Chapter 2

A Brief Introduction to Lévy Processes

In this chapter, we present a brief introduction to Lévy processes which underlie the main object of interest of the optimal stopping problems considered in this thesis.

We refer among others to Applebaum [5], Bertoin [13], Kyprianou [69], Protter [105], and Sato [111] for a detailed account on Lévy processes.

2.1 Introduction

Definition 2.1.1 (Lévy process) Let $\mathbb{P}$ be a probability measure on probability space $(\Omega, \mathcal{F})$. A process $X = (X_t, t \geq 0)$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a Lévy process if the paths of X are right continuous with left limits $\mathbb{P}$-almost surely, for every $s, t \geq 0$, the increment $X_{t+s} - X_t$ is independent of the process $(X_u, 0 \leq u \leq t)$ and has the same law as X_s. In particular, $\mathbb{P}(X_0 = 0) = 1$.

From now on, the law of the Lévy process started at $x \in \mathbb{R}$ will be denoted by $\mathbb{P}_x$. For convenience we write $\mathbb{P} = \mathbb{P}_0$ and we shall write $\mathbb{E}_x$ for the expectation operator associated with $\mathbb{P}_x$ and in the special case that $x = 0$ we write $\mathbb{E}$.

The characteristic exponent of X is given by the well known *Lévy-Khintchine formula* which shall be given by the following theorem (see for instance Theorem 1 in Chapter I of Bertoin [13]).

Theorem 2.1.2 (Lévy-Khintchine formula) *Suppose that $\mu \in \mathbb{R}$, $\sigma \geq 0$ and Π is a measure concentrated on $\mathbb{R}\backslash\{0\}$ such that $\int_{-\infty}^{\infty}(1 \wedge y^2)\Pi(dy) < \infty$. From this triple define for each $\theta \in \mathbb{R}$ a continuous function $\Psi : \mathbb{R} \to \mathbb{C}$ given by*

$$\Psi(\theta) \;=\; i\mu\theta + \frac{\sigma^2}{2}\theta^2 + \int_{-\infty}^{\infty} \big(1 - e^{i\theta y} + i\theta y 1_{\{|y| \leq 1\}}\big)\Pi(dy). \tag{2.1.1}$$

Then there exists a unique probability measure $\mathbb{P}$ on Ω under which $(X_t, t \geq 0)$ is a Lévy process with characteristic exponent Ψ, i.e.,

$$\mathbb{E}\big(e^{i\theta X_t}\big) \;=\; e^{-t\Psi(\theta)} \quad \textit{for } \theta \in \mathbb{R} \textit{ and } t \geq 0. \tag{2.1.2}$$

Moreover, the jump process of X, namely $\Delta X = (\Delta X_t, t \geq 0)$, is a Poisson point process with characteristic measure Π.

The measure Π is called the *Lévy measure* and the parameter σ the *Gaussian coefficient.* The Lévy-Khintchine formula has a simpler expression when the sample paths of the Lévy process have *bounded variation* on every compact time interval almost surely. For short, we will then say that the Lévy process has bounded variation. Specifically, a Lévy process has bounded variation if and only if $\sigma = 0$ and the Lévy measure Π satisfies the integral test $\int_{-\infty}^{\infty}(1 \wedge |y|)\Pi(dy) < \infty$. In that case, the mapping $\lambda \mapsto \int_{-\infty}^{\infty} \lambda x \mathbf{1}_{\{|y|<1\}}\Pi(dy)$ is a well-defined linear function and the characteristic exponent Ψ can be re-expressed for each $\theta \in \mathbb{R}$ as

$$\Psi(\theta) \ = \ -i\mathrm{d}\theta + \int_{-\infty}^{\infty} \big(1 - e^{i\theta y}\big)\Pi(dy), \tag{2.1.3}$$

for some $\mathrm{d} \in \mathbb{R}$ which is known as the *drift coefficient.* Moreover, if $\Delta = (\Delta_t, t \geq 0)$ is a Poisson point process with characteristic measure Π, then the process

$$X_t \ = \ \mathrm{d}t + \sum_{0 \leq s \leq t} \Delta_s, \quad \text{for } t \geq 0,$$

is a Lévy process of bounded variation (recall that the series is absolutely convergent almost surely if and only if the Lévy measure Π satisfies $\int_{-\infty}^{\infty}(1 \wedge |y|)\Pi(dy) < \infty$, see for instance Chapter I of Bertoin [13]) with characteristic exponent Ψ (2.1.3). It is clear that a compound Poisson process has bounded variation and, conversely, a Lévy process with bounded variation is a compound Poisson process if and only if its drift coefficient d is null and its Lévy measure Π has finite mass.

For every rapidly decreasing function[1] f, one can use the Lévy-Khintchine formula to get the infinitesimal generator $\mathcal{L}_X$ of the Lévy process X defined by

$$\begin{aligned} \mathcal{L}_X f(x) \ &:= \lim_{t \downarrow 0} t^{-1}\big(\mathbb{E}_x f(X_t) - f(x)\big) \\ &= \ \mu \frac{df}{dx}(x) + \frac{1}{2}\sigma^2 \frac{d^2 f}{dx^2}(x) + \int_{-\infty}^{\infty} \Big(f(x+y) - f(x) - y\frac{df}{dx}(x)1_{\{|y| \leq 1\}}\Big)\Pi(dy). \end{aligned}$$

We refer to Section 4.1 in Skorohod [114] for the details of the calculations.

To finish this section, let us introduce the notion of regularity of a point for an open or closed set $\mathcal{O}$ for a Lévy process. This notion becomes relevant to the discussions on the smooth and continuous pasting conditions later in this thesis.

Definition 2.1.3 (Regularity of a point for a Lévy process) For a Lévy process X, the point $x \in \mathbb{R}$ is said to be regular (respectively, irregular) for an open or closed set $\mathcal{O}$ if

$$\mathbb{P}_x\big(\tau_{\mathcal{O}} = 0\big) = 1 \ \text{ (respectively, 0)},$$

[1] We say a function $f(x)$ is *rapidly decreasing* if there are constants M_N such that $|f(x)| \leq M_N |x|^{-N}$ as $x \to \infty$ for $N = 1, 2, 3, \cdots$. See for example Gel'fand and Shilov [52] for more details.

thanks to Blumenthal's zero-one law, where the stopping time

$$\tau_{\mathcal{O}} = \inf\{t > 0 : X_t \in \mathcal{O}\}.$$

Intuitively speaking, x is regular for $\mathcal{O}$ if, when starting from x, the Lévy process hits $\mathcal{O}$ immediately.

2.2 The Wiener-Hopf factorization

In this section we discuss the fundamental notion of the Wiener-Hopf factorization in fluctuation theory of Lévy processes. The results presented in this section will be used later in Chapters 4, 5 and 6. What we shall say in this section mainly refers to the Chapters VI and 6 of the books of Bertoin [13] and Kyprianou [69], respectively.

To begin with, let us introduce a so-called the *supremum* and *infimum* processes

$$\overline{X}_t \;=\; \sup_{0\le s\le t} X_s \quad \text{and} \quad \underline{X}_t \;=\; \inf_{0\le s\le t} X_s.$$

We see that $\overline{X}$ and $-\underline{X}$ are two nonnegative increasing right-continuous processes, which are adapted to the filtration $\mathcal{F}_t = \sigma(X_s, s \le t)$. It is well known (see Proposition 1 in Chapter VI of Bertoin [13]) that the *reflected process at supremum* $\overline{X} - X$ is a Markov process in the filtration $\mathcal{F}_t$. (Note that $X-\underline{X}$, *the reflected process at infimum*, can also be viewed as the dual process $-X$ reflected at its supremum.)

We denote by $L = (L(t), t \ge 0)$ a local time of the reflected process $\overline{X} - X$ at zero and by $L^{-1}(t) = \inf\{s > 0 : L(s) > t\}$ its right-continuous inverse also known as the (ascending) *ladder time process*. Note that the range of the inverse local time L^{-1} corresponds to the set of real times at which new maxima occur. Recalling from Chapter IV of Bertoin [13] and Chapter 6 of Kyprianou [69], the support of the Stieltjes measure dL_t coincides with the closure of the zero set of the reflected process.

Next, let us introduce a *ascending ladder height process* H, using the inverse local time to time-change the supremum process. The process H is defined by

$$H(t) \;=\; \overline{X}_{L^{-1}(t)} \quad \text{if } L^{-1}(t) < \infty, \quad H(t) = \infty \text{ otherwise.} \tag{2.2.1}$$

The pair (L^{-1}, H) is known as the *ascending ladder process*. Analogously, the process $(\widehat{L}^{-1}, \widehat{H})$ constructed from the dual process $-X$ is called the *descending ladder process*. The law of the ladder process is characterized by the bivariate Laplace exponent κ and $\widehat{\kappa}$ defined by

$$e^{-\kappa(\alpha,\beta)} \;=\; \mathbb{E}\Big(e^{-\alpha L^{-1}(1)-\beta H(1)}\Big) \quad \text{and} \quad e^{-\widehat{\kappa}(\alpha,\beta)} \;=\; \mathbb{E}\Big(e^{-\alpha \widehat{L}^{-1}(1)-\beta \widehat{H}(1)}\Big)$$

for $\alpha, \beta \ge 0$. It is therefore important to evaluate explicitly the quantities κ and $\widehat{\kappa}$ as they play fundamental role in the fluctuation theory of Lévy processes.

The random variables of interest in fluctuation theory are the following. Let $\mathbf{e}_q$ be an independent exponentially distributed random time with parameter $q \ge 0$. We shall work with the convention that when $q = 0$, the random variable $\mathbf{e}_q$ is understood to be equal to ∞ with probability one. Next, we define by

$$\overline{G}_{\mathbf{e}_q} \;=\; \sup\{t < \mathbf{e}_q : X_t = \overline{X}_t\} \quad \text{and} \quad \underline{G}_{\mathbf{e}_q} \;=\; \sup\{t < \mathbf{e}_q : X_t = \underline{X}_t\}$$

the last zero of the reflected processes before the exponential random time $\mathbf{e}_q$. According to Proposition VI.4 in Bertoin [13], it is known that, if X is not a compound Poisson process, $\overline{G}_{\mathbf{e}_q}$ is actually the unique instant time t in the random interval $[0, \mathbf{e}_q]$ such that $X_t = \overline{X}_{\mathbf{e}_q}$ or $X_{t-} = \overline{X}_{\mathbf{e}_q}$.

We move on now to introducing the fluctuation identity which provides many results concerning the distributional decomposition of the excursions of any Lévy process when sampled at an independent exponentially distributed random time.

Theorem 2.2.1 (The Wiener-Hopf factorization) [2] *Suppose that X is any Lévy process other than compound Poisson process.*

(i) *The pairs $(\overline{G}_{\mathbf{e}_q}, \overline{X}_{\mathbf{e}_q})$ and $(\mathbf{e}_q - \overline{G}_{\mathbf{e}_q}, \overline{X}_{\mathbf{e}_q} - X_{\mathbf{e}_q})$ are independent and infinitely divisible, yielding the factorization*

$$\mathbb{E}\Big(e^{i\vartheta \mathbf{e}_q + i\theta X_{\mathbf{e}_q}}\Big) \;=\; \frac{q}{q - i\vartheta + \Psi(\theta)} \;=\; \Psi_q^{(-)}(\vartheta,\theta).\Psi_q^{(+)}(\vartheta,\theta), \tag{2.2.2}$$

where $\vartheta, \theta \in \mathbb{R}$,

$$\Psi_q^{(-)}(\vartheta,\theta) \;=\; \mathbb{E}\Big(e^{i\vartheta \underline{G}_{\mathbf{e}_q} + i\theta \underline{X}_{\mathbf{e}_q}}\Big) \quad \textit{and} \quad \Psi_q^{(+)}(\vartheta,\theta) \;=\; \mathbb{E}\Big(e^{i\vartheta \overline{G}_{\mathbf{e}_q} + i\theta \overline{X}_{\mathbf{e}_q}}\Big).$$

The pair $\Psi_q^{(-)}(\vartheta,\theta)$ and $\Psi_q^{(+)}(\vartheta,\theta)$ are called the Wiener-Hopf factors.

(ii) *The Wiener-Hopf factors may themselves be identified in terms of the analytically extended Laplace exponent $\kappa(\alpha,\beta)$ and $\widehat{\kappa}(\alpha,\beta)$ via the Laplace transforms,*

$$\mathbb{E}\Big(e^{-\alpha \underline{G}_{\mathbf{e}_q} + \beta \underline{X}_{\mathbf{e}_q}}\Big) \;=\; \frac{\widehat{\kappa}(q,0)}{\widehat{\kappa}(q+\alpha,\beta)} \quad \textit{and} \quad \mathbb{E}\Big(e^{-\alpha \overline{G}_{\mathbf{e}_q} - \beta \overline{X}_{\mathbf{e}_q}}\Big) \;=\; \frac{\kappa(q,0)}{\kappa(q+\alpha,\beta)}$$

for every complex numbers α, β having positive real part.

(iii) *The Laplace exponent $\kappa(\alpha,\beta)$ and $\widehat{\kappa}(\alpha,\beta)$ may also be identified in terms of the law of X in the following way,*

$$\kappa(\alpha,\beta) \;=\; k \exp\Big(\int_0^\infty \int_{(0,\infty)} \big(e^{-t} - e^{-\alpha t - \beta x}\big) t^{-1} \mathbb{P}(X_t \in dx) dt\Big) \tag{2.2.3}$$

and

$$\widehat{\kappa}(\alpha,\beta) \;=\; \widehat{k} \exp\Big(\int_0^\infty \int_{(-\infty,0)} \big(e^{-t} - e^{-\alpha t + \beta x}\big) t^{-1} \mathbb{P}(X_t \in dx) dt\Big), \tag{2.2.4}$$

where $\alpha, \beta \in \mathbb{R}$ and k and $\widehat{k}$ are strictly positive constants.

[2] We refer to Theorem 6.16 of Chapter 6 of Kyprianou [69].

(iv) *By setting $\vartheta = 0$ and taking limits as q tends to zero in (2.2.2), we obtain*

$$k'\Psi(\theta) \;=\; \kappa(0,-i\theta)\widehat{\kappa}(0,i\theta)$$

for some constants $k' > 0$ (which may be taken equal to unity by a suitable normalization of local time).

We conclude this section with an important result of Theorem 2.2.1. Recall that Ψ is the characteristic exponent of X so that, for $q > 0$, $q/(q+\Psi(\theta))$ is the characteristic function of $X_{\mathbf{e}_q}$. Theorem 2.2.1 yields the following remarkable fluctuation identity:

$$\mathbb{E}\big(e^{i\theta X_{\mathbf{e}_q}}\big) \;=\; \frac{q}{q+\Psi(\theta)} \;=\; \Psi_q^{(-)}(\theta)\Psi_q^{(+)}(\theta), \tag{2.2.5}$$

for $q > 0$, where $\Psi_q^{(-)}(\theta)$ and $\Psi_q^{(+)}(\theta)$ are respectively the characteristic function of the random variable $\underline{X}_{\mathbf{e}_q}$ and $X_{\mathbf{e}_q} - \underline{X}_{\mathbf{e}_q}$ defined by

$$\Psi_q^{(-)}(\theta) \;=\; \mathbb{E}\big(e^{i\theta \underline{X}_{\mathbf{e}_q}}\big), \tag{2.2.6}$$

and

$$\Psi_q^{(+)}(\theta) \;=\; \mathbb{E}\big(e^{i\theta (X_{\mathbf{e}_q} - \underline{X}_{\mathbf{e}_q})}\big) \;=\; \mathbb{E}\big(e^{i\theta \overline{X}_{\mathbf{e}_q}}\big). \tag{2.2.7}$$

Notice that $\Psi_q^{(-)}(\theta)$ (respectively, $\Psi_q^{(+)}(\theta)$) admits the analytic continuation into the lower half-plane $\mathfrak{Im}(\theta) < 0$ (respectively, upper half-plane $\mathfrak{Im}(\theta) > 0$), and does not vanish there. We refer among others to Applebaum [5], Bertoin [13], Kyprianou [69], and Sato [111] for more details.

2.3 Some important classes of Lévy processes

In the next section, we outline some important class of Lévy processes for which the two factors $\Psi_q^{(-)}(\lambda)$ and $\Psi_q^{(+)}(\lambda)$ of the Wiener-Hopf factorization formula (2.2.5) have explicit expressions. These class of Lévy processes can be found among others in Bertoin [13], [15], Kyprianou [69], Mordecki [87], [88], [89], and Asmussen et al. [6].

Working under these classes of Lévy processes, numerical computation for the problem discussed in Chapters 4, 5 and 6 can be performed quite easily.

2.3.1 Lévy processes with no positive jumps

This class of processes has a great interest from a practical point of view, because they are processes for which fluctuation theory takes the nicest form and can be developed explicitly to its full extent. The degenerate case when X is either the negative of a subordinator or a deterministic drift has no interest and will not be discussed throughout. What we shall say here is based on Chapter VII in Bertoin [13].

Due to the absence of the positive jumps, the characteristic function $\theta \mapsto \mathbb{E}(e^{i\theta X_t})$ $(\theta \in \mathbb{R})$ can be extended to define an analytic function in the complex lower half-plane

$(\mathfrak{Im}(\theta) \leq 0)$. Because of the fact that the Lévy measure vanishes on the positive half-line, the Lévy-Khintchine formula shows that the characteristic exponent $\Psi(\theta)$ is well defined and analytic on $(\mathfrak{Im}(\theta) \leq 0)$. Hence, it is therefore sensible to define

$$\kappa(\theta) \;=\; -\Psi(-i\theta) \;=\; -\mu\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{(-\infty,0)} \big(e^{\theta y} - 1 - \theta y \mathbf{1}_{\{y>-1\}}\big)\Pi(dy),$$

and, hence, we see that the identity $\mathbb{E}(\exp\{\theta X_t\}) = \exp\{t\kappa(\theta)\}$ holds whenever $\mathfrak{Re}(\theta) \geq 0$. The function $\kappa : [0,\infty) \to (-\infty,\infty)$ also called as the *Laplace exponent* of X is zero at the origin and is strictly convex with $\lim_{\theta\uparrow\infty}\kappa(\theta) = \infty$. Next we denote by $\Phi(\alpha)$ the largest solution of the equation

$$\kappa(p) \;=\; \alpha \quad \text{for all } \alpha \geq 0.$$

Note that due to the convexity of κ, there exists at most two solutions for a given α and precisely one root when $\alpha > 0$. A special feature of spectrally negative Lévy processes is that $\overline{X}_{\mathbf{e}_q}$ is known to have exponential law with parameter $\Phi(q)$, namely

$$\mathbb{P}\big(\overline{X}_{\mathbf{e}_q} \in dx\big) \;=\; \Phi(q)e^{-\Phi(q)x}dx, \tag{2.3.1}$$

whose Laplace transform is given by

$$\kappa_q^+(\lambda) \;=\; \mathbb{E}\big(e^{-\lambda \overline{X}_{\mathbf{e}_q}}\big) \;=\; \int_0^\infty e^{-\lambda x}\mathbb{P}\big(\overline{X}_{\mathbf{e}_q} \in dx\big) \;=\; \frac{\Phi(q)}{\lambda + \Phi(q)}, \tag{2.3.2}$$

for all $\lambda \geq 0$ which in turn by the Wiener-Hopf factorization (2.2.5) yields

$$\kappa_q^-(\lambda) \;=\; \mathbb{E}\big(e^{\lambda \underline{X}_{\mathbf{e}_q}}\big) \;=\; \int_0^\infty e^{-\lambda x}\mathbb{P}\big(-\underline{X}_{\mathbf{e}_q} \in dx\big) \;=\; \frac{q}{\Phi(q)}\frac{(\lambda - \Phi(q))}{(\kappa(\lambda) - q)}. \tag{2.3.3}$$

In principle, there is no difficulty to invert the above equation numerically. However, by introducing a special class of functions known as scale functions, the definition of which is given below, the measure $\mathbb{P}\big(-\underline{X}_{\mathbf{e}_q} \in dx\big)$ can be recovered theoretically in terms of such functions.

Definition 2.3.1 (q-Scale function) For a given spectrally negative Lévy process X with Laplace exponent κ, there exists for every $q \geq 0$ a right-continuous function $W^{(q)} : [0,\infty) \to [0,\infty)$, called the q-scale function, with Laplace transform given by

$$\int_0^\infty e^{-\lambda x}W^{(q)}(x)dx \;=\; \frac{1}{\kappa(\lambda) - q}, \quad \text{for } \lambda > \Phi(q), \tag{2.3.4}$$

where $\Phi(q)$ was defined previously. We shall write for short $W^{(0)} = W$.

Following the definition of $W^{(q)}$ introduced above and by applying Laplace inversion method to (2.3.3), the measure of the random variable $-\underline{X}_{\mathbf{e}_q}$ is given by

$$\mathbb{P}\big(-\underline{X}_{\mathbf{e}_q} \in dx\big) \;=\; \frac{q}{\Phi(q)}dW^{(q)}(x) - qW^{(q)}(x)dx. \tag{2.3.5}$$

If the Lévy measure Π has no atoms, it is known that the q-scale function $W^{(q)}$ is at least C^1 smooth, see for instance Lambert [75] and Chan and Kyprianou [28] for more details. For some spectrally negative Lévy processes, the q-scale functions $W^{(q)}$ are available in explicit form. In general, numerical methods are required to compute $W^{(q)}$ from the equation (2.3.4). Further discussion on the property of scale functions and their numerical computation are given in Chapters 6 and 7, respectively.

2.3.2 Lévy processes with mixed exponential jumps

Consider now a Lévy process with Lévy measure Π given by

$$\Pi(dy) \;=\; \mathbf{1}_{(y>0)}\pi(dy) + \mathbf{1}_{(y<0)}\lambda \sum_{k=1}^{n} a_k \alpha_k e^{\alpha_k y} dy, \tag{2.3.6}$$

where π is an arbitrary Lévy measure on $(0, \infty)$, $0 < \alpha_1 < ... < \alpha_n$, $a_k > 0$, for $k = 1, ..., n$ and $\sum_{k=1}^{n} a_k = 1$. The magnitude of the negative jumps of X is mixed exponentially distributed, with parameter α_k chosen with probability a_k, and they occur at the times of a Poisson process with rate λ. Simple computations give

$$\Psi(\theta) \;=\; i\eta\theta - \frac{\sigma^2}{2}\theta^2 + \int_0^\infty \big(e^{i\theta y} - 1 - i\theta h(y)\big)\pi(dy) - \lambda \sum_{k=1}^{n} a_k \frac{i\theta}{\alpha_k + i\theta}, \tag{2.3.7}$$

where $h(y) = y\mathbf{1}_{\{0<y<1\}}$ and η is given by

$$\eta \;=\; \mu + \lambda \sum_{k=1}^{n} \frac{a_k}{\alpha_k}\big(1 - (1 + \alpha_k)e^{-\alpha_k}\big).$$

Considered as a function with complex domain, the characteristic exponent $iq \mapsto \Psi(q)$ in (2.3.7) can be extended analytically to the complex strip $\{z = p + iq : p \in (-\alpha_1, 0]\}$ and, for $-\alpha_1 < p \leq 0$, we have the Laplace exponent of X defined by

$$\kappa(p) \;=\; \eta p + \frac{\sigma^2}{2}p^2 + \int_0^\infty \big(e^{py} - 1 - ph(y)\big)\pi(dy) - \lambda \sum_{k=1}^{n} a_k \frac{p}{\alpha_k + p}. \tag{2.3.8}$$

Due to the convexity of $\kappa(p)$ on $(-\alpha_1, 0]$, it follows when $\sigma > 0$ that under the condition

$$\kappa'(0-) \;=\; \lim_{p \to 0-} \frac{1}{p}\kappa(p) \;=\; \eta + \int_1^\infty y\pi(dy) - \lambda \sum_{k=1}^{n} \frac{a_k}{\alpha_k} \;>\; 0, \tag{2.3.9}$$

(where the integral can take the value ∞), the equation $\kappa(p) = 0$ has $n+1$ negative roots $-p_j$, $j = 1, \cdots, n+1$, that satisfy

$$0 < p_1 < \alpha_1 < p_2 < ... < p_n < \alpha_n < p_{n+1}. \tag{2.3.10}$$

Furthermore, observe that when $\gamma > 0$, the *Cramèr-Lundberg equation*

$$\kappa(p) \;=\; \gamma, \quad \text{for } \gamma > 0 \tag{2.3.11}$$

has always $n+1$ negative roots $\{-p_j, j = 1, ..., n+1\}$ satisfying (2.3.10).

Denote by $\mathbf{e}_\gamma$ an exponential random variable with parameter $\gamma \geq 0$, independent of X, and for $\gamma = 0$, it is understood that $\mathbf{e}_\gamma = \infty$ with probability 1. Assuming that the condition (2.3.9) holds when $\gamma = 0$, Mordecki [87], [88] shows that

$$\Psi_\gamma^{(-)}(s) \;=\; \sum_{j=1}^{n+1} A_j \frac{p_j}{s+p_j},$$

where $-p_1, ..., -p_{n+1}$ are the negative roots of the equation (2.3.11) and the constants $A_1, ..., A_{n+1}$ are given by

$$A_j \;=\; \frac{\prod_{k=1}^{n}(1 - p_j/\alpha_k)}{\prod_{k=1, k\neq j}^{n+1}(1 - p_j/p_k)}, \quad \text{for } j = 1, ..., n+1.$$

By applying Laplace inversion to $\Psi_\gamma^{(-)}(s)$, Mordecki [87], [88] shows that

$$\mathbb{P}\big(-\underline{X}_{\mathbf{e}_\gamma} \in dx\big) \;=\; \sum_{j=1}^{n+1} A_j p_j e^{-p_j x} dx, \quad x \geq 0.$$

2.3.3 Lévy processes with jumps of phase-type

This class of Lévy processes includes and generalizes exponential jumps Lévy processes detailed earlier. A phase-type Lévy process is constructed by independent sum of a spectrally positive process with a compound Poisson process having negative phase-type jumps. We refer to Mordecki [89] and Asmussen et al. [6] for more details.

A distribution F on $(0, \infty)$ is said to be *phase-type* if it is the distribution of the absorption time in a finite state continuous time Markov process $\{J_t : t \geq 0\}$ with one state Δ absorbing and the remaining ones $1, ..., m$ transient. The parameters of this system are m, the restriction $\mathbf{T}$ of the full intensity matrix to the m transient states and the initial probability (row) vector $\mathbf{a} = (a_1, ..., a_m)$, where $a_i = \mathbb{P}(J_0 = i)$. For any $i = 1, ..., m$, let t_i be the intensity of the transition $i \to \Delta$ and write $\mathbf{t}$ for the column vector of intensities. It follows that F has a density given by $f(x) = \mathbf{a}e^{\mathbf{T}x}\mathbf{t}$ and its Laplace transform is given for $s > 0$ by $\widehat{F}(s) = \int_0^\infty e^{-sx} f(x)dx = \mathbf{a}(s\mathbf{I} - \mathbf{T})^{-1}\mathbf{t}$ which can be analytically extended to the complex plane except at a finite number of poles (the eigen values of $\mathbf{T}$). The phase-type Lévy process has the representation

$$X_t \;=\; X_t^+ - \sum_{j=1}^{N(t)} U_j, \quad t \geq 0,$$

where $\{X_t^{(+)} : t \geq 0\}$ is a spectrally positive Lévy process, $\{N_t : t \geq 0\}$ is a Poisson process with rate λ and $\{U_j : j \geq 1\}$ are i.i.d random variables with a common distribution F; all of the objects mentioned above are mutually independent.

The corresponding Lévy-Khintchine exponent, Ψ, can be analytically extended to the complex plane $\{z \in \mathbb{C} : \mathfrak{Re}(z) \leq 0\}$ with the exception of a finite number of poles (the eigenvalues of $\mathbf{T}$). Define, for each $\alpha > 0$, the finite set of roots, with negative real part, of the Cramèr-Lundberg equation $\Psi(p_i) = \alpha$, i.e.,

$$\mathcal{I}_\alpha \;=\; \{p_i : \Psi(p_i) = \alpha, \mathfrak{Re}(p_i) \;<\; 0\},$$

where multiple roots are counted individually. Next, define, for each $\alpha > 0$, a second set of roots with negative real part

$$\mathcal{J}_\alpha \;=\; \{q_i : \frac{\alpha}{\alpha - \Psi(q_i)} = 0, \mathfrak{Re}(q_i) \;<\; 0\},$$

again taking into account of multiplicity, Mordecki [89] and Asmussen et al. [6] show that

$$\Psi_\alpha^{(-)}(s) \;=\; \frac{\prod_{j\in\mathcal{J}_\alpha}(s - q_j)}{\prod_{j\in\mathcal{J}_\alpha}(-q_j)} \frac{\prod_{j\in\mathcal{I}_\alpha}(-p_j)}{\prod_{j\in\mathcal{I}_\alpha}(s - p_j)},$$

which can be analytically extended to the whole complex plane $\mathbb{C}$ except for the poles at $\{p_j \in \mathcal{I}_\alpha\}$. Applying Laplace inversion to $\Psi_\alpha^{(-)}(s)$, it was shown in [89] and [6] that

$$\mathbb{P}\big(-\underline{X}_{\mathbf{e}_\alpha} \in dx\big) \;=\; \sum_{j=1}^{n}\sum_{k=1}^{m(j)} A_{j,k}\frac{(-p_j x)^{k-1}}{(k-1)!}e^{p_j x}dx, \quad x \geq 0,$$

where $m(j)$ is the multiplicity of root p_j, n is the number of distinct roots and

$$A_{j,k} \;=\; \frac{1}{(m-k)!}\frac{d^{m-k}}{ds^{m-k}}\Big(\frac{\Psi_\alpha^{(-)}(s)(s-p_j)^m}{(-p_j)^k}\Big)\Big|_{s=p_j}.$$