Chapter 5

An Approach for Solving Perpetual Optimal Stopping Problems Driven by Lévy Processes[1]

Abstract

In this chapter, we propose an approach for solving perpetual optimal stopping problems for a general class of payoff functions under Lévy processes. This approach was inspired by the work of Boyarchenko and Levendorski [21]. In contrast to [21], our approach does not appeal to a free boundary problem associated to the optimal stopping problem nor to the theory of pseudodifferential operators to solve the problem. Instead, we introduce an averaging problem from which we obtain, using the Wiener-Hopf factorization, a fluctuation identity for first passage of Lévy processes. This identity constitutes the main principle in solving the optimal stopping problem. If a solution to the averaging problem can be found and has certain monotonicity properties, we show using the fluctuation identity that an optimal solution to the optimal stopping problem can be written in terms of such a monotone function.

Using the optimal solution, we give sufficient and necessary condition for the C^1 smooth pasting condition to occur in the considered problem. Our conclusion over the smooth pasting condition extends further the recent result of Alili and Kyprianou [3] into a more general payoff function.

Furthermore, assuming that the moment generating function of the Lévy process exists on an open set containing zero, we give an estimate for the value function of the finite maturity American put option problem in terms of the value function of the perpetual American put option problem.

5.1 Introduction and problem formulation

Let $X = \{X_t; t \geq 0\}$ be a Lévy process defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ satisfying the usual conditions, see Chapter 2 for more details. For

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a given Lévy process X, we consider the following optimal stopping problem which consists of finding the *optimal value function*

$$V(x) \;=\; \sup_{\tau \in \mathcal{T}_{[0,\infty]}} \mathbb{E}_x\Big(e^{-q\tau} G(X_\tau)\mathbf{1}_{(\tau\,<\,\infty)}\Big), \tag{5.1.1}$$

where G is a measurable function, and the supremum is taken over the class $\mathcal{T}_{[0,\infty]}$ of Markov stopping times taking values in $[0,\infty]$ with respect to the filtration $\mathcal{F}_t = \sigma(X_s, s \leq t)$. We say that a stopping time $\tau^\star$ is optimal if

$$V(x) \;=\; \mathbb{E}_x\Big(e^{-q\tau^\star} G(X_{\tau^\star})\mathbf{1}_{(\tau^\star\,<\,\infty)}\Big). \tag{5.1.2}$$

From general results for the optimal stopping problem of diffusion processes (see for instance Shiryaev [113]), it is well-known that if the stopping time

$$\tau^\star \;=\; \inf\{t > 0 : X_t \in \mathcal{S}\}, \tag{5.1.3}$$

where $\mathcal{S}$ is the *stopping region* in which the optimal value function V is equal to the payoff G, is finite $\mathbb{P}_x$ a.s. for any $x \in \mathbb{R}$ then under very general assumptions on the payoff G it is optimal in the class $\mathcal{T}_{[0,\infty]}$ of Markov stopping times. Thus finding such a stopping time $\tau^\star$ completely determines the value function V in (5.1.1). For diffusion processes, it was shown in Shiryaev and Grigelionis [55], van Moerbeke [86], and Shiryaev [113] that the boundary $\partial\mathcal{S}$ of $\mathcal{S}$ is determined by using the *smooth pasting principle*, and solving the optimal stopping problem (5.1.1) is then reduced to solving a corresponding Stefan's free boundary problem. However, when the sample paths of X are not continuous, this smooth pasting principle may break down. This observation over the breakdown of the smooth pasting was studied by Peskir and Shiryaev in [96] for the problem of sequential testing for compound Poisson processes, by Boyarchenko and Levendorskii in [21], Hirsa and Madan [59], Matache et al. [81], Almendral and Oosterlee [4], and Alili and Kyprianou [3] for the problem of pricing the American put option under a Lévy process, and Kyprianou and Surya [73] for an American call-type optimal stopping problem with integer power function of Lévy processes. (Note that in [21], [59], [81], and [4] the solution to the problem (5.1.1) was obtained by solving the free boundary problem without imposing the smooth pasting condition). Observe that even though the stopping time (5.1.3) characterizes the value function V, it presents only qualitative features of the solution to the problem (5.1.1), but it does not present an effective way of finding the value function or for constructing the optimal stopping boundary explicitly.

In this chapter we propose an effective approach for solving the problem (5.1.1) in a general setting. This approach was inspired by the work of Boyarchenko and Levendorski [21] on perpetual American put-type optimal stopping problem for payoff function with exponential growth under stable like Lévy processes. In contrast to [21], our approach does not appeal to a free boundary problem associated to the problem (5.1.1) nor to the theory of pseudodifferential operators to solve the problem. Instead, we introduce an averaging problem from which we obtain, using the Wiener-Hopf factorization, a fluctuation identity for first passage of Lévy processes. This fluctuation identity represents the main principle in obtaining an optimal solution to the problem (5.1.1). This identity gives a generic link to some known identities which have been used to solve the problem (5.1.1) for special payoff functions. See for instance Darling et al. [33], Mordecki [87], Asmussen et al. [6], Alili and Kyprianou [3], Novikov and Shiryaev [91], and Kyprianou and Surya [73]. If a solution to the averaging problem can be found and has certain monotonicity properties, we show using the fluctuation identity that an optimal solution to the problem (5.1.1) can be written explicitly in terms of such monotone function.

Using our approach, we are able to reproduce the special results of those discussed among others by Darling et al. [33], Mordecki [87], Boyarchenko and Levendorskii [21], Alili and Kyprianou [3], Novikov and Shiryaev [91], and Kyprianou and Surya [73]. Using the optimal solution to the problem (5.1.1), we show that the C^1 smooth pasting condition exists if and only if the optimal stopping boundary is *regular* for the interior points of the stopping region for the Lévy process. But, in the case when the optimal boundary is *irregular* for the interior points of the stoping region for the Lévy process, we replace the principle of smooth pasting by a principle of *continuous pasting* in determining the optimal boundary. Our observation on the smooth pasting principle extends further the recent work of Alili and Kyprianou [3] and Kyprianou and Surya [73], into a more general payoff function.

In addition, assuming that the moment generating function of a Lévy process exists on an open set containing zero, we obtain an estimate for the value function of the finite maturity American put option problem in terms of the value function of the perpetual American put option problem. The estimate allows us to have a quick access of information about an estimate of what the arbitrage-free price $V(t,x)$ of the finite maturity American put option would be at time t given the initial value x of the stock price process.

The outline of this chapter is as follows. Building upon the Wiener-Hopf factorization introduced in Chapter 2, we present in Section 2 an averaging problem and fluctuation identity for first passage of Lévy processes which form the main principle in obtaining an optimal solution to the problem (5.1.1) in a general setting. The result is presented in Section 3. In Section 4, we discuss sufficient and necessary conditions for the C^1 smooth pasting to occur in the considered problem. In Section 5, we use our approach to reproduce the results of the aforementioned authors. Section 6 presents details of derivation of the main results of Sections 2-4. We exemplify in Section 7 the main results by means of numerical examples for pricing the perpetual American put and call options driven by tempered stable Lévy processes with downward jumps. The estimate for the the arbitrage-free price of the finite maturity American put option is given in Section 8. Finally, Section 9 concludes this chapter.

5.2 Preliminary results

Before establishing our general solution to the problem (5.1.1), we present in this section our main tool needed to obtain the main results.

5.2.1 An averaging problem

Suppose that X is a Lévy process with the assumption that

$$\text{either } q > 0 \text{ or } \Big(q = 0 \text{ and } \mathbb{P}\big(\liminf_{t\to\infty} X_t > -\infty\big) \;=\; 1\Big). \tag{H1}$$

The problem consists of finding a function $\mathscr{P}_G^{(q)}$ such that for a given continuous function G and $q \geq 0$, we have for every $x \in \mathbb{R}$ that

$$\mathbb{E}\Big(\mathscr{P}_G^{(q)}\big(x + \underline{X}_{\mathbf{e}_q}\big)\Big) \;=\; G(x). \tag{5.2.1}$$

In general, this problem may or may not have a solution in the class $\mathcal{C}_b(\mathbb{R})$ of bounded continuous functions $f : \mathbb{R} \to \mathbb{R}$, and the solution of which may not necessarily be unique.

However, there are examples for which the problem (5.2.1) is solved. We give below some solutions to this problem for exponential, linear combination of exponential, polynomials, and sufficiently regular functions.

In the sequel below let us remind ourself that $\underline{X}_{\mathbf{e}_q}$ is a negative valued random variable and therefore for a positive $\theta \in \mathbb{R}$ we see that the Wiener-Hopf factor $\Psi_q^{(-)}(-i\theta) = \mathbb{E}\big(e^{\theta \underline{X}_{\mathbf{e}_q}}\big)$ (see Section 2.2 of Chapter 2) is finite and, in particular, for $q = 0$ it is strictly positive due to the assumption (H1) imposed on the Lévy process.

Example 5.2.1 (Exponential) Suppose that X is a Lévy process with property $(H1)$. Define, for a given $\theta \geq 0$, a function $G(x) = e^{\theta x}$. Then, a solution to the problem (5.2.1) is given for every $x \in \mathbb{R}$ by

$$\mathscr{P}_G^{(q)}(x) \;=\; \frac{e^{\theta x}}{\Psi_q^{(-)}(-i\theta)}. \tag{5.2.2}$$

It is clear to see for each $q \geq 0$ and every $x \in \mathbb{R}$ that $\mathbb{E}\big(\mathscr{P}_G^{(q)}(x + \underline{X}_{\mathbf{e}_q})\big) = G(x)$.

Example 5.2.2 (Linear combination of exponential) Suppose that X is a Lévy process with property $(H1)$. Define, for a given $m = 1, 2, \cdots$, a function $G(x) = \sum_{j=1}^{m} c_j e^{\theta_j x}$ with $\theta_j \geq 0$. Then, a solution to the problem (5.2.1) is given for each $q \geq 0$ and every $x \in \mathbb{R}$ by

$$\mathscr{P}_G^{(q)}(x) \;=\; \sum_{j=1}^{m} c_j \frac{e^{\theta_j x}}{\Psi_q^{(-)}(-i\theta_j)}, \quad \text{with } \theta_j \geq 0. \tag{5.2.3}$$

It is clear to see for each $q \geq 0$ and every $x \in \mathbb{R}$ that $\mathbb{E}\big(\mathscr{P}_G^{(q)}(x + \underline{X}_{\mathbf{e}_q})\big) = G(x)$.

Example 5.2.3 (Polynomials) Suppose that X is a Lévy process with property $(H1)$. Define, for a given $n = 1, 2, \cdots$, a function $G(x) = x^n$.

Let us consider the *Esscher transform*

$$\frac{e^{\theta x}}{\mathbb{E}\big(e^{\theta \underline{X}_{\mathbf{e}_q}}\big)} \;=\; \sum_{n=0}^{\infty} \frac{\theta^n}{n!} Q_n(x), \quad \text{with } \theta \;\geq\; 0. \tag{5.2.4}$$

In literature, $Q_n(x)$, $n = 1, 2, \cdots$, are known as the *Appell polynomials* generated by the random variable $\underline{X}_{\mathbf{e}_q}$. We refer to Schoutens [112] for more details. For the polynomials x^n, $n = 1, 2, \cdots$, a solution to the problem (5.2.1) is given by

$$\mathscr{P}_G^{(q)}(x) \;=\; Q_n(x). \tag{5.2.5}$$

Following the Esscher transform (5.2.4), it is not difficult to check that for each $q \geq 0$ and every $x \in \mathbb{R}$ we have $\mathbb{E}\big(\mathscr{P}_G^{(q)}(x + \underline{X}_{\mathbf{e}_q})\big) = G(x)$.

Example 5.2.4 (Sufficiently regular function) Denote by $\mathscr{R}$ a subset of L_1- integrable functions $h : \mathbb{R} \to \mathbb{R}$ within which the *Fourier transform* $\widehat{h}$ of h, defined by

$$\widehat{h}(\lambda) \;=\; \int_{-\infty}^{\infty} e^{-i\lambda x} h(x) dx, \tag{5.2.6}$$

satisfies the integrability condition

$$\int_{-\infty}^{\infty} \big(1 + |\lambda|^3\big)\big|\widehat{h}(\lambda)\big| d\lambda \;<\; \infty. \tag{5.2.7}$$

From the fact that the function h and its Fourier transform $\widehat{h}$ are L_1-integrable, every function in $\mathscr{R}$ can be decomposed into the *Fourier integral* representation

$$h(x) \;=\; \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda x} \widehat{h}(\lambda) d\lambda. \tag{5.2.8}$$

We refer to Rudin [109] for more details of discussion. We call throughout $\mathscr{R}$ as the set of *sufficiently regular functions.* It is clear that the set $\mathscr{R}$ belongs to the class $\mathcal{C}_b^3$ of continuously differentiable functions bounded with its derivatives f^j with $j = 1, 2, 3$ and contains the *Schwartz class*[2] $\mathcal{S}(\mathbb{R})$ of *rapidly decreasing* functions and the class $\mathcal{C}_0^\infty$ of infinitely differentiable functions which tend to zero at infinity.

[2]If h is in the *Schwartz class* $\mathcal{S}(\mathbb{R})$ of *rapidly decreasing* functions, then using integration by parts it can be checked straightforwardly from (5.2.6) that the function $\widehat{h}$ admits the estimate $|\widehat{h}(\lambda)| \leq C\big((1+|\lambda|)^{-N}\big)$, for $C > 0$, as $\lambda \longrightarrow \infty$ for any integer $N = 1, 2, 3....$ This is the reason that the class $\mathcal{S}(\mathbb{R})$ is useful in studying Fourier transform since $\widehat{h} \in \mathcal{S}(\mathbb{R})$ whenever $h \in \mathcal{S}(\mathbb{R})$. We refer to Hörmander [61] for more details on general theory of Fourier integral operators.

Lemma 5.2.5 *Suppose that the Wiener-Hopf factor $\Psi_q^{(-)}(\lambda)$ (2.2.6) is nowhere zero. For a function $G \in \mathscr{R}$ and a fixed $q > 0$, the problem (5.2.1) has a unique solution within the class of C_b^1 given for each $q > 0$ and every $x \in \mathbb{R}$ by*

$$\mathscr{P}_G^{(q)}(x) \; = \; \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x}\frac{\widehat{G}(\lambda)}{\Psi_q^{(-)}(\lambda)}d\lambda. \tag{5.2.9}$$

The proof of this claim can be found in Section 6.

5.2.2 Fluctuation identity for first passage of Lévy processes

We present in this section a fluctuation identity for the first passage above or below a certain level of a Lévy process. To a fixed level $y \in \mathbb{R}$ we associate the first strict passage time τ_y^- (resp. τ_y^+) below (resp. above) y defined by

$$\tau_y^- \; = \; \inf\{t > 0 : X_t \; < \; y\} \quad \text{and} \quad \tau_y^+ \; = \; \inf\{t > 0 : X_t \; > \; y\}. \tag{5.2.10}$$

The identity is stated as follows.

Lemma 5.2.6 *Let X be a Lévy process under the hypothesis $(H1)$. Suppose that $\mathscr{P}_G^{(q)}$ solves the problem (5.2.1). Then for every $x, y \in \mathbb{R}$ such that $x \geq y$, we have*

$$\mathbb{E}_x\Big(e^{-q\tau_y^-}G(X_{\tau_y^-})\mathbf{1}_{(\tau_y^- \, < \, \infty)}\Big) \; = \; \mathbb{E}_x\Big(\mathscr{P}_G^{(q)}(\underline{X}_{\mathbf{e}_q})\mathbf{1}_{(\underline{X}_{\mathbf{e}_q}<y)}\Big). \tag{5.2.11}$$

Proof The proof is mainly based on the fact that conditionally on $\mathcal{F}_{\tau_y^-}$ and on the event $\{\mathbf{e}_q > \tau_y^-\}$, $\underline{X}_{\mathbf{e}_q} - X_{\tau_y^-}$ is independent of $\mathcal{F}_{\tau_y^-}$, and has the same distribution as $\underline{X}_{\mathbf{e}_q}$, thanks to the lack of memory property of exponential random variable $\mathbf{e}_q$ and the stationary independent increment of X. Combined with the fact that the function $\mathscr{P}_G^{(q)}$ solves the problem (5.2.1), we see that

$$\begin{aligned}
\mathbb{E}_x\Big(\mathscr{P}_G^{(q)}(\underline{X}_{\mathbf{e}_q})\mathbf{1}_{(\underline{X}_{\mathbf{e}_q}<y)}\Big) \; &= \; \mathbb{E}_x\Big(\mathscr{P}_G^{(q)}(\underline{X}_{\mathbf{e}_q})\mathbf{1}_{(\mathbf{e}_q>\tau_y^-)}\Big) \\
&= \; \mathbb{E}_x\Big(\mathbb{E}\big(\mathscr{P}_G^{(q)}(\underline{X}_{\mathbf{e}_q})\mathbf{1}_{(\mathbf{e}_q>\tau_y^-)}\big|\mathcal{F}_{\tau_y^-}\big)\Big) \\
&= \; \mathbb{E}_x\Big(\mathbf{1}_{(\mathbf{e}_q>\tau_y^-)}\mathbb{E}\big(\mathscr{P}_G^{(q)}(\underline{X}_{\mathbf{e}_q})\big|\mathcal{F}_{\tau_y^-}\big)\Big) \\
&= \; \mathbb{E}_x\Big(\mathbf{1}_{(\mathbf{e}_q>\tau_y^-)}\mathbb{E}\big(\mathscr{P}_G^{(q)}(\underline{X}_{\mathbf{e}_q} - X_{\tau_y^-} + X_{\tau_y^-})\big|\mathcal{F}_{\tau_y^-}\big)\Big) \\
&= \; \mathbb{E}_x\Big(\mathbf{1}_{(\mathbf{e}_q>\tau_y^-)}\mathbb{E}_{X_{\tau_y^-}}\big(\mathscr{P}_G^{(q)}(\underline{X}_{\mathbf{e}_q} - X_{\tau_y^-})\big|\mathcal{F}_{\tau_y^-}\big)\Big) \\
&= \; \mathbb{E}_x\Big(\mathbf{1}_{(\mathbf{e}_q>\tau_y^-)}\mathbb{E}_{X_{\tau_y^-}}\big(\mathscr{P}_G^{(q)}(\underline{X}_{\mathbf{e}_q})\big)\Big) \\
&= \; \mathbb{E}_x\Big(\mathbf{1}_{(\mathbf{e}_q>\tau_y^-)}G(X_{\tau_y^-})\Big) \\
&= \; \mathbb{E}_x\Big(e^{-q\tau_y^-}G(X_{\tau_y^-})\Big),
\end{aligned}$$

which indeed establishes our claim. □

Example 5.2.7 Now let us consider, for a given $\theta \geq 0$, the function $G(x) = e^{\theta x}$. Then using (5.2.2), we deduce from the foregoing theorem that

$$\mathbb{E}_x\Big(e^{-q\tau_y^- + \theta X_{\tau_y^-}} \mathbf{1}_{(\tau_y^- < \infty)}\Big) = \frac{\mathbb{E}_x\big(e^{\theta \underline{X}_{\mathbf{e}_q}} \mathbf{1}_{(\underline{X}_{\mathbf{e}_q} < y)}\big)}{\mathbb{E}\big(e^{\theta \underline{X}_{\mathbf{e}_q}}\big)}. \tag{5.2.12}$$

This identity can be analytically extended to $\theta \in \mathbb{C}$ with $\mathfrak{Re}(\theta) > 0$. This particular fluctuation identity goes back to the work of Darling et al. [33] for random walks, and has been extended to continuous-time, among others, by Alili and Kyprianou [3], Asmussen et al. [6], and Mordecki [87], and was used to solve the optimal stopping problem (5.1.1) with payoff function $G(x) = (K - e^x)^+$.

By replacing X with its dual $\widehat{X} = -X$ and y with $-y$, the problem of first exit above a level y for X can be transformed into the problem of first exit of $\widehat{X}$ below a level $-y$. The following result is the dual form of Lemma 5.2.6.

Corollary 5.2.8 *Let X be a Lévy process with the assumption that*

$$\text{either } q > 0 \text{ or } \Big(q = 0 \text{ and } \mathbb{P}\big(\limsup_{t\to\infty} X_t < \infty\big) = 1\Big). \tag{H2}$$

Suppose that for a given continuous function G and $q \geq 0$, $\mathscr{C}_G^{(q)}$ solves the problem

$$\mathbb{E}\Big(\mathscr{C}_G^{(q)}\big(x + \overline{X}_{\mathbf{e}_q}\big)\Big) = G(x), \quad \text{for every } x \in \mathbb{R}. \tag{5.2.13}$$

Then for every $x, y \in \mathbb{R}$ such that $x \leq y$, we have

$$\mathbb{E}_x\Big(e^{-q\tau_y^+} G(X_{\tau_y^+})\mathbf{1}_{(\tau_y^+ < \infty)}\Big) = \mathbb{E}_x\Big(\mathscr{C}_G^{(q)}(\overline{X}_{\mathbf{e}_q})\mathbf{1}_{(\overline{X}_{\mathbf{e}_q} > y)}\Big). \tag{5.2.14}$$

Apart from exponential, linear combination of exponential, polynomials, and sufficiently regular functions, we provide here another example of solution to the problem (5.2.13).

Example 5.2.9 (Appell functions with index $\nu < 0$ and $\nu > 0$) It was shown recently by Novikov and Shiryaev [92] that it is possible to construct Appell functions $Q_\nu(x)$ associated with the random variable $\overline{X}_{\mathbf{e}_q}$ with index $\nu < 0$ and $\nu > 0$. What we shall say below is based on [92]. The construction is based on the *Esscher-Mellin transform*

$$\mathscr{C}_G^{(q)}(x;\nu) = \frac{1}{\Gamma(-\nu)} \int_0^\infty \lambda^{-\nu-1} \frac{e^{-\lambda x}}{\mathbb{E}\big(e^{-\lambda \overline{X}_{\mathbf{e}_q}}\big)} d\lambda, \quad \text{for } \nu < 0. \tag{5.2.15}$$

Following this, we see for $x > 0$ and $\nu < 0$ that

$$\frac{d\mathscr{C}_G^{(q)}}{dx}(x;\nu) = \frac{\nu}{\Gamma(1-\nu)} \int_0^\infty \lambda^{-\nu} \frac{e^{-\lambda x}}{\mathbb{E}\big(e^{-\lambda \overline{X}_{\mathbf{e}_q}}\big)} d\lambda = \nu \mathscr{C}_G^{(q)}(x;\nu - 1), \tag{5.2.16}$$

and (5.2.15) solves the problem (5.2.13) for $G(x) = x^\nu$. That is to say

$$\mathbb{E}\Big(\mathscr{C}_G^{(q)}(x+\overline{X}_{\mathbf{e}_q};\nu)\Big) = \frac{1}{\Gamma(-\nu)}\int_0^\infty \lambda^{-\nu-1}\frac{\mathbb{E}\big(e^{-\lambda(x+\overline{X}_{\mathbf{e}_q})}\big)}{\mathbb{E}\big(e^{-\lambda\overline{X}_{\mathbf{e}_q}}\big)}d\lambda \\ = x^\nu, \quad \text{for } x>0. \tag{5.2.17}$$

An analytical continuation of the function $\nu \mapsto \mathscr{C}_G^{(q)}(x;\nu)$, $\nu < 0$, to the region $\nu > 0$ can be constructed with the help of (5.2.16), see Novikov and Shiryaev [92] for more details. Thus, we see that the Appell function $\mathscr{C}_G^{(q)}(x;\nu)$ is a solution to the problem (5.2.13) for $G(x) = x^\nu$.

Example 5.2.10 Let $\zeta = \mathbf{1}_{(q=0)}$. Assume that the assumption $(H2)$ holds and that the Lévy measure Π satisfies the integrability condition

$$\int_{(1,\infty)} x^{n+\zeta}\Pi(dx) < \infty.$$

For a fixed $n = 1, 2, \cdots$, let us consider a function $G(x) = x^n$. Then using (5.2.15), we deduce for every $x \in \mathbb{R}$ and $n = 1, 2, \cdots$ that

$$\mathbb{E}_x\Big(e^{-q\tau_y^+}X^n_{\tau_y^+}\mathbf{1}_{(\tau_y^+<\infty)}\Big) = \mathbb{E}_x\Big(Q_n(\overline{X}_{\mathbf{e}_q})\mathbf{1}_{(\overline{X}_{\mathbf{e}_q}>y)}\Big),$$

where Q_n, $n = 1, 2, \cdots$, are now the Appell polynomials generated by the random variable $\overline{X}_{\mathbf{e}_q}$. This particular fluctuation identity goes back to the work of Darling et al. [33] for random walks, and was used recently by Novikov and Shiryaev [91] and Kyprianou and Surya [73] to solve the optimal stopping problem (5.1.1) with integer power function $G(x) = (x^+)^n$ of random walks and Lévy processes, respectively.

If the function $\mathscr{P}_G^{(q)}$ (resp. $\mathscr{C}_G^{(q)}$), solving the problem (5.2.1) (resp. (5.2.13)), has a certain monotonicity property, using the fluctuation identity (5.2.11) (resp. (5.2.14)), we will show in the next section that the optimal solution to the problem (5.1.1), with payoff function G, can be written in terms of the function $\mathscr{P}_G^{(q)}$ (resp. $\mathscr{C}_G^{(q)}$).

5.3 General results on optimal stopping problems

In this section, we present a general solution to the perpetual optimal stopping problem (5.1.1). The solution is expressed in terms of the function $\mathscr{P}_G^{(q)}$ (resp. $\mathscr{C}_G^{(q)}$) that solves the problem (5.2.1) (resp. (5.2.13)).

5.3.1 American put-type optimal stopping problems

A general solution to the problem (5.1.1) is given by the following theorem.

Theorem 5.3.1 (General solution) *Suppose that $\mathscr{P}_G^{(q)}$ is a continuous function that solves the problem (5.2.1), and there exists $\widehat{x} \in \mathbb{R}$ such that $\mathscr{P}_G^{(q)}(\widehat{x}) = 0$, $\mathscr{P}_G^{(q)}(x)$ is non-increasing for $x < \widehat{x}$, and $\mathscr{P}_G^{(q)}(x) \leq 0$ for $x > \widehat{x}$, under the assumption that $(H1)$ holds. Denote by $x^\star$ the smallest root of the equation*

$$\mathscr{P}_G^{(q)}(x) \ = \ 0. \tag{5.3.1}$$

Then the optimal solution to the problem (5.1.1), with payoff G, is given by

$$V_{x^\star}(x) \ = \ \mathbb{E}\Big(\mathscr{P}_G^{(q)}(x + \underline{X}_{\mathbf{e}_q})\mathbf{1}_{\{x+\underline{X}_{\mathbf{e}_q} < x^\star\}}\Big), \tag{5.3.2}$$

for every $x \in \mathbb{R}$ while the optimal stopping time is given by

$$\tau_{x^\star}^- \ = \ \inf\{t > 0 : X_t \ < \ x^\star\}. \tag{5.3.3}$$

That is to say that

$$V_{x^\star}(x) \ = \sup_{\tau \in \mathcal{T}_{[0,\infty]}} \mathbb{E}_x\Big(e^{-q\tau}G(X_\tau)\mathbf{1}_{(\tau\,<\,\infty)}\Big) \ = \ \mathbb{E}_x\Big(e^{-q\tau_{x^\star}^-}G(X_{\tau_{x^\star}^-})\mathbf{1}_{(\tau_{x^\star}^-\,<\,\infty)}\Big).$$

The result of the previous theorem still holds true while the payoff G is replaced by the function $\widetilde{G}(x) = \max\{G(x), 0\}$. The following lemma establishes this claim.

Lemma 5.3.2 *Let $\widetilde{V}(x)$ be the value function of the problem (5.1.1) under the payoff function $\widetilde{G}(x) = \max\{G(x), 0\}$. Then for all $x \in \mathbb{R}$, we see that $V(x) \ = \ \widetilde{V}(x)$.*

To obtain the main result in Theorem 5.3.1, the following lemma is needed.

Lemma 5.3.3 (Candidate solution) *Suppose that $\mathscr{P}_G^{(q)}$ is a continuous function that solves the problem (5.2.1) and fulfills the requirements of Theorem 5.3.1. Define for every $y \in \mathbb{R}$ and $q \geq 0$ a candidate solution to the problem (5.1.1) as*

$$V_y(x) \triangleq \mathbb{E}\Big(\mathscr{P}_G^{(q)}(x + \underline{X}_{\mathbf{e}_q})\mathbf{1}_{\{x+\underline{X}_{\mathbf{e}_q} < y\}}\Big). \tag{5.3.4}$$

Let $x^\star$ be the smallest root of the equation (5.3.1). Then it holds true that

(i) *for all $x, y \in \mathbb{R}$ such that $x < y$, we have*

$$V_y(x) \ = \ G(x);$$

(ii) *for any $x \in \mathbb{R}$, we have*

$$V_{x^\star}(x) \ \geq \ G(x);$$

(iii) *and $\{e^{-qt}V_{x^\star}(X_t), t \geq 0\}$ is a $\mathbb{P}_x$-supermartingale.*

As a result of the optimality of the function $V_{x^\star}$, we have the following result.

Proposition 5.3.4 *For every $x, y \in \mathbb{R}$, it is then true that*

$$V_{x^\star}(x) \;\geq\; V_y(x); \tag{5.3.5}$$

and if $y < x^\star$, then there exists x such that

$$V_y(x) \;<\; G(x). \tag{5.3.6}$$

5.3.2 American call-type optimal stopping problems

By replacing X with its dual $\widehat{X} = -X$ and y with $-y$, the problem of first exit above a level y for X can be transformed into the problem of first exit of $\widehat{X}$ below a level $-y$. Thus, the results below are the obvious dual forms with respect to those achieved previously for the American put-type optimal stopping problem.

Theorem 5.3.5 (General solution) *Suppose that $\mathscr{C}_G^{(q)}$ is a continuous function that solves the problem (5.2.13), and there exists $\widehat{x} \in \mathbb{R}$ such that $\mathscr{C}_G^{(q)}(\widehat{x}) = 0$, $\mathscr{C}_G^{(q)}(x)$ is non-decreasing for $x > \widehat{x}$, and $\mathscr{C}_G^{(q)}(x) \leq 0$ for $x < \widehat{x}$, under the assumption that $(H2)$ holds. Denote by $x^\star$ the largest root of the equation*

$$\mathscr{C}_G^{(q)}(x) \;=\; 0. \tag{5.3.7}$$

Then the optimal solution to the problem (5.1.1), with payoff G, is given by

$$V_{x^\star}(x) \;=\; \mathbb{E}\Big(\mathscr{C}_G^{(q)}(x + \overline{X}_{\mathbf{e}_q})\mathbf{1}_{\{x+\overline{X}_{\mathbf{e}_q} > x^\star\}}\Big), \tag{5.3.8}$$

for every $x \in \mathbb{R}$ while the optimal stopping time is given by

$$\tau^+_{x^\star} \;=\; \inf\{t > 0 : X_t \; > x^\star\}. \tag{5.3.9}$$

That is to say that

$$V_{x^\star}(x) \;=\; \sup_{\tau \in \mathcal{T}_{[0,\infty]}} \mathbb{E}_x\Big(e^{-q\tau} G(X_\tau)\mathbf{1}_{(\tau\,<\,\infty)}\Big) \;=\; \mathbb{E}_x\Big(e^{-q\tau^+_{x^\star}} G(X_{\tau^+_{x^\star}})\mathbf{1}_{(\tau^+_{x^\star}\,<\,\infty)}\Big).$$

It can be shown in the similar way as before that the result of the previous theorem still holds true when the payoff G is replaced by $\widetilde{G}(x) = \max\{G(x), 0\}$.

5.4 The continuous and smooth pasting principles

In this section, we discuss the behaviour of the candidate solution (5.3.4) of an American put-type optimal stopping problem at a stopping boundary $y \in \mathbb{R}$. The behaviour of the candidate solution $V_y(x) = \mathbb{E}\big(\mathscr{C}_G^{(q)}(x + \overline{X}_{\mathbf{e}_q})\mathbf{1}_{\{x+\overline{X}_{\mathbf{e}_q} > y\}}\big)$ of an American call-type optimal stopping problem can be obtained similarly.

Firstly, assume that the solution is evaluated at the optimal stopping boundary $x^\star$ (5.3.1). Then we have the following results.

Theorem 5.4.1 *Suppose that the functions G and $\mathscr{P}_G^{(q)}$ are continuously differentiable. Then, the optimal value function (5.3.2) of the problem (5.1.1) is continuous at the optimal stopping boundary $x^\star$, i.e.,*

$$V_{x^\star}(x^\star) \;=\; G(x^\star),$$

and has the property that

$$\frac{dV_{x^\star}}{dx}(x^\star) \;=\; \frac{dG}{dx}(x^\star) - \mathbb{P}\big(-\underline{X}_{\mathbf{e}_q} = 0\big)\frac{d\mathscr{P}_G^{(q)}}{dx}(x^\star). \tag{5.4.1}$$

Hence there is C^1 smooth pasting at $x^\star$ if and only if

$$\mathbb{P}(-\underline{X}_{\mathbf{e}_q} = 0) \;=\; 0,$$

that is when 0 is regular for the lower half-line $(-\infty, 0)$ for the Lévy process X.

The theory of Lévy processes offers the opportunity to specify when regularity of 0 for the lower half-line $(-\infty, 0)$ (resp. for the upper half-line $(0, \infty)$) for the Lévy process X occurs in terms of the triplet (μ, σ, Π) of the Lévy-Khintchine exponent (2.1.1). When X has bounded variation, it will be more convenient to write (2.1.1) in the form

$$\Psi(\theta) \;=\; -i\mathrm{d}\theta + \int_{-\infty}^{\infty}\big(1 - e^{i\theta x}\big)\Pi(dx).$$

Theorem 5.4.2 (Regularity of half-line for Lévy processes) *Suppose that X is any Lévy process other than a compound Poisson process. Denote the upper and lower tails $\overline{\Pi}^{\pm}$ of the Lévy measure Π by*

$$\overline{\Pi}^+(x) = \Pi((x,\infty)), \quad \textit{and} \quad \overline{\Pi}^-(x) = \Pi((-\infty, x)).$$

We have that 0 is regular for $(-\infty, 0)$ (respectively, for $(0, \infty)$) for X if and only if one of the following conditions[3] *is satisfied:*

(i) *X has bounded variation with $\mathrm{d} < 0$ (respectively, with $\mathrm{d} > 0$).*

(ii) *X has bounded variation, $\mathrm{d} = 0$, and the Lévy measure Π satisfies*

$$\int_{-1}^{0-} \frac{|x|\Pi(dx)}{\int_0^{|x|}\overline{\Pi}^+(y)dy} \;=\; \infty, \quad \left(\textit{respectively, } \int_0^1 \frac{x\Pi(dx)}{\int_0^x \overline{\Pi}^-(-y)dy} \;=\; \infty\right)$$

(iii) *X has unbounded variation.*

On the other hand, when the candidate solution (5.3.4) to the problem (5.1.1) is evaluated at a stopping boundary $y \in \mathbb{R}$ other than the optimal stopping boundary $x^\star$ (5.3.1), we have the following results.

[3] See for instance [16], [3], [73], and the literature therein for more details on regularity of half-line for Lévy processes.

Theorem 5.4.3 *Suppose that the functions G and $\mathscr{P}_G^{(q)}$ are continuously differentiable. Let us denote by $P_q^{(-)}(x) := \mathbb{P}\big(-\underline{X}_{\mathbf{e}_q} \leq x\big)$ the distribution function of the random variable $-\underline{X}_{\mathbf{e}_q}$. Consider the candidate solution (5.3.4) of the problem (5.1.1)*

$$V_y(x) \; = \; \mathbb{E}\Big(\mathscr{P}_G^{(q)}(x+\underline{X}_{\mathbf{e}_q})\mathbf{1}_{\{x+\underline{X}_{\mathbf{e}_q}<y\}}\Big).$$

If the limit

$$p_q^{(-)}(x) \; \triangleq \lim_{h\downarrow 0}\frac{1}{h}\Big(P_q^{(-)}(x+h) - P_q^{(-)}(x)\Big) \tag{5.4.2}$$

exists for every $x \in \mathbb{R}$, then we see at $x = y$ that

$$V_y(y) \; = \; G(y) - \mathbb{P}\big(-\underline{X}_{\mathbf{e}_q} = 0\big)\mathscr{P}_G^{(q)}(y), \tag{5.4.3}$$

and the derivative at $x = y$ of the function $V_y(x)$ is given by

$$\frac{dV_y}{dx}(y) \; = \frac{dG}{dx}(y) - \mathbb{P}\big(-\underline{X}_{\mathbf{e}_q} = 0\big)\frac{d\mathscr{P}_G^{(q)}}{dx}(y) - p_q^{(-)}(0)\mathscr{P}_G^{(q)}(y). \tag{5.4.4}$$

Hence, when $y \neq x^\star$ we see that there is discontinuity for the candidate solution $V_y(x)$ and its derivative at $x = y$ when X is a Lévy process for which 0 is irregular for $(-\infty, 0)$ for X. In the regular case, there is only discontinuity for the derivative. Additionally, if $|p_q^{(-)}(0)| = \infty$, then there is an infinite gradient of the candidate solution $V_y(x)$ at the point $x = y$.

Below are examples of Lévy processes for which $p_q^{(-)}(0) = \infty$.

Lemma 5.4.4 *Suppose that the Lévy measure Π of X has no atoms and $\Pi(0,\infty) = 0$ so that the Laplace exponent of X exists and is given by*

$$\kappa(\lambda) \; = \; -\Psi(-i\lambda) \; = \; \mathrm{d}\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{(-\infty,0)}\big(e^{\lambda x} - 1 - \lambda x\mathbf{1}_{\{x>-1\}}\big)\Pi(dx).$$

For every $q \geq 0$ the value of the limit (5.4.2) at zero is given by

$$p_q^{(-)}(0) = \begin{cases} \frac{2q}{\sigma^2\Phi(q)}, & \text{when } X \text{ has unbounded variation and } \sigma \neq 0, \\ \infty, & \text{when } X \text{ has unbounded variation with } \sigma = 0, \\ \infty, & \text{when } X \text{ has bounded variation and } \Pi(-\infty,0) = \infty, \\ \frac{q(\Pi(-\infty,0)+q)}{\mathrm{d}^2\Phi(q)} - \frac{q}{\mathrm{d}}, & \text{when } X \text{ has bounded variation and } \Pi(-\infty,0) < \infty, \end{cases}$$

where $\Phi(q) = \sup\{\lambda : \kappa(\lambda) = q\}$ is the largest root of the equation $\kappa(x) - q = 0$.

Example 5.4.5 (Regular Lévy process of exponential type [21]) Suppose that X is a regular Lévy process of exponential type[4] with the Wiener-Hopf factor

$$\Psi_q^{(-)}(\lambda) \ = \ C(\alpha + i\lambda)^{-\beta} + \widehat{f}(\lambda), \tag{5.4.5}$$

where α and C are positive constants, $\beta \in (0,1]$, and $\widehat{f}(\lambda) = \mathcal{O}((1+|\lambda|)^{-s})$ as $\lambda \to \infty$ for some $s > 1$. Hence, f, the inverse Fourier transform of $\widehat{f}$, is a bounded continuous function. Since it is known that

$$\int_{-\infty}^{\infty} e^{-i\lambda x} x^{\nu-1} e^{-\alpha x} \mathbf{1}_{(0,\infty)}(x) dx \ = \ \Gamma(\nu)(\alpha + i\lambda)^{-\nu}, \quad \text{for } \nu > 0,$$

following the Wiener-Hopf factorization (2.2.6), discussed in Section 2.2 of Chapter 2, we deduce from equation (5.4.5) that the function

$$p_q^{(-)}(x) \ = \ C\Gamma(\beta)^{-1} x^{\beta-1} e^{-\alpha x} \mathbf{1}_{(0,\infty)}(x) + f(x),$$

is continuous on $(0, +\infty)$ and is unbounded as $x \to 0$. On noticing the fact that $p_q^{(-)}$ represents the density of the distribution function $\mathbb{P}(-\underline{X}_{\mathbf{e}_q} \le x)$, it is clear following Theorem 5.4.3 that there exists an infinite gradient for the candidate solution $V_y(x)$ at a stopping boundary y, unless $y = x^\star$ or $\beta = 1$.

5.5 Consistency with existing literature

In this section, we use our approach to reproduce the special results of those discussed, among others, by Darling et al. [33], Mordecki [87], Boyarchenko and Levendorskii [21], Novikov and Shiryaev [91], [92], and Kyprianou and Surya [73].

Example 5.5.1 (Option with a relatively general payoff) In [21], Boyarchenko and Levendorskii considered perpetual optimal stopping problems of American put option type with a relatively general payoff function G of the form

$$G(x) \ = \ \sum_{j=1}^{m} c_j e^{\theta_j x}, \quad \text{with } \theta_j \ge 0, \tag{5.5.1}$$

for a class of regular Lévy processes of exponential type which includes normal inverse Gaussian, hyperbolic processes, tempered stable processes, and Variance Gamma processes. Using the result (5.2.3) of Section 2.1, we see that the function

$$\mathscr{P}_G^{(q)}(x) \ = \ \sum_{j=1}^{m} c_j \Psi_q^{(-)}(-i\theta_j)^{-1} e^{\theta_j x}$$

[4] This is a process of pure jumps whose characteristic exponent is given for $c > 0$, $\kappa_- < 0 < \kappa_+$, $\nu \in (0,2]$ and $\nu_1 < \nu$ by $\Psi(\lambda) = -i\mu\lambda + \phi(\lambda)$ where $\phi(\lambda) = c|\lambda|^\nu + \mathcal{O}(|\lambda|^{\nu_1})$ as $\lambda \to \infty$ in the strip $\mathfrak{Im}(\lambda) \in [\kappa_-, \kappa_+]$. This type of Lévy process was considered by Boyarchenko and Levendorskii [21]. They showed in [21] that under some regularity conditions imposed on $\phi(\lambda)$ the Wiener-Hopf factor $\Psi_q^{(-)}(\lambda)$ (2.2.6) is of the form (5.4.5).

solves for a given $q \geq 0$ and the payoff function G the averaging problem (5.2.1). Denote by $x^\star$ the root of the equation $\mathscr{P}_G^{(q)}(x) = 0$. Suppose that the measure $\mathbb{P}\big(-\underline{X}_{\mathbf{e}_q} \in dx\big)$ is absolutely continuous w.r.t the Lebesque measure dx. (This assumption was used in Theorem 4.6 of [21] to prove the optimality of the stopping time $\tau_{x^\star}^-$).

By applying Fourier transform, with $\mathfrak{Im}(\lambda) = \sigma$, for some $\sigma > 0$, to the optimal value function (5.3.2), we come to rest at the following expression

$$\int_{-\infty}^{\infty} e^{-i\lambda x} V_{x^\star}(x)dx \;=\; \widehat{w}(\lambda)\Psi_q^{(-)}(\lambda), \tag{5.5.2}$$

where $\widehat{w}(\lambda)$ is the Fourier transform, with $\mathfrak{Im}(\lambda) = \sigma$, for some $\sigma > 0$, of the function $x \mapsto \mathscr{P}_G^{(q)}(x)\mathbf{1}_{(-\infty,x^\star)}(x)$ defined by

$$\widehat{w}(\lambda) \;=\; \int_{-\infty}^{\infty} e^{-i\lambda x}\mathscr{P}_G^{(q)}(x)\mathbf{1}_{(-\infty,x^\star)}(x)dx.$$

The result in (5.5.2) was given by Boyarchenko and Levendorskii [21] (see Section 4.2 of [21]). They obtained the result by reducing the problem (5.1.1) using potential theory of Lévy processes and Dynkin's formula to a free boundary problem and solving the latter using the standard theory of pseudodifferential operators.

Example 5.5.2 (Perpetual American put option) Let us consider an optimal stopping problem (5.1.1) with the payoff function $G(x) = K - e^x$ under the hypothesis $(H1)$. Applying the result in (5.2.2), we see that $\mathscr{P}_G^{(q)}(x) = K - \Psi_q^{(-)}(-i)^{-1}e^x$ solves the averaging problem (5.2.1) for a given $q \geq 0$ and the payoff function G.

According to Theorem 5.3.1, the optimal stopping boundary $y = x^\star$ is determined as the smallest root of the equation

$$0 \;=\; \mathscr{P}_G^{(q)}(x) \;=\; K - \Psi_q^{(-)}(-i)^{-1}e^x.$$

That is to say that $e^{x^\star} \;=\; K\Psi_q^{(-)}(-i) \;=\; K\mathbb{E}\big(e^{\underline{X}_{\mathbf{e}_q}}\big)$. The rational price can be calculated using the explicit pricing formula (5.3.2) and is given by

$$\begin{aligned} V_{x^\star}(x) &= \mathbb{E}\Big(\mathscr{P}_G^{(q)}(x+\underline{X}_{\mathbf{e}_q})\mathbf{1}_{\{x+\underline{X}_{\mathbf{e}_q}<x^\star\}}\Big) \\ &= \mathbb{E}\Big(\big(K - \Psi_q^{(-)}(-i)^{-1}e^{(x+\underline{X}_{\mathbf{e}_q})}\big)\mathbf{1}_{\{x+\underline{X}_{\mathbf{e}_q}<x^\star\}}\Big) \\ &= \frac{\mathbb{E}\big(K\mathbb{E}\big(e^{\underline{X}_{\mathbf{e}_q}}\big) - e^{(x+\underline{X}_{\mathbf{e}_q})}\big)^+}{\mathbb{E}\big(e^{\underline{X}_{\mathbf{e}_q}}\big)}. \end{aligned} \tag{5.5.3}$$

This expression for the rational price was given by Mordecki [87].

Example 5.5.3 (Perpetual American call option) Let us consider the optimal stopping problem (5.1.1) with the payoff function $G(x) = e^x - K$ under the hypothesis $(H2)$. Similar to (5.2.2), it is clear that $\mathscr{C}_G^{(q)}(x) \;=\; \Psi_q^{(+)}(-i)^{-1}e^x - K$ solves the averaging problem (5.2.13).

According to Theorem 5.3.5, the optimal stopping boundary $y = x^\star$ is determined as the largest root of the equation

$$\mathscr{C}_G^{(q)}(x) \ = \ \Psi_q^{(+)}(-i)^{-1}e^x - K = 0.$$

That is to say that $e^{x^\star} \ = \ K\Psi_q^{(+)}(-i) \ = \ K\mathbb{E}\big(e^{\overline{X}_{\mathbf{e}_q}}\big)$. The rational price can be calculated using the explicit pricing formula (5.3.8) and is given by

$$\begin{aligned} V_{x^\star}(x) &= \mathbb{E}\Big(\mathscr{C}_G^{(q)}(x + \overline{X}_{\mathbf{e}_q})\mathbf{1}_{\{x+\overline{X}_{\mathbf{e}_q} > x^\star\}}\Big) \\ &= \mathbb{E}\Big(\big(\Psi_q^{(+)}(-i)^{-1}e^{x+\overline{X}_{\mathbf{e}_q}} - K\big)\mathbf{1}_{\{x+\overline{X}_{\mathbf{e}_q} > x^\star\}}\Big) \\ &= \frac{\mathbb{E}\big(e^{(x+\overline{X}_{\mathbf{e}_q})} - K\mathbb{E}\big(e^{\overline{X}_{\mathbf{e}_q}}\big)\big)^+}{\mathbb{E}\big(e^{\overline{X}_{\mathbf{e}_q}}\big)}. \end{aligned} \tag{5.5.4}$$

This solution was given by Darling et al [33] for random walks and by Mordecki in [87] for continuous time.

Example 5.5.4 (Option with integer power function) This is a special type of optimal stopping problem where the payoff is an integer power function $G(x) = (x^+)^n$, $n = 1, 2, ...,$ of the underlying process. For random walks, this problem was introduced by Novikov and Shiryaev [91] and was extended to continuous time by Kyprianou and Surya [73]. Similar to (5.2.15), it is clear for $\nu = n = 1, 2, ...$ that

$$\mathscr{C}_G^{(q)}(x) \ = \ Q_\nu(x).$$

According to Theorem 5.3.5, the optimal boundary $y = x^\star$ is determined as the largest root of the equation $Q_n(x) = 0$ and the optimal value function is given by

$$\begin{aligned} V_{x^\star}(x) &= \mathbb{E}\Big(\mathscr{C}_G^{(q)}(x + \overline{X}_{\mathbf{e}_q})\mathbf{1}_{\{x+\overline{X}_{\mathbf{e}_q} > x^\star\}}\Big) \\ &= \mathbb{E}\Big(Q_n(x + \overline{X}_{\mathbf{e}_q})\mathbf{1}_{\{x+\overline{X}_{\mathbf{e}_q} > x^\star\}}\Big). \end{aligned}$$

This result is equal to the one given by Novikov and Shiryaev [91] for discrete time and to the one in Kyprianou and Surya [73] for continuous time.

In particular, for $n = 1$, we have $Q_1(x) \ = \ x - \mathbb{E}\big(\overline{X}_{\mathbf{e}_q}\big)$ and the optimal boundary is given by $x^\star = \mathbb{E}\big(\overline{X}_{\mathbf{e}_q}\big)$. The optimal value function is given by

$$\begin{aligned} V_{x^\star}(x) &= \mathbb{E}\Big(\big(x + \overline{X}_{\mathbf{e}_q} - \mathbb{E}\big(\overline{X}_{\mathbf{e}_q}\big)\big)\mathbf{1}_{\{x+\overline{X}_{\mathbf{e}_q} > x^\star\}}\Big) \\ &= \mathbb{E}\Big(x + \overline{X}_{\mathbf{e}_q} - \mathbb{E}\big(\overline{X}_{\mathbf{e}_q}\big)\Big)^+. \end{aligned}$$

This result was also given by Darling et al [33] for payoff function $G(x) = x^+$ under random walks.

Remark 5.5.5 It was shown recently by Novikov and Shiryaev [92] that it is possible to extend the result of Darling et al. [33] to the function x^ν, for $\nu < 0$ and $\nu > 0$, as a payoff function of random walks and Lévy processes. Our results presented in Section 4 show no contradiction with their results for the case of Lévy processes.

Thus, we have seen that our results in Theorems 5.3.1 and 5.3.5 are consistent with those provided in the above mentioned literature.

5.6 Proofs and main calculations

Proof of Lemma 5.2.5

Proving uniqueness of the solution

To see that (5.2.9) is the solution to the problem (5.2.1), let us first show that the integral in (5.2.9) exists. Due to the regularity assumption (5.2.7) imposed on the payoff function G, using the Wiener-Hopf factorization (2.2.5), we see for $q > 0$ that

$$\begin{aligned}\int_{-\infty}^{\infty}\Big|\frac{\widehat{G}(\lambda)}{\Psi_q^{(-)}(\lambda)}\Big|d\lambda &= \int_{-\infty}^{\infty}\Big|\frac{\Psi_q^{(+)}(\lambda)\widehat{G}(\lambda)}{\Psi_q^{(-)}(\lambda)\Psi_q^{(+)}(\lambda)}\Big|d\lambda \\ &= q^{-1}\int_{-\infty}^{\infty}\big|\big(q+\Psi(\lambda)\big)\big|\big|\Psi_q^{(+)}(\lambda)\big|\big|\widehat{G}(\lambda)\big|d\lambda \\ &\le q^{-1}\int_{-\infty}^{\infty}\big|\big(q+\Psi(\lambda)\big)\big|\big|\widehat{G}(\lambda)\big|d\lambda \\ &\le \int_{-\infty}^{\infty}\big|\widehat{G}(\lambda)\big|d\lambda + q^{-1}\int_{-\infty}^{\infty}\big|\Psi(\lambda)\big|\big|\widehat{G}(\lambda)\big|d\lambda.\end{aligned} \tag{5.6.1}$$

In view of (5.2.7), it is clear that the first integral in (5.6.1) is finite. To see that the second integral is finite, we need to take account on the fact that

$$\big|e^{i\lambda x}-1-i\lambda x\mathbf{1}_{\{|x|\le 1\}}\big| \;\le\; \frac{1}{2}|\lambda|^2|x|^2\mathbf{1}_{\{|x|\le 1\}}+2\mathbf{1}_{\{|x|>1\}},$$

which, following (2.1.1), implies that

$$\big|\Psi(\lambda)\big| \le \mu|\lambda|+\frac{1}{2}|\lambda|^2\Big(\sigma^2+\int_{\{|y|\le 1\}}|y|^2\Pi(dy)\Big)+2\int_{\{|y|>1\}}\Pi(dy),$$

where the Lévy measure Π satisfies the integrability condition

$$\int_{-\infty}^{\infty}(1\wedge|y|^2)\Pi(dy) \;<\; \infty.$$

On observing that

$$\begin{aligned}\int_{-\infty}^{\infty}|\lambda|\big|\widehat{G}(\lambda)\big|d\lambda &= \int_{\{|\lambda|\le 1\}}|\lambda|\big|\widehat{G}(\lambda)\big|d\lambda+\int_{\{|\lambda|>1\}}|\lambda|\big|\widehat{G}(\lambda)\big|d\lambda \\ &\le \int_{-\infty}^{\infty}\big|\widehat{G}(\lambda)\big|d\lambda+\int_{-\infty}^{\infty}|\lambda|^3\big|\widehat{G}(\lambda)\big|d\lambda \;<\; \infty,\end{aligned}$$

and similarly

$$\int_{-\infty}^{\infty} |\lambda|^2 \big|\widehat{G}(\lambda)\big| d\lambda \;<\; \infty, \tag{5.6.2}$$

we see that the integral in (5.2.9) is convergent in absolute value.

We move now to showing that the function (5.2.9) is the solution to the averaging problem (5.2.1).

Taking account of the fact that every sufficiently regular function in $\mathscr{R}$ can be decomposed as the Fourier integral representation (5.2.8), and the Wiener-Hopf factor $\Psi_q^{(-)}(\lambda)$ is nowhere zero, we see for every $x \in \mathbb{R}$ that

$$\begin{aligned}
G(x) \;&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x}\widehat{G}(\lambda)d\lambda \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x}\mathbb{E}\big(e^{i\lambda \underline{X}_{\mathbf{e}_q}}\big)\frac{\widehat{G}(\lambda)}{\Psi_q^{(-)}(\lambda)}d\lambda \\
&= \mathbb{E}\Big(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda (x+\underline{X}_{\mathbf{e}_q})}\frac{\widehat{G}(\lambda)}{\Psi_q^{(-)}(\lambda)}d\lambda\Big) \\
&= \mathbb{E}\Big(\mathscr{P}_G^{(q)}(x+\underline{X}_{\mathbf{e}_q})\Big),
\end{aligned}$$

where the third equality was obtained by applying (in view of the integrability conditions (5.6.1)-(5.6.2)) Fubini's theorem.

Proving boundedness and continuous differentiability

Following (5.6.1)-(5.6.2), it is clear that the function $\mathscr{P}_G^{(q)}(x)$ is bounded in $\mathbb{R}$. To see that the function $\mathscr{P}_G^{(q)}(x)$ is continuous in $\mathbb{R}$, let us take $a \in \mathbb{R}$ and $\epsilon > 0$ arbitrarily such that for the chosen $\epsilon > 0$ there exists an $R > 1$ such that

$$\int_{-\infty}^{-R} \Big|\frac{\widehat{G}(\lambda)}{\Psi_q^{(-)}(\lambda)}\Big| d\lambda < \frac{\epsilon}{10} \quad \text{and} \quad \int_{R}^{\infty} \Big|\frac{\widehat{G}(\lambda)}{\Psi_q^{(-)}(\lambda)}\Big| d\lambda < \frac{\epsilon}{10}. \tag{5.6.3}$$

The existence of such an R is guaranteed by the fact that the integral (5.6.1) is finite. For such an R, we see for all $x \in \mathbb{R}$ that

$$\begin{aligned}
\Big|\int_{-\infty}^{-R} \frac{\widehat{G}(\lambda)}{\Psi_q^{(-)}(\lambda)} e^{i\lambda x} d\lambda\Big| \;&\le\; \int_{-\infty}^{-R} \Big|\frac{\widehat{G}(\lambda)}{\Psi_q^{(-)}(\lambda)}\Big|\Big|\cos(\lambda x)\Big| d\lambda + \int_{-\infty}^{-R} \Big|\frac{\widehat{G}(\lambda)}{\Psi_q^{(-)}(\lambda)}\Big|\Big|\sin(\lambda x)\Big| d\lambda \\
&\le 2\int_{-\infty}^{-R} \Big|\frac{\widehat{G}(\lambda)}{\Psi_q^{(-)}(\lambda)}\Big| d\lambda \;<\; \frac{\epsilon}{5}.
\end{aligned}$$

Likewise, following the similar arguments, we see that

$$\Big|\int_{R}^{\infty} \frac{\widehat{G}(\lambda)}{\Psi_q^{(-)}(\lambda)} e^{i\lambda x} d\lambda\Big| \;<\; \frac{\epsilon}{5}.$$

Hence, following (5.2.9), we see that

$$\begin{aligned}
\Big|\mathscr{P}_G^{(q)}(x)-\mathscr{P}_G^{(q)}(a)\Big| &= \Big|\int_{-\infty}^{\infty}\Big(\frac{\widehat{G}(\lambda)}{\Psi_q^{(-)}(\lambda)}\Big)e^{i\lambda x}d\lambda-\int_{-\infty}^{\infty}\Big(\frac{\widehat{G}(\lambda)}{\Psi_q^{(-)}(\lambda)}\Big)e^{i\lambda a}d\lambda\Big| \\
&< \frac{4\epsilon}{5}+\Big|\int_{-R}^{R}\Big(\frac{\widehat{G}(\lambda)}{\Psi_q^{(-)}(\lambda)}\Big)e^{i\lambda x}d\lambda-\int_{-R}^{R}\Big(\frac{\widehat{G}(\lambda)}{\Psi_q^{(-)}(\lambda)}\Big)e^{i\lambda a}d\lambda\Big| \\
&\le \frac{4\epsilon}{5}+\int_{-R}^{R}\Big|\frac{\widehat{G}(\lambda)}{\Psi_q^{(-)}(\lambda)}\Big|\Big|\cos(\lambda x)-\cos(\lambda a)\Big|d\lambda \\
&\quad +\int_{-R}^{R}\Big|\frac{\widehat{G}(\lambda)}{\Psi_q^{(-)}(\lambda)}\Big|\Big|\sin(\lambda x)-\sin(\lambda a)\Big|d\lambda.
\end{aligned}$$

Let us now define

$$\delta \;=\; \Big(2R^2\int_{-R}^{R}\Big|\frac{\widehat{G}(\lambda)}{\Psi_q^{(-)}(\lambda)}\Big|d\lambda\Big)^{-1}\frac{\epsilon}{10}.$$

Making use of the inequalities

$$\big|\cos(\lambda x)-\cos(\lambda a)\big| \;\le\; 2\big(1\wedge|\lambda(x-a)|^2\big),$$

and

$$\big|\sin(\lambda x)-\sin(\lambda a)\big| \;\le\; 2\big(1\wedge|\lambda(x-a)|\big),$$

where $a\wedge b=\min\{a,b\}$, it is easy to see for all $x\in\mathbb{R}$, with $|x-a|<\delta$, that

$$\int_{-R}^{R}\Big|\frac{\widehat{G}(\lambda)}{\Psi_q^{(-)}(\lambda)}\Big|\Big|\cos(\lambda x)-\cos(\lambda a)\Big|d\lambda \;<\; \frac{\epsilon}{10},$$

and

$$\int_{-R}^{R}\Big|\frac{\widehat{G}(\lambda)}{\Psi_q^{(-)}(\lambda)}\Big|\Big|\sin(\lambda x)-\sin(\lambda a)\Big|d\lambda \;<\; \frac{\epsilon}{10}. \tag{5.6.4}$$

Thus, combining the results of (5.6.3)-(5.6.4), we see for all $x\in\mathbb{R}$, with $|x-a|<\delta$, that

$$\Big|\mathscr{P}_G^{(q)}(x)-\mathscr{P}_G^{(q)}(a)\Big| \;<\; \epsilon.$$

By replacing $\widehat{G}(\lambda)$ with $\lambda\widehat{G}(\lambda)$ in (5.6.3)-(5.6.4) we see that

$$\int_{-\infty}^{\infty}\Big|\frac{\lambda\widehat{G}(\lambda)}{\Psi_q^{(-)}(\lambda)}\Big|d\lambda \;<\; \infty, \tag{5.6.5}$$

and therefore the function $\frac{d\mathscr{P}_G^{(q)}}{dx}$ is bounded in $\mathbb{R}$. Taking account of the condition (5.6.5), it can be shown following the similar steps as before that

$$\Big|\frac{d\mathscr{P}_G^{(q)}}{dx}(x)-\frac{d\mathscr{P}_G^{(q)}}{dx}(a)\Big| \;<\; \epsilon,$$

for all $x\in\mathbb{R}$ with $|x-a|<\delta$. Thus, our claim that the function $\mathscr{P}_G^{(q)}$ and its derivative $\frac{d\mathscr{P}_G^{(q)}}{dx}$ are both continuous and bounded in $\mathbb{R}$ is then proved. □

Proof of Lemma 5.3.3

To establish the results, let us recall that the function $\mathscr{P}_G^{(q)}$ solves the problem (5.2.1) and our notation for the first passage time below a level y of X is given by

$$\tau_y^- = \inf\{t > 0 : X_t < y\}.$$

Proof of (i)-(iii)

(i) Let us now consider the function (5.3.4):

$$V_y(x) = \mathbb{E}\Big(\mathscr{P}_G^{(q)}(x + \underline{X}_{\mathbf{e}_q})\mathbf{1}_{\{x+\underline{X}_{\mathbf{e}_q}<y\}}\Big). \tag{5.6.6}$$

Since $\underline{X}_{\mathbf{e}_q} \le 0$ almost surely, it is obvious from equations (5.6.6) and (5.2.1) that $V_y(x) = G(x)$ for all $x, y \in \mathbb{R}$ such that $x < y$.

(ii) Majorant property We want to show that the function $V_{x^\star}(x)$ (5.3.2) is majorant to the payoff function $G(x)$, namely $V_{x^\star}(x) \ge G(x)$ for every $x \in \mathbb{R}$.

On noticing the fact that $\mathscr{P}_G^{(q)}(x) \le 0$ for all $x \ge x^\star$, we see for each $q \ge 0$ and every $x \in \mathbb{R}$ that

$$\begin{aligned} V_{x^\star}(x) &= \mathbb{E}\Big(\mathscr{P}_G^{(q)}(x + \underline{X}_{\mathbf{e}_q})\mathbf{1}_{\{x+\underline{X}_{\mathbf{e}_q}<x^\star\}}\Big) \\ &= \mathbb{E}\Big(\mathscr{P}_G^{(q)}(x + \underline{X}_{\mathbf{e}_q})\Big) \\ &\quad - \mathbb{E}\Big(\mathscr{P}_G^{(q)}(x + \underline{X}_{\mathbf{e}_q})\mathbf{1}_{\{x+\underline{X}_{\mathbf{e}_q}\ge x^\star\}}\Big) \\ &\ge \mathbb{E}\Big(\mathscr{P}_G^{(q)}(x + \underline{X}_{\mathbf{e}_q})\Big) \\ &= G(x), \end{aligned}$$

where the inequality is due to the fact that $\mathscr{P}_G^{(q)}(x) \le 0$ for all $x \ge x^\star$, while the last equality is based on the fact that $\mathscr{P}_G^{(q)}$ solves the problem (5.2.1). Thus, the claim that the function $V_{x^\star}$ is majorant to the payoff function G is then proved.

(iii) Supermartingale property Let us now show that the function $V_{x^\star}$ (5.3.2) has the supermartingale property. The proof is obtained by noticing the fact that conditionally on the event $\{\mathbf{e}_q > t\}$, the following identity

$$\underline{X}_{\mathbf{e}_q} = \underline{X}_t \wedge (\mathcal{I} + X_t)$$

holds, and conditionally on the filtration $\mathcal{F}_t$, the random variable $\mathcal{I}$ has the same distribution as $\underline{X}_{\mathbf{e}_q}$. Following this and the fact that the function $x \mapsto \mathscr{P}_G^{(q)}(x)$ is non-increasing on the interval $(-\infty, x^\star]$, we see for every $q \geq 0$ that

$$\begin{aligned} V_{x^\star}(x) &= \mathbb{E}_x\Big(\mathscr{P}_G^{(q)}(\underline{X}_{\mathbf{e}_q})\mathbf{1}_{\{\underline{X}_{\mathbf{e}_q}<x^\star\}}\Big) \\ &= \mathbb{E}_x\Big(\mathbb{E}\big(\mathscr{P}_G^{(q)}(\underline{X}_{\mathbf{e}_q})\mathbf{1}_{\{\underline{X}_{\mathbf{e}_q}<x^\star\}}\Big|\mathcal{F}_t\big)\Big) \\ &\geq \mathbb{E}_x\Big(\mathbf{1}_{\{\mathbf{e}_q>t\}}\mathbb{E}\big(\mathscr{P}_G^{(q)}(X_t+\mathcal{I})\mathbf{1}_{\{X_t+\mathcal{I}<x^\star\}}\Big|\mathcal{F}_t\big)\Big) \\ &= \mathbb{E}_x\Big(\mathbf{1}_{\{\mathbf{e}_q>t\}}\mathbb{E}_{X_t}\big(\mathscr{P}_G^{(q)}(\underline{X}_{\mathbf{e}_q})\mathbf{1}_{\{\underline{X}_{\mathbf{e}_q}<x^\star\}}\big)\Big) \\ &= \mathbb{E}_x\Big(\mathbf{1}_{\{\mathbf{e}_q>t\}}V_{x^\star}(X_t)\Big) \\ &= \mathbb{E}_x\Big(e^{-qt}V_{x^\star}(X_t)\Big). \end{aligned}$$

Thus, the supermartingale property of the process $\{e^{-qt}V_{x^\star}(X_t), t \geq 0\}$ is established. □

Proof of Theorem 5.3.1

The proof of the theorem is mainly based on the fluctuation identity (5.2.11), the majorant and supermartingale properties of the function $V_{x^\star}$ (5.3.2), see Lemma 5.3.3. On noticing the fact that τ is arbitrary in $\mathcal{T}_{[0,\infty]}$ and the function $V_{x^\star}$ is lower bounded by the payoff G and has the supermartingale property, we see for every $x \in \mathbb{R}$ that

$$V_{x^\star}(x) \geq \sup_{\tau\in\mathcal{T}_{[0,\infty]}} \mathbb{E}_x\Big(e^{-q\tau}V_{x^\star}(X_\tau)\mathbf{1}_{(\tau<\infty)}\Big) \geq \sup_{\tau\in\mathcal{T}_{[0,\infty]}} \mathbb{E}_x\Big(e^{-q\tau}G(X_\tau)\mathbf{1}_{(\tau<\infty)}\Big).$$

On the other hand, rather trivially, we have for every $x \in \mathbb{R}$ that

$$\sup_{\tau\in\mathcal{T}_{[0,\infty]}} \mathbb{E}_x\Big(e^{-q\tau}G(X_\tau)\mathbf{1}_{(\tau<\infty)}\Big) \geq \mathbb{E}_x\Big(e^{-q\tau_{x^\star}^-}G(X_{\tau_{x^\star}^-})\mathbf{1}_{(\tau_{x^\star}^-<\infty)}\Big) = V_{x^\star}(x),$$

where the equality is due to the fluctuation identity (5.2.11).

Thus, all the inequalities are equalities and hence

$$V_{x^\star}(x) = \sup_{\tau\in\mathcal{T}_{[0,\infty]}} \mathbb{E}_x\Big(e^{-q\tau}G(X_\tau)\mathbf{1}_{(\tau<\infty)}\Big) = \mathbb{E}_x\Big(e^{-q\tau_{x^\star}^-}G(X_{\tau_{x^\star}^-})\mathbf{1}_{(\tau_{x^\star}^-<\infty)}\Big).$$

Thus, the value function $V(x)$ of the optimal stopping problem (5.1.1) coincides for every $x \in \mathbb{R}$ with the function $V_{x^\star}(x)$, and the optimal stopping time is given by $\tau_{x^\star}^-$. Thus, the claim that the function $V_{x^\star}(x)$ (5.3.2) is the optimal solution to the problem (5.1.1) is then proved. □

Proof of Proposition 5.3.4

The proof that the candidate solution $V_y(x)$ (5.3.4) satisfies the first claim (5.3.5) follows from applying the results of Lemma 5.2.6 and Theorem 5.3.1.

The proof of the other claim (5.3.6) is established as follows. Following the assumption $(H1)$ imposed on X, we see for a fixed $y \in \mathbb{R}$ that

$$\lim_{x\to\infty} V_y(x) \;=\; \lim_{x\to\infty} \mathbb{E}\Big(\mathscr{P}_G^{(q)}(x+\underline{X}_{\mathbf{e}_q})\mathbf{1}_{(\underline{X}_{\mathbf{e}_q}<y-x)}\Big) \;=\; 0.$$

Notice also that the function $V_{x^\star}(x)$ (5.3.2) is lower bounded for every $x \in \mathbb{R}$ by the payoff function $G(x)$, and for $y < x^\star$ we have that $0 \leq V_y(x) < V_{x^\star}(x)$ for every $x \in \mathbb{R}$. Taking into account of the fact that $V_y(x) = G(x)$ for every $x < y$, we see that there exists $x \in \mathbb{R}$ such that for each $y < x^\star$ we have $V_y(x) < G(x)$. Thus, the claim that the candidate solution $V_y(x)$ (5.3.4) satisfies the inequalities (5.3.5) and (5.3.6) is then proved. □

Proof of Lemma 5.3.2

Let $\widetilde{V}$ be the optimal value function of the problem (5.1.1) under the payoff function $\widetilde{G}(x) = \max\{G(x), 0\}$. Since $\widetilde{G}(x) \geq G(x)$ for all $x \in \mathbb{R}$, we see that

$$\widetilde{V}(x) \;=\; \sup_{\tau\in\mathcal{T}_{[0,\infty]}} \mathbb{E}_x\Big(e^{-q\tau}\widetilde{G}(X_\tau)\Big) \;\geq\; \sup_{\tau\in\mathcal{T}_{[0,\infty]}} \mathbb{E}_x\Big(e^{-q\tau}G(X_\tau)\Big) \;=\; V_{x^\star}(x). \tag{5.6.7}$$

However, following the positivity of the optimal value function $V_{x^\star}$ and the majorant property of $V_{x^\star}$ over the payoff G, it is straightforward to verify that $V_{x^\star}(x) \geq \widetilde{G}(x)$ for all $x \in \mathbb{R}$. Thus, from the supermartingale property of $V_{x^\star}$, we then obtain

$$V_{x^\star}(x) \;\geq\; \sup_{\tau\in\mathcal{T}_{[0,\infty]}} \mathbb{E}_x\Big(e^{-q\tau}V_{x^\star}(X_\tau)\Big) \;\geq\; \sup_{\tau\in\mathcal{T}_{[0,\infty]}} \mathbb{E}_x\Big(e^{-q\tau}\widetilde{G}(X_\tau)\Big) \;=\; \widetilde{V}(x).$$

Hence, combining with the inequality (5.6.7), our claim is then established. □

Proof of Theorem 5.4.3

In this section, we provide details of calculations for the proof of Theorem 5.4.3. By evaluating the candidate solution $V_y(x)$ (5.3.4) at the point $x = y$, we have

$$V_y(y) \;=\; G(y) - \mathbb{P}\big(-\underline{X}_{\mathbf{e}_q} = 0\big)\mathscr{P}_G^{(q)}(y). \tag{5.6.8}$$

Since $\mathscr{P}_G^{(q)}(y) \neq 0$, it is clear from the foregoing relation above that $V_y(y) \neq G(y)$ whenever $\mathbb{P}\big(-\underline{X}_{\mathbf{e}_q} = 0\big) > 0$, the case where 0 is irregular for $(-\infty, 0)$ for X.

The other claim is achieved as follows. Since the function $\mathscr{P}_G^{(q)}$ solves the problem (5.2.1), the candidate solution $V_y(x)$ (5.3.4) can be rewritten as follows

$$V_y(x) \;=\; G(x) \;-\; H_y(x), \tag{5.6.9}$$

where the function $H_y(x)$ is defined for every $x \in \mathbb{R}$ by

$$H_y(x) \;=\; \mathbb{E}\Big(\mathscr{P}_G^{(q)}(x+\underline{X}_{\mathbf{e}_q})\mathbf{1}_{(x+\underline{X}_{\mathbf{e}_q}\geq y)}\Big).$$

The proof will be done once we show that

$$\frac{dH_y}{dx}(y) \;= \mathbb{P}\big(-\underline{X}_{\mathbf{e}_q}=0\big)\frac{d\mathscr{P}^{(q)}_G}{dx}(y)+p^{(-)}_q(0)\mathscr{P}^{(q)}_G(y).$$

After some algebra, we have

$$\begin{aligned}\frac{H_y(x)-H_y(y)}{x-y} &= \mathbb{E}\Big(\Big(\frac{\mathscr{P}^{(q)}_G(x+\underline{X}_{\mathbf{e}_q})-\mathscr{P}^{(q)}_G(y+\underline{X}_{\mathbf{e}_q})}{x-y}\Big)\mathbf{1}_{(-\underline{X}_{\mathbf{e}_q}\le x-y)}\Big)\\ &+\;\mathbb{E}\Big(\mathscr{P}^{(q)}_G(y+\underline{X}_{\mathbf{e}_q})\Big(\frac{\mathbf{1}_{(-\underline{X}_{\mathbf{e}_q}\le x-y)}-\mathbf{1}_{(-\underline{X}_{\mathbf{e}_q}=0)}}{x-y}\Big)\Big)\\ &+\;\mathbb{E}\Big(\frac{\mathscr{P}^{(q)}_G(y+\underline{X}_{\mathbf{e}_q})\mathbf{1}_{(-\underline{X}_{\mathbf{e}_q}=0)}-\mathscr{P}^{(q)}_G(y)\mathbf{1}_{(-\underline{X}_{\mathbf{e}_q}=0)}}{x-y}\Big).\end{aligned}$$

On noticing the fact that the third term is zero, the above expression now reduces to

$$\begin{aligned}\frac{H_y(x)-H_y(y)}{x-y} &= \mathbb{E}\Big(\Big(\frac{\mathscr{P}^{(q)}_G(x+\underline{X}_{\mathbf{e}_q})-\mathscr{P}^{(q)}_G(y+\underline{X}_{\mathbf{e}_q})}{x-y}\Big)\mathbf{1}_{(-\underline{X}_{\mathbf{e}_q}\le x-y)}\Big)\\ &+\;\mathbb{E}\Big(\mathscr{P}^{(q)}_G(y+\underline{X}_{\mathbf{e}_q})\Big(\frac{\mathbf{1}_{(-\underline{X}_{\mathbf{e}_q}\le x-y)}-\mathbf{1}_{(-\underline{X}_{\mathbf{e}_q}=0)}}{x-y}\Big)\Big).\end{aligned}\tag{5.6.10}$$

Since the functions G and $\mathscr{P}^{(q)}_G$ are assumed to be continuously differentiable, we see using the Lebesque dominated convergence theorem that

$$\frac{H_y(x)-H_y(y)}{x-y}\longrightarrow\mathbb{P}\big(-\underline{X}_{\mathbf{e}_q}=0\big)\frac{d\mathscr{P}^{(q)}_G}{dx}(y)+p^{(-)}_q(0)\mathscr{P}^{(q)}_G(y),\tag{5.6.11}$$

as $x\downarrow y$. Thus, following (5.6.9)-(5.6.11), we see that

$$\frac{dV_y}{dx}(y) \;=\; \frac{dG}{dx}(y)-\mathbb{P}\big(-\underline{X}_{\mathbf{e}_q}=0\big)\frac{d\mathscr{P}^{(q)}_G}{dx}(y)-p^{(-)}_q(0)\mathscr{P}^{(q)}_G(y).\tag{5.6.12}$$

Hence, while $y\neq x^\star$, we see that there is discontinuity at $x=y$ for the candidate solution $V_y(x)$ and its derivative when X is a Lévy processes for which 0 is *irregular* for $(-\infty,0)$ for X. In the regular case, there is only discontinuity for the derivative when $p^{(-)}_q(0)\neq 0$. The infinite gradient only exists if and only if $|p^{(-)}_q(0)|=\infty$. □

Proof of Theorem 5.4.1

Following (5.6.12) above, we see at $y=x^\star$ that

$$\frac{dV_{x^\star}}{dx}(x^\star) \;=\; \frac{dG}{dx}(x^\star)-\mathbb{P}\big(-\underline{X}_{\mathbf{e}_q}=0\big)\frac{d\mathscr{P}^{(q)}_G}{dx}(x^\star)-p^{(-)}_q(0)\mathscr{P}^{(q)}_G(x^\star).$$

Since the optimal stopping boundary $x^\star$ solves the equation $\mathscr{P}^{(q)}_G(x)=0$, we see following the previous equation that

$$\frac{dV_{x^\star}}{dx}(x^\star) \;=\; \frac{dG}{dx}(x^\star)-\mathbb{P}\big(-\underline{X}_{\mathbf{e}_q}=0\big)\frac{d\mathscr{P}^{(q)}_G}{dx}(x^\star).$$

Hence, the smooth pasting condition holds if and only if $\mathbb{P}\big(-\underline{X}_{\mathbf{e}_q}=0\big)=0$. □

Proof of Lemma 5.4.4

From Section 2.3 of Chapter 2, we see that by applying integration by part to (2.3.3) the Stieltjes measure $d\mathbb{P}\big(-\underline{X}_{\mathbf{e}_q} \le x\big)$, associated to the function $\mathbb{P}\big(-\underline{X}_{\mathbf{e}_q} \le x\big)$, can be written in terms of the q-scale function $W^{(q)}(x)$, defined in (2.3.4), as

$$d\mathbb{P}\big(-\underline{X}_{\mathbf{e}_q} \le x\big) \;=\; \frac{q}{\Phi(q)} dW^{(q)}(x) - qW^{(q)}(x)dx. \tag{5.6.13}$$

By defining $P_q^{(-)}(x) = \mathbb{P}\big(-\underline{X}_{\mathbf{e}_q} \le x\big)$, we have from (5.6.13) that

$$P_q^{(-)}(x) \;=\; \mathbb{P}\big(-\underline{X}_{\mathbf{e}_q} = 0\big) + \frac{q}{\Phi(q)}\big(W^{(q)}(x) - W^{(q)}(0)\big) - q\int_0^x W^{(q)}(y)dy.$$

Therefore, following (5.4.2) we see that

$$\begin{aligned} p_q^{(-)}(0) &= \lim_{h\downarrow 0}\frac{1}{h}\big(P_q^{(-)}(h) - P_q^{(-)}(0)\big) \\ &= \frac{q}{\Phi(q)}\lim_{h\downarrow 0}\frac{1}{h}\big(W^{(q)}(h) - W^{(q)}(0)\big) - q\lim_{h\downarrow 0}\frac{1}{h}\int_0^h W^{(q)}(y)dy \\ &= \frac{q}{\Phi(q)}\frac{dW^{(q)}}{dx}(0+) - qW^{(q)}(0), \end{aligned}$$

where the last equality is due to the fact that the Lévy measure has no atoms so that the q-scale function $W^{(q)}(x)$ is differentiable (we refer to Lambert [75] and Chan and Kyprianou [28]) and, hence, a right derivative at zero of $W^{(q)}(x)$ exists. Using the result of Lemma 2 in Section 6.4 of Chapter 6, our claim is then proved. □

5.7 Numerical examples: the arbitrage-free pricing of American options under tempered stable processes with downward jumps

In this section we verify our main results in Theorems 5.3.1 and 5.3.1 for the optimal stopping problem (5.1.1) with payoffs $(K-e^x)^+$ and $(e^x-K)^+$ under tempered stable processes with no positive jumps, so its Lévy measure Π has support in $(-\infty, 0]$.

5.7.1 Tempered stable processes with downward jumps

A tempered stable process is obtained by taking a one-dimensional stable process and multiplying the Lévy measure with a decreasing exponential on each half of the real axis. This exponential softening keeps the initial stable-like behaviour whereas the large jumps become much less heavy tailed. A tempered stable process is thus a Lévy process with no Gaussian component and a Lévy measure of the form

$$\Pi(dx) \;=\; C\frac{e^{-\lambda|x|}}{|x|^{1+\alpha}}\mathbf{1}_{\{x<0\}}dx \tag{5.7.1}$$

where $C > 0$, $\lambda > 0$ and $\alpha < 2$. Unlike the case of stable processes, which can only be defined for $\alpha > 0$, in the tempered stable process there is no natural lower bound on α and the expression in (5.7.1) yields a Lévy measure for $\alpha < 2$. In fact, taking negative values of α we obtain compound Poisson models with a rich structure. We refer among others to Cont and Tankov [32] for more details. It is clear that tempered stable process is of compound Poisson type if $\alpha < 0$ and has paths of bounded variation if $\alpha < 1$. Because of exponential decay of the tails of the Lévy measure, it is then more convenient to work with tempered stable process without truncation of big jumps. To compute the Laplace exponent, we consider the case that $\alpha \neq 1$ and $\alpha \neq 0$. The integration maybe performed in the following way:

$$\begin{aligned}\int_0^\infty \big(e^{-\theta x} - 1 + \theta x\big)\frac{e^{-\lambda x}}{x^{1+\alpha}}dx &= \sum_{n=2}^{\infty}\frac{(-\theta)^n}{n!}\int_0^\infty x^{n-1-\alpha}e^{-\lambda x}dx \\ &= \sum_{n=2}^{\infty}\frac{(-\theta)^n}{n!}\lambda^{\alpha-n}\Gamma(n-\alpha) \\ &= \lambda^\alpha\Gamma(-\alpha)\Big\{\Big(1+\frac{\theta}{\lambda}\Big)^\alpha - 1 - \frac{\theta\alpha}{\lambda}\Big\}. \qquad (5.7.2)\end{aligned}$$

Note that interchanging the sum and integral and the convergence of the power series are possible if $|\theta| < \lambda$. By analytic continuation, it is clear that the expression (5.7.2) exists for all values of θ such that $\mathfrak{Re}(\theta) < \lambda$.

Performing similar calculation for the case $\alpha = 1$, we obtain

$$\int_0^\infty \big(e^{-\theta x} - 1 + \theta x\big)\frac{e^{-\lambda x}}{x^{1+\alpha}}dx = -\theta + (\lambda+\theta)\log\Big(1+\frac{\theta}{\lambda}\Big),$$

and for $\alpha = 0$, we have

$$\int_0^\infty \big(e^{-\theta x} - 1 + \theta x\big)\frac{e^{-\lambda x}}{x^{1+\alpha}}dx = \frac{\theta}{\lambda} + \log\Big(\frac{\lambda}{\lambda+\theta}\Big).$$

Following the Lévy-Khintchine formula, the Laplace exponent κ of a tempered stable process having no positive jumps is given by the following proposition.

Proposition 5.7.1 *Let X be a tempered stable process having no positive jumps. In the general case ($\alpha \neq 1$ and $\alpha \neq 0$) the Laplace exponent of X is given by*

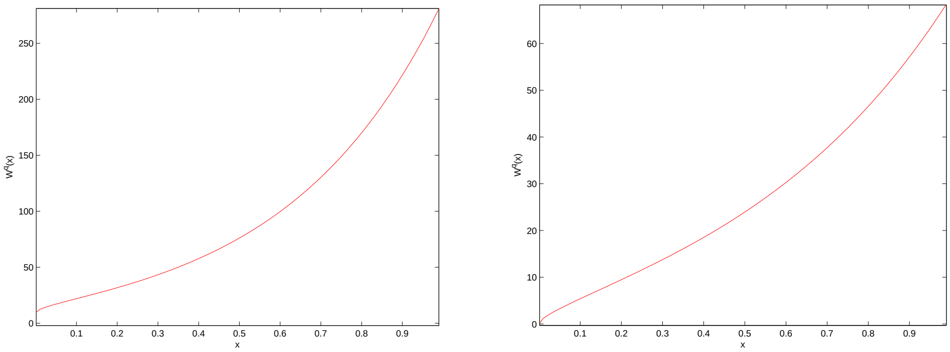
$$\kappa(\theta) = \mu\theta + \Gamma(-\alpha)\lambda^\alpha C\Big\{\Big(1+\frac{\theta}{\lambda}\Big)^\alpha - 1 - \frac{\theta\alpha}{\lambda}\Big\}. \qquad (5.7.3)$$

If $\alpha = 1$, then

$$\kappa(\theta) = (\mu - C)\theta + C(\lambda+\theta)\log\Big(1+\frac{\theta}{\lambda}\Big), \qquad (5.7.4)$$

and if $\alpha = 0$, then

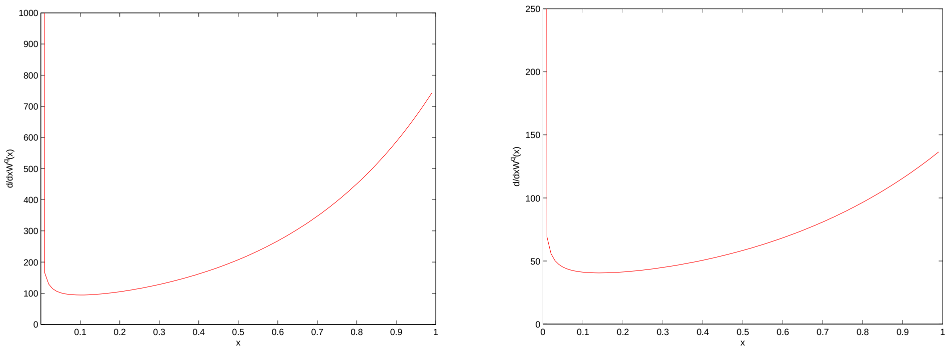
$$\kappa(\theta) = \mu\theta - C\Big\{-\frac{\theta}{\lambda} + \log\Big(1+\frac{\theta}{\lambda}\Big)\Big\}. \qquad (5.7.5)$$



(a) Tempered stable process with $\alpha = 0.5$.

(b) Tempered stable process with $\alpha = 1.5$

Figure 5.1: Numerical plots of the scale function $W^{(q)}(x)$ for tempered stable processes of bounded and unbounded variations.



(a) Tempered stable process with $\alpha = 0.5$.

(b) Tempered stable process with $\alpha = 1.5$

Figure 5.2: Numerical plots of the derivative of the scale function $W^{(q)}(x)$ for tempered stable processes of bounded and unbounded variations.

5.7.2 The rational price of perpetual American options

5.7.2.1 Perpetual American put option

Let us now consider the perpetual American put option problem

$$V(x) \;=\; \sup_{\tau \in \mathcal{T}_{[0,\infty]}} \mathbb{E}_x\Big(e^{-q\tau}\big(K - e^{X_\tau}\big)^+ \mathbf{1}_{(\tau<\infty)}\Big), \tag{5.7.6}$$

under the hypothesis that $(H1)$ holds. To get a better understanding of the problem (5.7.6) both analytically and numerically, let us consider, for a given stopping boundary $y \geq 0$, a candidate solution $V_y(x)$ to the problem (5.7.6) defined by

$$V_y(x) \;=\; \mathbb{E}\Big(\big[K - \Psi_q^{(-)}(-i)^{-1} e^{x+\underline{X}_{\mathbf{e}_q}}\big]\mathbf{1}_{\{x+\underline{X}_{\mathbf{e}_q}<y\}}\Big).$$

In the sequel below we write $V(x;y) \triangleq V_y(x)$. Using the measure (2.3.5), we can rewrite the function $V(x;y)$ explicitly in terms of the q-scale function[5] $W^{(q)}$ as

$$\begin{aligned} V(x;y) \;&=\; K - e^x + \mathbf{1}_{\{x\geq y\}}\Big(\frac{(\kappa(1)-q)}{(1-\Phi(q))}e^y - K\frac{q}{\Phi(q)}\Big)W^{(q)}(x-y) \\ &+ (\kappa(1)-q)e^x\mathbf{1}_{\{x\geq y\}}\int_0^{x-y} e^{-z}W^{(q)}(z)dz + Kq\mathbf{1}_{\{x\geq y\}}\int_0^{x-y} W^{(q)}(z)dz. \end{aligned} \tag{5.7.7}$$

Let us now define for each $q\geq 0$ a function

$$W_{\Phi(q)}(x) \;:=\; e^{-\Phi(q)x}W^{(q)}(x), \tag{5.7.8}$$

where $\Phi(q)$ is the largest root of the equation $\kappa(\theta)=q$. Due to the convexity of the Laplace exponent κ, there exists at most two solutions for a given q and precisely one root when $q>0$. As will be shown later in Chapters 6 and 7, the scale function $W_{\Phi(q)}(x)$ is increasing and corresponds to the role of the scale function $W^{(0)}(x)$ when X is taken under the measure $\mathbb{P}^{\Phi(q)}$ defined by the *Esscher transform*

$$\frac{d\mathbb{P}^{\Phi(q)}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} \;=\; e^{\Phi(q)X_t - qt} \quad \text{for all } t\geq 0.$$

Using the transformation (5.7.8) in (5.7.7) and varying the value of the stopping boundary y in the interval $(-\infty,x]$, we obtain after some calculations that

$$\frac{dV}{dy}(x;y) \;=\; -\Big(\frac{(\kappa(1)-q)}{(1-\Phi(q))}e^y - K\frac{q}{\Phi(q)}\Big)e^{\Phi(q)(x-y)}W'_{\Phi(q)}(x-y)\mathbf{1}_{\{x\geq y\}}.$$

Next, let us define $x^\star = \log\big(K\frac{q}{\Phi(q)}\frac{(1-\Phi(q))}{(\kappa(1)-q)}\big)$, i.e., $x^\star = \log\big(K\mathbb{E}(e^{\underline{X}_{\mathbf{e}_q}})\big)$. Since the scale function $W_{\Phi(q)}(x)$ is increasing, we see from the foregoing expression that

$$\frac{dV}{dy}(x;y) \;>(<)\; 0 \;\text{ for }\; y \;<(>)\; x^\star \;\text{ and }\; \frac{dV}{dy}(x;x^\star)=0.$$

Hence, we deduce that $y=x^\star$ is the level at which the function $y\mapsto V(x;y)$ attains its maximum value. As explained previously in Section 3, it is known that the level $y=\log(K\mathbb{E}(e^{\underline{X}_{\mathbf{e}_q}}))$ corresponds to the optimal stopping boundary for the optimal stopping problem (5.7.6) and the function $V(x;x^\star)$ coincides for every $x\in\mathbb{R}$ with the value function $V(x)$ of the problem (5.5.3).

Furthermore, by evaluating (5.7.7) at the point $x=y$, we see that

$$V(y;y) \;=\; K - e^y + \Big(\frac{(\kappa(1)-q)}{(1-\Phi(q))}e^y - K\frac{q}{\Phi(q)}\Big)W^{(q)}(0), \tag{5.7.9}$$

while its derivative at $x=y$ is defined by

$$\begin{aligned} \frac{dV}{dx}(y;y) \;&=\; -\,e^y + \Big(\frac{(\kappa(1)-q)}{(1-\Phi(q))}e^y - K\frac{q}{\Phi(q)}\Big)\frac{dW^{(q)}}{dx}(0) \\ &+\; \big((\kappa(1)-q)e^y + Kq\big)W^{(q)}(0). \end{aligned} \tag{5.7.10}$$

[5]Using excursion theory of spectrally negative Lévy processes, Avram et al [7], Pistorius [101] obtained the expression (5.7.7) as the candidate solution to the problem (5.7.6).

Thus, discontinuity or infinite gradient of the function $V(\bullet; y)$ only exists if

$$W^{(q)}(0) \neq 0 \quad \text{or} \quad \frac{dW^{(q)}}{dx}(0) = \infty,$$

respectively. The latter happens when 0 is regular for $(-\infty, 0)$ for X with zero Gaussian component and X has paths of bounded variation with $\Pi(-\infty, 0) = \infty$, see Lemma 5.4.4. Therefore, following (5.7.9) and (5.7.10), we observe that the optimal value function $V(x)$ is continuous at the point $x = x^\star$ and there exists smooth pasting if and only if $W^{(q)}(0) = 0$, the case when 0 is regular[6] for the lower half-line $(-\infty, 0)$ for X.

5.7.2.2 Perpetual American call option

Let us now consider the perpetual American call option problem

$$V(x) = \sup_{\tau \in \mathcal{T}_{[0,\infty]}} \mathbb{E}_x\Big(e^{-q\tau}\big(e^{X_\tau} - K\big)^+ \mathbf{1}_{(\tau<\infty)}\Big), \tag{5.7.11}$$

under the hypothesis that $(H2)$ holds. To obtain a better understanding of the problem (5.7.11) both analytically and numerically, let us consider, for a given stopping boundary $y \geq 0$, a candidate solution $V_y(x)$ to the problem (5.7.11) defined by

$$V_y(x) = \mathbb{E}\Big(\big(\Psi_q^{(+)}(-i)^{-1} e^{x+\overline{X}_{\mathbf{e}_q}} - K\big)\mathbf{1}_{\{x+\overline{X}_{\mathbf{e}_q}>y\}}\Big). \tag{5.7.12}$$

Again, we will write $V(x; y) \triangleq V_y(x)$. Using the measure (2.3.1), the expression for $V(x; y)$ can be simplified further as

$$\begin{aligned} V(x; y) &= e^x - K + K\big(1 - e^{-\Phi(q)(y-x)}\big)\mathbf{1}_{\{y\geq x\}} \\ &\quad + e^x\big(e^{-(\Phi(q)-1)(y-x)} - 1\big)\mathbf{1}_{\{y\geq x\}}. \end{aligned} \tag{5.7.13}$$

By varying the value of boundary y in the interval $[x, \infty)$, we obtain from (5.7.13) that

$$\frac{dV}{dy}(x; y) = -\Big((\Phi(q) - 1)e^y - K\Phi(q)\Big)e^{-\Phi(q)(y-x)}\mathbf{1}_{\{y\geq x\}}.$$

By defining $x^\star = \log\big(\frac{K\Phi(q)}{(\Phi(q)-1)}\big)$, i.e., $x^\star = \log(K\mathbb{E}(e^{\overline{X}_{\mathbf{e}_q}}))$, it is clear that

$$\frac{dV}{dy}(x; y) > (<)\ 0 \ \text{ for } \ y < (>)\ x^\star \ \text{ and } \ \frac{dV}{dy}(x; x^\star) = 0.$$

Hence, we deduce that $y = x^\star$ is the level at which the function $y \mapsto V(x; y)$ attains its maximum value. As explained previously in Section 3, it is known that the level $y = \log(K\mathbb{E}(e^{\overline{X}_{\mathbf{e}_q}}))$ corresponds to the optimal boundary for the stopping problem

[6] By applying integration by parts and a Tauberian theorem to the Laplace transforms (2.3.3) and (2.3.4), it can be shown that $\mathbb{P}(-\underline{X}_{\mathbf{e}_q} = 0) = \frac{q}{\Phi(q)} W^{(q)}(0)$.

(5.7.11) and the function $V(x;x^\star)$ coincides for every $x \in \mathbb{R}$ with the value function $V(x)$ of the optimal stopping problem (5.5.4).

Furthermore, by evaluating the expression (5.7.13) at the point $x = y$, we see that

$$V(y;y) \;=\; e^y - K,$$

and the derivative at $x = y$ of the value function $V(x;y)$ is given by

$$\frac{dV}{dx}(y;y) \;= e^y \;-\; \Phi(q)\Big(\big(\frac{\Phi(q)}{(\Phi(q)-1)}\big)^{-1} e^y - K\Big).$$

On noticing the fact that $e^{x^\star} = \frac{K\Phi(q)}{(\Phi(q)-1)}$, we see that the candidate solution $V(x;y)$ obeys the smooth pasting condition at the stopping boundary $y = x^\star$. While $y \neq x^\star$, we observe that there exists discontinuity at the point $x = y$ for the derivative of the candidate solution $V(x;y)$ with no infinite gradient.

The results of numerical computation for the value functions $V(x;y)$ (5.7.7) and (5.7.13) of the perpetual American put and call option problems will be discussed in more details in the section below. In particular, for the perpetual American put option, the computation boils down to numerically produce the q-scale function $\{W^{(q)}(x) : q \geq 0, x \in \mathbb{R}_+\}$ of the Lévy process $(X, \mathbb{P})$. Further details of the computation will be elaborated in more details later in Chapter 7.

5.7.3 Numerical results

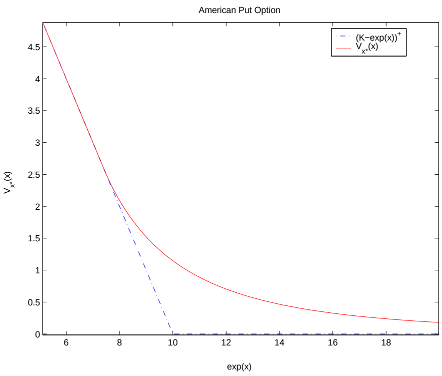
This section deals with pricing the perpetual American put and call options (5.7.6) and (5.7.11) on the stock price process S_t whose dynamics are given under a chosen martingale measure $\mathbb{P}$ by an exponential Lévy process

$$S_t(x) \;=\; xe^{X_t}. \tag{5.7.14}$$

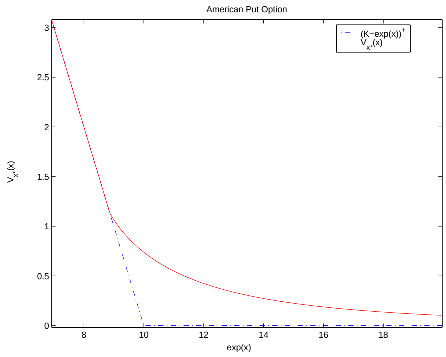
We assume that a default-free asset exists that pays a continuous interest rate $r > 0$ and denote by δ the total payout rate of dividend. Furthermore, we assume under the measure $\mathbb{P}$ that the discounted stock price process $e^{-(r-\delta)t}S_t(x)$ is $\mathbb{P}$-martingale which implies that

$$\mathbb{E}\Big(e^{-(r-\delta)t}S_t(x)\Big) \;=\; x. \tag{5.7.15}$$

For the purpose of numerical computation, we use generalized tempered stable process for X whose Laplace exponent is given in Proposition 5.7.1. The numerical computation is carried out using MATLAB6.5. The parameter setting for interest rate r, dividend rate δ, the strike value K, and the jump rate λ are set to be 0.1, 0.07, 10, and 2.5, respectively. In the case where X has path of bounded variation we choose $\alpha = 0.5$ and the relative frequency of downward jumps C to be 0.075. In the case of unbounded variation X, we set $\alpha = 1.5$ and $C = 0.05$ and the other parameters have the same value. The drift in the Laplace exponent κ is chosen so that the martingale condition (5.7.15) is satisfied.

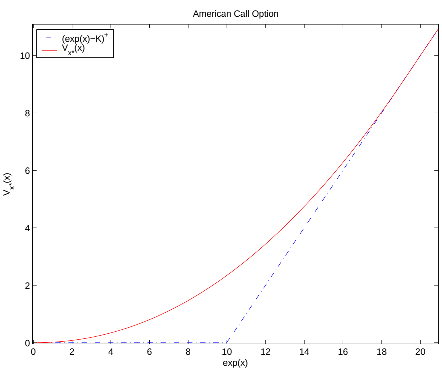


(a) Tempered stable process with $\alpha = 1.5$. The optimal stopping boundary $e^{x^\star} = 6.8531$.

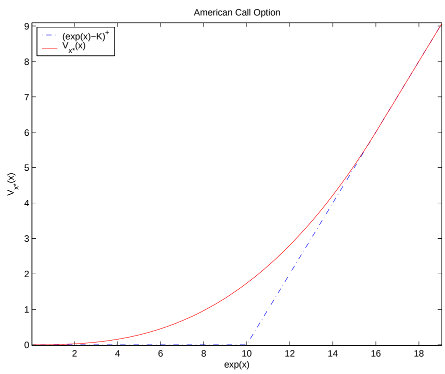


(b) Tempered stable process with $\alpha = 0.5$. The optimal stopping boundary $e^{x^\star} = 8.8686$.

Figure 5.3: The shape of the rational price $V_{x^\star}(x)$ of the American put option.



(a) Tempered stable process with $\alpha = 1.5$. The optimal stopping boundary $e^{x^\star} = 20.8455$.

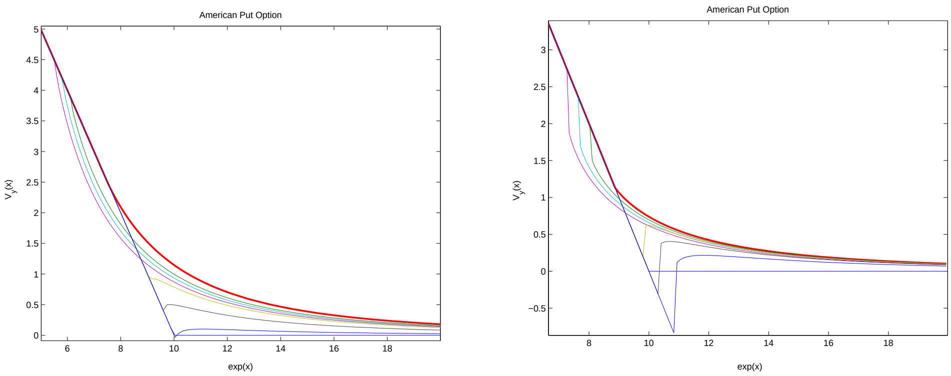


(b) Tempered stable process with $\alpha = 0.5$. The optimal stopping boundary $e^{x^\star} = 16.1082$.

Figure 5.4: The shape of the rational price $V_{x^\star}(x)$ of the American call option.

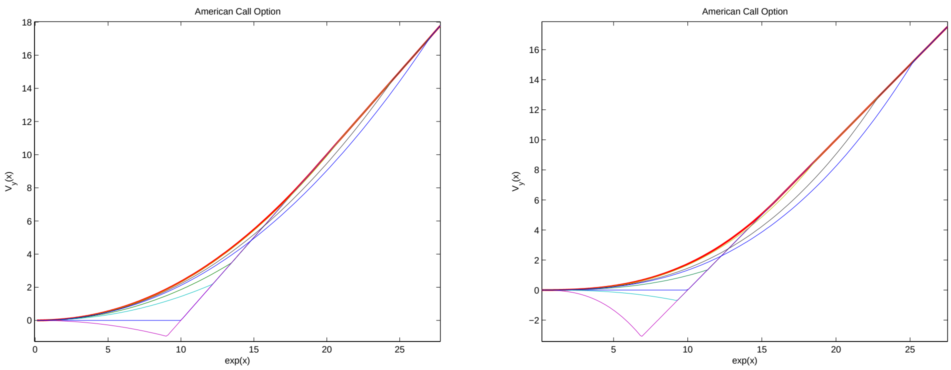
We present in Figure 5.3 plots of the value function $V_{x^\star}$ (5.5.3) of the American put option problem (5.7.6). From this figure, we observe that the value function $V_{x^\star}$ satisfies the smooth pasting condition at the optimal stopping boundary $x^\star = \log(K\mathbb{E}(e^{\underline{X}_{\mathbf{e}_q}}))$ for both Lévy processes, except for the case of $\alpha = 0.5$. Since for this case X has path of bounded variation with positive drift and hence 0 is irregular for $(-\infty, 0)$ for X, we see from figure 5.3(b) that the smooth pasting condition does not hold. All of the plots exhibit the general types of behaviour found recently by Hirsa and Madan [59], Matache et al. [81], and Almendral and Oosterlee [4].

Figure 5.4 shows plots of the value function $V_{x^\star}$ (5.5.4) of the American call option



(a) Tempered stable process with $\alpha = 1.5$. (b) Tempered stable process with $\alpha = 0.5$.

Figure 5.5: The shape of a candidate solution $V_y(x)$ of the American put option problem for different values of stopping boundary y.

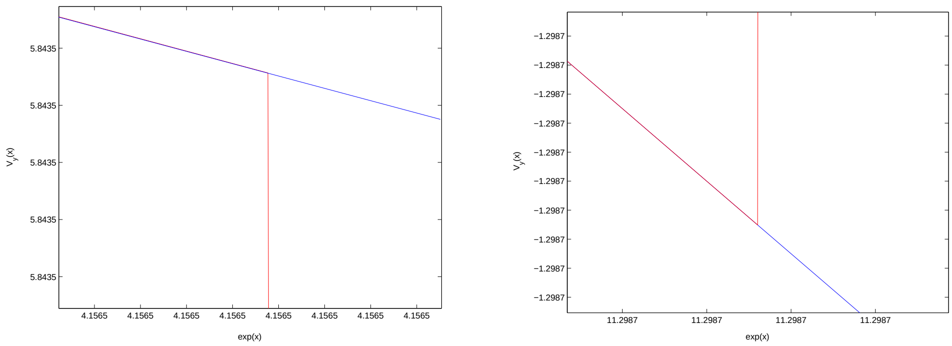


(a) Tempered stable process with $\alpha = 1.5$. (b) Tempered stable process with $\alpha = 0.5$.

Figure 5.6: The shape of a candidate solution $V_y(x)$ for different values of stopping boundary y of the American call option problem.

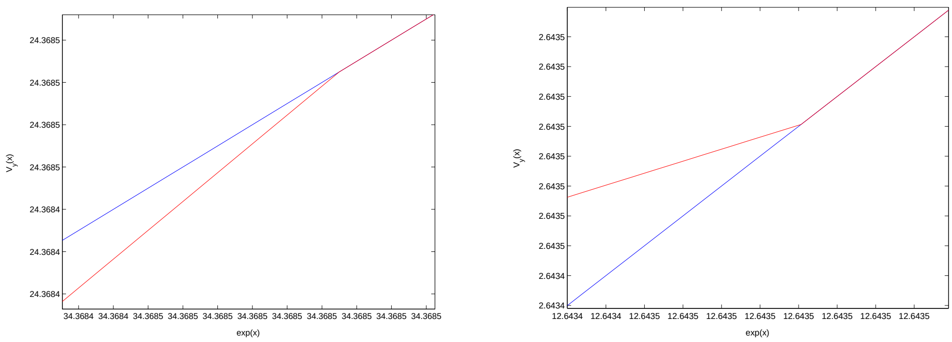
problem (5.7.11). From this figure, we observe for both Lévy processes that the value function $V_{x^\star}$ satisfies the smooth pasting condition at the optimal stopping boundary $x^\star = \log(K\mathbb{E}(e^{\overline{X}_{\mathbf{e}_q}}))$, the fact that follows from regularity of 0 for $(0,\infty)$ for both Lévy processes (see Theorem 5.4.2 for more details).

Next in Figures 5.5(a) and 5.5(b) we present plots of the candidate solution $x \mapsto V_y(x)$ (5.7.7) of the American put option problem (5.7.6) for different values of stopping boundary y. From these figures we observe that $V_y(x) = G(x)$ for every $x < y$, $V_{x^\star} \geq G(x)$ for all $x \in \mathbb{R}$, and for $y < x^\star$ we see that there exists x such that $V_y(x) < G(x)$. These are the features specified previously in Lemma 5.3.3. In particular, we observe from these figures that all of the curves $V_y(x)$ are seen to be



(a) Tempered stable process with $\alpha = 1.5$. (b) Tempered stable process with $\alpha = 1.5$.

Figure 5.7: The shape of a candidate solution $V_y(x)$ of the American put option problem at $x = y$ for $y \neq x^\star$.



(a) Tempered stable process with $\alpha = 1.5$. (b) Tempered stable process with $\alpha = 1.5$.

Figure 5.8: The shape of a candidate solution $V_y(x)$ of the American call option problem at $x = y$ for $y \neq x^\star$.

upper bounded by the curve $V_{x^\star}$ of the value function (the one associated with the stopping boundary $x^\star = \log(K\mathbb{E}(e^{\underline{X}_{\mathbf{e}_q}})))$ (see Proposition 5.3.4). Hence, the claim that $x^\star$ is the optimal stopping boundary is numerically justified.

Furthermore, In the irregular case of $\alpha = 0.5$ we notice from Figure 5.5(b) that the candidate solution $V_y(x)$ exhibits a jump of size $\big(\frac{(\kappa(1)-q)}{(1-\Phi(q))}e^y - K\frac{q}{\Phi(q)}\big)W^{(q)}(0)$ at the point $x = y$, with $y \neq x^\star$. In the regular case of $\alpha = 1.5$, we see from figures 5.5(a) that there is discontinuity only for the derivative of the candidate solution $V_y(x)$ at a stopping boundary $y \neq x^\star$; the smooth pasting only exists at the optimal stopping boundary $y = x^\star$. Moreover, since the sample path of X contains no Gaussian component, we observe from Figure 5.7 that there exists an infinite gradient at a stopping boundary $y \neq x^\star$ for the candidate solution $V_y(x)$ of the American put option problem (5.7.6). The different behaviour of the (candidate) solution of the problem (5.7.6) is the principle point of difference between the numerical results of Hirsa and Madan [59], Matache et al. [81], and Almendral and Oosterlee [4] and ours. In other respects, the results associated with the optimal boundary $y = x^\star$ are qualitatively similar. This numerical observation agrees with our claim stated previously in Theorem 5.4.3.

In the final plot, Figure 5.6 displays numerical plots of the candidate solution $x \mapsto V_y(x)$ (5.7.13) of the American call option problem (5.7.11) for various values of stopping boundary y. All of the curves are seen to be dominated by the curve $V_{x^\star}$ of the value function (the one associated with the stopping boundary $x^\star = \log(K\mathbb{E}(e^{\overline{X}_{\mathbf{e}_q}})))$. This is to say that $x^\star$ is indeed the optimal stopping boundary of the problem (5.7.11). From the plots, we also observe that $V_y(x) = G(x)$ for every $x \geq y$, $V_{x^\star} \geq G(x)$ for all $x \in \mathbb{R}$, and for $y \geq x^\star$ we see that there exists x such that $V_y(x) \leq G(x)$. These are the features specified by Lemma 5.3.3 in its dual form. In complement to the plots of the candidate solution of the American put option problem (5.7.6), we observe that there exists only discontinuity for the derivative of the curve $V_y(x)$ with no infinite gradient at the stopping boundary $y \neq x^\star$ (see Figure 5.8).

To summarize this section, we have shown that, by working with a completely general spectrally negative Lévy process, it is possible to verify both analytically and numerically the main results of Sections 3 and 4.

5.8 Connection to the finite maturity American put option problem

Let us now consider the finite maturity American put option problem

$$V(t,x) \;=\; \sup_{0\leq\tau\leq t} \mathbb{E}\Big(e^{-\alpha\tau}\big(K - S_\tau(x)\big)^+\Big) \tag{5.8.1}$$

for $\alpha \geq 0$ and all $(t,x) \in [0,T] \times \mathbb{R}_+$, where τ is a stopping time of the stock price process S whose dynamics are given under a measure $\mathbb{P}$ by

$$S_t(x) \;=\; xe^{(\alpha+\omega)t+X_t}, \tag{5.8.2}$$

where X is a Lévy process with $X_0 = 0$ under the measure $\mathbb{P}$.

We assume that the moment generating function

$$\Psi(\theta) \;=\; t^{-1}\log\mathbb{E}(e^{\theta X_t}) \text{ exists on the interval } (-\eta_1,\eta_2) \text{ with } \eta_1,\eta_2 \geq 1. \tag{H3}$$

The discount rate α is chosen so that

$$\alpha \;\geq\; (\Psi(-1) + \Psi(1)). \tag{H4}$$

Furthermore, we assume under the measure $\mathbb{P}$ that the discounted stock price process $\big(e^{-rt}S_t(x), t \geq 0\big)$ is $\mathbb{P}$-martingale, which implies that

$$\mathbb{E}\big(e^{-\alpha t}S_t(x)\big) \;=\; x.$$

The latter condition requires the parameter ω in (5.8.2) to be equal to

$$\omega \;=\; -t^{-1}\log \mathbb{E}(e^{X_t}) \;=\; -\Psi(1).$$

The problem of interest is to give an estimate for the value function V of the problem (5.8.1) in terms of the rational price of the perpetual American put option.

Remark 5.8.1 For finite maturity optimal stopping problem with payoff function $G(x) = (x^+)^n$, $n = 1, 2, ...$ or $G(x) = 1 - e^{-x^+}$ for random walks, an estimate for the value function V (5.8.1) was given recently by Novikov and Shiryaev in [91].

In the sequel below it should be understood that $V(\infty, x)$ corresponds to the value function of the perpetual American put option problem (5.7.6) and will be written simply by $V(x)$. Next, let us define for a fixed level $y \in \mathbb{R}$ a first passage time under the measure $\mathbb{P}$ of the stock price process S below a level e^y:

$$\tau_y^- \;=\; \inf\{t > 0 : S_t(x) \;\leq\; e^y\}. \tag{5.8.3}$$

Since the moment generating function $\Psi(\theta)$ of the underlying Lévy process is assumed to exist on an open set containing zero, we have an estimate for the value function of the finite maturity American put option problem (5.8.1) in terms of the value function of the perpetual American put option problem (5.7.6). The result is given by the following theorem.

Theorem 5.8.2 *Suppose that the assumptions* $(H3)$ *and* $(H4)$ *are satisfied. Assume that* $\tau_{b^\star}^-$ *is the optimal stopping time for the perpetual optimal stopping problem associated to (5.8.1). Then for each* $x \in \mathbb{R}_+$ *and all* $t > 0$ *we have the following estimate*[7]

$$\max\left\{V(x) - Ke^{-(\log(x)-b^\star)} \times e^{-(\alpha-(\Psi(1)+\Psi(-1)))t}, 0\right\} \;\leq\; V(t,x) \;\leq\; V(x). \tag{5.8.4}$$

From (5.8.4) we obtain the asymptotic value for the value function $V(t,x)$ as

$$\lim_{t\uparrow\infty} V(t,x) \;=\; V(x) \quad \text{for every } x \in \mathbb{R}.$$

and

$$\lim_{x\uparrow\infty} V(t,x) \;=\; 0 \quad \text{for every } t \geq 0.$$

The latter is quite straightforward from the equations (5.8.1) and (5.8.2), and from the fact that $\lim_{x\uparrow\infty} V(x) = 0$ (see the proof of Proposition 1 and also Figure 5.5).

Proof Following (5.8.1), it is clear that $V(t,x) \geq 0$ for all $t \geq 0$ and every $x \in \mathbb{R}$. Moreover, by the nature of the increasing property of the function $t \mapsto V(t,x)$, we see that

$$V(x) - V(t,x) \;\geq\; 0 \quad \text{for all } (t,x) \in [0,T] \times \mathbb{R}_+. \tag{5.8.5}$$

[7] Note that the result in Theorem 5.8.2 can be extended to a non-negative bounded payoff function G for which the problem (5.2.1) has a solution $\mathscr{P}_G^{(q)}$ for each $q \geq 0$ given.

Furthermore, from the optimal stopping problem (5.8.1), we see for all Markov stopping time τ, taking values in $[0,t]$, that

$$\begin{aligned} V(t,x) &\geq \mathbb{E}\Big(e^{-\alpha\tau}(K-S_\tau(x))^+\mathbf{1}_{(\tau\,\leq\, t)}\Big) \\ &= \mathbb{E}\Big(e^{-\alpha\tau}(K-S_\tau(x))^+\mathbf{1}_{(\tau\,<\infty)}\Big) \\ &\quad -\mathbb{E}\Big(e^{-\alpha\tau}(K-S_\tau(x))^+\mathbf{1}_{(t<\tau<\infty)}\Big). \end{aligned} \tag{5.8.6}$$

Since the level $e^{b^\star}$ is assumed to be the optimal boundary for the perpetual counterpart of the problem (5.8.1) with the associated stopping time $\tau^-_{b^\star}$, we see that

$$V(x) = \mathbb{E}\Big(e^{-\alpha\tau^-_{b^\star}}(K-S_{\tau^-_{b^\star}}(x))^+\mathbf{1}_{(\tau^-_{b^\star}\,<\infty)}\Big).$$

Following the inequality (5.8.6), we then obtain

$$\begin{aligned} V(x)-V(t,x) &\leq \mathbb{E}\Big(e^{-\alpha\tau^-_{b^\star}}(K-S_{\tau^-_{b^\star}}(x))^+\mathbf{1}_{(t<\tau^-_{b^\star}<\infty)}\Big) \\ &\leq K\mathbb{E}\Big(\mathbf{1}_{(t<\tau^-_{b^\star}<\infty)}\Big) \\ &\leq K\mathbb{P}(t<\tau^-_{b^\star}<\infty). \end{aligned}$$

The proof is completed once we show that

$$\mathbb{P}(t<\tau^-_{b^\star}<\infty) \leq e^{-(\log(x)-b^\star)}\times e^{-(\alpha-(\Psi(1)+\Psi(-1)))t}. \tag{5.8.7}$$

To complete the proof, let us introduce the *Esscher transform*[8]:

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = e^{-X_t-\Psi(-1)t} \quad \text{for all } t\geq 0.$$

Using this Esscher transform, we see that

$$\begin{aligned} \widetilde{\mathbb{P}}(t<\tau^-_{b^\star}<\infty) &= \widetilde{\mathbb{E}}\Big(\mathbf{1}_{(t<\tau^-_{b^\star}<\infty)}\Big) \\ &= \mathbb{E}\Big(\mathbf{1}_{(t<\tau^-_{b^\star}<\infty)}e^{-X_{\tau^-_{b^\star}}-\Psi(-1)\tau^-_{b^\star}}\Big) \\ &\geq \mathbb{E}\Big(\mathbf{1}_{(t<\tau^-_{b^\star}<\infty)}e^{(\log(x)-b^\star)+(\alpha-\Psi(1))\tau^-_{b^\star}-\Psi(-1)\tau^-_{b^\star}}\Big) \\ &= \mathbb{E}\Big(\mathbf{1}_{(t<\tau^-_{b^\star}<\infty)}e^{(\log(x)-b^\star)+(\alpha-(\Psi(1)+\Psi(-1)))\tau^-_{b^\star}}\Big) \\ &\geq \mathbb{E}\Big(\mathbf{1}_{(t<\tau^-_{b^\star}<\infty)}e^{(\log(x)-b^\star)+(\alpha-(\Psi(1)+\Psi(-1)))t}\Big) \\ &= e^{(\log(x)-b^\star)+(\alpha-(\Psi(1)+\Psi(-1)))t}\mathbb{E}\Big(\mathbf{1}_{(t<\tau^-_{b^\star}<\infty)}\Big) \\ &= e^{(\log(x)-b^\star)+(\alpha-(\Psi(1)+\Psi(-1)))t}\mathbb{P}(t<\tau^-_{b^\star}<\infty), \end{aligned}$$

[8]The Esscher transform is by now standard methodology in mathematical insurance, gradually however its appearance within mathematical finance is becoming more and more prominent, see for instance Gerber and Shiu [54] and the references and discussions therein.

which in turn leads to the inequality

$$\mathbb{P}(t < \tau_{b^\star}^- < \infty) \ \le \ e^{-(\log(x)-b^\star)-(\alpha-(\Psi(1)+\Psi(-1)))t}\widetilde{\mathbb{P}}(t < \tau_{b^\star}^- < \infty).$$

Since $\widetilde{\mathbb{P}}(t < \tau_{b^\star}^- < \infty) \le 1$, the claim that the value function $V(t,x)$ of the optimal stopping problem (5.8.1)-(5.8.2) satisfies the bounds (5.8.4) is then established. □

5.9 Conclusion and remarks

We have presented in this chapter an effective approach for solving perpetual optimal stopping problem (5.1.1) in a general setting. The approach is based on finding a solution to an averaging problem from which we obtain, using the Wiener-Hopf factorization, a fluctuation identity for first passage of Lévy processes. This fluctuation identity constitutes the main principle in obtaining an optimal solution of (5.1.1). This identity gives a generic link to some known identities used to solve the problem (5.1.1) with special payoff G, see for instance Darling et al. [33], Mordecki [87], Asmussen et al. [6], Alili and Kyprianou [3], Novikov and Shiryaev [91], and Kyprianou and Surya [73]. If a solution to the averaging problem can be found and has certain monotonicity properties, we showed that an optimal solution to the problem (5.1.1) can be written in terms of such monotone function.

Using the proposed approach, we are able to reproduce the special results of those discussed, among others, by Darling et al. [33], Mordecki [87], Boyarchenko and Levendorski [21], Alili and Kyprianou [3], Novikov and Shiryaev [91], and Kyprianou and Surya [73]. Using the optimal solution, we show that the C^1 smooth pasting condition holds if and only if the optimal stopping boundary is regular for the interior points of the stoping region for the Lévy process. Our conclusion over the smooth pasting condition extends further the recent work of Alili and Kyprianou [3] and Kyprianou and Surya [73] into a more general payoff function.

Furthermore, this conclusion shows no contradiction to the current numerical work, among others, of Hirsa and Madan [59], Matache et al.[81], and Almendral and Oosterlee [4]. Furthermore, assuming that the moment generating exists on an open set containing zero, we provided an upper and lower bounds for the value function of the finite maturity American put option problem in terms of the value function of the perpetual American put option problem.

Throughout this chapter we have assumed that the optimal stopping time belongs to a class of first passage times. This assumption boils down to computing the joint Laplace transform of the time of first exit of Lévy process below a certain level and its overshoot. It should also be possible to extend the problem (5.1.1) into the case where the class of Markov stopping times have values in finite time interval $[0,T]$. This problem amounts to solving the problem of first exit below a moving barrier for the Lévy process and solving this problem will be a challenging task both theoretically and numerically. We keep this task for possible future work.