_1)}}{2xl_1} \sum_{j=-N}^{N} e^{-ij\pi/l_1} \widehat{f}\Big(\frac{A_1}{2xl_1} - \frac{ij\pi}{l_1 x}\Big). \tag{7.4.14}$$

The raw value of S_N may not be a very good approximation, but by using *Euler summation* to smooth the values of the (nearly) alternating sums, we were able to obtain good accuracy. The approximation to $f(x)$ finally is given by

$$f(x) \;\doteq\; \sum_{n=0}^{M} 2^{-M} \binom{M}{n} S_{N+n}(x).$$

This is the formula we used in the thesis to invert numerically a univariate Laplace transform for the scale function $W_{\Phi(q)}(x)$ (the role of the scale function $W(x)$ under the measure $\mathbb{P}^{\Phi(q)}$) with Laplace transform given by (7.3.3).

7.5 Numerical examples

For our numerical examples, we consider four different Lévy processes. Firstly, we assume that X is generated by α-stable processes with Laplace exponent

$$\kappa(\lambda) \;=\; K\lambda^{\alpha}, \quad \text{for } \alpha \in (0, 2]. \tag{7.5.1}$$

Secondly, we consider jump diffusion processes where the jump component of X is contributed by a compound Poisson process having independent downward jumps with exponential $\exp(b)$ distribution occurring at the times of a Poisson process with rate a; that is to say that X has Laplace exponent

$$\kappa(\lambda) \;=\; \mathrm{d}\lambda + \frac{\sigma^2}{2}\lambda^2 - \frac{a\lambda}{b+\lambda}. \tag{7.5.2}$$

Thirdly, we consider X to be a one sided tempered stable process. In the general case ($\alpha \neq 1$ and $\alpha \neq 0$) the Laplace exponent of X is given by

$$\kappa(\lambda) \;= \mathrm{d}\lambda + \Gamma(-\alpha)\beta^{\alpha} C\Big\{\Big(1 + \frac{\lambda}{\beta}\Big)^{\alpha} - 1 - \frac{\lambda\alpha}{\beta}\Big\}. \tag{7.5.3}$$

(See Section 5.7.1 of Chapter 5 for further details.) Tempered stable processes were used in Section 5.7 of Chapter 5 for the computation of the arbitrage-free price of the perpetual American put and call options.

Finally, we consider the case where X is a gamma process perturbed by diffusion process. The Laplace exponent of X is described by

$$\kappa(\lambda) \;=\; \mathrm{d}\lambda + \frac{\sigma^2}{2}\lambda^2 - \psi(\lambda) \tag{7.5.4}$$

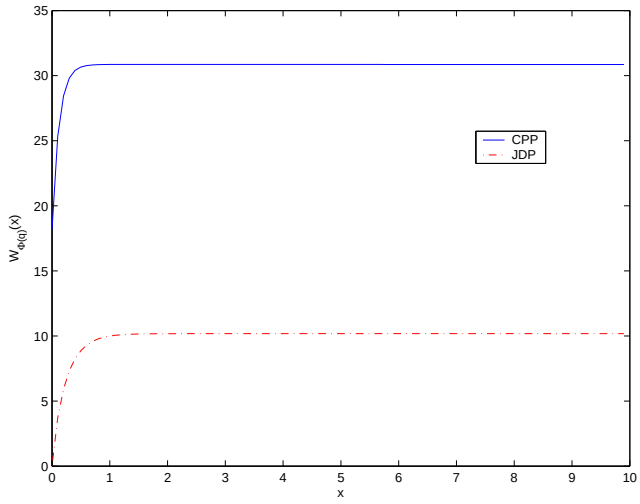

Figure 7.1: The shape of the scale function $W_{\Phi(q)}(x)$ for compound Poisson and jump diffusion processes. All the curves are bounded by $1/\kappa'(\Phi(q))$, equal to 30.8640 and 10.1787, respectively. From the plot, we notice that $W_{\Phi(q)}(0)$ is respectively positive for compound Poisson process and zero for jump diffusion process.

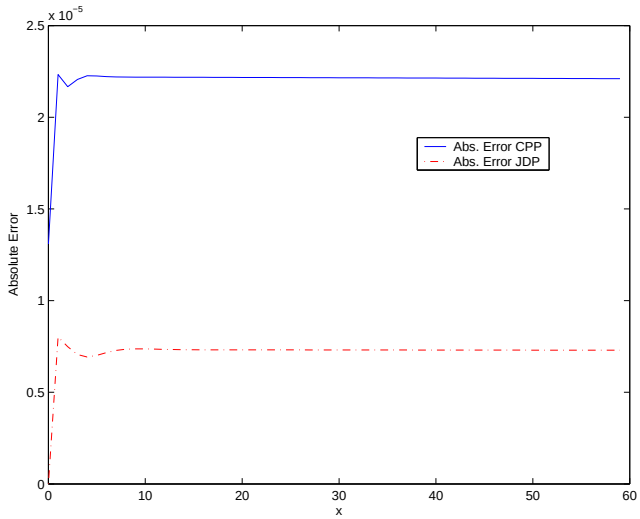

Figure 7.2: The absolute error $\left|\widehat{W}_{\Phi(q)}(x) - W_{\Phi(q)}(x)\right|$ between numerical and theoretical results for the scale function $W_{\Phi(q)}(x)$ for compound Poisson (CPP) and jump diffusion (JDP) processes. For CPP (resp. JDP) the error is bounded by 2.5664×10^{-5} (resp. 8.4639×10^{-6}). See inequality (7.5.5) for the error estimate.

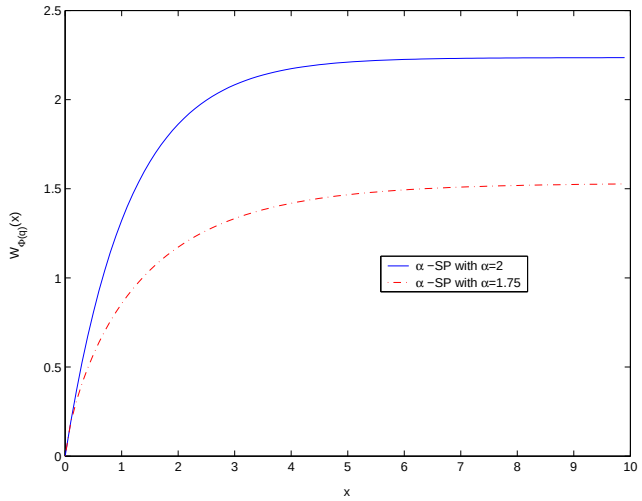

Figure 7.3: The shape of the scale function $W_{\Phi(q)}(x)$ of $\alpha-$ stable Lévy processes with indexes $\alpha = 2$ and $\alpha = 1.75$. All the curves are bounded by $1/\kappa'(\Phi(q))$, equal to 2.2361 and 1.5330, respectively. From the plot, we notice that $W_{\Phi(q)}(0)$ has zero value for both processes.

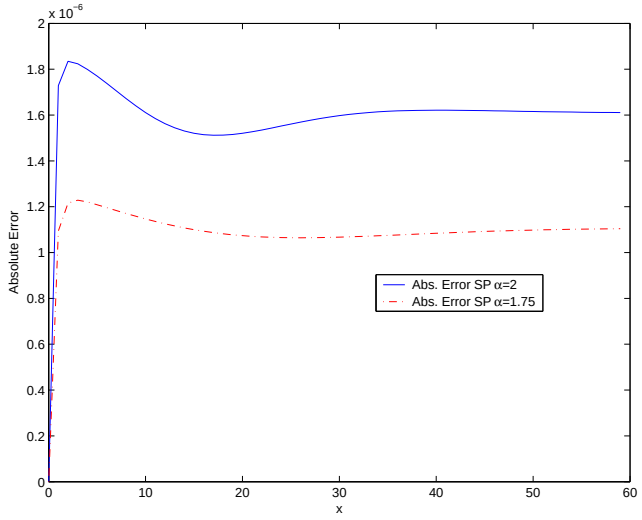

Figure 7.4: The absolute error $\left|\widehat{W}_{\Phi(q)}(x) - W_{\Phi(q)}(x)\right|$ between numerical and theoretical results for the scale function $W_{\Phi(q)}(x)$ of $\alpha-$ stable Lévy processes with indexes $\alpha = 2$ and $\alpha = 1.75$. For $\alpha = 2$ (resp. $\alpha = 1.75$) the error is bounded by 1.8594×10^{-6} (resp. 1.2747×10^{-6}). See inequality (7.5.5) for the error estimate.

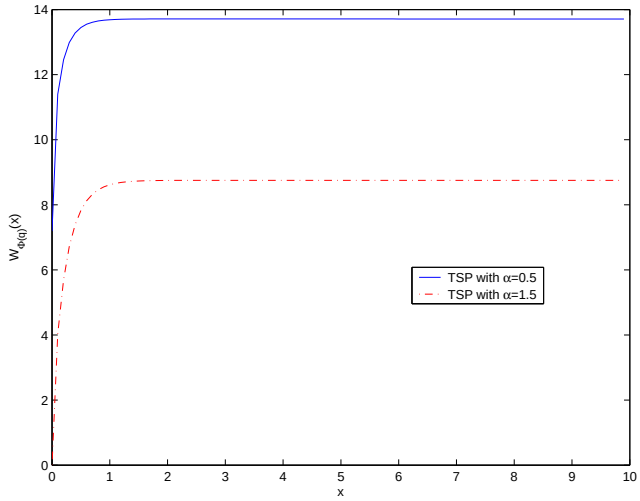

Figure 7.5: The shape of the scale function $W_{\Phi(q)}(x)$ of tempered stable Lévy processes with indexes $\alpha = 0.5$ and $\alpha = 1.5$. All the curves are bounded by $1/\kappa'(\Phi(q))$, equal to 13.7137 and 8.7532, respectively. From the plot, we notice that $W_{\Phi(q)}(0)$ is positive for $\alpha = 0.5$ and zero for $\alpha = 1.5$.

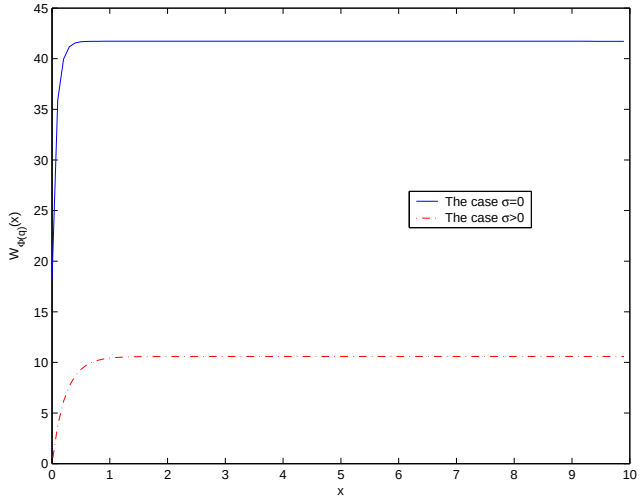

Figure 7.6: The shape of the scale function $W_{\Phi(q)}(x)$ of gamma process perturbed by diffusion process. All the curves are bounded by $1/\kappa'(\Phi(q))$, equal to 10.5823 (when $\sigma > 0$) and 41.7222 (when $\sigma = 0$), respectively. From the plot, we notice that $W_{\Phi(q)}(0)$ is positive when $\sigma = 0$ and zero when $\sigma > 0$.

where $\psi(\lambda)$ is the Laplace exponent of a gamma process S defined by

$$\psi(\lambda) \;=\; \int_0^\infty (1-e^{-\lambda x})ax^{-1}e^{-bx}dx \;=\; a\log\big(1+\frac{\lambda}{b}\big), \quad \text{for } a,b>0.$$

Note that the perturbed gamma process (7.5.4) is a slight generalization of the model studied by Dufresne and Gerber [38] and has been used in risk theory quite extensively. For related references, see for instance Dufresne and Gerber [38], Gerber [53], Gerber and Shiu [54], Yang and Zhang [119], and the literature therein.

For all computations, we fix some values of parameters: $N = 11$, $M = 9$, $A1 = 14.0$, $l1 = 1$; for jump diffusion and gamma processes we set $\sigma = 0.2$, d $= 0.055$, $a = 0.5$, $b = 9$; for α-stable process we set $K = 0.5$ for $\alpha = 2$ (a standard Brownian motion) and $K = 1$ for $\alpha = 1.75$. In the case where X is tempered stable process with $\alpha = 0.5$ (resp. $\alpha = 1.5$) we choose the relative frequency of downward jumps C to be 0.075 (resp. $C = 0.05$) and the jump rate β to be 2.5 (resp. $\beta = 2.5$).

The numerical results of the scale function $W_{\Phi(q)}(x)$ (7.3.3) for $q = 0.1$ are presented in Figures 7.1-7.6. We observe from these plots that all of the curves are increasing and bounded from above by $1/\kappa'(\Phi(q))$. These numerical outcomes support our theoretical results given previously in Lemma 7.3.3. In particular, for the case where X has path of bounded and unbounded variation we see respectively that $W_{\Phi(q)}(0) > 0$ and $W_{\Phi(q)}(0) = 0$, (see Section 6.4 of Chapter 6 for more discussions).

Using the explicit form of the scale function $W_{\Phi(q)}(x)$ of α-stable (7.5.1) and jump diffusion (7.5.2) processes, given in Section 6.4.1 of Chapter 6, we present in Figures 7.2 and 7.4 plots of the absolute error between the theoretical curve $W_{\Phi(q)}(x)$ and the numerical curve $\widehat{W}_{\Phi(q)}(x)$ produced by the proposed numerical method. Following (7.4.13) we observe that the absolute error is bounded by $\frac{1}{\kappa'(\Phi(q))}\frac{e^{-A_1}}{(1-e^{-A_1})}$, i.e.,

$$\big|\widehat{W}_{\Phi(q)}(x) - W_{\Phi(q)}(x)\big| \;\le\; \frac{1}{\kappa'(\Phi(q))}\frac{e^{-A_1}}{(1-e^{-A_1})}. \tag{7.5.5}$$

Since the (tuning) parameter A_1 was chosen relatively big (with $A_1 = 14$), we see that the absolute error is relatively small and hence our numerical method performs very good job in the computation of the scale function $W_{\Phi(q)}(x)$.

The final plot, Figures 7.5-7.6 show the shape of the scale function $W_{\Phi(q)}(x)$ for tempered stable processes (7.5.3) with indexes $\alpha = 0.5$ and $\alpha = 1.5$, and perturbed gamma process (7.5.4). For these stochastic processes, we notice that an explicit expression for the scale function $W_{\Phi(q)}(x)$ is not available. But nonetheless, it exhibits the important properties of the scale function as specified in Lemma 7.3.3.

7.6 MATLAB program code

We present in this section a simple MATLAB program used to compute the function $W_{\Phi(q)}(x)$ by means of numerical inversion of the Laplace transform (7.3.3). The algorithm is based on the framework of Abate and Whitt [1] and Choudhury et al. [31] described earlier. However, to handle matrix multiplication, we use the approach of Rogers [107].

```
function G=ILT(F,X,P)

% This program is for numerical inversion
% of univariate Laplace transform

% F is the Laplace transform function,
% P is a vector of parameters sitting in F

% Setting the parameters values 11, N, M and A1

N=11; M=9; A1=14.0; l1=1;

% Creating the weights to be used in the
% Euler summation of the partial sums:

mx=pascal(M+1); my=fliplr(mx); bn=diag(my)*2^(-M);
weight=ones([2*N+1 1]); head=cumsum(bn); tail=1-cumsum(bn);
tail(M+1,:)=[]; head(M+1,:)=[]; weight=[head;weight;tail];

% Setting the values of the arguments at which the
% transfor series to be evaluated:

val1=-(N+M):(N+M); val1=(i*pi*val1+A1/2)/l2; X_inv=1./X;
X_args=kron(X_inv,val1);

% Evaluating the integrand at all the points.

integrand=feval(F,X_args,P)
            .*exp(X_args*diag(kron(X,ones(1,1+2*N+2*M))));

% Preparing the matrix which will
% post-multiply the integrand

right=kron(diag(X_inv),weight)/(2*l2);
```

```
% and finally the answer is given by

G=real(integrand*right);
----------------------------------------
```

The following MATLAB m-files are needed to compute the scale function $W_{\Phi(q)}(x)$ of α-stable, jump diffusion, tempered stable, and gamma processes whose Laplace exponents are given in equations (7.5.1)-(7.5.4).

```
function g=LEXP(z,A)

% Non zero value of the parameter A is needed to compute
% the largest root of the Laplace exponent

% For Spectrally Negative (SN) alpha stable process with alpha <1
alpha=0.5; K=1; f=K.*z-z.^(alpha); g=f-A;

% For SN alpha stable processes with alpha >=1
alpha=1.75; K=1; f=K.*z.^(alpha); g=f-A;

% For SN alpha stable process with alpha =2
f=z.^2/2; g=f-A;

% For SN Jump Diffusion Processes (JDP)
sigma=0.2; a=0.5; c=9; d=0.055;
f=d.*z+0.5.*(sigma.^2).*(z.^2)-a.*z./(c+z); g=f-A;

% For SN bounded variation (BV) tempered stable process
alpha=0.5; lambda=2.5; C=0.075; d=0.0550;

% For SN unbounded variation (UBV) tempered stable process
alpha=1.5; lambda=2.5; C=0.05; d=0.0550;

f=d.*z+gamma(-alpha).*(lambda).^(alpha).*(C)...
   .*((1+z./(lambda)).^(alpha)-1-(alpha).*z./lambda); g=f-A;

% For Gamma process perturbed by diffusion process
sigma=0.2; d=0.0550; a=0.5; b=9;
f=d.*z+0.5.*(sigma.^2).*(z.^2)-a.*log(1+z./b); g=f-A;
----------------------------------------

function f=Phir(q)
```

```
% Computing the largest root of the Laplace exponent

x0=1; f=fsolve('LEXP',x0,optimset('MaxFunEvals',100),q);
-----------------------------------------

function f=funcscale(lambda,q)

% The right hand side of the equation (7.4.9)

A=0; g=(LEXP(lambda+Phir(q),A)-q); f=1./g;
-----------------------------------------
```

Finally, the scale function $W_{\Phi(q)}(x)$ is given by

```
function W=Scale(x,q)

% Producing the scale function

W=ILT('funcscale',x,q);
-----------------------------------------
```

Having solved the problem (7.3.3), the q-scale function $W^{(q)}(x)$ is given by

$$W^{(q)}(x) \;=\; \exp(\text{Phir(q)*x}).*\text{W}.$$