Chapter 3

A Change of Variable Formula with Local Time-Space for Bounded Variation Lévy Processes with Application to Solving the American Put Option Problem[1]

Abstract

We establish a change of variable formula with local time-space for 'ripped' functions of Lévy processes of bounded variation. Our results complement the recent work of Föllmer et al. [50], Eisenbaum ([43], [44]), Peskir ([97], [98]) and Elworthy et al. [46] in which generalized versions of Itô's formula were established with local time-space. The result is applied to solving the American put option problem driven by bounded variation Lévy processes.

3.1 Lévy processes of bounded variation and local time-space

In this chapter we shall establish a change of variable formula for 'ripped' time-space functions of Lévy processes of bounded variation at the cost of an additional integral with respect to local time-space in the formula. Roughly speaking, by a ripped function, we mean here a time-space function which is $C^{1,1}$ on either side of a time-dependent barrier and which may exhibit a discontinuity along the barrier itself. Such functions have appeared in the theory of optimal stopping problems for Markov processes of bounded variation (cf. Peskir and Shiryaev ([95], [96]), Chan ([26], [27]), Avram et al. [7]). Our starting point is to give a brief review of the relevant features of Lévy processes of bounded variation and what is meant by local time-space for these processes.

Suppose that $(\Omega, \mathcal{F}, \mathbb{F}, P)$ is a filtered probability space with filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ satisfying the usual conditions of right continuity and completion. In this text,

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we take as our definition of a Lévy process on $(\Omega, \mathcal{F}, \mathbb{F}, P)$, the strong Markov, $\mathbb{F}$-adapted process $X = \{X(t) : t \geq 0\}$ with paths that are right continuous with left limits (càdlàg) having the properties that $P(X(0) = 0) = 1$ and for each $0 \leq s \leq t$, the increment $X(t) - X(s)$ is independent of $\mathcal{F}_s = \sigma(X_u, u \leq s)$ and has the same distribution as $X(t-s)$. On each finite time interval, X has paths of bounded variation (or just X has bounded variation for short) if and only if for each $t \geq 0$,

$$X(t) \ = \ \mathrm{d}t + \sum_{0<s\leq t} \Delta_s, \tag{3.1.1}$$

where $\mathrm{d} \in \mathbb{R}$ and $\{(s, \Delta_s) : s \geq 0\}$ is a Poisson point process on $[0, \infty) \times (\mathbb{R}\backslash\{0\})$ with (time-space) intensity measure $dt \times \Pi\,(dx)$ satisfying

$$\int_{-\infty}^{\infty} (1 \wedge |x|)\, \Pi\,(dx) \ < \ \infty.$$

Note that the later integrability condition is necessary and sufficient for the convergence of $\sum_{0<s\leq t} |\Delta_s|$. The process X is further a compound Poisson process with drift if and only if $\Pi\,(\mathbb{R}\backslash\{0\}) < \infty$.

For any such Lévy process we say that 0 *is irregular for itself* if

$$P(T = 0) \ = \ 0$$

where T is the first visit of X to the origin,

$$T \ = \ \inf\left\{t > 0 : X_t = 0\right\}$$

with the usual definition $\inf \emptyset = \infty$ being understood in the present context as corresponding to the case that Y never visits the origin over the time interval $(0, \infty)$. Standard theory allows us to deduce that T is a stopping time. With the exception of a compound Poisson process, 0 is always irregular for itself within the class of Lévy processes of bounded variation. Further, again excluding the case of a compound Poisson process, we have that

$$P(T < \infty) > 0 \Longleftrightarrow \mathrm{d} \ \neq \ 0. \tag{3.1.2}$$

We refer to Bertoin [14] for a much deeper account of regularity properties Lévy processes. For the purpose of this text we need to extend the idea of irregularity for points to irregularity of time-space curves.

Definition 3.1.1 Given a Lévy process X with finite variation, a measurable time-space curve $b : [0, \infty) \to \mathbb{R}$ is said to be *irregular for itself for* X if for all $\infty > T \geq s \geq 0$,

$$P_{(s,b(s))}(\#\{t \in (s, T] : X(t) = b(t)\} < \infty) = 1,$$

and $t \in \{s \geq 0 : X(s) = b(s)\}$ if and only if $\lim_{s\uparrow t} |X(s) - b(s)| = 0$.

A curve b which is irregular for itself for X allows for the construction of the almost surely finite counting measure

$$L^b : \mathcal{B}[0,\infty) \to \mathbb{N}$$

defined by

$$L^b[0,t] \;=\; 1 + \sum_{0<s\leq t} \delta_{(X(s)=b(s))}(s) \tag{3.1.3}$$

where δ_x is the Dirac unit mass at point x. Further, $L^b[0,\infty]$ is almost surely 1 if and only if $\mathrm{d} = 0$. We call the right continuous process

$$L^b \;=\; \{L^b_t := L[0,t] : t \geq 0\}$$

local time-space for the curve b. Our choice of terminology here is motivated by Peskir [97] who gave the name *local time-space* for an analogous object defined for continuous semi-martingales.

There seems to be little known about local times of Lévy processes of bounded variation (see however Fitzsimmons and Port [49]) and hence a full classification of all such curves b which are irregular for themselves for X remains an open question. The definition as given is not empty however as we shall now show with the following simple examples.

Example 3.1.2 Suppose simply that $b(t) = x$ for all $t \geq 0$ and some $x \in \mathbb{R}$ and that X is not a compound Poisson process. In this case, the local time process is nothing more than the number of visits to x plus one which is a similar definition to the one given in Fitzsimmons and Port [49]. As can be deduced from the above introduction to Lévy processes of bounded variation, if $\mathrm{d} = 0$ then $L_t = 1$ for all $t > 0$. If on the other hand $\mathrm{d} \neq 0$ then since X has the property that $\{0\}$ is irregular for itself for X then the number of times X hits x in each finite time interval is almost surely finite. Further, X hits x by either creeping upwards over it or creeping downwards below according to the respective sign of d. (Creeping both upwards and downwards is not possible for Lévy processes which do not posses a Gaussian component). Creeping upwards above x occurs at first passage time T if and only if $\lim_{s\uparrow T} X(s) = x$. Since the same statement is true of downward creeping and X may only creep in at most one direction, it follows with the help of the Strong Markov Property that $t \in \{s > 0 : X(s) = x\}$ if and only if $\lim_{s\uparrow t} |X(s) - x| = 0$.

Example 3.1.3 More generally, if $\Pi(\mathbb{R}\backslash\{0\}) = \infty$ then an argument similar to the above shows that if b satisfying $b(0+) = b(0)$ and $|b'(0+)| < \infty$, belongs to the class $C^1(0,\infty)$, then it is also irregular for itself for X. One needs to take advantage in this case of the fact that b has locally linear behaviour. Furthermore, one sees that points t for which $b'(t) = \mathrm{d}$ cannot be hit. We have excluded $\Pi(\mathbb{R}\backslash\{0\}) < \infty$ in order to avoid simple pathological examples such as the case of the compound Poisson process and $b(t) = 0$ for all $t > 0$.

3.2 A generalization of the change of variable formula

In this section we state our result. The idea is to take the change of variable formula and to weaken the assumption on the class of functions to which it applies. For clarity, let us first state the Itô's change of variable formula in the special form that it takes for bounded variation Lévy processes. The proof can be found in a standard text book on semimartingales, see for instance Revuz and Yor [106], Protter [105], and Jacod and Shiryaev [63] (Theorem I.4.57).

Theorem 3.2.1 (Itô's Change of Variable Formula) *Suppose that the time-space function $f \in C^{1,1}([0,\infty)\times\mathbb{R})$. Then for any Lévy process X of bounded variation of the form (3.1.1), it holds almost surely that*

$$\begin{aligned} f(t,X(t)) - f(0,X(0)) &= \int_0^t \frac{\partial f}{\partial t}(s,X(s-))ds + \mathrm{d}\int_0^t \frac{\partial f}{\partial x}(s,X(s-))ds \\ &\quad + \sum_{0<s\le t}\Big\{f(s,X(s)) - f(s,X(s-))\Big\}. \end{aligned} \tag{3.2.1}$$

Remark 3.2.2 By inspection of the proof of the change of variable formula it is also clear that if for some random time T, $X_t \in D$ for all $t < T$ where D is an open set, then the change of variable formula as given above still holds on the event $\{t \le T\}$ for functions $f \in C^{1,1}([0,\infty), D)$.

The generalization we are interested in consists of weakening the class $C^{1,1}([0,\infty)\times\mathbb{R})$ in the Change of Variable formula to the following class.

Definition 3.2.3 Suppose that $b : [0,\infty) \to \mathbb{R}$ is a measurable function. A function f is said to be $C^{1,1}([0,\infty)\times\mathbb{R})$ *ripped along* b if

$$f(t,x) = \begin{cases} f^{(1)}(t,x) & x > b(t),\ t \ge 0 \\ f^{(2)}(t,x) & x < b(t),\ t \ge 0 \end{cases} \tag{3.2.2}$$

where $f^{(1)}$ and $f^{(2)}$ each belong to the class $C^{1,1}([0,\infty)\times\mathbb{R})$.

We shall prove the following theorem.

Theorem 3.2.4 *Suppose that b is a measurable function which is irregular for itself for X and f is $C^{1,1}([0,\infty)\times\mathbb{R})$ ripped along b. Then for any Lévy process of bounded variation, X, it holds almost surely that*

$$\begin{aligned} f(t,X(t)) - f(0,X(0+)) &= \int_0^t \frac{\partial f}{\partial t}(s,X(s-))ds + \mathrm{d}\int_0^t \frac{\partial f}{\partial x}(s,X(s-))ds \\ &\quad + \sum_{0<s\le t}\Big\{f(s,X(s)) - f(s,X(s-))\Big\} \qquad (3.2.3) \\ &\quad + \int_0^t \big\{f(s,X(s+)) - f(s,X(s-))\big\}dL_s^b, \end{aligned}$$

where dL^b_s refers to integration with respect to $s \mapsto L^b_s$.

Note, the term $f(0, X(0+))$ is deliberate in place of $f(0, X(0))$ as, in the case that $X(0) = b(0)$, it is possible that the process $f(\cdot, X(\cdot))$ starts with a jump.

This result complements the recent results of Peskir [97] which concern an extension of Itô's formula for continuous semi-martingales. Peskir accommodates for the case that the time-space function, f, to which Itô's formula is applied has a disruption in its smoothness along a *continuous* space time barrier of *bounded variation.* In particular, on either side of the barrier, the function is equal to a $C^{1,2}(\mathbb{R} \times [0, \infty))$ time-space function but, unlike the case here, it is assumed that there is continuity in f across the barrier. The formula that Peskir obtained has an additional integral with respect to the semi-martingale local time at zero of the distance of the underlying semi-martingale from the boundary (this is again a semi-martingale) which he calls *local time-space.* As mentioned above, we have chosen for obvious reasons to refer to the integrator in the additional term obtained in Theorem 3.2.4 as local time-space also. Peskir's results build further on those of Föllmer et al. [50] and Eisenbaum [43] for Brownian motion and in this sense our results now bring the discussion into the particular and somewhat simpler class of bounded variation semimartingales that we study here. Eisenbaum [44], Elworthy et al. [46] and Peskir [97] all have further results for general and special types of semi-martingales. However, the present study is currently the only one which considers discontinuous functions and hence the necessity to introduce a local time-space as a counting measure rather than an occupation density at zero of the semimartingale $X - b$ as one normally sees. Note in the case at hand, the semimartingale definition of local time at zero of $X - b$ is in fact identically zero (cf. Protter [105]). Other definitions of local time-space may be possible in order to work with more general classes of curves than those given in Definition 1 and hence the current presentation merely scratches the surface of the problem considered.

3.3 Proofs and main calculations

Proof of Theorem 3.2.4

Proof The essence of the proof is based around a telescopic sum which we shall now describe. Define the inverse local time process $\tau = \{\tau_t : t \geq 0\}$ where

$$\tau_t \; = \; \inf \left\{ s > 0 : L^b_s > t \right\}$$

for each $t \geq 0$. Note the second strict inequality in the definition ensures that τ is a càdlàg process and since $L^b_0 = 1$ by definition, it follows that $\tau_0 = 0$. The process τ is nothing more than a step function which increases on the integers $k = 1, 2, 3, ...$ by an amount corresponding to the length of the excursion of X from b whose right end point corresponds to the k-th crossing of b by X. Note that even when $X_0 \neq b(0)$ we count the section of the path of X until it first meets b as an (incomplete) excursion.

The increment in $\{f(s,X(s)) : s \geq 0\}$ between $s = 0+$ and $s = t$ can be seen as the accumulation of the increments incurred by X crossing the boundary b, the excursions of X from b and the final increment between the last time of contact of X with b and time t. We have

$$\begin{aligned} f(t,X(t)) - f(0,X(0+)) \;=\; & \int_0^t \big\{f(s,X(s+)) - f(s,X(s-))\big\} dL_s^b \\ & + \sum_{s \leq L_t^b} \big\{f(\tau_s, X_{\tau_s}) - f(\tau_{s-}, X_{\tau_{s-}})\big\} \mathbf{1}_{\{|\Delta\tau_s|>0\}} \\ & + \big\{f(t,X(t)) - f(\tau_{L_t^b}, X_{\tau_{L_t^b}+})\big\}. \end{aligned} \tag{3.3.1}$$

The proof is then completed once we know that the increments in the curly brackets of the second and third term on the right hand side of (3.3.1) observe the same development as the change of variable formula. Indeed, taking into account of the Strong Markov Property, it would suffice to prove that under the given assumptions on f we have that for all $t \in (0,\infty]$

$$\begin{aligned} & f(t \wedge \eta, X(t \wedge \eta)) - f(0, X(0+)) \\ = \; & \int_0^{t\wedge\eta} \frac{\partial f}{\partial t}(s, X(s-))ds + \mathrm{d}\int_0^{t\wedge\eta} \frac{\partial f}{\partial x}(s, X(s-))ds \\ & + \sum_{0<s\leq t\wedge\eta} \big\{f(s,X(s)) - f(s,X(s-))\big\}. \end{aligned} \tag{3.3.2}$$

Note that η is the first strictly positive time that $X - b = 0$, that is to say that

$$\eta \;=\; \inf\big\{t > 0 : X_t - b(t) \;=\; 0\big\}.$$

The statement in (3.3.2) is intuitively appealing since up to the stopping time η the process X does not intersect with the boundary b and hence the discontinuity in f should not appear in a development of the function $f(\cdot, X(\cdot))$. The result is proved in the lemma below and thus concludes the proof of the main result.

Lemma 3.3.1 *Under the assumptions of Theorem 3.2.4, the identity (3.3.2) holds for all $t \in (0,\infty]$.*

Proof First fix some $\kappa > 0$, define

$$\sigma_{\kappa,0} \;=\; \inf\big\{t \geq 0 : |X(t) - b(t)| > \kappa\big\}.$$

and $\Omega_\kappa = \{\omega \in \Omega : \sigma_{\kappa,0} < \eta\}$. Next define for each $j \geq 1$ the stopping times

$$\sigma_{\kappa,j} \;=\; \inf\Big\{t > \sigma_{\kappa,j-1} : |X(t) - b(t)| < \frac{1}{2}\big|X(\sigma_{\kappa,j-1}) - b(\sigma_{\kappa,j-1})\big|\Big\}$$

where we again work with the usual definition $\inf\emptyset = \infty$. On the set $\Omega_\kappa \cap \{\eta < \infty\}$ we have that

$$\limsup_{j\uparrow\infty} \big|X(\sigma_{\kappa,j}) - b(\sigma_{\kappa,j})\big| \;\leq\; \lim_{j\uparrow\infty} \left(\frac{1}{2}\right)^j \big|X_{\sigma_{\kappa,0}}\big| = 0$$

and hence by the definition of irregularity of b for itself for X,

$$\lim_{j\uparrow\infty}\sigma_{\kappa,j} \;=\; \eta \tag{3.3.3}$$

where the limit is interpreted to be infinite on the set $\{\eta=\infty\}$. It is also clear that, since X has right continuous paths,

$$\lim_{\kappa\downarrow 0}P\left(\Omega_\kappa\right) \;=\; 1. \tag{3.3.4}$$

Over the time interval $[\sigma_{\kappa,j-1},\sigma_{\kappa,j})$ the process X does not enter a tube of positive, $\mathcal{F}_{\sigma_{\kappa,j-1}}$-measurable radius around the curve b, we may appeal to then standard Change of Variable Formula to deduce that on Ω_κ

$$\begin{aligned}
&f\left(\sigma_{\kappa,j}\wedge t, X_{\sigma_{\kappa,j}\wedge t}\right)-f\left(\sigma_{\kappa,j-1}\wedge t, X_{\sigma_{\kappa,j-1}\wedge t}\right)\\
&=\int_{\sigma_{\kappa,j-1}\wedge t}^{\sigma_{\kappa,j}\wedge t}\frac{\partial f}{\partial t}(s,X(s-))ds+\mathrm{d}\int_{\sigma_{\kappa,j-1}\wedge t}^{\sigma_{\kappa,j}\wedge t}\frac{\partial f}{\partial x}(s,X(s-))ds\\
&\quad+\sum_{\sigma_{\kappa,j-1}\wedge t<s\le\sigma_{\kappa,j}\wedge t}\big\{f(s,X(s))-f(s,X(s-))\big\}.
\end{aligned}$$

Hence on Ω_κ we have

$$\begin{aligned}
&f(\eta\wedge t,X(\eta\wedge t))-f(\sigma_{\kappa,0},X(\sigma_{\kappa,0}))\\
&=\sum_{j\ge1}\big\{f\left(\sigma_{\kappa,j}\wedge t,X(\sigma_{\kappa,j}\wedge t)\right)-f\left(\sigma_{\kappa,j-1}\wedge t,X(\sigma_{\kappa,j-1}\wedge t)\right)\big\}\\
&=\sum_{j\ge1}\int_0^{\eta\wedge t}\Big\{\frac{\partial f}{\partial t}(s,X(s-))+\mathrm{d}\frac{\partial f}{\partial x}(s,X(s-))\Big\}\mathbf{1}_{(\sigma_{\kappa,j-1}\wedge t<s\le\sigma_{\kappa,j}\wedge t)}ds\\
&\quad+\sum_{j\ge1}\sum_{0<s\le\eta\wedge t}\big\{f(s,X(s))-f(s,X(s-))\big\}\mathbf{1}_{(\sigma_{\kappa,j-1}\wedge t<s\le\sigma_{\kappa,j}\wedge t)}\\
&=\int_0^{\eta\wedge t}\frac{\partial f}{\partial t}(s,X(s-))ds+\mathrm{d}\int_0^{\eta\wedge t}\frac{\partial f}{\partial x}(s,X(s-))ds\\
&\quad+\sum_{0<s\le\eta\wedge t}\big\{f(s,X(s))-f(s,X(s-))\big\},
\end{aligned}$$

where the final equality follows from (an almost sure version of) Fubini's theorem which in turn appeals to the assumption that the limits of f, $\partial f/\partial t$ and $\partial f/\partial x$ all exist and are finite when approaching any point on the curve b. In particular, to deal with the final term in the second equality, note that an almost sure uniform bound of the form

$$\big|f(s,X(s))-f(s,X(s-))\big|\le C\big|\triangle X(s)\big|$$

holds (for random C) because of the assumptions on $\partial f/\partial x$ and hence the double sum converges (as X is a process of bounded variation). Since κ may be chosen arbitrarily small, (3.3.4) shows that (3.3.2) is true almost surely on Ω. □

3.4 Application to solving the American put option problem

We present in this section a connection between the change of variable formula developed in the previous section with the problem of finding the arbitrage-free price of the finite maturity American put option. The option confers the right to sell a unit of stock at any time up to a finite time horizon T at a strike price K.

We assume that the stock pays no dividends during the lifetime of the option and the evolution of the stock price process $S_t = e^{X_t}$ is driven under a chosen martingale measure $\mathbb{P}_x$ by a bounded variation Lévy process of the form

$$X_t \;=\; x + (r+\omega)t + J_t, \tag{3.4.1}$$

where $(J_t, t \geq 0)$ is a pure jump Lévy process of bounded variation defined as $J_t = \sum_{0<s\leq t} \Delta_s$ and $\{(s, \Delta_s) : s \geq 0\}$ is a Poisson point process on $[0,\infty) \times (\mathbb{R}\backslash\{0\})$ with (time-space) intensity measure $dt \times \Pi\,(dx)$ (see the expression (3.1.1)). Throughout the remaining of this section, we assume that the Lévy measure Π satisfies

$$\int_{-\infty}^{\infty} (e^{|y|} - 1)\Pi(dy) \;<\; \infty, \tag{3.4.2}$$

and the rate $(r+\omega)$ in (3.4.1) is assumed to be strictly positive. Furthermore, we assume that the discounted process $(e^{-rt}S_t, t \geq 0)$ is $\mathbb{P}_x$-martingale, implying that

$$\mathbb{E}_x\big(e^{-rt}S_t\big) \;=\; e^x.$$

This condition implies that the parameter ω in (3.4.1) is given by

$$\omega \;=\; -\int_{-\infty}^{\infty} \big(e^y - 1\big)\Pi(dy), \tag{3.4.3}$$

which is well-defined due to the integral test (3.4.2). Note that under the martingale condition (3.4.3) and the integral test (3.4.2), it can be shown using the formula (3.2.1) that the stock price process S_t fulfills the *arbitrage-free* condition

$$\mathbb{E}_x\Big(dS_t - rS_{t-}dt\Big) \;=\; 0.$$

The problem of interest in this section is to characterize the arbitrage-free price of the finite maturity American put option

$$V(t,x) \;=\; \sup_{0\leq\tau\leq t} \mathbb{E}_x\Big(e^{-r\tau}\big(K - e^{X_\tau}\big)^+\Big) \tag{3.4.4}$$

for all $(t,x) \in [0,T] \times \mathbb{R}_+$ where τ is a Markov stopping time of X.

Adapting arguments of Peskir [98], we derive using the change of variable formula (3.2.3) a nonlinear integral equation for optimal stopping boundary of the problem (3.4.4) within a bounded variation Lévy process and show that the optimal value function V is continuous across the boundary. Taking into account the continuous pasting condition, we give a proof similar to Jacka [62] and Peskir [98] for the uniqueness of the nonlinear integral equation and show that the value function V solves uniquely a free boundary problem of parabolic integro-differential type [2].

The results are given by the following two theorems.

Theorem 3.4.1 (Free boundary problem) *Assume that the Lévy measure* Π *of* X *(3.4.1) satisfies the integrability condition (3.4.2) and* $b : [0,T] \to (-\infty, \log(K)]$ *is a curved boundary which is irregular for itself for* X*. Suppose that* (U, b) *is a solution pair, with* $U \in C^{1,1}([0,T] \times \mathbb{R})$ *ripped along the curved boundary* b*, to the problem:*

$$\Big(-\frac{\partial}{\partial t} + \mathcal{L}_X - r\Big)U(t,x) = 0 \quad \textit{for } (t,x) \in \mathcal{C} \tag{3.4.5}$$

$$U(0,x) = (K - e^x)^+ \quad \textit{for all } x \in \mathbb{R}, \tag{3.4.6}$$

$$U(t,x) = (K - e^x)^+ \quad \textit{for } x = b(t) \ \textit{(continuous fit)}, \tag{3.4.7}$$

$$U(t,x) > (K - e^x)^+ \quad \textit{for } (t,x) \in \mathcal{C} \tag{3.4.8}$$

$$U(t,x) = (K - e^x)^+ \quad \textit{for } (t,x) \in \mathcal{D} \tag{3.4.9}$$

where the continuation region $\mathcal{C}$ *and the stopping region* $\mathcal{S} = \overline{\mathcal{D}}$ *are defined by*

$$\mathcal{C} = \big\{(t,x) \in [0,T] \times \mathbb{R} \;\big|\; x > b(t)\big\} \tag{3.4.10}$$

and

$$\mathcal{D} = \big\{(t,x) \in [0,T] \times \mathbb{R} \;\big|\; x < b(t)\big\}, \tag{3.4.11}$$

and the infinitesimal generator $\mathcal{L}_X$ *of* X *(3.4.1) is defined by*

$$\mathcal{L}_X U(t,x) = (r+\omega)\frac{\partial U}{\partial x}(t,x) + \int_{-\infty}^{\infty}\Big(U(t,x+y) - U(t,x)\Big)\Pi(dy). \tag{3.4.12}$$

Then, the curved boundary b *solves for all* $t \in (0,T]$ *the nonlinear integral equation*[3]

$$K - e^{b(t)} = e^{-rt}\mathbb{E}_{b(t)}\big(K - e^{X_t}\big)^+ + rK\int_0^t e^{-ru}\mathbb{P}_{b(t)}\big(X_{u-} \le b(t-u)\big)du. \tag{3.4.13}$$

Theorem 3.4.2 (Uniqueness) *If the value function* V *of the problem (3.4.4) solves the free boundary problem (3.4.5)-(3.4.12) and the optimal stopping time is the first passage time* τ_b^- *of* X *below a curved boundary* b *solving the integral equation (3.4.13), then* (V, b) *represents the unique pair solution to the problem (3.4.5)-(3.4.12).*

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[2]For jump-diffusion processes, the uniqueness of a free boundary problem of parabolic integro-differential type associated to the optimal stopping problem (3.4.4) was discussed in Pham [100].

[3]For exponential of a linear Brownian motion $S_t(x) = x\exp\big(\sigma B_t + (r - 1/2\sigma^2)t\big)$, it was shown in Kim [67], El Karoui and Karatzas [45], Jacka [62], Myneni [90], Carr et al. [23], and Peskir [98] that the optimal boundary $h(t)$ of the stopping problem $V(t,x) = \sup_{0\le\tau\le t}\mathbb{E}\big(e^{-r\tau}\big(K - S_\tau(x)\big)^+\big)$ solves a nonlinear integral equation of the similar form:

$$K - h(t) = e^{-rt}\mathbb{E}\big(K - S_t(h(t))\big)^+ + rK\int_0^t e^{-ru}\mathbb{P}\big(S_u(h(t)) \le h(t-u)\big)du.$$

3.4.1 Proof and main calculations of Theorem 3.4.1

To start with, let us remind ourself that the curved boundary b is irregular for itself for the Lévy process (3.4.1) and T is a finite maturity time. Next, let us consider for a fixed $t \in (0,T]$ a function $f : [0,t] \times \mathbb{R} \to \mathbb{R}_+$ defined by

$$f(u,x) \;=\; e^{-ru}U(t-u,x),$$

where U is a $C^{1,1}$ function, ripped along the curved boundary b, that solves the free boundary problem (3.4.5)-(3.4.12). Observe that the functions f and U have the same number of continuous derivatives w.r.t t and x. Since U is ripped along b, we have by the construction that f is also a $C^{1,1}$ function that is ripped along b. Therefore, we can now apply the change of variable formula (3.2.3) to get

$$\begin{aligned} f(s,X_s) \;&=\; f(0,X_0) + \int_0^s \frac{\partial f}{\partial u}(u,X_{u-})du + (r+\omega)\int_0^s \frac{\partial f}{\partial x}(u,X_{u-})du \\ &\quad + \sum_{0<u\le s}\Big\{f(u,X_{u-}+\Delta X_u) - f(u,X_{u-})\Big\} \qquad (3.4.14) \\ &\quad + \int_0^s \big(f(u,X(u+)) - f(u,X(u-))\big)dL_u^b(X). \end{aligned}$$

Using the chain rule for partial differentiation, we see that

$$\frac{\partial f}{\partial u} \;=\; -\Big(re^{-ru}U + e^{-ru}\frac{\partial U}{\partial t}\Big) \quad\text{and}\quad \frac{\partial f}{\partial x} \;=\; e^{-rt}\frac{\partial U}{\partial x}.$$

Inserting these expressions in (3.4.14), we obtain

$$\begin{aligned} &e^{-rs}U(t-s,X_s) \\ &\quad = U(t,x) + \int_0^s e^{-ru}\Big(-rU - \frac{\partial U}{\partial t} + (r+\omega)\frac{\partial U}{\partial x}\Big)(t-u,X_{u-})du \\ &\qquad + \sum_{0<u\le s} e^{-ru}\Big\{U(t-u,X_{u-}+\Delta X_u) - U(t-u,X_{u-})\Big\} \qquad (3.4.15) \\ &\qquad + \int_0^s e^{-ru}\{U(t-u,X_{u+}) - U(t-u,X_{u-})\}dL_u^b(X). \end{aligned}$$

Notice that the sum in the foregoing expression converges in absolute value due to the assumptions on U and the fact that X has path of bounded variation.

On recalling the fact that for a Borel set $\Lambda \subset \mathbb{R}$, with $0 \notin \Lambda$, we have for every (bounded) measurable function h that

$$\int_0^t\int_\Lambda h(u,y)\nu(dy,du) \;=\; \sum_{0<u\le t} h(u,\Delta X_u)\mathbf{1}_\Lambda(\Delta X_u), \qquad (3.4.16)$$

where $\nu(dy,du)$ is a Poisson random measure, with the space-time compensator $\Pi(dy)\times du$, defined by

$$\nu(dy,du) \;=\; \sum_{s\ge 0}\mathbf{1}_{\{\Delta X_s\neq 0\}}\delta_{(\Delta X_s,s)}(dy,du),$$

where δ_x denotes the Dirac measure at point x, see for instance Proposition 1.16 in Chapter II of Jacod and Shiryaev [63] for details. By adding and subtracting

$$\int_0^s \int_{\mathbb{R}} e^{-ru}\Big(U(t-u, X_{u-}+y) - U(t-u, X_{u-})\Big)\Pi(dy)du$$

in the equation (3.4.15) we finally obtain

$$\begin{aligned} & e^{-rs}U(t-s, X_s) \\ =& U(t,x) + \int_0^s e^{-ru}\Big(-\frac{\partial}{\partial t} + \mathcal{L}_X - r\Big)U(t-u, X_{u-})du + \mathcal{M}_s \\ &+ \int_0^s e^{-ru}\big\{U(t-u, X_{u+}) - U(t-u, X_{u-})\big\}dL_u^b(X), \end{aligned} \tag{3.4.17}$$

where $\mathcal{L}_X$ is the infinitesimal generator of the Lévy process (3.4.1) defined earlier in (3.4.12) and the stochastic process $(\mathcal{M}_s)_{0\le s\le t}$ is a $\mathbb{P}_x$-(local) martingale defined by

$$\begin{aligned} \mathcal{M}_s \ =& \int_0^s \int_{\mathbb{R}} e^{-ru}\Big(U(t-u, X_{u-}+y) - U(t-u, X_{u-})\Big)\nu(dy, du) \\ & - \int_0^s \int_{\mathbb{R}} e^{-ru}\Big(U(t-u, X_{u-}+y) - U(t-u, X_{u-})\Big)\Pi(dy)du. \end{aligned} \tag{3.4.18}$$

Using the fact that U is $C^{1,1}$ ripped at b and is continuous at b (see (3.4.7)), we see that there exists for all $u \in [0, s]$ a positive constant C_s such that

$$\big|U(u, x+y) - U(u, x)\big| \ \le\ C_s\big|y\big| \quad \text{for all } x, y \in \mathbb{R}.$$

Note that the constant C_s depends on the interval of time over which u is considered, i.e., the estimate has a uniform constant C_s for all u in the interval $[0, s]$. Hence, in view of the integrability condition (3.4.2), we see that the last double integral in (3.4.18) converges in absolute value, and therefore we have that

$$\begin{aligned} \mathbb{E}\big|\mathcal{M}_s\big| \ \le& \ \mathbb{E}\int_0^s \int_{\mathbb{R}} e^{-ru}\Big|U(t-u, X_{u-}+y) - U(t-u, X_{u-})\Big|\nu(dy, du) \\ &+ \ \mathbb{E}\int_0^s \int_{\mathbb{R}} e^{-ru}\Big|U(t-u, X_{u-}+y) - U(t-u, X_{u-})\Big|\Pi(dy)du \\ \le& \ \frac{2C_s}{r}\big(1 - e^{-rs}\big)\int_{\mathbb{R}} \big|y\big|\Pi(dy). \end{aligned} \tag{3.4.19}$$

Moreover, by recalling that U is a $C^{1,1}$ function and is continuous along the curved boundary b and that the Lévy process X has paths of bounded variation, we can apply the *compensation formula* (see for instance Theorem 4.4 in Kyprianou [69]) to have that

$$\begin{aligned} & \mathbb{E}\Big(\int_0^s \int_{\mathbb{R}} e^{-ru}\Big(U(t-u, X_{u-}+y) - U(t-u, X_{u-})\Big)\nu(dy, du)\Big) \\ &\quad = \ \mathbb{E}\Big(\int_0^s \int_{\mathbb{R}} e^{-ru}\Big(U(t-u, X_{u-}+y) - U(t-u, X_{u-})\Big)\Pi(dy)du\Big). \end{aligned} \tag{3.4.20}$$

Hence, in view of (3.4.19) it follows that the process $(\mathcal{M}_s)_{0\le s\le t}$ is an L_1-integrable $\mathbb{P}_x$-martingale and hence vanishes after taking expectation under the measure $\mathbb{P}_x$.

On noticing the fact that $(-\frac{\partial}{\partial t}+\mathcal{L}_X-r)U(t,x)=0$ for all $(t,x)\in\mathcal{C}$ and

$$\Big(-\frac{\partial}{\partial t}+\mathcal{L}_X-r\Big)(K-e^x) \;=\; -rK,$$

we have following (3.4.17) that

$$\begin{aligned} e^{-rs}U(t-s,X_s) \;&=\; U(t,x)-rK\int_0^s e^{-ru}\mathbf{1}_{(X_{u-}\le b(t-u))}du+\mathcal{M}_s \\ &+\int_0^s e^{-ru}\big\{U(t-u,X_{u+})-U(t-u,X_{u-})\big\}dL_u^b(X), \end{aligned} \tag{3.4.21}$$

holds $\mathbb{P}_x$ almost surely. Note that we have used in (3.4.21) the fact that the curved boundary b is bounded from above by $\log(K)$. Since $U(0,x)=(K-e^x)^+$ for all $x\in\mathbb{R}$ we have after inserting $s=t$ in (3.4.21) and taking expectation under $\mathbb{P}_x$ that

$$\begin{aligned} U(t,x) =& e^{-rt}\mathbb{E}_x\Big(K-e^{X_t}\Big)^+ + rK\int_0^t e^{-ru}\mathbb{P}_x\big(X_{u-}\le b(t-u)\big)du \\ &-\mathbb{E}_x\Big(\int_0^t e^{-ru}\big\{U(t-u,X_{u+})-U(t-u,X_{u-})\big\}dL_u^b(X)\Big) \end{aligned} \tag{3.4.22}$$

for all $(t,x)\in[0,T)\times\mathbb{R}$. On recalling that $U(t,x)=(K-e^x)^+$ for $(t,x)\in\overline{\mathcal{D}}$, we see that

$$\begin{aligned} (K-e^x)^+ =& e^{-rt}\mathbb{E}_x\Big(K-e^{X_t}\Big)^+ + rK\int_0^t e^{-ru}\mathbb{P}_x\big(X_{u-}\le b(t-u)\big)du \\ &-\mathbb{E}_x\Big(\int_0^t e^{-ru}\big\{U(t-u,X_{u+})-U(t-u,X_{u-})\big\}dL_u^b(X)\Big). \end{aligned} \tag{3.4.23}$$

Since U is a $C^{1,1}$ function ripped along the curved boundary b and is continuous at b (see (3.4.7)), the second expectation on the right hand side of (3.4.23) vanishes. Hence, we deduce that b must solve the integral equation

$$(K-e^x)^+ = e^{-rt}\mathbb{E}_x\Big(K-e^{X_t}\Big)^+ + rK\int_0^t e^{-ru}\mathbb{P}_x\big(X_{u-}\le b(t-u)\big)du,$$

for all $x\le b(t)$ and all $t\in(0,T]$. By inserting $x=b(t)$ in the foregoing equation, we come to rest at a free-boundary equation that the boundary b has to solve:

$$\big(K-e^{b(t)}\big)^+ \;= e^{-rt}\mathbb{E}_{b(t)}\Big(K-e^{X_t}\Big)^+ + rK\int_0^t e^{-ru}\mathbb{P}_{b(t)}\big(X_{u-}\le b(t-u)\big)du.$$

Thus, the claim that the curved boundary b solves the nonlinear integral equation (3.4.13) is then established. □

3.4.2 Proof and main calculations of Theorem 3.4.2

To start with let us denote by τ_h^- the first exit time of X below a curved boundary h defined by

$$\tau_h^- \; = \; \inf\{u > 0 : X_u \leq h(t-u)\} \wedge t. \tag{3.4.24}$$

Suppose that (W, c) is a solution pair, with $W \in C^{1,1}([0,T]\times\mathbb{R})$ ripped along a curved boundary c, to the free boundary problem (3.4.5)-(3.4.12). By applying the formula (3.2.3) to the function W subject to the pasting condition (3.4.7), we have following the similar calculations as before that

$$e^{-rt}W(0,X_t) \; = \; W(t,x) - rK\int_0^t e^{-ru}\mathbf{1}_{(X_{u-}\leq c(t-u))}du + \mathcal{M}_t \tag{3.4.25}$$

where $\mathcal{M}$ is in principle a $\mathbb{P}_x$ (local) martingale process, but in view of (3.4.19) and (3.4.20) one can argue that it is $\mathbb{P}_x$-martingale. Recall that $W(0,x) = (K-e^x)^+$ for all $x \in \mathbb{R}_+$ and $W(t,x) = (K-e^x)^+$ for all $x \leq c(t)$. Following these two facts, one can deduce following the same arguments as before that the curved boundary c solves the integral equation (3.4.13). Moreover, by replacing t with stopping time τ_c^- in the expression (3.4.25) and taking expectation under $\mathbb{P}_x$, we have for all $(t,x) \in [0,T]\times\mathbb{R}$ that

$$W(t,x) \; = \; \mathbb{E}_x\Big(e^{-r\tau_c^-}(K-e^{X_{\tau_c^-}})^+\Big). \tag{3.4.26}$$

Since the value function V of the problem (3.4.4) is assumed to solve the free boundary problem (3.4.5)-(3.4.12) and the optimal stopping time is the first exit of X below a curved boundary b solving (3.4.13), we have that

$$V(t,x) \; = \; \sup_{0\leq\tau\leq t} \mathbb{E}_x\Big(e^{-r\tau}(K-e^{X_\tau})^+\Big) \; = \mathbb{E}_x\Big(e^{-r\tau_b^-}(K-e^{X_{\tau_b^-}})^+\Big). \tag{3.4.27}$$

Following (3.4.26) and (3.4.27), we see for every $t > 0$ and $x \in \mathbb{R}$ that

$$V(t,x) \; \geq \; W(t,x). \tag{3.4.28}$$

This inequality implies that

$$c(t) \; \geq \; b(t) \quad \text{for all } t \in [0,T]. \tag{3.4.29}$$

Suppose that there exists $t \in (0,T)$ such that $c(t) > b(t)$. Next, let us take for a given $t \in (0,T)$ a point $x \in (b(t), c(t))$. By replacing s and t with the stopping time τ_b^- in (3.4.25) and (3.4.21), respectively, we obtain after taking expectation under the measure $\mathbb{P}_x$ that

$$\mathbb{E}_x\Big(e^{-r\tau_b^-}(K-e^{X_{\tau_b^-}})^+\Big) \; = \; W(t,x) - rK\mathbb{E}_x\Big(\int_0^{\tau_b^-} e^{-ru}\mathbf{1}_{(X_{u-}\leq c(t-u))}du\Big),$$

and

$$\mathbb{E}_x\Big(e^{-r\tau_b^-}(K-e^{X_{\tau_b^-}})^+\Big) \; = \; V(t,x).$$

On remarking that $V(t,x) \geq W(t,x)$ for all $(t,x) \in [0,T] \times \mathbb{R}$ (see equation (3.4.28)), we deduce from the two foregoing equations that

$$\mathbb{E}_x\Big(\int_0^{\tau_b^-} e^{-ru}\mathbf{1}_{(X_{u-}\leq c(t-u))}du\Big) \;\leq\; 0,$$

which can not be true. Hence, in absence of the existence of such a point x, it follows that

$$b(t) \;=\; c(t) \quad \text{for all } t \in [0,T].$$

As a result, having shown that the integral equation (3.4.13) admits a unique solution for the optimal stopping boundary of the problem (3.4.4), we deduce following the two expressions (3.4.26) and (3.4.27) that

$$W(t,x) \;=\; \mathbb{E}_x\Big(e^{-r\tau_b^-}(K - e^{X_{\tau_b^-}})^+\Big) \;=\; V(t,x)$$

for all $(t,x) \in [0,T] \times \mathbb{R}$. Thus, the claim that the integral equation (3.4.13) and the free boundary problem (3.4.5)-(3.4.12) admit a unique solution is then proved. □

Remark 3.4.3 In fact using the change of variable formula (3.2.3), it can be shown in the similar calculations as before that a solution to the free boundary problem (3.4.5)-(3.4.12) coincides with the value function V of the optimal stopping problem (3.4.4), and the optimal stopping time of (3.4.4) is the first exit time τ_b^- of X below a curved boundary b that solves the integral equation (3.4.13).

3.5 Concluding remarks

To summarize, we have seen using the free boundary problem (3.4.5)-(3.4.12) that the change of variable formula (3.2.3) with local time-space on a irregular curved boundary b, developed earlier in Section 2, has been able to deliver three important things. Firstly, we show using the formula that the smallest superharmonic majorant property of the value function V (3.4.4) is simplified to the analytic condition of continuous pasting at the stopping boundary b. By imposing the continuous pasting condition, we derive using the change of variable formula a nonlinear integral equation for b and show that b is the optimal stopping boundary for the problem (3.4.4). Secondly, given the continuous pasting condition, we show using (3.2.3) that such a nonlinear integral equation admits a unique solution. Thirdly, as a final result of the use of the formula (3.2.3), we see that it is possible to show that (V,b) is the unique solution pair to the free boundary problem.