Chapter 4

On the Novikov-Shiryaev Optimal Stopping Problems in Continuous Time[1]

Abstract

Novikov and Shiryaev [91] give explicit solutions to a class of optimal stopping problems for random walks based on other similar examples given in Darling et al. [33]. We give the analogue of their results when the random walks are replaced by Lévy processes. Further we show that the solutions show no contradiction with the conjecture given in Alili and Kyprianou [3] that there is smooth pasting at the optimal boundary if and only if the boundary of the stopping region is irregular for the interior of the stopping region.

4.1 Introduction

Let $X = \{X_t; t \geq 0\}$ be a Lévy process defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ satisfying the usual conditions, see Chapter 2 for more details. Consider for a given Lévy process X an optimal stopping problem of the form

$$V(x) \;=\; \sup_{\tau \in \mathcal{T}_{[0,\infty]}} \mathbb{E}_x\Big(e^{-q\tau} G(X_\tau)\mathbf{1}_{(\tau\,<\,\infty)}\Big), \tag{4.1.1}$$

where $q \geq 0$ and $\mathcal{T}_{[0,\infty]}$ is the family of stopping times with respect to the filtration $\mathcal{F}_t = \sigma(X_s, s \leq t)$. The purpose of this chapter is to characterize the solution to the problem (4.1.1) for the choices of gain functions

$$G(x) \;=\; (x^+)^n, \quad \text{for } n = 1, 2, 3, \ldots$$

under the hypothesis that

$$\text{either } q > 0 \text{ or } q = 0 \text{ and } \limsup_{t \uparrow \infty} X_t < \infty. \tag{H}$$

[1]This chapter is the extended version of: Kyprianou, A.E. and Surya, B. A. On the Novikov-Shiryaev optimal stopping problems in continuous time. *Elect. Comm. in Probab.*, **10** (2005), 146-154.

Note that when $q = 0$ and $\limsup_{t\uparrow\infty} X_t < \infty$ it is clear that it is never optimal to stop in (4.1.1) for the given choices of gain function G.

This chapter thus verifies that the results of Novikov and Shiryaev [91] for random walks carry over into the context of the Lévy process as predicted by the aforementioned authors. Novikov and Shiryaev [91] write:

"The results of this paper can be generalized to the case of stochastic processes with continuous time parameter (that is for Lévy processes instead of random walk). This generalization can be done by passage of limit from discrete time case (similarly to the techniques used in Mordecki [87] for pricing American options) or by use of the technique of pseudo-differential operators (described e.g. in the monograph Boyarchenko and Levendorskii [20] in the context of Lévy processes)".

We appeal to neither of the two methods referred to by Novikov and Shiryaev however. Instead we work with fluctuation theory of Lévy processes which is essentially the direct analogue of the random walks counterpart used in Novikov and Shiryaev [91]. In this sense our proofs are loyal to those of the latter. Minor additional features of our proofs are that we also allow for discounting as well avoiding the need to modify the gain function in order to obtain the solution. Truncation techniques are also avoided as much as possible. Undoubtedly however, the link with Appell polynomials as laid out by Novikov and Shiryaev remains the driving force of the solution. In addition we show that the solutions show no contradiction with the conjecture given in Alili and Kyprianou [3] that there is smooth pasting at the optimal boundary if and only if the boundary of the stopping region is regular for the interior of the stopping region.

4.2 Main results

In order to state the main results we need to introduce one of the tools identified by Novikov and Shiryaev to be instrumental in solving the optimal stopping problems at hand.

Definition 4.2.1 (Appell Polynomials) Suppose that Y is a non-negative random variable with n-th cummulant given by $\kappa_n \in (0, \infty]$ for $n = 1, 2, ...$ Then define the Appell polynomials iteratively as follows. Take $Q_0(x) = 1$ and assuming that $\kappa_n < \infty$ (equivalently Y has an n-th moment) given $Q_{n-1}(x)$ we define $Q_n(x)$ via

$$\frac{d}{dx} Q_n(x) = n Q_{n-1}(x). \tag{4.2.1}$$

This defines Q_n up to a constant. To pin this constant down we insist that $\mathbb{E}(Q_n(Y)) = 0$. The first three Appell polynomials are given for example by

$$\begin{aligned} Q_0(x) = 1,\ Q_1(x) = x - \kappa_1,\ Q_2(x) = (x - \kappa_1)^2 - \kappa_2 \\ Q_3(x) = (x - \kappa_1)^3 - 3\kappa_2(x - \kappa_1) - \kappa_3, \end{aligned} \tag{4.2.2}$$

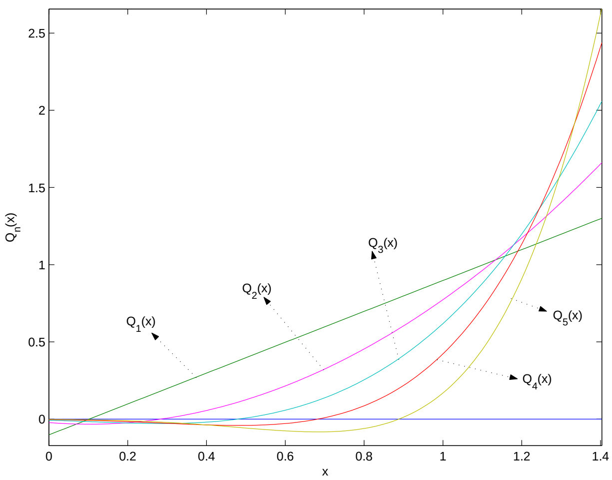

Figure 4.1: Plots of the first five Appell polynomials $Q_n(x)$, $n = 1, 2, \ldots, 5$, generated by upward jumps compound Poisson process having drift $\mathrm{d} = -0.1$.

under the assumption that $\kappa_3 < \infty$ (see Figure 4.1 above for the plots of $Q_n(x)$, $n = 1, 2, ..5$). We refer to Schoutens [112] for further details of Appell polynomials.

In the following theorem, we shall work with the Appell polynomials generated by the random variable $Y = \overline{X}_{\mathbf{e}_q}$ where for each $t \in [0, \infty]$, $\overline{X}_t = \sup_{s\in[0,t]} X_s$ and $\mathbf{e}_q$ is an exponentially distributed random variable which is independent of the Lévy process X. We shall work with the convention that when $q = 0$, the variable $\mathbf{e}_q$ is understood to be equal to ∞ with probability 1.

Theorem 4.2.2 *Let $\zeta = \mathbf{1}_{(q=0)}$. Fix $n \in \{1, 2, ...\}$. Suppose that the assumption (H) holds as well as*

$$\int_{(1,\infty)} x^{n+\zeta}\Pi(dx) \; < \; \infty.$$

Then $Q_n(x)$ has finite coefficients and there exists $x_n^\star \in [0, \infty)$ being the largest root of the equation $Q_n(x) = 0$. Let

$$\tau_n^\star \;=\; \inf\{t \geq 0 : X_t \geq x_n^\star\}.$$

Then $\tau_n^\star$ is an optimal strategy to (4.1.1) with $G(x) = (x^+)^n$. Further,

$$V_n(x) \;=\; \mathbb{E}_x\Big(Q_n(\overline{X}_{\mathbf{e}_q})\mathbf{1}_{(\overline{X}_{\mathbf{e}_q} \geq x_n^\star)}\Big).$$

Theorem 4.2.3 *For each $n = 1, 2, ...$ the solution V_n to the optimal stopping problem in the previous theorem is continuous and has the property that*

$$\frac{d}{dx}V_n(x_n^\star-) \;=\; \frac{d}{dx}V_n(x_n^\star+) - \frac{d}{dx}Q_n(x_n^\star)\mathbb{P}(\overline{X}_{\mathbf{e}_q} = 0).$$

Hence there is smooth pasting at $x_n^\star$ if and only if 0 *is regular for* $(0,\infty)$ *for* X.

Remark 4.2.4 (Regularity of 0 for $(0,\infty)$ for Lévy processes) Suppose that X is any Lévy process other than a compound Poisson process. The theory of Lévy processes offers us the opportunity to specify when regularity of 0 for $(0,\infty)$ for X occurs in terms of the triple (a,σ,Π) appearing the Lévy-Khintchine exponent (2.1.1). When X has bounded variation it will be more convenient to write (2.1.1) in the form

$$\Psi(\theta) \;=\; -i\mathrm{d}\theta + \int_{-\infty}^{\infty}(1-e^{i\theta x})\Pi(dx) \tag{4.2.3}$$

where $\mathrm{d}\in\mathbb{R}$ is known as the *drift coefficient*. We have that the point 0 is regular for $(0,\infty)$ for X (i.e., $\mathbb{P}(\overline{X}_{\mathbf{e}_q}=0)=0$) if and only if one of the following three conditions are fulfilled.
(i) $\int_{(-1,1)}|x|\Pi(dx)=\infty$ (so that X has unbounded variation).
(ii) $\int_{(-1,1)}|x|\Pi(dx)<\infty$ (so that X has bounded variation) and in the representation (4.2.3) we have $\mathrm{d}>0$.
(iii) $\int_{(-1,1)}|x|\Pi(dx)<\infty$ (so that X has bounded variation) and in the representation (4.2.3) we have $\mathrm{d}=0$ and further

$$\int_{(0,1)}\frac{x}{\int_{(0,x)}\Pi(-\infty,-y)dy}\Pi(dx) \;=\; \infty.$$

The latter conclusions being collectively due to Rogozin [108], Shtatland [116] and Bertoin [16].

Intuitively, the conditions $(i)-(iii)$ can be explained as follows. In case (i) when $\sigma>0$ regularity follows as a consequence of the presence of Brownian motion whose behavior on the small time scale always dominates the path of the Lévy process. If on the other hand $\sigma=0$, the condition $\int_{(-1,1)}|x|\Pi(dx)=\infty$ causes small jumps to have behavior on the small time scale close to Brownian motion. The case (ii) says that when the Poisson point process of jumps fulfills the condition $\int_{(-1,1)}|x|\Pi(dx)<\infty$, over small time scales, the sum of the jumps grows sublinearly in time almost surely. Therefore if the drift is present, this dominates the initial movement of the path. In case (iii) when there is no dominant drift, the integral test may be thought of as a statement about what the 'relative weight' of the small positive jumps compared to the small negative jumps needs to be in order for regularity to occur.

4.3 Preliminary lemmas

We need some preliminary results given in the following series of lemmas. All have previously been dealt with in Novikov and Shiryaev [91] for the case of random walks. For some of these lemmas we include slightly more direct proofs which work equally well for random walks (for example avoiding the use of truncation methods).

Lemma 4.3.1 (Moments of the supremum) *Let $\zeta = \mathbf{1}_{(q=0)}$. Fix $n > 0$ and $q \geq 0$. Suppose that the Lévy process X has jump measure Π satisfying*

$$\int_{(1,\infty)} x^{n+\zeta}\Pi(dx) \ < \ \infty. \tag{4.3.1}$$

Then $\mathbb{E}\big((X_1^+)^{n+\zeta}\big) < \infty$. Suppose further that (H) holds. Then $\mathbb{E}\big(\overline{X}_{\mathbf{e}_q}^n\big) < \infty$.

Although the analogue of this lemma is well known for random walks, it seems that one cannot find so easily the equivalent statement for Lévy processes in the existing literature; in particular the final statement of the lemma. None the less the proof can be extracted from a number of well known facts concerning Lévy processes.

Proof The fact that $\mathbb{E}\big((X_1^+)^{n+\zeta}\big) < \infty$ follows from the integral condition (4.3.1) can be seen by combining the result of Theorem 25.3 and Proposition 25.4 of Sato [111]. The remaining statement follows when $q \geq 0$ by Theorem 25.18 of the same book.

To see this let us denote by X^K the Lévy process with the same characteristic as X except that the Lévy measure Π is replaced by the truncated one Π^K defined as

$$\Pi^K(dx) \ = \ \Pi(dx)\mathbf{1}_{(x>-K)} + \delta_{-K}(dx)\Pi(-\infty, -K).$$

In other words, the paths of the Lévy process X^K are an adjustment of the paths of X in that all negative jumps of magnitude K or greater are replaced by a negative jump of precisely magnitude K. To establish our claim, first suppose that $q > 0$. The Wiener-Hopf factorization (see Theorem 2.2.1 in Chapter 2) gives us

$$\mathbb{E}\Big(e^{-i\theta\overline{X}^K_{\mathbf{e}_q}}\Big) \ = \ \mathbb{E}\Big(e^{i\theta X^K_{\mathbf{e}_q}}\Big) \times \frac{\widehat{\kappa}^K(q, i\theta)}{\widehat{\kappa}^K(q, 0)} \tag{4.3.2}$$

where $\mathbf{e}_q$ is an independent and exponentially distributed random variable with mean $1/q$ and $\widehat{\kappa}^K$ is the Laplace-Fourier exponent of the bivariate descending ladder process $\widehat{H}$ (see Section 2.2). Note that the descending ladder height process of $\widehat{H}$ cannot have jumps of size greater than K as X^K cannot jump downwards by more than K. Hence the Lévy measure of the descending ladder height process of X^K has bounded support which with the help of Theorem 25.3 and Proposition 25.4 of Sato [111] imply that all moments of the aforementioned process exist. Since $\mathbb{E}\big((X_1^+)^n\big) < \infty$, it implies that $\mathbb{E}(|X_t^K|^n) < \infty$ for all $t > 0$. The latter implies that the right-hand side of (4.3.2) has a Maclaurin expansion up to order n. Specifically this means that $\mathbb{E}\big((\overline{X}^K_{\mathbf{e}_q})^n\big) < \infty$. Due to the truncation, we finally have $\overline{X}_{\mathbf{e}_q} < \overline{X}^K_{\mathbf{e}_q}$ and hence $\mathbb{E}(\overline{X}^n_{\mathbf{e}_q}) < \infty$.

Now suppose that $\limsup_{t\uparrow\infty} X_t < \infty$ and $q = 0$ so that $\overline{X}_{\mathbf{e}_q} = \overline{X}_\infty$. In the absence of the killing constant, we assume[2] that $\mathbb{E}\big((X_1^+)^{n+1}\big) < \infty$. This condition follows from the integral condition (4.3.1) combining the result of Theorem 25.3 and

[2]This is a sufficient condition used in [91] to prove that $\mathbb{E}(\overline{X}^n_\infty) < \infty$ for random walk. In our case, this condition is required in case $\Psi^K(\theta)$ and $\widehat{\kappa}^K(0, i\theta)$ has a factor θ cancelling in (4.3.3).

Proposition 25.4 of Sato [111]. As before, we appeal to the Wiener-Hopf factorization for X^K in the form (up to a multiplicative constant)

$$\kappa^K(0,-i\theta) \;=\; \frac{\Psi^K(\theta)}{\widehat{\kappa}^K(0,i\theta)} \tag{4.3.3}$$

where κ^K and Ψ^K are obviously defined. The same reasoning in the previous paragraphs shows that the Maclaurin expansion on the right-hand side above exists up to order n and hence the same is true for the left-hand side. We make the truncation level K large enough so that it is still the case that $\lim_{t\to\infty} X_t^K = -\infty$. This is possible by choosing K sufficiently large so that $\mathbb{E}(X_1^K) < 0$.

We now have that $\widehat{\kappa}^K(0,0) = 0$ and that $\widehat{\kappa}^K(0,i\theta)$ has an infinite Maclaurin expansion. The assumption $\mathbb{E}\big((X_1^+)^{n+1}\big) < \infty$ implies that $\Psi^K(\theta)$ has Maclaurin expansion up to order $n+1$ and as a matter of fact $\Psi^K(0) = 0$. It now follows that the ratio $\Psi^K(\theta)/\widehat{\kappa}^K(0,i\theta)$ has a Maclaurin expansion up to order n. Since $\kappa^K(0,-i\theta)$ is the cumulative generating function of the ascending ladder height process of X^K it follows that the aforementioned process has finite nth moments. Since $\overline{X}_\infty^K$ is equal in law to the ascending ladder height process of X^K stopped at an independent and exponentially distributed random time, we have that $\mathbb{E}\big((\overline{X}_\infty^K)^n\big) < \infty$. Finally we have $\mathbb{E}\big((\overline{X}_\infty)^n\big) < \infty$ since $\overline{X}_\infty \le \overline{X}_\infty^K$, which can be shown similar to above. □

Lemma 4.3.2 (Mean value property) *Fix $n \in \{1,2,...\}$. Suppose that Y is a non-negative random variable satisfying $\mathbb{E}(Y^n) < \infty$. Then if Q_n is the n-th Appell polynomial generated by Y then we have that*

$$\mathbb{E}\Big(Q_n(x+Y)\Big) \;=\; x^n \quad \text{for all } x \in \mathbb{R}.$$

Proof As remarked in Novikov and Shiryaev [91], this result can be obtained by truncation of the variable Y. However, it can also be derived from the definition of Q_n given in (4.2.1). Indeed note the result is trivially true for $n = 1$. Next suppose the result is true for Q_{n-1}. Then using dominated convergence we have from (4.2.1)

$$\frac{d}{dx}\mathbb{E}\Big(Q_n(x+Y)\Big) = \mathbb{E}\left(\frac{d}{dx}Q_n(x+Y)\right) = n\mathbb{E}\Big(Q_{n-1}(x+Y)\Big) = nx^{n-1}.$$

Solving together with the requirement that $\mathbb{E}(Q_n(Y)) = 0$ we have the result. □

Lemma 4.3.3 (Fluctuation identity) *Let $\zeta = \mathbf{1}_{(q=0)}$. Fix $n \in \{1,2,...\}$ and suppose that*

$$\int_{(1,\infty)} x^{n+\zeta}\Pi(dx) \;<\; \infty,$$

and that (H) holds. Define $\tau_a^+ = \inf\{t \ge 0 : X_t > a\}$. Then for all $a > 0$ and $x \in \mathbb{R}$

$$\mathbb{E}_x\Big(e^{-q\tau_a^+}X_{\tau_a^+}^n\mathbf{1}_{(\tau_a^+<\infty)}\Big) = \mathbb{E}_x\Big(Q_n(\overline{X}_{\mathbf{e}_q})\mathbf{1}_{(\overline{X}_{\mathbf{e}_q}>a)}\Big).$$

Proof Note that on the event $\{\tau_a^+ < \mathbf{e}_q\}$ we have that $\overline{X}_{\mathbf{e}_q} = X_{\tau_a^+} + S$ where S is independent of $\mathcal{F}_{\tau_a^+}$ and has the same distribution as $\overline{X}_{\mathbf{e}_q}$. It follows that

$$\mathbb{E}_x\Big(Q_n(\overline{X}_{\mathbf{e}_q})\mathbf{1}_{(\overline{X}_{\mathbf{e}_q}>a)}\Big|\mathcal{F}_{\tau_a^+}\Big) = \mathbf{1}_{(\tau_a^+<\mathbf{e}_q)}h(X_{\tau_a^+})$$

where $h(x) = \mathbb{E}_x(Q_n(\overline{X}_{\mathbf{e}_q}))$. From Lemma 4.3.2 with $Y = \overline{X}_{\mathbf{e}_q}$ one also has that $h(x) = x^n$. We see then by taking expectations again in the previous calculation that

$$\mathbb{E}_x\Big(Q_n(\overline{X}_{\mathbf{e}_q})\mathbf{1}_{(\overline{X}_{\mathbf{e}_q}>a)}\Big) = \mathbb{E}_x\Big(e^{-q\tau_a^+}X^n_{\tau_a^+}\mathbf{1}_{(\tau_a^+<\infty)}\Big)$$

as required. □

Lemma 4.3.4 (Largest positive root) *Let* $\zeta = \mathbf{1}_{(q=0)}$. *Fix* $n \in \{1, 2, ...\}$ *and suppose that*

$$\int_{(1,\infty)} x^{n+\zeta}\Pi(dx) \ < \ \infty.$$

Suppose that (H) *holds and* Q_n *is generated by* $\overline{X}_{\mathbf{e}_q}$. *Then* Q_n *has a unique positive root* x_n^* *such that* $Q_n(x)$ *is negative on* $[0, x_n^*)$ *and positive and increasing on* $[x_n^*, \infty)$.

Proof The proof follows proof of the same statement given for random walks in Novikov and Shiryaev [91] with minor modifications. (It is important to note that in following their proof, it is not necessary to make an approximation of the Lévy process by a random walk). Notice that the statement of the lemma is straightforward for $n = 1$. The proof for $n > 1$ is done using induction arguments.

The first step is to show that $Q_n(0) \le 0$. To start with let us denote by

$$\tau_a^+ \ = \ \inf\{t \ge 0 : X_t \ge a\}$$

the first time X goes above a level a and

$$\gamma(a, n) \ = \ \mathbb{E}\Big(e^{-q\tau_a^+}X^n_{\tau_a^+}\mathbf{1}_{\tau_a^+<\infty}\Big).$$

Note that $\gamma(a, n) \ge 0$ for all $a \ge 0$ and $n = 1, 2, ...$ On the other hand, we see that

$$\begin{aligned}
\gamma(a, n) &= \mathbb{E}\Big(Q_n(\overline{X}_{\mathbf{e}_q})\mathbf{1}_{(\overline{X}_{\mathbf{e}_q}\ge a)}\Big) \\
&= -\mathbb{E}\Big(Q_n(\overline{X}_{\mathbf{e}_q})\mathbf{1}_{(\overline{X}_{\mathbf{e}_q}<a)}\Big) \\
&= -\mathbb{P}\big(\overline{X}_{\mathbf{e}_q} < a\big)Q_n(0) \\
&\quad +\mathbb{E}\Big((Q_n(0) - Q_n(\overline{X}_{\mathbf{e}_q}))\mathbf{1}_{(\overline{X}_{\mathbf{e}_q}<a)}\Big)
\end{aligned}$$

where the first equality follows from applying Lemma 4.3.3 while the second equality follows from using Lemma 4.3.2. Following the definition

$$Q_n(x) \ = \ Q_n(0) + n\int_0^x Q_{n-1}(y)dy \qquad (4.3.4)$$

for all $x \geq 0$ we have the estimate

$$\left|\mathbb{E}\Big((Q_n(0) - Q_n(\overline{X}_{\mathbf{e}_q}))\mathbf{1}_{(\overline{X}_{\mathbf{e}_q}<a)}\Big)\right| \leq na \sup_{y\in[0,a]}\Big|Q_{n-1}(y)\Big|\mathbb{P}\big(\overline{X}_{\mathbf{e}_q} < a\big),$$

which tends to zero as $a \downarrow 0$. Thus, we then deduce that

$$0 \leq \gamma(a,n) \leq -\mathbb{P}\big(\overline{X}_{\mathbf{e}_q} < a\big)\big[Q_n(0) + o(a)\big]$$

as a approaches zero. This implies that $Q_n(0) \leq 0$. Under the induction hypothesis for Q_{n-1}, we see from (4.3.4) together with the fact that $Q_n(0) \leq 0$ that Q_n is negative and decreasing on the interval $[0, x^\star_{n-1})$. The point $x^\star_{n-1}$ corresponds to the minimum of Q_n thanks to the positivity and the monotonicity of $Q_{n-1}(x)$ for $x > x^\star_{n-1}$. In particular, $Q_n(x)$ tends to infinity from its minimum point and hence there must be a unique strictly positive root of the equation $Q_n(x) = 0$. Thus, our claim that Q_n has a unique positive root is then established. □

4.4 Proofs of theorems

Proof of Theorem 4.2.2

Proof In light of the Novikov-Shiryaev optimal stopping problems and their solutions, we verify that the analogue of their solution, namely the one proposed in Theorem 4.2.2, is also a solution for (4.1.1) for $G(x) = (x^+)^n$, $n = 1, 2, ...$

To this end, fix $n \in \{1, 2, ...\}$ and define

$$V_n(x) = \mathbb{E}_x\Big(Q_n(\overline{X}_{\mathbf{e}_q})\mathbf{1}_{(\overline{X}_{\mathbf{e}_q}>x^*_n)}\Big).$$

First note from Lemma 4.3.3 that

$$V_n(x) = \mathbb{E}_x\Big(e^{-q\tau^*_n}(X^+_{\tau^*_n})^n\mathbf{1}_{(\tau^*_n<\infty)}\Big)$$

and hence the pairs (V_n, τ^*_n) are a candidate pair to solve the problem (4.1.1).

Secondly we prove that $V_n(x) \geq (x^+)^n$ for all $x \in \mathbb{R}$. Note that this statement is obvious for $x \in (-\infty, 0] \cup [x^*_n, \infty)$ just from the definition of V_n. Otherwise when $x \in (0, x^*_n)$ we have, using the mean value property in Lemma 4.3.2 that

$$\begin{aligned} V_n(x) &= \mathbb{E}_x\Big(Q_n(\overline{X}_{\mathbf{e}_q})\mathbf{1}_{(\overline{X}_{\mathbf{e}_q}>x^*_n)}\Big) \\ &= x^n - \mathbb{E}_x\Big(Q_n(\overline{X}_{\mathbf{e}_q})\mathbf{1}_{(\overline{X}_{\mathbf{e}_q}\leq x^*_n)}\Big) \\ &\geq (x^+)^n \end{aligned}$$

where the final inequality follows from Lemma 4.3.4 and specifically the fact that $Q_n(x) \leq 0$ on $(0, x^*_n]$. Note in particular, embedded in this argument is the statement that $V_n(x-) = (x^+)^n$ at $x = x^*_n$.

Thirdly, we have $\mathbb{P}_x$ almost surely that $Q_n(\overline{X}_{\mathbf{e}_q})\mathbf{1}_{(\overline{X}_{\mathbf{e}_q}>x_n^*)} \geq 0$. Using the latter together with the fact that, on the event that $\{\mathbf{e}_q > t\}$ we have $\overline{X}_{\mathbf{e}_q}$ is equal in distribution to $X_t + I$ where I is independent of $\mathcal{F}_t$ and equal in distribution to $\overline{X}_{\mathbf{e}_q}$, it follows that

$$\begin{aligned} V_n(x) &\geq \mathbb{E}_x\Big(\mathbf{1}_{(\mathbf{e}_q>t)}Q_n(\overline{X}_{\mathbf{e}_q})\mathbf{1}_{(\overline{X}_{\mathbf{e}_q}>x_n^*)}\Big) \\ &= \mathbb{E}_x\Big(\mathbf{1}_{(\mathbf{e}_q>t)}\mathbb{E}_x\Big(Q_n(X_t+\overline{X}_{\mathbf{e}_q})\mathbf{1}_{(X_t+\overline{X}_{\mathbf{e}_q}>x_n^*)}\Big|\mathcal{F}_t\Big)\Big) \\ &= \mathbb{E}_x\Big(e^{-qt}V_n(X_t)\Big). \end{aligned}$$

From this inequality together with the Markov property, it is easily shown that $\{e^{-qt}V_n(X_t) : t \geq 0\}$ is a supermartingale.

Finally we put these three facts together as follows to complete the proof. From the supermartingale property and the lower bound on V_n it follows that

$$V_n(x) \geq \sup_{\tau\in\mathcal{T}_{[0,\infty]}} \mathbb{E}_x\Big(e^{-q\tau}V_n(X_\tau)\mathbf{1}_{(\tau<\infty)}\Big) \geq \sup_{\tau\in\mathcal{T}_{[0,\infty]}} \mathbb{E}_x\Big(e^{-q\tau}(X_\tau^+)^n\mathbf{1}_{(\tau<\infty)}\Big). \tag{4.4.1}$$

On the other hand, rather trivially, we have

$$\sup_{\tau\in\mathcal{T}_{[0,\infty]}} \mathbb{E}_x\Big(e^{-q\tau}(X_\tau^+)^n\mathbf{1}_{(\tau<\infty)}\Big) \geq \mathbb{E}_x\Big(e^{-q\tau_n^*}(X_{\tau_n^*}^+)^n\mathbf{1}_{(\tau_n^*<\infty)}\Big) = V_n(x). \tag{4.4.2}$$

and the proof of the theorem follows. □

Proof of Theorem 4.2.3

Proof It has already been noted in the previous proof that there is continuity of V_n at the point x_n^*. To establish when there is a smooth pasting at this point, we calculate as follows. For $x < x_n^*$

$$\begin{aligned} \frac{V_n(x_n^*) - V(x)}{x_n^* - x} &= \frac{(x_n^*)^n - x^n}{x_n^* - x} + \frac{\mathbb{E}_x(Q_n(\overline{X}_{\mathbf{e}_q})\mathbf{1}_{(\overline{X}_{\mathbf{e}_q}\leq x_n^*)})}{x_n^* - x} \\ &= \frac{(x_n^*)^n - x^n}{x_n^* - x} + \frac{\mathbb{E}_x((Q_n(\overline{X}_{\mathbf{e}_q}) - Q_n(x_n^*))\mathbf{1}_{(\overline{X}_{\mathbf{e}_q}\leq x_n^*)})}{x_n^* - x} \end{aligned}$$

where the final equality follows because $Q_n(x_n^*) = 0$. Clearly

$$\lim_{x\uparrow x_n^*} \frac{(x_n^*)^n - x^n}{x_n^* - x} = \frac{dV_n}{dx}(x_n^*+).$$

However,

$$\begin{aligned} \frac{\mathbb{E}_x((Q_n(\overline{X}_{\mathbf{e}_q}) - Q_n(x_n^*))\mathbf{1}_{(\overline{X}_{\mathbf{e}_q}\leq x_n^*)})}{x_n^* - x} =& \frac{\mathbb{E}_x((Q_n(\overline{X}_{\mathbf{e}_q}) - Q_n(x))\mathbf{1}_{(x<\overline{X}_{\mathbf{e}_q}\leq x_n^*)})}{x_n^* - x} \\ &- \frac{\mathbb{E}_x((Q_n(x_n^*) - Q_n(x))\mathbf{1}_{(\overline{X}_{\mathbf{e}_q}\leq x_n^*)})}{x_n^* - x}, \end{aligned} \tag{4.4.3}$$

where in the first term on the right hand side we may restrict the expectation to $\{x < \overline{X}_{\mathbf{e}_q} \leq x_n^*\}$ as the atom of $\overline{X}_{\mathbf{e}_q}$ at zero gives zero mass to the expectation. Denote by A_x and B_x the two expressions on the right hand side of equation (4.4.3). We have that

$$\lim_{x \uparrow x_n^*} B_x = -\frac{dQ_n(x_n^*)}{dx}\mathbb{P}(\overline{X}_{\mathbf{e}_q} = 0).$$

Integration by parts also gives

$$\begin{aligned} A_x &= \int_{(0,x_n^*-x]} \frac{Q_n(x+y) - Q_n(x)}{x_n^* - x}\mathbb{P}(\overline{X}_{\mathbf{e}_q} \in dy) \\ &= \frac{Q_n(x_n^*) - Q_n(x)}{x_n^* - x}\mathbb{P}(\overline{X}_{\mathbf{e}_q} \in (0, x_n^* - x]) \\ &\quad - \frac{1}{x_n^* - x}\int_0^{x_n^*-x} \mathbb{P}(\overline{X}_{\mathbf{e}_q} \in (0,y])\frac{dQ_n}{dx}(x+y)dy. \end{aligned}$$

Hence it follows that

$$\lim_{x \uparrow x_n^*} A_x = 0.$$

In conclusion we have that

$$\lim_{x \uparrow x_n^*} \frac{V_n(x_n^*) - V(x)}{x_n^* - x} = \frac{dV_n}{dx}(x_n^*+) - \frac{dQ_n(x_n^*)}{dx}\mathbb{P}(\overline{X}_{\mathbf{e}_q} = 0)$$

which concludes the proof. □

4.5 Numerical examples

This section discusses some numerical examples of the results presented in Section 2.

For this numerical purposes, we consider two cases. Firstly, we choose X to be a spectrally negative Lévy process of bounded variation. Necessarily, it takes the form of a linear drift minus a subordinator. We take the drift d to be at rate 0.1 and the subordinator to be a compound Poisson process with exponentially distributed jumps; that is to say that X has Laplace exponent

$$\kappa(\lambda) \;=\; \mathrm{d}\lambda + \int_{-\infty}^0 ace^{cx}(e^{\lambda x} - 1)dx \;=\; \mathrm{d}\lambda - \frac{a\lambda}{c+\lambda}. \tag{4.5.1}$$

As explained in Section 2.3 of Chapter 2, it is known that the moment generating function $\Psi_q^{(+)}(\lambda)$ of the random variable $\overline{X}_{\mathbf{e}_q}$ is given for $q \geq 0$ and $\mathfrak{Re}(\lambda) \geq 0$ by

$$\Psi_q^{(+)}(\lambda) \;=\; \int_0^\infty e^{-\lambda x}\mathbb{P}(\overline{X}_{\mathbf{e}_q} \in dx) \;=\; \frac{\Phi(q)}{\lambda + \Phi(q)}.$$

Following this Laplace transform, we deduce using Tauberian theorem that

$$\mathbb{P}(\overline{X}_{\mathbf{e}_q} = 0) \;=\; \lim_{\lambda \uparrow \infty} \frac{\Phi(q)}{\lambda + \Phi(q)} = 0, \tag{4.5.2}$$

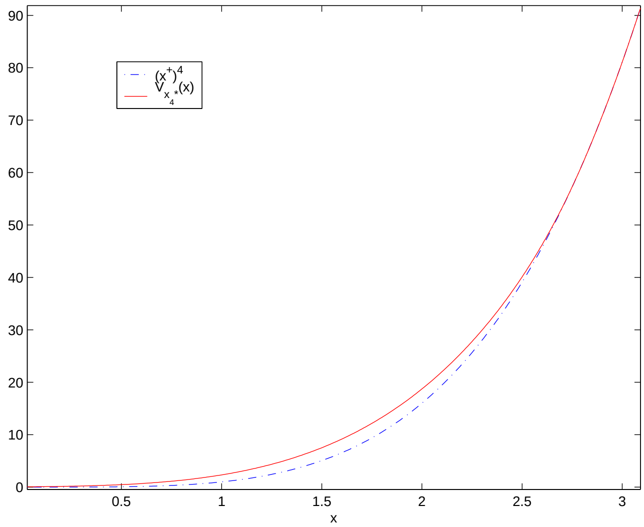

Figure 4.2: The shape of the value function of an optimal stopping problem with payoff function $G(x) = (x^+)^4$ driven by downward jumps compound Poisson process with drift $\mathrm{d} = 0.1$. The optimal stopping boundary is given by $x_4^\star = 2.7789$.

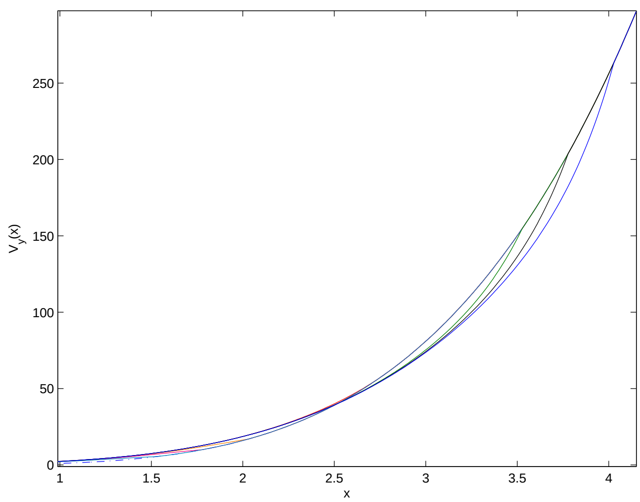

Figure 4.3: The shape of a candidate solution $V_y(x)$ for different values of boundary y of an optimal stopping problem with payoff function $G(x) = (x^+)^4$ driven by downward jumps compound Poisson process with drift $\mathrm{d} = 0.1$. The optimal stopping boundary is given by $x_4^\star = 2.7789$.

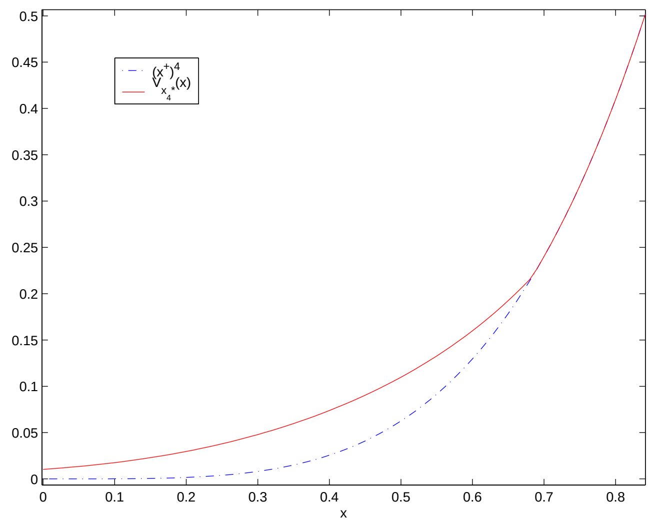

Figure 4.4: The shape of the value function of an optimal stopping problem with payoff function $G(x) = (x^+)^4$ driven by upward jumps compound Poisson process with drift $\mathrm{d} = -0.1$. The optimal stopping boundary is given by $x_4^\star = 0.6832$.

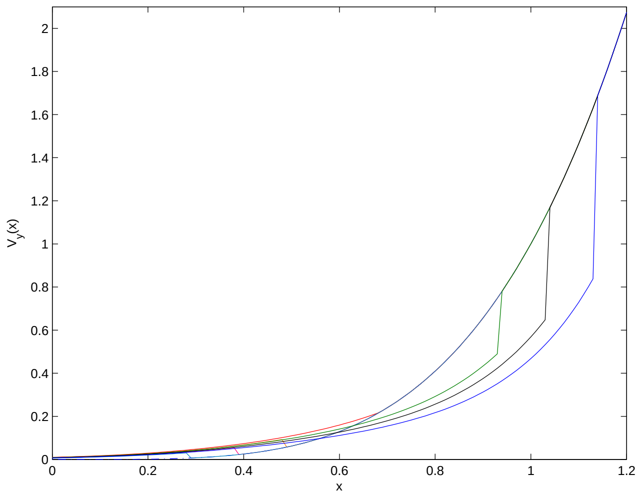

Figure 4.5: The shape of a candidate solution $V_y(x)$ for different values of boundary y of an optimal stopping problem with payoff function $G(x) = (x^+)^4$ driven by upward jumps compound Poisson process with drift $\mathrm{d} = -0.1$. The optimal stopping boundary is given by $x_4^\star = 0.6832$.

from which it follows that 0 is regular for the upper half-line $(0,\infty)$ for X.

Secondly, we consider the dual process $\widehat{X} = -X$ of the former spectrally negative compound Poisson process X. We denote by $\overline{\widehat{X}}_{\mathbf{e}_q} = \sup_{0\le s\le \mathbf{e}_q} \widehat{X}_s$ the running supremum of the dual process $\widehat{X}$ up to random time $\mathbf{e}_q$. Since $\overline{\widehat{X}}_{\mathbf{e}_q} = -\underline{X}_{\mathbf{e}_q}$, by duality arguments, it is known (see for instance Section 5 of Bingham [17]) that the moment generating function $\Psi_q^{(+)}(\lambda)$ of the random variable $\overline{\widehat{X}}_{\mathbf{e}_q}$ is given by

$$\Psi_q^{(+)}(\lambda) \;=\; \int_0^\infty e^{-\lambda x}\mathbb{P}(\overline{\widehat{X}}_{\mathbf{e}_q}\in dx) \;=\; \frac{q}{\Phi(q)}\Big(\frac{\lambda-\Phi(q)}{\kappa(\lambda)-q}\Big),$$

for $q>0$ and $\mathfrak{Re}(\lambda)\ge 0$, where $\Phi(a)$ is the largest root γ of the equation $\kappa(\gamma)=a$. (See also Section 2.3 of Chapter 2 for more details.) For numerical inversion of Laplace transform, we refer to Chapter 7 for further discussions. Applying the same arguments as before, we deduce from the foregoing expression that

$$\mathbb{P}(\overline{\widehat{X}}_{\mathbf{e}_q}=0) \;=\; \lim_{\lambda\uparrow\infty}\frac{q}{\Phi(q)}\Big(\frac{\lambda-\Phi(q)}{\kappa(\lambda)-q}\Big) \;=\; \frac{q}{\Phi(q)}\Big(\frac{1}{\mathrm{d}}\Big) \;>\; 0. \tag{4.5.3}$$

The expression (4.5.3) tells us that 0 is irregular for $(0,\infty)$ for the dual process $\widehat{X}$.

For all computations, we set $q=0.075$, $n=4$, $c=9$ and $a=0.5$. The numerical computation is carried out using MATLAB6.5.

For each of these two Lévy processes, we consider the function

$$V_y(x) \;=\; (x^+)^n - \mathbb{E}_x\Big(Q_4(\overline{X}_{\mathbf{e}_q})\mathbf{1}_{(\overline{X}_{\mathbf{e}_q}\le y)}\Big) \tag{4.5.4}$$

as a candidate solution to the optimal stopping problem (4.1.1). By varying the values of the boundary y, we present in Figures 4.2-4.5 the plots of the function $x\mapsto V_y(x)$ for $x\ge 0$ with steps $dx=0.01$. From these plots, we notice in all respects that all curves V_y are upper bounded by that of associated with the optimal boundary $y=x_4^\star$ (the largest root of the equation $Q_4(x)=0$). This majorant property of $V_{x_4^\star}$ can be explained using (4.4.1)-(4.4.2) and the result of Lemma 4.3.3 as follows

$$\begin{aligned} V_{x_4^\star}(x) \;&=\; \sup_{\tau\in\mathcal{T}_{[0,\infty]}} \mathbb{E}_x\Big(e^{-q\tau}(X_\tau^+)^n\mathbf{1}_{(\tau<\infty)}\Big) \quad \{\text{by (4.4.1) and (4.4.2)}\}\\ &\ge\; \mathbb{E}_x\Big(e^{-q\tau_y^+}(X_{\tau_y^+}^+)^n\mathbf{1}_{(\tau_y^+<\infty)}\Big)\\ &=\; \mathbb{E}_x\Big(Q_n(\overline{X}_{\mathbf{e}_q})\mathbf{1}_{(\overline{X}_{\mathbf{e}_q}>y)}\Big) \qquad \{\text{by Lemma 4.3.3}\}\\ &=\; V_y(x). \end{aligned}$$

For the spectrally negative compound Poisson process X (4.5.1), we observe from Figures 4.2-4.3 that the continuous pasting condition $V_y(x)=(x^+)^4$ holds at point $x=y$, for any $y\ge 0$. This is because by evaluating the candidate solution (4.5.4) at the point $x=y$, we see that

$$V_y(x) \;=\; (x^+)^4 - \mathbb{P}(\overline{X}_{\mathbf{e}_q}=0)Q_4(x) \quad \text{at } x=y. \tag{4.5.5}$$

Hence, since $\mathbb{P}(\overline{X}_{\mathbf{e}_q} = 0) = 0$ for this process (see the expression (4.5.2)) we obtain following (4.5.5) that $V_y(x) = (x^+)^4$ at the point $x = y$ of any stopping boundary $y \geq 0$. In addition, we observe also from Figures 4.2 and 4.3 that the smooth pasting condition $\frac{d}{dx}V_y(x) = \frac{d}{dx}(x^+)^4$ only holds at the point $x = y$ of the optimal boundary $y = x_4^\star$. In view of Theorem[3] 4.2.3, this observation is obvious following the fact that $\mathbb{P}(\overline{X}_{\mathbf{e}_q} = 0) = 0$ for this process. Thus, our claim in Theorem 4.2.3 is then verified.

In contrast to the first two plots, we observe from Figures 4.4-4.5 that the candidate solution $V_y(x)$ for the dual process $\widehat{X}$ satisfies the continuous pasting condition $V_y(x) = (x^+)^4$ only at the point $x = y$ of the optimal stopping boundary $y = x_4^\star$ (as displayed in Figure 4.4) and exhibit negative (resp., positive) jumps of magnitude $\mathbb{P}(\overline{\widehat{X}}_{\mathbf{e}_q} = 0)Q_4(y)$ when $y > x_4^\star$ (resp., $y < x_4^\star$) as Figure 4.5 shows. This phenomenon is well understood following the equation (4.5.5). Moreover, taking account of the fact that $\mathbb{P}(\overline{\widehat{X}}_{\mathbf{e}_q} = 0) > 0$ for the dual process $\widehat{X}$ (see expression (4.5.3)), it is clear following Theorem 4.2.3 that the smooth pasting condition $\frac{d}{dx}V_y(x) = \frac{d}{dx}(x^+)^4$ does not hold at the point $x = y$ of the optimal stopping boundary $y = x_4^\star$ as illustrated in Figure 4.4.

To summarize, we have seen that all the numerical results obtained in this section are found to be consistent with the main results of Section 2.

4.6 Concluding remarks

(i) As in Alili and Kyprianou [3] one can argue that the occurrence of continuous pasting for irregularity and smooth pasting for regularity appear as a matter of principle. The way to see this is to consider the candidate solutions (V_y, τ_y^+) where $\tau_y^+ = \inf\{t \geq 0 : X_t > y\}$ and $V_y(x) = \mathbb{E}_x\big(Q_n(\overline{X}_{\mathbf{e}_q})\mathbf{1}_{(\overline{X}_{\mathbf{e}_q} > y)}\big)$. By varying the value of y in $(0, \infty)$ one will find that, when there is irregularity, in general there is a discontinuity of V_y at y (as illustrated in Figure 4.5) and otherwise when there is regularity, there is always continuity at y (as Figure 4.5 displays). In both cases, let $\mathcal{C}$ be the class of $y > 0$ for which V_y is lower bounded by the gain and is superharmonic (it composes with X to make a supermartingale when discounted at rate q). When there is irregularity, the choice of $y = x_n^*$ is the unique point in $\mathcal{C}$ for which the discontinuity at y is closed and hence the function V_y turns out to be pointwise minimal. When there is regularity, the minimal curve indexed in $\mathcal{C}$ will occur by adjusting y so that the gradients either side of y match which again turns out to be the unique value $y = x_n^*$.

(ii) From arguments presented in Novikov and Shiryaev [91] together with the supporting arguments given in this chapter, it is now clear how to handle the gain function $G(x) = 1 - e^{x^+}$ for Lévy processes instead of random walks as well as how to handle the pasting principles at the optimal boundary.

[3] See also Theorem 5.4.3 in Chapter 5.