# Chapter 6

## Principles of Smooth and Continuous Fit in the Determination of Endogenous Bankruptcy Levels[1]

**Abstract**

The purpose of this chapter is threefold. Firstly to revisit the previous works of Leland [77], Leland and Toft [76] and Hilberink and Rogers [58] on optimal capital structure and show that the issue of determining an optimal endogenous bankruptcy level can be dealt with analytically and numerically when the underlying source of randomness is replaced by that of a general spectrally negative Lévy process. Secondly, by working with the latter class of processes we bring to light a new phenomenon, namely that, depending on the nature of the small jumps, the optimal bankruptcy level may be determined by a principle of *continuous pasting* as opposed to the usual *smooth pasting*. Thirdly, we are able to *prove* the optimality of the bankruptcy level according to the appropriate choice of pasting. This improves on the results of Hilberink and Rogers [58] who were only able to give a numerical justification for the case of smooth pasting. Our calculations are greatly eased by the recent perspective on fluctuation theory of spectrally negative Lévy processes in which many new identities are expressed in terms of the so called *scale functions*.

## 6.1 Introduction

We consider the following model for a firm based on the earlier works of Leland [77], Leland and Toft [76] and Hilberink and Rogers [58].

The firm is assumed to be partly financed by debt, whose maturity profile is kept constant through time, by the simultaneous issue of new debt and retirement of old debt. This debt is of equal seniority, and distributes a continuous stream of coupon

[1] This chapter is the extended version of: Kyprianou, A.E. and Surya, B. A. Principles of smooth and continuous fit in the determination of endogenous bankruptcy levels. To appear in *Finance and Stochastics*, Springer-Verlag.

payment to bondholders in a fixed amount. From this the firm also receives tax benefits which are also issued as a continuous stream, providing the value of its assets is above a certain threshold, at a fixed rate. The bankruptcy level is determined endogenously by the shareholders to maximize the firm's equity value. Note that most of the authors mentioned above consider the case where the coupon is paid at a constant rate to the bondholder rather than proportional to the value of the underlying asset and the tax rebates are accordingly received at a constant rate providing the value of the underlying asset is above a certain threshold.

In this chapter we shall assume that the value of underlying assets of the firm is modelled using a general exponential spectrally negative Lévy process. This was also the case in Hilberink and Rogers [58]; however, it was necessary for them after a certain point in their calculations to work with the special case of a spectrally negative Lévy process taking the form of an independent sum of a linear Brownian motion and a compound Poisson process with negative jumps (cf. formula (3.21) on p245). As advocated by Leland and Toft [76] and Hilberink and Rogers [58], the optimal bankruptcy level should be determined by applying the smooth-pasting condition. Although for the special subclass of spectrally negative processes considered by Hilberink and Rogers [58], no rigorous proof was given to show that smooth pasting leads to the optimal choice bankruptcy level; the authors relied instead on numerical observation. By working with a *completely general spectrally negative Lévy process* here, we not only show that an analytical treatment of the optimal bankruptcy level is possible, but we are able to show that the smooth-pasting condition is not always appropriate. We give an analytical *proof of the fact* that, depending on the path regularity of the underlying Lévy process, a principle of either smooth pasting or continuous pasting should be applied accordingly as the underlying Lévy process has unbounded or bounded variation, respectively.

Among the class of spectrally negative Lévy processes, we consider the *$\alpha$-stable process* with index $\alpha \in (0,1) \cup (1,2]$ for numerical examples. With the exception of the case $\alpha = 2$ which corresponds to linear Brownian motion, these are pure jump processes. Further, they have paths of *unbounded variation* when $\alpha \in (1,2]$ and paths of bounded variation when $\alpha \in (0,1)$. The numerical results for these processes give a significant differences from the jump diffusion processes considered by Hilberink and Rogers [58].

In other recent work, Chen and Kou [29] consider the same model as we do here except that the underlying source of randomness is a Lévy process which is the independent sum of a linear Brownian motion and a compound Poisson process with two-sided exponential jumps. They also succeed in *proving* that the optimal bankruptcy level is obtained by a principle of smooth pasting for the case considered there.

The chapter is organized as follows. In Section 2 we present in more mathematical terms the basic models for the evolution of the value of the firm's assets and the capital structures of the firm following Hilberink and Rogers [58]. Section 3 and 4 discuss some notions of fluctuation theory of (general) Lévy processes, including a number of identities expressed in terms of scale functions, from which we are able to give analytic expressions for the value and debt of a firm. In Section 5 we discuss the computation of the optimal endogenous bankruptcy level. The term structure of credit spreads are given in Section 6. The main conclusion of this section is that the term structure rapidly goes to zero as debt maturity approaches zero for the case where the Lévy process has no jumps and has positive value when there are jumps in the Lévy process. This observation confirms the result of Hilberink and Rogers [58] for the jump diffusion case with one-sided independent exponential jumps and Chen and Kou [29] for the jump diffusion case with two-sided independent exponential jumps. The computation of the term structure of credit spreads requires numerical inversion of a double Laplace transform. The numerical method for this is given in Section 7. In Section 8 we verify the main results of Sections 5 and 6 by means of numerical examples. Finally, Section 9 concludes this chapter.

## 6.2 The capital structure of the firm

Throughout this chapter we assume that Lévy processes will form the basis of the model for the value of a firm as we shall now describe. Note that with some exceptions, most of what we shall say below is fundamentally the model described in Duffie and Lando [37], Hilberink and Rogers [58], and Leland and Toft [76].

To start with, let $V(t)$ denote the value of the firm's assets at time $t$ whose dynamics are given by an exponential Lévy process

$$V(t) \;=\; Ve^{X_t}. \tag{6.2.1}$$

We assume the existence of a default-free asset that pays a continuous interest rate $r > 0$. Further, it is assumed that under $\mathbb{P}$, the discounted value $e^{-(r-\delta)t}V(t)$ of the firm's assets is $\mathbb{P}-$ martingale, that is to say that

$$\mathbb{E}\big(e^{-(r-\delta)t}V(t)\big) = V, \tag{6.2.2}$$

where $\delta > 0$ is the total payout rate to the firm's investors (including both bond and equity holders).

The firm is assumed to be partly financed by debt, which is being constantly retired and reissued in the following way. In a time interval $(t, t+dt)$, the firm issues new debt with face value $pdt$, and maturity profile $\varphi$, where $\varphi$ is non-negative and $\int_0^\infty \varphi(s)ds = 1$. Thus in the time the interval $(t, t+dt)$ it issues debt with face value $p\varphi(s)dtds$ maturing in the time interval $(t+s, t+s+ds)$. Therefore, at time 0 the face value of debt maturing in $(s, s+ds)$ is given by

$$\Big(\int_{-\infty}^{0} p\varphi(s-u)du\Big)ds \;=\; pF(s)ds, \tag{6.2.3}$$

where $F(s) \equiv \int_s^\infty \varphi(u)du$ is the tail of the maturity profile. Taking $s = 0$ in (6.2.3), we see that the face value of debt maturing in $(0, ds)$ is $pds$, the same as the face value of the newly-issued debt. Thus the face value of all debt is constant, equal to

$$P = p \int_0^\infty F(s)ds. \tag{6.2.4}$$

This is same the debt profile given in Hilberink and Rogers [58] and opposed to the paper of Leland and Toft [76] who take the Dirac delta-function at $T$ which means that all new debt is always issued with a maturity of $T$. As in both of the above papers, however, we take $\varphi(t) = me^{-mt}$ for some positive $m$. This has the direct implication that $P = p/m$.

All debt is of equal seniority and attracts coupons of an amount $\rho P$ at time $t$ until maturity, or until default if that occurs sooner, where $\rho > 0$. Default happens at the first time that the value of the firm's assets falls to some level $V_B$ or lower, i.e., at

$$\sigma_{V_B}^- = \inf\{t > 0 : V(t) < V_B\}. \tag{6.2.5}$$

As we shall show later, the value of $V_B$ can be determined endogenously for a general class of Lévy processes. At default, a fraction $\eta$ of the value of the firm's asset is also assumed to be lost in reorganization.

Let us now consider a bond issued at time 0 with face value 1 and maturity $t$, which continuously pays a constant coupon flow at a fixed rate $\rho > 0$. Let $\frac{1}{P}$ be the fraction of the asset value $V(\sigma_{V_B}^-)$ which debt of maturity $t$ receives in the event of bankruptcy. The value of the debt with maturity $t$ is given by

$$\begin{aligned} d(V; V_B, t) &= \mathbb{E}\Big(\int_0^{t \wedge \sigma_{V_B}^-} \rho e^{-rs} ds\Big) + \mathbb{E}\Big(e^{-rt} : t < \sigma_{V_B}^-\Big) \\ &\quad + \frac{1}{P}(1-\eta)\mathbb{E}\Big(e^{-r\sigma_{V_B}^-} V(\sigma_{V_B}^-) : \sigma_{V_B}^- < t\Big). \end{aligned} \tag{6.2.6}$$

The first term on the right-hand side of (6.2.6) represents the expected discounted value of all coupon payment until time $t$ or the default time $\sigma_{V_B}^-$, whichever is sooner. The second term represents the expected discounted value of the principle repayment, if this occurs before bankruptcy, and the final term must be the net present value of what is recovered upon bankruptcy, if this happens before maturity time $t$. Indeed, $V(\sigma_{V_B}^-)$ is the value of the firm's asset when bankruptcy occurs and $(1-\eta)V(\sigma_{V_B}^-)$ is the value of the remains after bankruptcy costs are deducted. Of this, the bondholder with face value 1 gets the fraction $\frac{1}{P}$, since his debt represents this fraction of the total debt outstanding. Notice that if the process $X$ were continuous, then $V(\sigma_{V_B}^-)$ would simply be the bankruptcy level $V_B$; but since we allow $X$ to have possible jumps, $V(\sigma_{V_B}^-)$ can be below the bankruptcy level $V_B$.

Let $D(V; V_B)$ denote the total value of debt. The fraction of the firm's asset value lost in bankruptcy is $\eta$. The remaining value $(1-\eta)V(\sigma_{V_B}^-)$ is distributed to debt holders so that the sum of all fractional claims $\frac{1}{P}$ for debt of all outstanding maturities equals $(1-\eta)$. We can now determine the total value at time 0 of all outstanding debt as

$$\begin{aligned} D(V;V_B) = &\int_0^\infty pe^{-mt} d(V;V_B,t)dt \\ =&\rho P\mathbb{E}\Big(\int_0^{\sigma_{V_B}^-} e^{-(r+m)t}dt\Big) + p\mathbb{E}\Big(\int_0^{\sigma_{V_B}^-} e^{-(r+m)t}dt\Big) \\ &+ (1-\eta)\mathbb{E}\Big(e^{-(r+m)\sigma_{V_B}^-}V(\sigma_{V_B}^-)\Big) \\ =&\frac{(\rho+m)P}{r+m}\mathbb{E}\Big(1-e^{-(r+m)\sigma_{V_B}^-}\Big) + (1-\eta)\mathbb{E}\Big(e^{-(r+m)\sigma_{V_B}^-}V(\sigma_{V_B}^-)\Big). \end{aligned} \tag{6.2.7}$$

We assume that there is a corporate tax rate $\tau > 0$ which depends on the value of the underlying risky asset in the following way. As introduced by Leland and Toft [76] (see also Hilberink and Rogers [58]), there exists a cutoff level $V_T$, whose effect is that the tax rebates are 0 while $V(t) < V_T$, and are $\tau\rho P dt$ when $V(t) \geq V_T$. Under this assumption, the value of the firm at time zero becomes

$$v(V;V_B) \;=\; V - \eta\mathbb{E}\Big(e^{-r\sigma_{V_B}^-}V(\sigma_{V_B}^-)\Big) + \tau\rho P\mathbb{E}\Big(\int_0^{\sigma_{V_B}^-} e^{-rt}\mathbf{1}_{\{V(t)\geq V_T\}}dt\Big). \tag{6.2.8}$$

In terms of (6.2.8) and (6.2.7), the value of the firm's equity is given by

$$E(V;V_B) = v(V;V_B) - D(V;V_B). \tag{6.2.9}$$

The expressions for the expectation in (6.2.7) and (6.2.8) cannot be written in closed form in general, although this is possible in the Brownian motion case of Leland and Toft [76]. These difficulties can be circumvented by modeling the dynamics of the firm's asset value by Lévy processes having downward jumps.

## 6.3 Lévy processes with no positive jumps

Now, let us return to the dynamics (6.2.1) for the value of the firm's assets. We assume throughout the remaining of this chapter that $X$ is a real-valued Lévy process having no positive jumps, that is, its Lévy measure $\Pi$ is concentrated on $(-\infty, 0)$. This class of processes has a great interest from theoretical point of view, because they are processes for which fluctuation theory can be developed to a fuller extent. As $X$ will be chosen from this class in our financial model, we devote a little time in this section and the next to an overview of a number of relevant results from the above-mentioned fluctuation theory. Unless otherwise stated, all of what follows in this section can be extracted from the books of Bertoin [13] or Kyprianou [69].

The degenerate case when $X$ is either the negative of a subordinator or a deterministic drift has no interest and will be excluded throughout. The Laplace exponent $\kappa$ of $X$ is given by

$$\mathbb{E}\big(e^{\lambda X_t}\big) \;=\; e^{t\kappa(\lambda)} \quad \text{for } \lambda, t \geq 0. \tag{6.3.1}$$

The function $\kappa : [0, \infty) \to (-\infty, \infty)$ is defined by

$$\kappa(\lambda) \;=\; -\Psi(-i\lambda) \;=\; \mu\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{(-\infty,0)} \big(e^{\lambda x} - 1 - \lambda x \mathbf{1}_{\{x>-1\}}\big)\Pi(dx). \quad (6.3.2)$$

It is easily shown that $\kappa$ is zero at the origin, tends to infinity at infinity and is strictly convex. We denote by $\Phi : [0, \infty) \to [0, \infty)$ the right continuous inverse of $\kappa(\lambda)$, so that

$$\Phi(\alpha) \;=\; \sup\{p > 0 : \kappa(p) = \alpha\}$$

and

$$\kappa(\Phi(\lambda)) \;=\; \lambda \quad \text{for all} \quad \lambda \geq 0.$$

Note that due to the convexity of $\kappa$, there exist at most two roots for a given $\alpha$ and precisely one root when $\alpha > 0$.

The class of spectrally negative Lévy processes is very rich. Among other things it allows for processes which have paths of both unbounded and bounded variations. The latter case occurs if and only if $\sigma = 0$ and

$$\int_{(-\infty,0)} |x|\Pi(dx) \;<\; \infty.$$

In that case one may rearrange (6.3.2) into the form

$$\kappa(\lambda) = \mathrm{d}\lambda - \int_{(-\infty,0)} \big(1 - e^{\lambda x}\big)\Pi(dx) \quad (6.3.3)$$

where necessarily $\mathrm{d} > 0$. This reflects the fact that a spectrally negative Lévy process of bounded variation must be the difference of a linear drift and a pure jump subordinator. If further it is assumed that $\Pi(-\infty, 0) < \infty$, then $X$ is nothing more than the difference of a linear drift and a compound Poisson subordinator.

The path variation for a spectrally negative Lévy process also dictates how the process moves away from its initial position. It can be shown that a general Lévy process has one of four types of behaviour in this respect which we shall now describe. Let

$$\sigma_0^+ \;=\; \inf\{t > 0 : X_t \;>\; 0\} \quad \text{and} \quad \sigma_0^- \;=\; \inf\{t > 0 : X_t \;<\; 0\}.$$

Then either

**(i)** $\mathbb{P}(\sigma_0^+ = 0) = \mathbb{P}(\sigma_0^- = 0) = 1$,

**(ii)** $\mathbb{P}(\sigma_0^+ = 0) = \mathbb{P}(\sigma_0^- > 0) = 1$,

**(iii)** $\mathbb{P}(\sigma_0^+ > 0) = \mathbb{P}(\sigma_0^- = 0) = 1$ or

**(iv)** $\mathbb{P}(\sigma_0^+ > 0) = \mathbb{P}(\sigma_0^- > 0) = 1$.

Note in particular that all probabilities are either zero or one (this follows by Blumenthal's zero-one law). Case (iv) is only fulfilled by compound Poisson processes. It is well known that a spectrally negative Lévy process necessarily obeys case (i) when it has paths of unbounded variation and case (iii) when it has paths of bounded variation. To some extent, it is clear that when a spectrally negative process has a Gaussian component ($\sigma > 0$) then (i) must hold on account of the dominant behavior of the latter. If however $\sigma = 0$, then the above conclusions tell us that when $\int_{(-1,0)} |x|\Pi(dx) = \infty$, the movement of $X$ is volatile enough that the process visits both the upper and lower half-lines immediately. If on the other hand $\int_{(-1,0)} |x|\Pi(dx) < \infty$ then, taking (6.3.3) into account, the accumulation of negative jumps in the first moments of time is not sufficient to counterbalance the upward linear motion with rate d, thus bringing $X$ immediately into the upper half line for a strictly positive period of time.

When $\mathbb{P}(\sigma_0^+ = 0) = 1 (= 0)$ we say that 0 is regular (irregular) for $(0, \infty)$. When $\mathbb{P}(\sigma_0^- = 0) = 1 (= 0)$ we say that 0 is regular (irregular) for $(-\infty, 0)$.

## 6.4 Scale functions and fluctuation identities

As mentioned in the previous section, spectrally negative Lévy processes form a general class of Lévy processes that enjoy a degree of analytic tractability. The purpose of this section is to give some exposure to explicit expressions for certain fluctuation identities which will be of use when considering the problem of determining the optimal endogenous bankruptcy level for the financial model described in Section 6.2.

The starting point is the so-called scale function which features invariably in almost all known identities (see [13] and [14] for the origin of this function).

### 6.4.1 Scale functions

**Definition 6.4.1 (Scale function)** For a given spectrally negative Lévy process $X$ with Laplace exponent $\kappa$, there exists for every $q \geq 0$ a function $W^{(q)} : \mathbb{R} \to [0, \infty)$ such that $W^{(q)}(x) = 0$ for all $x < 0$ and $W^{(q)}$ is differentiable on $[0, \infty)$, satisfying

$$\int_0^\infty e^{-\lambda x} W^{(q)}(x) dx = \frac{1}{\kappa(\lambda) - q} \quad \text{for} \quad \lambda > \Phi(q), \tag{6.4.1}$$

where $\Phi(q)$ was defined in the previous section. We write $W^{(0)} = W$ for short.

Smoothness properties of the scale functions $W^{(q)}$ are very closely related to the roughness of the underlying paths of the associated Lévy process. The following result, found in Lambert [75] and Chan and Kyprianou [28], gives necessary and sufficient conditions for the scale function on $(0, \infty)$ to belong to $C^1(0, \infty)$.

**Theorem 6.4.2** *Suppose that $X$ is a spectrally negative Lévy process. For each $q \geq 0$,*

**(i)** *if $X$ is of unbounded variation, then $W^{(q)}$ is continuously differentiable on $(0, \infty)$;*

**(ii)** *if $X$ is of bounded variation, then $W^{(q)}$ is continuously differentiable on $(0, \infty)$ if and only if $\Pi$ has no atoms.*

In addition the behavior of the scale function at the origin can also be established. In both lemmas below, recall that d is the drift coefficient appearing in the representation (6.3.3) of the Laplace exponent $\kappa$ when $X$ has bounded variation.

**Lemma 6.4.3** *At the point zero, the value of the scale function $W^{(q)}(x)$ is determined for every $q \geq 0$ by*

$$W^{(q)}(0+) = \begin{cases} 1/\mathrm{d}, & \textit{when } X \textit{ has bounded variation} \\ 0, & \textit{when } X \textit{ has unbounded variation} \end{cases}$$

*Proof* From (6.4.1) we have for $q \geq 0$ that

$$\int_{[0,\infty)} \lambda e^{-\lambda x} W^{(q)}(x) dx = \frac{\lambda}{\kappa(\lambda) - q} \qquad \text{for } \lambda > \Phi(q). \tag{6.4.2}$$

When $X$ has unbounded variation, a straightforward argument using the expression (6.3.2) shows that $\lim_{\lambda \uparrow \infty} \kappa(\lambda)/\lambda = \infty$. (In particular one can show that the integral in the expression for $\kappa$ is of order $\lambda^2$). Hence by the continuity of $W^{(q)}$ it follows by taking limits as $\lambda \uparrow \infty$ in (6.4.2) that $W^{(q)}(0+) = 0$. On the other hand, when $X$ is of bounded variation, then another straightforward argument shows that in fact $\lim_{\lambda \uparrow \infty} \kappa(\lambda)/\lambda = \mathrm{d}$. □

**Lemma 6.4.4** *Following Theorem 6.4.2, we see for every $q \geq 0$ that*

$$\frac{dW^{(q)}}{dx}(0+) = \begin{cases} 2/\sigma^2, & \textit{when } X \textit{ has unbounded variation and } \sigma \neq 0, \\ \infty, & \textit{when } X \textit{ has unbounded variation with } \sigma = 0, \\ \infty, & \textit{when } X \textit{ has bounded variation and } \Pi(-\infty, 0) = \infty, \\ \frac{(\Pi(-\infty,0)+q)}{\mathrm{d}^2}, & \textit{when } X \textit{ has bounded variation and } \Pi(-\infty, 0) < \infty. \end{cases}$$

*Proof* Integrating (6.4.1) by parts and noting from Definition 6.4.1 and Theorem 6.4.2 that a right derivative at zero always exists, we have for each $q \geq 0$

$$\frac{dW^{(q)}}{dx}(0+) = \lim_{\lambda \uparrow \infty} \int_0^\infty \lambda e^{-\lambda x} \frac{dW^{(q)}(x)}{dx} dx = \lim_{\lambda \uparrow \infty} \frac{\lambda^2}{\kappa(\lambda) - q}.$$

In the spirit of the previous proof, it is easy to show when $X$ has unbounded variation that $\lim_{\lambda \uparrow \infty} \kappa(\lambda)/\lambda^2 = \sigma^2/2$ (see also Proposition 2 of Section I in Bertoin [13]). This accounts for the first two cases. When $X$ has bounded variation, a little more care is needed. Integrating again (6.4.1) by parts, taking care to note that $W^{(q)}(0+) = \mathrm{d}^{-1}$, we have

$$\begin{aligned}
&\frac{dW^{(q)}}{dx}(0+) \\
= \;& \lim_{\lambda\uparrow\infty} \frac{\lambda^2}{\mathrm{d}\lambda - \lambda\int_0^\infty e^{-\lambda x}\Pi(-\infty,-x)dx - q} - \lambda W^{(q)}(0+) \\
= \;& \lim_{\lambda\uparrow\infty} \frac{\lambda^2(1 - W^{(q)}(0+)\mathrm{d} + W^{(q)}(0+)\int_0^\infty e^{-\lambda x}\Pi(-\infty,-x)dx) + q\lambda W^{(q)}(0+)}{\mathrm{d}\lambda - \int_0^\infty \lambda e^{-\lambda x}\Pi(-\infty,-x)dx + q} \\
= \;& \lim_{\lambda\uparrow\infty} \frac{1}{\mathrm{d}} \frac{\int_0^\infty \lambda e^{-\lambda x}\Pi(-\infty,-x)dx + q}{\mathrm{d} - \int_0^\infty e^{-\lambda x}\Pi(-\infty,-x)dx} \\
= \;& \frac{\Pi(-\infty,0) + q}{\mathrm{d}^2}.
\end{aligned}$$

In particular, if $\Pi(-\infty,0) = \infty$ then the right-hand side above is equal to $\infty$, and if $\Pi(-\infty,0) < \infty$, then $dW^{(q)}(0+)/dx$ is finite and equals to $(\Pi(-\infty,0) + q)/\mathrm{d}^2$. Thus our claim is then proved. □

It should be noted that the first of the last two lemmas is essentially not new but implicitly embedded in the literature for spectrally negative Lévy processes.

Due to the complexity of the Laplace exponent $\kappa$, the scale functions $W^{(q)}$ are not available in explicit form in general. However, it turns out that we have sufficient analytical information regarding these 'special' functions in order to achieve our main goal of establishing an optimal choice of $V_B$ via the imposition of an appropriate pasting condition.

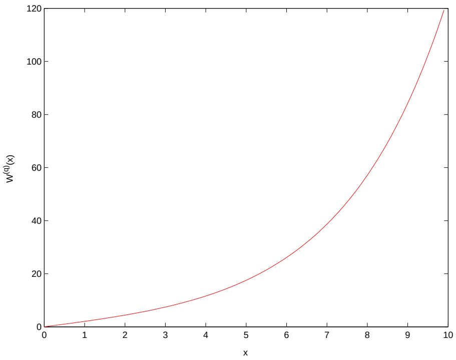
Numerical inversion of the Laplace transform (6.4.1) can always be used to compute the scale function numerically. We refer to Choudhury et al [31] for a general discussion on numerical inversion of Laplace transforms and to Surya [118] for a specific description of the case at hand (see also Chapter 7 for more details). For some spectrally negative Lévy processes, the scale functions $W^{(q)}$ are available explicitly. We consider four such examples below.

**Example 6.4.5** *Standard Brownian motion.* Taking $\kappa(\lambda) = \lambda^2/2$, it is a straightforward exercise to show that the scale function is given by

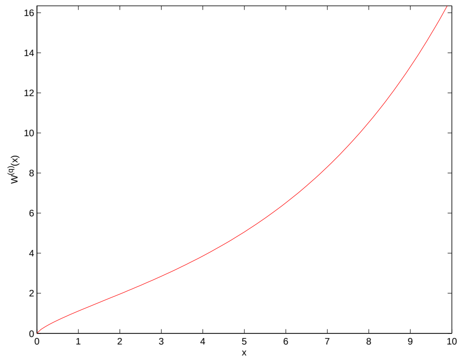
$$W^{(q)}(x) = \sqrt{\frac{2}{q}}\sinh(x\sqrt{2q}).$$

**Example 6.4.6** *Spectrally negative $\alpha$-stable process.* In this case $X$ has (up to a multiplicative constant which we take as equal to 1) Laplace exponent $\kappa(\lambda) = \lambda^\alpha$ with $\alpha \in (1,2)$. Due to [14], it is known that the scale function $W^{(q)}$ satisfies

$$\int_{[0,\infty)} e^{-\lambda x} W^{(q)}(x)dx = \frac{1}{\lambda^\alpha - q} \quad \text{for } \lambda > q^{1/\alpha},$$

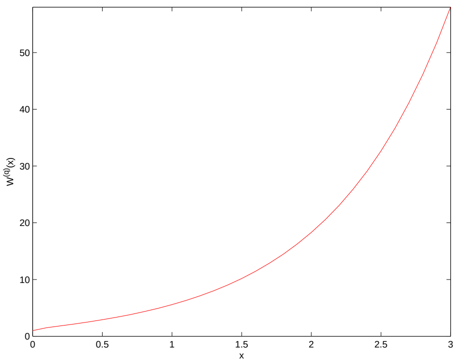



(a) $X$ is standard Brownian motion.

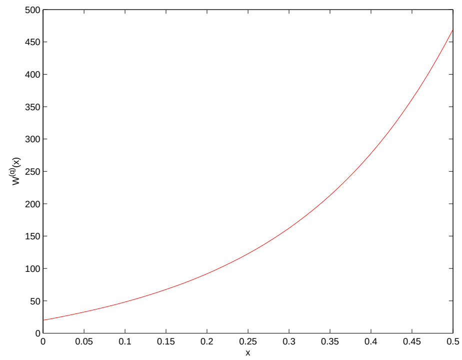



(b) $X$ is $\alpha$-stable process with $\alpha = 1.75$

Figure 6.1: The shapes of $W^{(q)}(x)$, $q = 0.075$, for unbounded variation $X$.




(a) $X$ is $\alpha$-stable process with $\alpha = 0.5$




(b) $X$ is compound Poisson process with drift $\mathrm{d} = 0.05$.

Figure 6.2: The shapes of $W^{(q)}(x)$, $q = 0.075$, for bounded variation $X$.

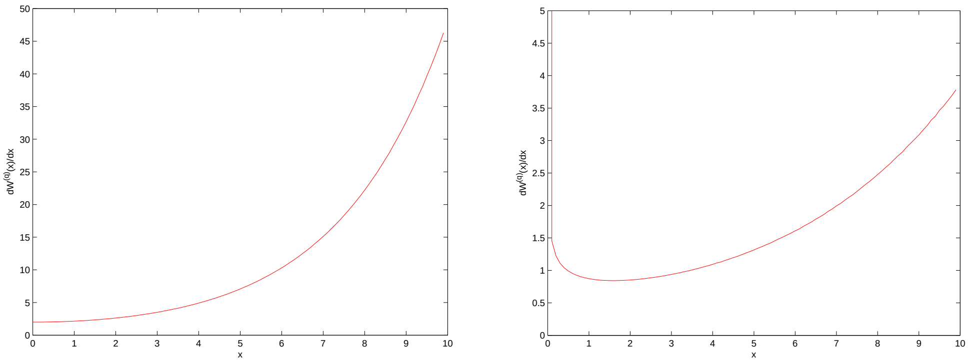
following which one can deduce that

$$W^{(q)}(x) \;=\; \alpha x^{\alpha-1} E'_{\alpha}(q x^{\alpha}) \quad \text{for } x \geq 0,$$

where $E_{\alpha}(.)$ is the Mittag-Leffler function of parameter $\alpha$ defined as

$$E_{\alpha}(y) \;=\; \sum_{n=0}^{\infty} \frac{y^n}{\Gamma(1+\alpha n)}, \quad y \in \mathbb{R}.$$

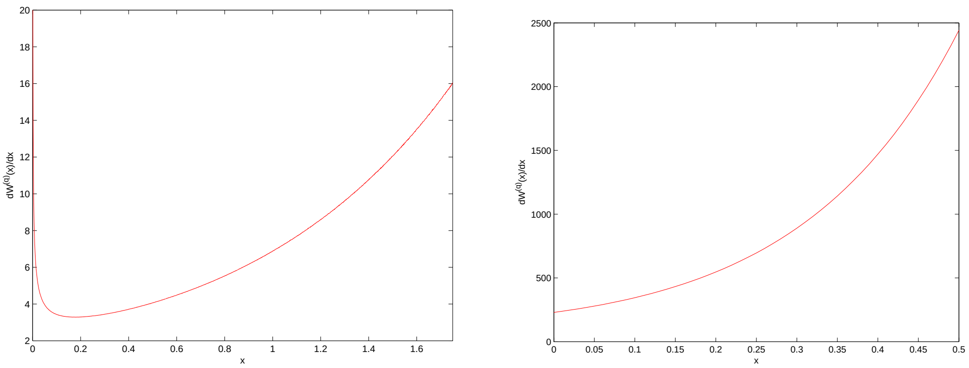
**Example 6.4.7** *Spectrally negative Lévy process of bounded variation drifting to infinity.* Suppose that $X_t = \mathrm{d}t - S_t$ where $\{S_t : t \geq 0\}$ is a subordinator with Lévy measure $\Pi$ having no atoms and $\mathbb{E}(X_1) > 0$ so that $\mathbb{P}(\lim_{t\uparrow\infty} X_t = \infty) = 1$. It can be




(a) $X$ is standard Brownian motion.

(b) $X$ is $\alpha$-stable process with $\alpha = 1.75$

Figure 6.3: The shapes of $\frac{d}{dx}W^{(q)}(x)$, $q = 0.075$, for unbounded variation $X$.




(a) $X$ is $\alpha$-stable process with $\alpha = 0.5$

(b) $X$ is compound Poisson process with drift $\mathrm{d} = 0.05$.

Figure 6.4: The shapes of $\frac{d}{dx}W^{(q)}(x)$, $q = 0.075$, for bounded variation $X$.

shown that the scale function $W(x)$ satisfies

$$\int_{[0,\infty)} e^{-\lambda x} W(x)dx = \frac{1}{\mathrm{d} - \int_{(0,\infty)} e^{-\lambda x}\Pi(x,\infty)dx},$$

from which we can deduce that

$$W(x) = \frac{1}{\mathrm{d}} \sum_{n\geq 0} \nu^{\star n}(x),$$

where $\nu^{\star n}$ denotes the $n$th convolution power of $\nu(x) = \mathrm{d}^{-1}\Pi(x,\infty)$ with $\nu^{\star 0}(x)$ being understood as $\delta_0(x)$.

**Example 6.4.8** *Compound Poisson process with exponential jumps with parameter* $\mu > 0$ *and rate* $\beta$. From the previous example one may deduce further that when $\mathrm{d}\mu - \beta > 0$, the scale function is given by

$$W(x) \;=\; \frac{1}{\mathrm{d}}\Big(1 + \frac{\beta}{\mathrm{d}\mu - \beta}\big(1 - e^{-(\mu - \mathrm{d}^{-1}\beta)x}\big)\Big).$$

The scale function $W^{(q)}$ can be determined by the formula

$$W^{(q)}(x) = e^{\Phi(q)x} W_{\Phi(q)}(x) \tag{6.4.3}$$

where $W_{\Phi(q)}(x)$ plays the role of $W(x)$ when $X$ is taken under the measure $\mathbb{P}^{\Phi(q)}$ defined by

$$\frac{d\mathbb{P}^{\Phi(q)}}{d\mathbb{P}}\bigg|_{\mathcal{F}_t} = e^{\Phi(q)X_t - qt}.$$

Note that it is known that under the latter change of measure, $(X, \mathbb{P}^{\Phi(q)})$ is still a spectrally negative Lévy process whose Laplace exponent has changed to

$$\kappa_{\Phi(q)}(\lambda) \;=\; \kappa(\lambda + \Phi(q)) - \kappa(\Phi(q)).$$

Various numerical plots of the scale function $W^{(q)}$ and its derivative $\frac{d}{dx}W^{(q)}$ can be found in Figures 6.1- 6.4 for each of the above examples when $q > 0$. Note that in each case the asymptotic behaviour is that of an exponential function. This is not surprising since for $\lambda > 0$ and $V^{(q)}(x) = e^{-\Phi(q)x}W^{(q)}(x)$,

$$\int_0^\infty e^{-\lambda x} V^{(q)}(dx) \;=\; \frac{\lambda}{\kappa(\lambda + \Phi(q)) - q},$$

and hence by taking limits as $\lambda \downarrow 0$ on the right hand side to obtain $1/\kappa'(\Phi(q))$, it follows from an application of the standard Tauberian theorem that

$$W^{(q)}(x) \sim \frac{e^{\Phi(q)x}}{\kappa'(\Phi(q))} \qquad \text{as } x \to \infty.$$

### 6.4.2 Fluctuation identities

Recall our notation,

$$\sigma_y^- \;=\; \inf\{t > 0 : X_t < y\}, \tag{6.4.4}$$

the first time that the Lévy process $X$ goes below a level $y$. Under the model described in Section 6.2, it is possible to write the equity (6.2.9) in terms of this stopping time. Via a number of fluctuation identities for spectrally negative processes this then allows us to write the equity in terms of scale functions. We devote this section to doing precisely this and we begin with quoting the necessary fluctuation identities. These come in the form of three lemmas. The first is due to Bertoin [15], the second is due to Emery [47] and the third is due to Bingham [17].

**Lemma 6.4.9** *Denote by* $\mathbf{e}_q$ *an independent exponential random variable with mean* $q^{-1}$. *We have for every* $x, y > 0$ *and* $q > 0$ *that*

$$q^{-1}\mathbb{P}_x(X_{\mathbf{e}_q} \in dy, \mathbf{e}_q < \sigma_0^-) = \Big(e^{-\Phi(q)y}W^{(q)}(x) - \mathbf{1}_{\{x\geq y\}}W^{(q)}(x-y)\Big)dy. \quad (6.4.5)$$

**Lemma 6.4.10** *For all* $q, \beta, x \geq 0$, *the joint Laplace transform under* $\mathbb{P}$ *of the stopping time* $\sigma_{-x}^-$ *and its overshoot* $X_{\sigma_{-x}^-}$ *is given by*

$$\begin{aligned}\mathbb{E}_x\Big(e^{-q\sigma_0^- + \beta X_{\sigma_0^-}}\Big) &= e^{\beta x} - \frac{(\kappa(\beta)-q)}{(\beta - \Phi(q))}W^{(q)}(x) \\ &- (\kappa(\beta)-q)\int_0^x e^{\beta(x-y)}W^{(q)}(y)dy.\end{aligned} \quad (6.4.6)$$

By applying Laplace transform in $x$ in the expression (6.4.6), we then end up with the well-known identity of Pecherskii and Rogozin (see for instance Bingham [17]).

**Lemma 6.4.11** *Let* $X$ *be a spectrally negative Lévy process. For every* $q, \lambda \geq 0$ *and* $\theta \in \mathbb{C}$ *with* $\mathfrak{Re}(\theta) \geq 0$ *we have*

$$\int_0^\infty \lambda e^{-\lambda x}\mathbb{E}_x\Big(e^{-q\sigma_0^- + \theta X_{\sigma_0^-}}\Big)dx = \frac{\lambda}{\lambda - \theta}\Big(1 - \frac{\kappa_q^{(-)}(\lambda)}{\kappa_q^{(-)}(\theta)}\Big), \quad (6.4.7)$$

*where* $\kappa_q^{(\pm)}(\lambda)$ *are the factors of the Wiener-Hopf factorization formula defined in equations (2.3.2) and (2.3.3) of Chapter 2.*

Throughout the rest of this chapter, we define

$$\begin{aligned}\gamma(x;q,\beta) &= \mathbb{E}\Big(1 - e^{-q\sigma_{-x}^- + \beta X_{\sigma_{-x}^-}}\Big) \\ g(x;q,b) &= \mathbb{E}\Big(\int_0^{\sigma_{-x}^-} e^{-qt}\mathbf{1}_{(X_t \geq b-x)}dt\Big).\end{aligned}$$

Writing $x = \log(V/V_B)$ and reconsidering (6.2.7), the total value of the debt can be re-expressed as follows

$$\begin{aligned}D(V;V_B) &= \frac{(\rho+m)P}{m+r}\mathbb{E}_x\Big(1 - e^{-(m+r)\sigma_0^-}\Big) + (1-\eta)V_B\mathbb{E}_x\Big(e^{-(r+m)\sigma_0^- + X_{\sigma_0^-}}\Big) \\ &= \frac{(\rho+m)P}{m+r}\gamma(x;m+r,0) + (1-\eta)V(1-\gamma(x;m+r,1)).\end{aligned} \quad (6.4.8)$$

The value of the firm (6.2.8) can be re-expressed as

$$\begin{aligned}v(V;V_B) &= V_Be^x + \tau\rho P\mathbb{E}_x\Big(\int_0^{\sigma_0^-} e^{-rt}\mathbf{1}_{\{X_t\geq b\}}dt\Big) - \eta V_B\mathbb{E}_x\Big(e^{-r\sigma_0^- + X_{\sigma_0^-}}\Big) \\ &= V(1-\eta) + \tau\rho Pg(x;r,b) + \eta V\gamma(x;r,1),\end{aligned} \quad (6.4.9)$$

where $b = \log(\frac{V_T}{V_B})$.

Following the expression in (6.4.6) one can easily deduce an explicit expression for the function $\gamma$ in terms of the scale function $W^{(q)}$.

**Lemma 6.4.12** *For* $x \in \mathbb{R}$, $q \geq 0$ *and* $\beta \geq 0$,

$$\gamma(x; q, \beta) \;=\; \frac{(\kappa(\beta) - q)}{(\beta - \Phi(q))} e^{-\beta x} W^{(q)}(x) + (\kappa(\beta) - q) \int_0^x e^{-\beta y} W^{(q)}(y) dy.$$

Using the resolvent density (6.4.5), the expression for $g$, a function that appears in the expression for the value of the firm, can also be deduced explicitly in terms of the scale function $W^{(q)}$. The expression for $g$ will be of use in the next section. The following lemma gives the expression of $g$.

**Lemma 6.4.13** *For* $x \in \mathbb{R}$, $q \geq 0$ *and* $b \in \mathbb{R}$,

$$g(x; q, b) = \frac{e^{-\Phi(q)(b \vee 0)}}{\Phi(q)} W^{(q)}(x) - \int_0^{x-(b\vee 0)} W^{(q)}(y) dy \qquad (6.4.10)$$

*Proof* Using (6.4.5) of Lemma 6.4.9, we see that

$$\begin{aligned}
\mathbb{E}_x\Big( \int_0^{\sigma_0^-} e^{-qt} \mathbf{1}_{\{X_t \geq b\}} dt \Big) &= q^{-1} \mathbb{P}_x \Big( X_{\mathbf{e}_q} \geq b, \mathbf{e}_q < \sigma_0^- \Big) \\
&= \int_{b\vee 0}^{\infty} \Big( e^{-\Phi(q) y} W^{(q)}(x) - \mathbf{1}_{\{x \geq y\}} W^{(q)}(x-y) \Big) dy \\
&= \int_{b\vee 0}^{\infty} e^{-\Phi(q) y} W^{(q)}(x) dy - \int_{b \vee 0}^{\infty} \mathbf{1}_{\{x \geq y\}} W^{(q)}(x-y) dy \\
&= \frac{e^{-\Phi(q)(b\vee 0)}}{\Phi(q)} W^{(q)}(x) - \int_0^{x - b\vee 0} W^{(q)}(y) dy,
\end{aligned}$$

where the last equality was obtained after changing variables in the integral. The required identity is then proved. □

To conclude this section, we may now write an explicit expression for the firm's equity values in terms of the scale function $W^{(q)}(x)$, namely

$$\begin{aligned}
E(V; V_B) &= V\Big( \eta \gamma(x; r, 1) + (1-\eta) \gamma(x; m+r, 1) \Big) \\
&\quad - \frac{(m+\rho) P}{m+r} \gamma(x; m+r, 0) + \tau \rho P g(x; r, b) \qquad (6.4.11)
\end{aligned}$$

where $x = \log(V/V_B)$ and $b = \log(V_T/V_B)$.

We now move on to determining an optimal bankruptcy level $V_B$.

## 6.5 Determining the bankruptcy level $V_B$

The expression for $E$ in (6.4.11) gives the firm's equity value as a function of the firm's initial asset value $V$ and the chosen bankruptcy-triggering asset level $V_B$. In determining the bankruptcy level $V_B$, the idea is to fix $V$ and maximize $E$ with respect to $V_B$ subject to the limited liability constraint that the equity $E(V; V_B)$ must always be worth uniformly non-negative for $V \geq V_B$. We refer to Leland [77], Leland and Toft [76], Hilberink and Rogers [58], and the literature therein for a more detailed discussion of the underlying economics.

The main claim of this chapter is that, observing this constraint, the bankruptcy level $V_B$ is determined in the following way.

**Theorem 6.5.1** *If the spectrally negative Lévy process $X$ has unbounded variation, so that $0$ is* regular *for the lower half-line $(-\infty, 0)$, then the bankruptcy-triggering asset level $V_B$ satisfies the condition of* smooth-pasting*; that is to say that $V_B$ is chosen to satisfy*

$$\frac{\partial E}{\partial V}(V_B+; V_B) \;=\; 0. \tag{6.5.1}$$

*However, if the spectrally negative Lévy process $X$ has bounded variation, so that $0$ is* irregular *for the lower half-line $(-\infty, 0)$, then $V_B$ satisfies the condition of continuous-pasting; that is to say that $V_B$ is chosen to satisfy*

$$E(V_B+; V_B) \;=\; 0. \tag{6.5.2}$$

*In both cases, it follows that $V_B$ is the unique solution to the equation*

$$x \;=\; \frac{\frac{(m+\rho)P}{\Phi(m+r)} - \frac{\tau\rho P}{\Phi(r)}\left(\left(\frac{x}{V_T}\right) \wedge 1\right)^{\Phi(r)}}{\left(\eta\frac{\left(r-\kappa(1)\right)}{\left(\Phi(r)-1\right)} + (1-\eta)\frac{\left(m+r-\kappa(1)\right)}{\left(\Phi(m+r)-1\right)}\right)}. \tag{6.5.3}$$

Before moving to the proof of this theorem, note that

$$f(x) = x - \frac{\frac{(m+\rho)P}{\Phi(m+r)} - \frac{\tau\rho P}{\Phi(r)}\left(\left(\frac{x}{V_T}\right) \wedge 1\right)^{\Phi(r)}}{\left(\eta\frac{\left(r-\kappa(1)\right)}{\left(\Phi(r)-1\right)} + (1-\eta)\frac{\left(m+r-\kappa(1)\right)}{\left(\Phi(m+r)-1\right)}\right)}$$

is continuous, strictly increasing in $x$, $f(0+) < 0$ and $f(\infty) = \infty$ so that there is a unique solution to the equation $f(x) = 0$ which we denote by $V_B^\star$. The equation (6.5.3) naturally agrees with the equation for determining the optimal $V_B$ in Hilberink and Rogers [58] who considered the case of a linear Brownian motion plus an independent spectrally negative compound Poisson process.

In fact Hilberink and Rogers [58] show that smooth pasting leads to the equation (6.5.3) for $V_B$ by using the Wiener-Hopf factorization where, in principle, they are working with a general spectrally negative Lévy process. However, close inspection of their calculations shows that they are implicitly assuming that $X$ has a Gaussian component. Specifically this is because of the assumed asymptotic behaviour of their functions $\varphi(x, \lambda)$ and $\gamma(x, \theta, \lambda)$ as $x \downarrow 0$ in the text following (3.16) on p. 244. One sees that this assumed asymptotic behaviour is equivalent to the assumption that the scale function $W^{(\lambda)}$ is zero with a finite derivative at the origin which in turn implies the presence of a Gaussian component.

We also note that in Chen and Kou [29], where the underlying Lévy process takes the form of an independent sum of a linear Brownian motion and a compound Poisson process with two-sided exponential jumps, it was proved that the optimal bankruptcy level follows as a consequence of the smooth-pasting condition. Their result is consistent with the above theorem in the sense that, for the Lévy process considered there, 0 is regular for $(-\infty, 0)$ on account of the presence of the Gaussian term.

We now move to the proof of Theorem 6.5.1. Without further elaboration, we shall make use of a number of facts and notions from the theory of spectrally negative Lévy processes which are well documented in the literature. We refer to Chapter 8 of Kyprianou [69] for a recent review in which all of the used concepts are addressed.

To establish our claim in the Theorem 6.5.1, the following lemma is needed.

**Lemma 6.5.2** *For each fixed $x \geq 0$, the quantity*

$$\Theta^{(q)}(x) \ := \ W^{(q)\prime}(x) - \Phi(q)W^{(q)}(x), \tag{6.5.4}$$

*is non-negative for all $x, q \geq 0$ and monotone decreasing in $q$.*

*Proof* It is known that the $q$-resolvent measure of the descending ladder height process, $\widehat{H} = \{\widehat{H}_t : t \geq 0\}$ (see Section 2.2 of Chapter 2), of $X$ can be identified as

$$\mathbb{E}\left(\int_0^\infty e^{-qt}\mathbf{1}_{\{\widehat{H}_t \in A\}}dt\right) = c\int_A \Theta^{(q)}(y)dy,$$

where $q \geq 0$ and $c > 0$ is a meaningless constant (determined by the normalization of local time at the minimum to generate the descending ladder height process $\widehat{H}$) and $A$ is a Borel set in $[0, \infty)$. See for example Millar [85] and Pistorius [102]. It is immediately obvious from this relation, in particular on account of the arbitrary choice of $A$, that $\Theta^{(q)}(x)$ is non-negative and for each fixed $x \geq 0$ it is also monotone decreasing in $q$.

The fact that $\Theta^{(q)}(x) \geq 0$ for all $q, x \geq 0$ is a simple consequence of its definition as a resolvent density. Thus the claim that $\Theta^{(q)}(x) \geq 0$ is non-negative for all $x, q \geq 0$ and monotone decreasing in $q$ is then proved. □

*Proof of Theorem 6.5.1* We split the proof into two: the cases that $X$ has paths of unbounded and bounded variation.

Firstly, we assume that $X$ has paths of unbounded variation. Since the scale function $W^{(q)}(x)$ is continuous and equal to zero at $x = 0$, it is easy to check that the continuous pasting condition (6.5.2) is always satisfied for any bankruptcy level $V_B > 0$. We look instead at choosing $V_B^\star$ by the criterion (6.5.1).

Assume temporarily that $\sigma > 0$. By differentiating the firm's equity value $E$ with respect to $V$, we see after a rather long calculation that

$$\frac{\partial E}{\partial V}(V_B+;V_B) = \frac{2}{\sigma^2 V_B}\Big(\eta\frac{(r-\kappa(1))}{(\Phi(r)-1)} + (1-\eta)\frac{(m+r-\kappa(1))}{(\Phi(m+r)-1)}\Big)f(V_B).$$

From the remarks following the statement of Theorem 6.5.1 we can now see that

$$\frac{\partial E}{\partial V}(V_B+;V_B) > (<)\, 0 \quad \text{for } V_B > (<)\ V_B^\star \quad \text{and} \quad \frac{\partial E}{\partial V}(V_B^\star+, V_B^\star) \ = \ 0.$$

In the case that $\sigma = 0$ one may similarly check with the help of the second case in the conclusion of Lemma 6.4.4 that if $V_B$ is chosen strictly greater than $V_B^\star$ then in fact $\frac{\partial E}{\partial V}(V_B+;V_B) = \infty$ and similarly if $V_B$ is chosen strictly less than $V_B^\star$ then $\frac{\partial E}{\partial V}(V_B+;V_B) = -\infty$. When $V_B = V_B^\star$ it conveniently turns out that $\frac{\partial E}{\partial V}(V_B+;V_B) = 0$.

Taking account of the limited liability constraint that the equity curve must be uniformly non-negative for all $V \geq V_B$, the calculations lead to the conclusion that the bankruptcy level $V_B$ must be at least as big as $V_B^\star$, i.e., $V_B \geq V_B^\star$. We should now like to prove that $V_B^\star$ is the optimal bankruptcy level. We do this by showing that for each fixed $V > V_B^\star$, the function $V_B \mapsto E(V;V_B)$ is monotone decreasing in $V_B$. To this end, we note that it can be shown after some algebra that for each fixed $V > V_B^\star$ and $V_B \in [V_B^\star, V]$,

$$\begin{aligned}\frac{\partial E}{\partial V_B}(V;V_B) \quad &= \quad -\eta\frac{\big(r-\kappa(1)\big)}{\big(\Phi(r)-1\big)}\Big\{\Theta^{(r)}(x) - \Theta^{(r+m)}(x)\Big\} \\ &\quad - \frac{\tau\rho P e^{-\Phi(r)(b\vee 0)}}{\Phi(r)V_B}\Big\{\Theta^{(r)}(x) - \Theta^{(r+m)}(x)\Big\} \qquad (6.5.5) \\ &\quad - \frac{\Theta^{(r+m)}(x)}{V_B}\Big\{\eta\frac{\big(r-\kappa(1)\big)}{\big(\Phi(r)-1\big)} + (1-\eta)\frac{\big(m+r-\kappa(1)\big)}{\big(\Phi(m+r)-1\big)}\Big\}f(V_B),\end{aligned}$$

where $x = \log(V/V_B)$ and the function $\Theta^{(q)}(x)$ is defined in (6.5.4). Note that in computing this derivative it is worth reminding oneself that

$$\begin{aligned}\frac{\partial\gamma}{\partial V_B}(x;q,\beta) \quad &= \quad -\frac{1}{V_B}\frac{\partial\gamma}{\partial x}(x;q,\beta) \\ &= \quad -\frac{1}{V_B}\frac{(q-\kappa(\beta))}{(\Phi(q)-\beta)}e^{-\beta x}\Theta^{(q)}(x),\end{aligned}$$

and that

$$\begin{aligned}\frac{\partial g}{\partial V_B}(x;q,b) \quad &= \quad -\frac{1}{V_B}\frac{\partial g}{\partial x}(x;q,b) - \frac{1}{V_B}\frac{\partial g}{\partial b}(x;q,b) \\ &= \quad -\frac{e^{-\Phi(q)(b\vee 0)}}{V_B\Phi(q)}\Theta^{(q)}(x),\end{aligned}$$

where special care should be taken in the derivatives of $g$ accordingly with the sign of the value $b$.

Our objective now is to show that each of the three terms on the right-hand side of (6.5.5) is non-positive.

Combined with the result of Lemma 6.5.2, we see that the first two terms on the right-hand side of (6.5.5) are non-positive. The monotonicity of $f$ also implies that the third expression is non-positive. In conclusion we see that for each fixed $V \geq V_B^\star$,

$$\frac{\partial E}{\partial V_B}(V; V_B) \ < \ 0$$

when $V_B \in [V_B^\star, V]$ thus justifying the claim that $V_B^\star$ is optimal.

Now consider the case that $X$ has paths of bounded variation. In that case the arguments above do not apply due to the fact that, for any given choice of $V_B$, $0 = E(V_B-; V_B)$ is not necessarily equal to $E(V_B+, V_B)$. To see this, one can show with the help of Lemma 6.4.3 that

$$E(V_B+; V_B) = \frac{f(V_B)}{\mathrm{d}}\Big(\eta\frac{(r-\kappa(1))}{(\Phi(r)-1)} + (1-\eta)\frac{(m+r-\kappa(1))}{(\Phi(m+r)-1)}\Big).$$

The monotonicity of $f$ in $V_B$ now implies that

$$E(V_B+, V_B) \ > \ (<)\, 0 \quad \text{for} \ \ V_B \ > \ (<)\, V_B^\star \quad \text{and} \ \ E(V_B^\star+, V_B^\star) \ = \ 0. \qquad (6.5.6)$$

The constraint of non-negativity of the equity curve thus implies that we must choose $V_B \geq V_B^\star$. Exactly the same analysis of the partial derivative $\frac{\partial E}{\partial V_B}(V; V_B)$ as for the unbounded variation case shows that in fact $V_B^\star$ must be optimal as $E(V; V_B)$ is decreasing in $V_B$ for each fixed $V$. Thus, our claim is then established. □

## 6.6 The term structure of credit spreads

In this section we discuss the term structure of credit spreads. Following Hilberink and Rogers [58], we identify the credit spreads as what coupon would be required to induce an investor to lend one dollar to the firm until maturity time $T$. This is the interpretation that one would put on a reported credit spreads curve for a given firm.

To start the discussion, let us return to the expression (6.2.6) and compute for a fixed $t > 0$ the value $\rho^\star$ of $\rho$ for which $d(V_0; V_B, t) = 1$. We denote by $\sigma_y^-$ the time of first exit of $X$ below a level $y$ defined in (6.4.4). By putting $x = \log\big(V_0/V_B\big)$, we can rewrite the equation (6.2.6) as

$$\begin{aligned} f(t,x) \ &\equiv \ d(V_B e^x; V_B, t) \\ &= \ \mathbb{E}\Big(\int_0^{t\wedge\sigma_{-x}^-} \rho e^{-rs} ds\Big) + \mathbb{E}_x\Big(e^{-rt} : t < \sigma_0^-\Big) \qquad (6.6.1) \\ &\quad + \frac{(1-\eta)}{P} V_B \mathbb{E}_x\Big(e^{-r\sigma_0^- + X_{\sigma_0^-}} : \sigma_0^- < t\Big). \end{aligned}$$

Taking Laplace transform in $t$, we have after some calculations that

$$\begin{aligned}\int_0^\infty e^{-\lambda t} f(t,x)dt &= \frac{\rho}{\lambda(\lambda+r)}\mathbb{E}\Big(1-e^{-(\lambda+r)\sigma^-_{-x}}\Big) + \frac{1}{\lambda+r}\mathbb{E}\Big(1-e^{-(\lambda+r)\sigma^-_{-x}}\Big)\\ &+ \frac{(1-\eta)V_B e^x}{\lambda P}\mathbb{E}\Big(e^{-(\lambda+r)\sigma^-_{-x}+X_{\sigma^-_{-x}}}\Big).\end{aligned}$$

If we take again Laplace transform in $x$, we see using (6.4.7) that

$$\begin{aligned}\widehat{f}(\lambda,\beta) &\equiv \int_0^\infty dx e^{-\beta x}\int_0^\infty e^{-\lambda t} f(t,x)dt\\ &= \frac{\rho}{\beta\lambda(\lambda+r)}\kappa^{(-)}_{\lambda+r}(\beta) + \frac{\kappa^{(-)}_{\lambda+r}(\beta)}{\beta(\lambda+r)} + \frac{(1-\eta)V_B}{\lambda P(\beta-1)}\Big(1-\frac{\kappa^{(-)}_{\lambda+r}(\beta)}{\kappa^{(-)}_{\lambda+r}(1)}\Big)\\ &= \rho\widehat{f}_1(\lambda+r,\beta)+\widehat{f}_2(\lambda+r,\beta), \qquad (6.6.2)\end{aligned}$$

where the two Laplace transforms $\widehat{f}_1$ and $\widehat{f}_2$ are defined subsequently by

$$\widehat{f}_1(\lambda+r,\beta) = \frac{1}{\beta\lambda(\lambda+r)}\kappa^{(-)}_{\lambda+r}(\beta),$$

and

$$\widehat{f}_2(\lambda+r,\beta) = \frac{\kappa^{(-)}_{\lambda+r}(\beta)}{\beta(\lambda+r)} + \frac{(1-\eta)V_B}{\lambda P(\beta-1)}\Big(1-\frac{\kappa^{(-)}_{\lambda+r}(\beta)}{\kappa^{(-)}_{\lambda+r}(1)}\Big).$$

By applying double inversion of Laplace transform, we obtain

$$f(t,x) = \mathcal{L}^{-1}_\beta\mathcal{L}^{-1}_\lambda[\widehat{f}](t,x) = \int\frac{d\lambda}{2\pi i}\int\frac{d\beta}{2\pi i}e^{t(\lambda-r)+\beta x}\widehat{f}(\lambda-r,\beta), \qquad (6.6.3)$$

from which the credit spreads is given by

$$\text{Credit spreads} = \rho^\star - r = \frac{1-\mathcal{L}^{-1}_\beta\mathcal{L}^{-1}_\lambda[\widehat{f}_2](V_B e^x;V_B,t)}{\mathcal{L}^{-1}_\beta\mathcal{L}^{-1}_\lambda[\widehat{f}_1](V_B e^x;V_B,t)} - r. \qquad (6.6.4)$$

This expression is not available in analytic form in general. Thus, numerical inversion to compute the inverse Laplace transform (6.6.3) is needed. Further technical discussion on this will be given later in Section 7.

### 6.6.1 Non-zero credit spreads for very short maturity bonds

This section discusses an analytical expression of the credit spreads for very short maturity bonds. It appears that credit spreads have strictly positive values.

By finding the value of $\rho = \rho^\star$ for which the right-hand side of (6.2.6) equals 1 when $t = T$ and $V(T) = V$, we find the spread $\rho^\star - r$ for borrowing with fixed maturity $T$. Rearrangement of (6.2.6) yields the following expression for the spreads:

$$\text{Credit spreads} = \frac{1-e^{-rT}+\mathbb{E}\Big(e^{-rT}-\frac{1-\eta}{P}V(\sigma^-_{V_B})e^{-r\sigma^-_{V_B}};\sigma^-_{V_B}\le T\Big)}{\frac{1}{r}\mathbb{E}\Big(1-e^{-r(T\wedge\sigma^-_{V_B})}\Big)} - r. \qquad (6.6.5)$$

To understand the asymptotic of the credit spreads (6.6.5) as the maturity $T$ approaches zero, let us denote by $\nu(dx, dt)$ the Poisson random measure associated with the jumps of a Lévy process $X$ and by $\sigma_{(-\epsilon,\epsilon)^c}$, with $\epsilon > 0$, the first entrance time of a jump of $X$ in the set $\{\mathbb{R}\backslash(-\epsilon,\epsilon)\}$. It is known (see Proposition 2 on page 7 of Bertoin [13]) that $\sigma_{(-\epsilon,\epsilon)^c}$ is exponentially distributed with parameter $\Pi(\mathbb{R}\backslash(-\epsilon,\epsilon))$ since

$$\mathbb{P}(\sigma_{(-\epsilon,\epsilon)^c} > t) \; = \; \mathbb{P}(\nu(\{\mathbb{R}\backslash(-\epsilon,\epsilon)\} \times [0,t]) = 0) \; = \; \exp\big(-t\Pi(\mathbb{R}\backslash(-\epsilon,\epsilon))\big).$$

(See also page 143 in Kyprianou [69].) Note that this expression can be rewritten as

$$\mathbb{P}(\sigma_{(-\epsilon,\epsilon)^c} \leq t) \; = \; 1 - \exp\big(-t\Pi(\mathbb{R}\backslash(-\epsilon,\epsilon))\big). \tag{6.6.6}$$

The expression in (6.6.6) tells us that if $\Pi(\mathbb{R}) = \infty$ then it becomes more and more probable to have jumps of size greater than $\epsilon > 0$ as $t \downarrow 0$. Thus the jumps have very significant influence over the initial behavior of the sample path of $X$, i.e., any contribution of the continuous part such as the drift and the Brownian motion to the movement of $X$ can be ignored. Thus, by ignoring the contribution of the continuous part, a spectrally negative Lévy process could have gone below the bankruptcy level $x = \log(\frac{V_B}{V}) < 0$ only made possible by a jump. Hence, following (6.6.6) we see for a very short maturity $T$ that

$$\mathbb{P}\big(\sigma_x^- \leq T\big) \; = \; T\overline{\Pi}^-(x) + o(T) \quad \text{as} \quad T \downarrow 0,$$

where $\overline{\Pi}^-(x) = \Pi(-\infty, x)$ and $o(T)$ is the probability of having more than one jump in the very short period $[0, T]$ of time. Given that $\sigma_x^- \leq T$, the law of $\log(V(\sigma_x^-))$ will be the law of a single jump conditioned to have gone below the level $x$, and therefore

$$\mathbb{E}\big(V(\sigma_x^-)\big|\sigma_x^- \leq T\big) \; = \; \frac{1}{\overline{\Pi}^-(x)} \int_{-\infty}^{x} Ve^y\Pi(dy) \; \equiv \; \overline{V}.$$

Since the denominator of (6.6.5) is asymptotically equal to $rT$ as $T \downarrow 0$, it is easily seen that

$$\text{Credit spreads} \to \overline{\Pi}^-(x)\Big(1 - \frac{(1-\eta)}{P}\overline{V}\Big) \tag{6.6.7}$$

as $T \downarrow 0$. Observe that when the process $X$ is continuous, that is when $\Pi = 0$, the credit spreads go to zero as $T \downarrow 0$. Thus, as a summary, the limiting spreads have strictly positive values as $T \downarrow 0$ except when the process $X$ is continuous. This conclusion agrees with the recent result of Hilberink and Rogers [58] and Chen and Kou [29] for jump diffusion processes and may be extended to cover a broader class of Lévy processes. To exemplify this observation over non-zero credit spreads for very short maturity bonds, we give some numerical examples in the next section.

## 6.7 Numerical inversion of double Laplace transform

This section discusses numerical inversion of double Laplace transforms of Abate and Whitt [1] and Choudhury et al [31] used to determine the term structure of credit spreads (6.6.4) expressed in terms of inversion of double Laplace transforms.

To begin with, let $f(t,x)$ be a complex-valued function on $\mathbb{R}^2_+$ whose double Laplace transform is given by

$$\widehat{f}(\lambda,\beta) = \int_0^\infty \int_0^\infty e^{-(\lambda t+\beta x)} f(t,x)dtdx, \tag{6.7.1}$$

which we assume to be well defined (see for example Ditkin and Prudnikov [35]). In (6.7.1), $\lambda$ and $\beta$ are complex variables with $\mathfrak{Re}(\lambda) > 0$ and $\mathfrak{Re}(\beta) > 0$. In this section, we discuss how to calculate $f(t,x)$ using the values of $\widehat{f}(\lambda,\beta)$.

Let $F$ be a complex valued function on $\mathbb{R}^2$ with a well-defined *double Fourier transform*

$$\phi(\lambda,\beta) \ = \ \int_{-\infty}^\infty \int_{-\infty}^\infty e^{i(\lambda t+\beta x)} F(t,x)dtdx. \tag{6.7.2}$$

If $F$ is a probability density function, then $\phi$ is known as its characteristic exponent, see for instance equation (2.1.1). Under regularity conditions, $F$ can be recovered using the *Fourier inversion formula*

$$F(t,x) \ = \ \frac{1}{4\pi^2}\int_{-\infty}^\infty \int_{-\infty}^\infty e^{-i(t\lambda+x\beta)} \phi(\lambda,\beta)d\lambda d\beta. \tag{6.7.3}$$

Our task is to compute the double integral numerically. The numerical approximation can be obtained using the two-dimensional *Poisson summation formula*

$$\begin{aligned}\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} F\Big(t+\frac{2\pi j}{h_1}, x+\frac{2\pi k}{h_2}\Big)& \\ = \frac{h_1h_2}{4\pi^2}\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} e^{-i(jh_1t+kh_2x)}\phi(jh_1,kh_2).&\end{aligned} \tag{6.7.4}$$

The left-hand side of (6.7.4) is constructed by *aliasing* to be a periodic function of $t$ and $x$ with periods $h_1^{-1}$ and $h_2^{-1}$, respectively. Assuming that the series on the left in (6.7.4) converges and that this periodic function has a proper *Fourier series*, the Fourier series is given by the right side of (6.7.4).

The key point in the inversion problem is that the Fourier transform values $\phi(jh_1,kh_2)$ from (6.7.2) appear as the Fourier coefficients in (6.7.4); see equation (5.47) in Abate and Whitt [1] and Champeney [25] page 163. Note that the right-hand side of (6.7.4) can be regarded as a *trapezoidal rule* form of numerical integration applied to the inversion integral (6.7.3).

In order to control the aliasing error, we apply *exponential damping*; that is, if $f$ is our original function of interest in the equation (6.7.1), then we replace $F(t,x)$

by the function $f(t,x)e^{-(a_1 t + a_2 x)}$ when $t, x \geq 0$ and 0 elsewhere. Then we have that $\phi(\lambda, \beta) = \widehat{f}(a_1 - i\lambda, a_2 - i\beta)$ for $\widehat{f}$ in (6.7.1), and the right-hand side of (6.7.4) can be expressed in terms of Laplace transform values. If, furthermore, we let $h_1 = \pi/(tl_1)$ and $h_2 = \pi/(xl_2)$, with $l_1, l_2 \geq 1$, and take $a_1 = A_1/(2tl_1)$ and $a_2 = A_2/(2xl_2)$, we obtain

$$\begin{aligned} f(t,x) &= \frac{\exp\big(A_1/(2l_1) + A_2/(2l_2)\big)}{4tl_1 x l_2} \\ &\quad \times \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} e^{-i(j\pi/l_1 + k\pi/l_2)} \widehat{f}\Big(\frac{A_1}{2tl_1} - \frac{ij\pi}{l_1 t}, \frac{A_2}{2xl_2} - \frac{ik\pi}{l_2 x}\Big) - \overline{e}_\infty. \end{aligned}$$

where the error $\overline{e}_\infty$ is given by

$$\overline{e}_\infty \equiv \sum_{\substack{0 \leq j,k \leq \infty \\ \text{not } j = k = 0}} e^{-(jA_1 + kA_2)} f\big((1+2jl_1)t, (1+2kl_2)x\big).$$

The term $\overline{e}_\infty$ can be regarded as the error term, which will not be explicitly computed. If $|f(t,x)| \leq C$ for some constant $C$ and all $t, x$ ($C = 1$ if $f(t,x)$ is a probability distribution), then the error can be bounded as

$$|\overline{e}_\infty| \leq \frac{C(e^{-A_1} + e^{-A_2} - e^{-(A_1+A_2)})}{(1-e^{-A_1})(1-e^{-A_2})} \approx C(e^{-A_1} + e^{-A_2}).$$

Therefore, a good approximation of the function $f(t,x)$ is given by

$$\begin{aligned} S_N(t,x) &= \frac{\exp\big(A_1/(2l_1) + A_2/(2l_2)\big)}{4tl_1 x l_2} \\ &\quad \times \sum_{j=-N}^{N} \sum_{k=-N}^{N} e^{-i(j\pi/l_1 + k\pi/l_2)} \widehat{f}\Big(\frac{A_1}{2tl_1} - \frac{ij\pi}{l_1 t}, \frac{A_2}{2xl_2} - \frac{ik\pi}{l_2 x}\Big). \end{aligned}$$

The raw value of $S_N$ may not be a very good approximation; but by using Euler summation to smooth the values of the (nearly) alternating sums, we were able to obtain good accuracy. The approximation to $f(t,x)$ finally is given by

$$f(t,x) \doteqdot \sum_{n=0}^{M} 2^{-M} \binom{M}{n} S_{N+n}(t,x).$$

This is the formula we used in the thesis to invert numerically a double Laplace transform for the term structure of credit spreads (6.6.4).

## 6.8 Numerical examples

We verify the results of Sections 5 and 6 by means of numerical examples. Our main objective is to show that the bankruptcy level $V_B^\star$ is the one that maximizes the equity value $E(V;V_B)$. For our numerical examples, we pay attention to two cases. Firstly, we assume that the underlying dynamics of $X$ is generated by $\alpha$-stable processes with Laplace exponent

$$\kappa(\lambda) \;=\; K\lambda - \lambda^{\alpha}, \quad \text{and} \quad \kappa(\lambda) \;=\; K\lambda^{\alpha},$$

respectively. For the first (second) Laplace exponent, we choose $\alpha = 0.5$ ($\alpha = 1.75$). Secondly, we consider *jump diffusion processes* where the jump component of $X$ is contributed by a compound Poisson process having independent downward jumps with exponential $\exp(c)$ distribution occurring at the times of a Poisson process with rate $a$, i.e., $X$ has Laplace exponent

$$\begin{aligned} \kappa(\lambda) \;&=\; \mathrm{d}\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{-\infty}^{0} ace^{cx}(e^{\lambda x} - 1)dx \\ &=\; \mathrm{d}\lambda + \frac{1}{2}\sigma^2\lambda^2 - \frac{a\lambda}{c+\lambda}. \end{aligned} \tag{6.8.1}$$

This special case of spectrally negative Lévy process was considered by Hilberink and Rogers in [58]. For all computations, we fix some values of parameters: we set $r = 7.5\%$, $\delta = 7\%$, $\eta = 50\%$ and $\tau = 35\%$, $\sigma = 0.2$, $a = 0.5$, $c = 9$, which are the values used in [77],[76], and [58]. We shall also assume as in [76] and [58] that $V_T = \rho P/\delta$. The parameters in the Laplace exponent $\kappa$ are chosen such that they match the martingale condition

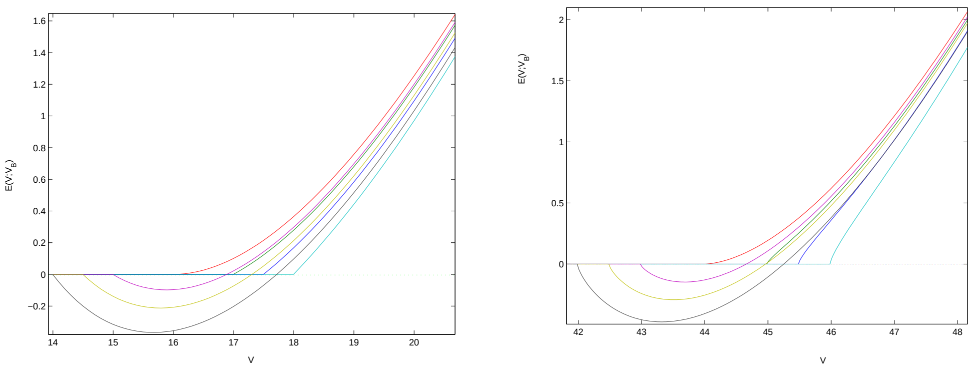
$$\mathbb{E}\Big(e^{-(r-\delta)t}V(t)\Big) \;=\; V.$$

Since our modeling for capital structures of a firm depends on the bankruptcy level $V_B$, we need to do the following in order to get one point on the curves (for the firm's (equity) values and debt values). Once a firm has been set up, the face value of the debt $P$ and the coupon rate $\rho$ are calculated for a fixed $m > 0$ in such a way that the equation (6.5.3) for the bankruptcy level $V_B$ holds,

$$D(V;V_B) \;=\; P \quad \text{and} \quad L \;=\; \frac{P}{v(V;V_B)},$$

for some positive constants *leverage* $L$ running from 5% to 95% in steps of 5%. The firm's value $v(V;V_B)$ and the total debt outstanding value $D(V;V_B)$ at time zero are defined in (6.2.8) and (6.2.7), respectively. The numerical results for the equity curves $E(V;V_B)$ are reported in Figures 6.5 and 6.6.

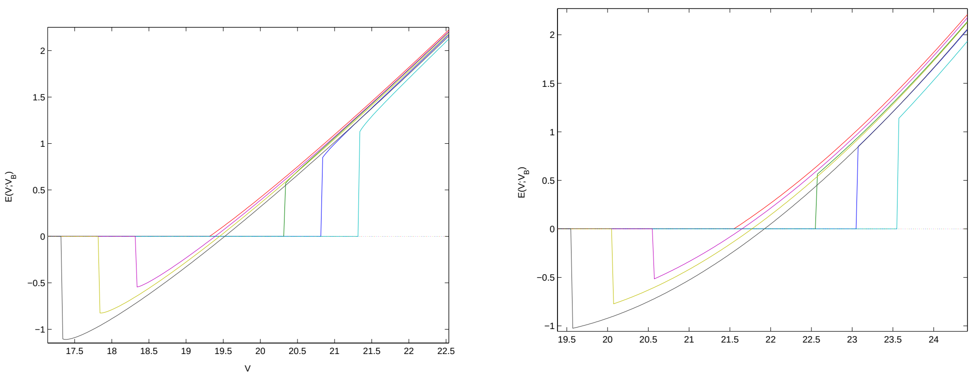
We present the numerical outcomes in Figures 6.5 for the case where the underlying dynamics $X$ of the firm asset has path of unbounded variation. The first picture is for the case where $X$ is a jump diffusion process and the other is for $\alpha$-stable process with $\alpha = 1.75$. The latter process is a process of pure jumps with no Gaussian component. We see that all the curves of the equity value $E(V;V_B)$ has zero values for all $V \leq V_B$. The curves with negative (positive) gradient at $V = V_B$ correspond with bankruptcy



(a) $X$ is a jump diffusion process with drift $\mathrm{d} = r - \delta - \sigma^2/2 + a/(1+c)$. The optimal bankruptcy level is $V_B^\star = 15.9964$.

(b) $X$ is $\alpha$-stable process with Laplace exponent $\kappa(\lambda) = K\lambda^\alpha$, $\alpha = 1.75$, and $K = r - \delta$. The optimal bankruptcy level is $V_B^\star = 43.9815$.

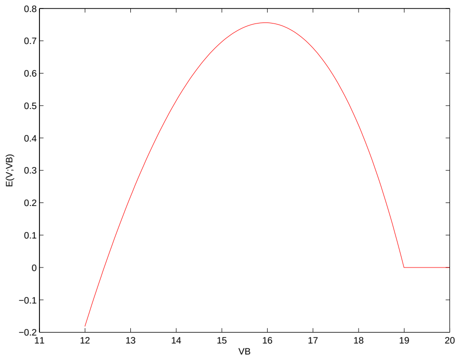
Figure 6.5: Various shapes of the equity curves $V \mapsto E(V; V_B)$ for different values of bankruptcy level $V_B$ for unbounded variation Lévy processes. The curve with zero gradient (smooth pasting) at $V = V_B$ (horizontal axis) corresponds to $V_B = V_B^\star$; those with negative (positive) gradient correspond to $V_B <$ ($>$) $V_B^\star$.



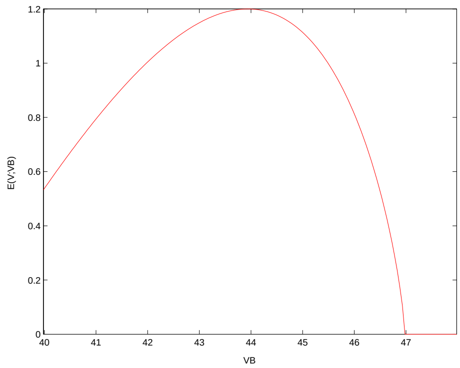
(a) $X$ is $\alpha$-stable process with Laplace exponent $\kappa(\lambda) = K\lambda - \lambda^\alpha$, $\alpha = 0.5$, and $K = 1 + r - \delta$. The optimal bankruptcy level is $V_B^\star = 19.3159$.

(b) $X$ is compound Poisson process with drift $\mathrm{d} = r - \delta + a/(1+c)$. The optimal bankruptcy level is $V_B^\star = 21.5487$.

Figure 6.6: Various shapes of the equity curves $V \mapsto E(V; V_B)$ for different values of bankruptcy level $V_B$ for bounded variation Lévy processes. The curve with zero value (continuous pasting) at $V = V_B$ (horizontal axis) corresponds to $V_B = V_B^\star$; those with negative (positive) jumps correspond to $V_B <$ ($>$) $V_B^\star$.

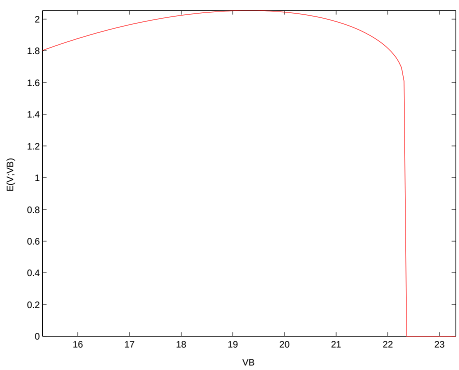



(a) $X$ is a jump diffusion process with drift $\mathrm{d} = r-\delta-\sigma^2/2+a/(1+c)$. The optimal bankruptcy level is $V_B^\star = 15.9964$.

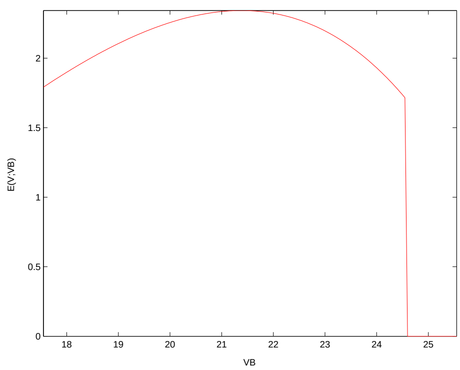



(b) $X$ is $\alpha$-stable process with Laplace exponent $\kappa(\lambda) = K\lambda^\alpha$, $\alpha = 1.75$, and $K = r - \delta$. The optimal bankruptcy level is $V_B^\star = 43.9815$.

Figure 6.7: The shape of the equity curves $V_B \mapsto E(V; V_B)$ for a fixed initial value $V$ of the firm's asset for unbounded variation Lévy processes. The curve achieves its maximum value at $V_B = V_B^\star$.




(a) $X$ is $\alpha$-stable process with Laplace exponent $\kappa(\lambda) = K\lambda - \lambda^\alpha$, $\alpha = 0.5$, and $K = 1 + r - \delta$. The optimal bankruptcy level is $V_B^\star = 19.3159$.




(b) $X$ is compound Poisson process with drift $\mathrm{d} = r - \delta + a/(1 + c)$. The optimal bankruptcy level is $V_B^\star = 21.5487$.

Figure 6.8: The shape of the equity curves $V_B \mapsto E(V; V_B)$ for a fixed initial value $V$ of the firm's asset for bounded variation Lévy processes. The curve achieves its maximum value at $V_B = V_B^\star$.

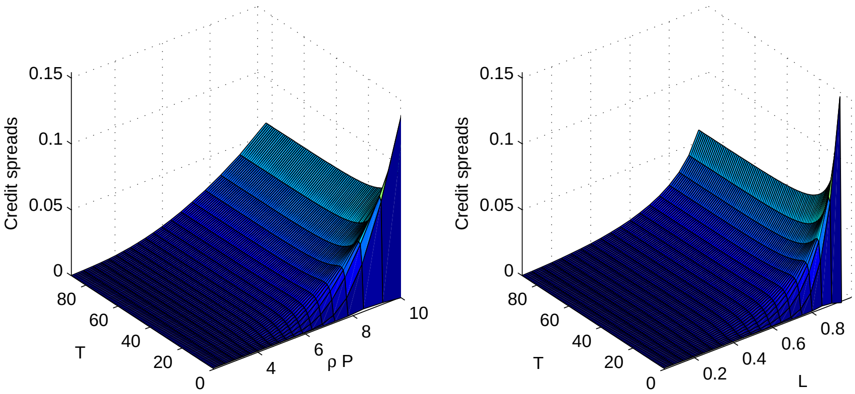



Figure 6.9: The shape of the credit spreads of a firm with debt maturity profile $m = 10$. The case where $X$ is a pure Brownian motion. Credit spreads are zero for very short maturity bonds.

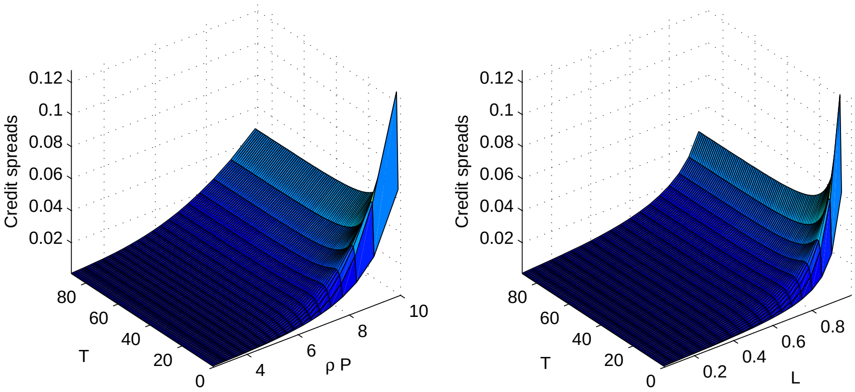



Figure 6.10: The shape of the credit spreads of a firm with debt maturity profile $m = 10$. The case where $X$ is $\alpha-$ stable process with index $\alpha = 1.75$. Credit spreads are strictly positive for very short maturity bonds.

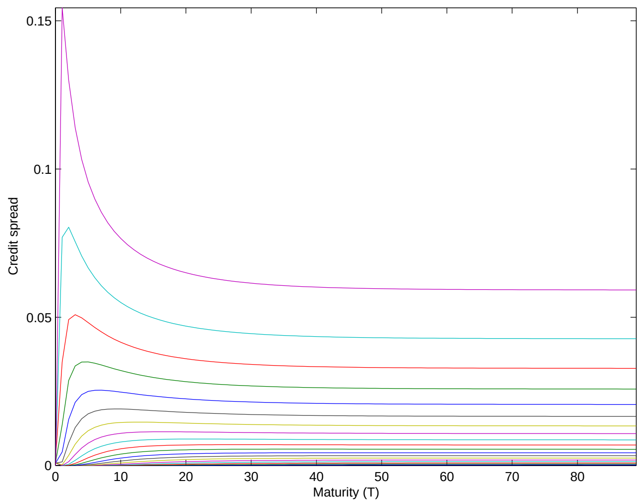



Figure 6.11: Credit spreads as a function of maturity, for different values of leverage, running from 5% to 75% in steps of 5%. The higher the leverage, the higher the spread. The case where $X$ is a pure Brownian motion. Credit spreads are zero for very short maturity bonds.

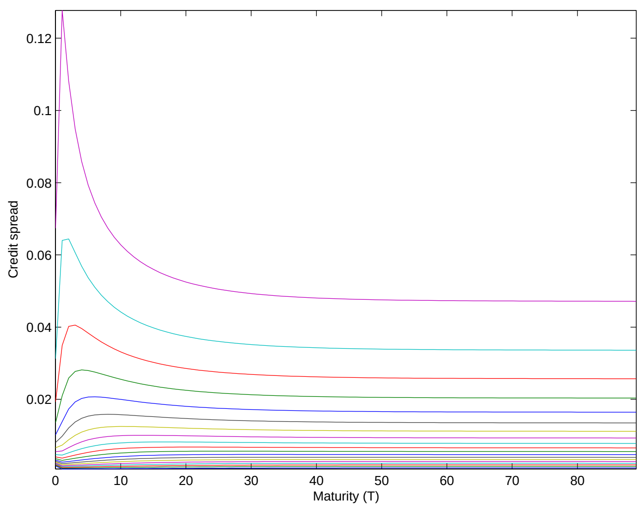



Figure 6.12: Credit spreads as a function of maturity, for different values of leverage, running from 5% to 75% in steps of 5%. The higher the leverage, the higher the spread. The case where $X$ is $\alpha-$ stable process with index $\alpha = 1.75$. Credit spreads are strictly positive for very short maturity bonds.

level $V_B < (>) V_B^\star$. The only curve which has zero gradient (smooth pasting) at $V = V_B$ corresponds to the one with the bankruptcy level $V_B = V_B^\star$. In addition, while $X$ has no Gaussian component we observe also that there are infinite gradient at $V = V_B$ for $V_B \neq V_B^\star$ for the equity curves.

For the case where $X$ has paths of bounded variation, the numerical outcomes are presented in Figure 6.6. We see that all the curves of the equity value $E(V; V_B)$ have zero values for all $V < V_B$. From the picture we observe that at the bankruptcy level $V_B < (>) V_B^\star$ the equity curves $E(V; V_B)$ exhibit negative (positive) jumps. The only curve which has no jumps (continuous pasting) at $V = V_B$ corresponds to the one with the bankruptcy level $V_B = V_B^\star$.

It is seen from the two figures that the equity curve associated with $V_B = V_B^\star$ seems to dominate the other curves, even without the constraint of positive equity. This is to say that the bankruptcy level $V_B^\star$ is indeed the optimal level of bankruptcy at which the firm's equity value is maximized. This conclusion concerning the optimality of the bankruptcy level $V_B^\star$ is illustrated in Figures 6.7 and 6.8 from which we see that $V_B^\star$ is the only bankruptcy value at which, for a fixed initial value $V$ of the firm's asset, the firm's equity value $E(V; V_B)$ is optimal. These numerical findings confirm our theoretical results given in Section 5.

The final plot, Figure 6.9-6.12, shows various shapes of the credit spreads as a function of maturity for a range of different values of leverage taken from 5% to 75% increasing in steps of 5%. Compare the continuous case, a pure Brownian motion, see Figures 6.9 and 6.11, with the other case with jumps, $\alpha-$ stable process with $\alpha = 1.75$ (see Figures 6.10 and 6.12). We notice that the credit spreads go to zero as the time to maturity $T$ tends to zero in the pure Brownian motion case, but seem to have positive limiting values in the other case. In other respects, the numerical results obtained resemble the similar type of behavior found previously by Sarig and Warga [110], Pitts and Shelby [103], Leland [77], Leland and Toft [76], Hilberink and Rogers [58], and Chen and Kou [29].

## 6.9 Conclusion and remarks

We have built on the work of Leland [77], Leland and Toft [76] and Hilberink and Rogers [58] showing that one may push the model considered by these authors fully into the case that the underlying source of randomness is a spectrally negative Lévy process. We have done this by giving an analytical treatment using scale functions. This has lead to the discovery that the optimal bankruptcy level is not always achieved by a smooth pasting condition, but instead continuous pasting is sufficient according to the path regularity of the underlying Lévy process. Moreover, our justification for the pasting principles goes further than numerical observation and we give a formal proof of this fact.