Chapter 1

Introduction and Preliminaries

1.1 Motivation

The American put option problem

The valuation of contingent claims has been a widely known topic in the theory of modern finance. Typical claims such as call and put options have been playing significant role not only in the theory but also in the real financial markets. A put (call) option is the "right" but not the obligation to sell (buy) a certain asset at a specified price until or at a predetermined maturity date in the future. If the option specifies that the option holder may exercise this right only at the given future date, the claim is termed *European*.

The pricing of European puts and calls on stocks has an interesting history, dating back to the work of Bachelier [9]. In 1900 Bachelier was the first to use a linear Brownian motion to model the movement of stock price fluctuations. The theory reaches a milestone with the celebrated papers of Black and Scholes [18] and Merton [83] in which the principles of *hedging* and *arbitrage-free pricing* were introduced for the first time. These idea were formalized and extended further by Harrison and Kreps [56] and Harrison and Pliska [57] by applying the fundamental concepts of stochastic integrals and the Girsanov theorem in stochastic calculus. Based on the important principle of hedging, Black and Scholes [18] derived the now famous formula for the value of the European call option, which bears their name and which was extended by Merton [83] in a variety of very significant ways. For this foundational work, Robert Merton and Myron Scholes were awarded the 1997 Nobel Prize in economics.

It is worth noting that most of the traded options, however, are of *American style* (or in the sequel, *American options*)-that is, the holder has the right to exercise an option at any instant before the option's expiry. It is the added feature of early exercise which makes the American options more interesting and complex to evaluate. According to the theory of modern finance[1], the arbitrage-free price of the American

[1] See for instance Karatzas and Shreve [66] and Myneni [90] for extensive review of the theory

put option with strike price K coincides with the value function V of an optimal stopping problem with *payoff function* $G(x) = (K - x)^+$. That is to say that the *arbitrage-free* price of the American put option is given by

$$V(t, x) \;=\; \sup_{0 \le \tau \le t} \mathbb{E}\Big(e^{-r\tau}\big(K - S_\tau(x)\big)^+\Big), \tag{1.1.1}$$

for all $(t, x) \in [0, T] \times \mathbb{R}_+$, where T is the maturity of the option and τ is a stopping time of the stock price process S the evolution of which is given by exponential of a linear Brownian motion

$$S_t(x) \;=\; xe^{(r+\omega)t+\sigma B_t}, \tag{1.1.2}$$

taken under a chosen martingale measure $\mathbb{P}$ (with associated expectation operator $\mathbb{E}$) under which $S_0 = x$. The parameter ω is chosen to be $-\frac{1}{2}\sigma^2$ so that the discounted stock price process $e^{-rt}S_t(x)$ is $\mathbb{P}$-*martingale*, implying that

$$\mathbb{E}\Big(e^{-rt}S_t(x)\Big) \;=\; x.$$

Although the American put option problem was treated as an optimal stopping problem, a financial justification using hedging arguments was given only later by Bensoussan [12] and Karatzas [64], [65]. The optimal stopping time in the American put option problem (1.1.1) is the first time when the stock price process S goes below a time-dependent boundary b. When the maturity time T of the option is finite, the problem (1.1.1) is essentially two-dimensional in the sense that it consists of finding the value function V and the optimal stopping boundary b simultaneously; that is to say that the value function can be seen as a function of the unknown stopping boundary. Therefore, from an analytical point of view, solving the problem is difficult.

The first and one of the most penetrating mathematical analysis of the problem (1.1.1) was due to McKean [82]. There the problem was transformed into a *free boundary problem* for the value function V and the boundary b. Solving the free boundary problem, McKean obtained the American option price explicitly in terms of the boundary. McKean's work was taken further by van Moerbeke [86]. Motivated by the physical problem of *the condition of heat balance* (i.e., the law of conservation of energy), van Moerbeke [86] introduced a so-called the *smooth pasting* condition to determine the boundary and specify the value function. This condition dictates that the value and the payoff functions must join smoothly at the boundary.

The derivation of the smooth pasting condition for diffusion processes are given by Grigelionis and Shiryaev [55], Shiryaev [113], Chernoff [30], McKean [82] and Myneni [90] using Taylor approximation of the value function around the boundary and by Bather [11] and van Moerbeke [86] using Taylor expansion of the payoff function around the boundary plus the assumption that the boundary is *regular*[2] for the interior of the stopping region for the underlying process. Since the value function is not

and methods of pricing American type options for diffusion processes.

[2]Starting at the boundary, the underlying process makes an immediate visit to the interior points of the stopping region. See Definition 2.1.3 in Chapter 2.

known a priori, the approach of Bather [11] and van Moerbeke [86] is more satisfactory than the others.

As an alternative to the Taylor expansion method, Peskir [98] introduced a probabilistic approach to prove the smooth pasting condition. The main approach of the proof is based on a *change of variable formula with local time-space* on curves which he derived recently in [97]. This formula extends further the Itô-Tanaka formula for convex functions (see for instance Revusz and Yor [106]). Using the change of variable formula and the free boundary problem, Peskir [98] derived the smooth pasting condition. (See also Peskir and Shiryaev [99] for more discussion on recent development of local time-space calculus in the theory of optimal stopping.)

Based on the free boundary problem formulation of the optimal stopping problem (1.1.1), with continuous and smooth pasting conditions in place, and combining with the Itô-Doob-Meyer decomposition of the value function of the problem (1.1.1) into martingale and potential processes, van Moerbeke [86], Myneni [90], El Karoui and Karatzas [45], Jacka [62], Carr et al. [23], and later Peskir [98] showed that the optimal stopping boundary can be characterized as a solution to a nonlinear integral equation. Such an equation was already obtained earlier by Friedman [51] in 1959 for a one-dimensional free boundary problem of ice melting. This nonlinear integral equation for the optimal boundary is known as the *Riesz decomposition* for the value function of the problem (1.1.1) and has a clear economical meaning to the *early exercise premium* representation of the value function. We refer among others to Kim [67], Myneni [90] and Carr et al. [23] and the literature therein for details.

The existence and local uniqueness of a solution to the nonlinear integral equation for the boundary was proved by Friedman [51] and van Moerbeke [86] using the fixed point theorem (contraction principle) first for a small time interval and extending it to any interval of time using induction arguments. The result of applying the fixed point theorem is that the nonlinear equation involves continuous differentiability of the curve boundary, a condition that is needed to be proved a priori, and results in a long computation and strong condition imposed on the boundary. In contrast to the fixed point method, Jacka [62] and later Peskir [98] introduced a probabilistic approach to prove the existence and uniqueness of a solution to the nonlinear integral equation. The key ingredient of the proof is based on the smooth pasting condition and the Itô-Doob-Meyer decomposition of the value function of the optimal stopping problem (1.1.1). (Note that the Itô-Doob-Meyer decomposition underlies the basic principle of the theory of optimal stopping developed earlier by Snell [115], Dynkin [39] and Dynkin and Yushkevich [41].) However, the incorporation of the smooth pasting condition in the proof was made clear by Peskir [98] using his change of variable formula.

Alternative modelling for underlying processes

Until now we have discussed exponential of a linear Brownian motion as the continuous time model for the evolution of the stock price process (1.1.2). In recent years, there has been a lot of interest in looking for alternative models for the evolution of the stock price process which gives a better fit to the real data. Empirical study of financial data reveals the fact that the distribution of the log-return of stock price exhibits features which cannot be captured by the normal distribution such as heavy tails and asymmetry. For the purpose of replicating more effectively these features, there has been a general shift in the literature to modelling with exponential *Lévy process* as an alternative to exponential of a linear Brownian motion.

A Lévy process is a stochastic process with stationary independent increments whose paths are right-continuous and have left limits. Most recent examples of Lévy processes used in modelling the evolution of the stock price process we refer among others to the normal inverse Gaussian model of Bandorff-Nielsen [10], the hyperbolic model of Eberlein and Keller [42], the variance gamma model of Madan and Seneta [80], the CGMY model of Carr et al. [24], and tempered stable process first introduced by Koponen [68] and extended further by Boyarchenko and Levendorski [21].

Working with a Lévy process leads to many intriguing mathematical issues which need to be resolved to completely settle the problem of valuing American options. In a market where the underlying dynamics for the stock price process is driven by the exponential of a linear Brownian motion, as discussed before, the valuation is transformed into a free boundary problem. The critical value (the stopping boundary) of the stock price process is determined by imposing continuous and smooth pasting conditions as optimality criterion for choosing the stopping boundary. However, by allowing jumps in the sample paths of the underlying dynamics of the stock price process, the smooth pasting *may* break down at the stopping boundary as the stock price process may jump over the boundary. As a result, the *continuous pasting* condition is perhaps the only criterion for determining the stopping boundary.

When maturity T is infinite and the underlying is a general Markov process, the optimal stopping problem (1.1.1) could be solved without necessarily being transformed into a free boundary problem and using the smooth pasting condition. The solution can be obtained using probabilistic approach. This approach was first introduced by Darling et al. [33] for random walks and was extended further using similar arguments in [33] to continuous time among others by Mordecki [87], Asmussen et al. [6], and Alili and Kyprianou [3]. Taking the result of Mordecki [87], it was shown recently by Alili and Kyprianou [3] that the existence of the smooth pasting condition for the problem (1.1.1) is determined by the *regularity* of the sample paths of the underlying process; for the problem considered there the smooth pasting occurs if and only if 0 is regular for the lower half-line $(-\infty, 0)$ for the process itself.

However, the solution to a perpetual optimal stopping problem with a more general payoff function was not discussed by the aforementioned authors. This problem was addressed by Boyarchenko and Levendorski [21]. There they considered the problem of solving perpetual optimal stopping for payoff functions with exponential growth. Their approach is much more sophisticated in which potential theory of Lévy processes and the theory of pseudo-differential operators are heavily used to solve the problem. See for instance [20] for recent work in this direction. Working under a particular class of Lévy processes with stable like characteristic exponent, Boyarchenko and Levendorski [21] gave an integral test for the smooth pasting condition to occur.

1.2 The main contribution of this thesis

This section outlines the main contribution of this thesis to the theory of optimal stopping problems driven by Lévy processes. The aim is to propose a framework by which semi-explicit solutions can be obtained. The solutions are given for both finite and infinite maturity and are obtained without using the continuous and smooth pasting conditions. Using the semi-explicit solutions in the problems we consider, we give sufficient and necessary conditions for the pasting conditions to be fulfilled.

The thesis consists of seven self-contained chapters. The content of the chapters is outlined in what follows.

Chapter 1 This chapter overviews some past and recent developments in the theory of optimal stopping and outlines some points that have not been discussed in the literature. The missing gaps in the theory are explained in this chapter and constitute the main source of motivation of the writing of this thesis.

Chapter 2 This chapter provides a brief introduction to Lévy processes and the Wiener-Hopf factorization formula which underlies the fluctuation theory of Lévy processes and forms one of the two main principles for solving an infinite horizon optimal stopping problem under Lévy processes. We also discuss in this chapter some important classes of Lévy processes for which the two factors of the Wiener-Hopf factorization have explicit expressions. Among theses classes, we use spectrally negative Lévy processes for the numerical computation performed in the last four chapters.

Chapter 3 In this chapter we establish a change of variable formula for 'ripped' time-space functions of Lévy processes of bounded variation at the cost of an additional integral with respect to local time-space in the formula. Roughly speaking, by a ripped function, we mean here a time-space function which is $C^{1,1}$ on either side of a time dependent barrier and which may exhibit a discontinuity along the barrier itself. Such functions have appeared in the theory of optimal stopping problems for Markov processes of bounded variation (cf. Peskir and Shiryaev ([95], [96]), Chan ([26], [27]), Avram et al. [7]. This result complements the recent work of Föllmer et al. [50], Eisenbaum ([43], [44]) and Peskir ([97], [98]) and Elworthy et al. [46] in which generalized versions of Itô's formula were established with local time-space. Using the change of variable formula, we address the finite maturity American put option problem where the evolution of the stock price process is driven by a bounded variation Lévy process. Combining this with Itô-Doob-Meyer decomposition of the value process of the American put option problem into martingale and potential processes, we show that the optimal stopping boundary can be characterized as a solution to a nonlinear integral equation. Taking account of the continuous pasting condition, we show using the change of variable formula that such integral equation admits, under some conditions, an unique solution for the optimal boundary. By the uniqueness of such solution, we show that the value function of the American put option problem and the optimal stopping boundary represent an unique pair solution to a free boundary problem of parabolic integro-differential type.

Chapter 4 This chapter discusses a relatively new optimal stopping problem where the payoff is an integer power function. This problem was first introduced by Novikov and Shiryaev [91] for random walks based on other similar examples given by Darling et al [33]. We give the analogue of their results when the random walks are replaced by Lévy processes. The main ingredient of solving this problem is central to using Appell polynomials and fluctuation theory of Lévy processes.

Chapter 5 In this chapter, we generalize the recent work of Boyarchenko and Levendorski [21] on a perpetual optimal stopping problem under Lévy processes. Unlike their approach, we do not appeal to the theory of pseudo-differential operators to solve the problem. We work with a more general class of Lévy processes and we allow for a more general class of payoffs. The solution is obtained by reducing the problem into an averaging problem from which we obtain, using the *Wiener-Hopf* factorization, a *fluctuation identity* for overshoots of Lévy processes. This fluctuation identity relates the solution of the averaging problem with the expected value of discounted payoff function up to a first passage time and is the key element in obtaining the value function and the optimal boundary of the stopping problem. Using our approach, we are able to verify the smooth pasting condition analytically and to reproduce the special results of those discussed among others by Darling et al. [33], Mordecki [87], Boyarchenko and Levendorskii [21], Alili and Kyprianou [3], Novikov and Shiryaev [91], and Kyprianou and Surya [73] (also presented in Chapter 4). Furthermore, assuming that the moment generating function of the underlying Lévy process exists on an open set containing zero, we obtain a lower and upper bounds for the arbitrage-free price of the finite maturity American put option in terms of the value function of the perpetual American put option problem.

Chapter 6 In this chapter we consider an endogenous bankruptcy problem. This problem is closely related to a perpetual type optimal stopping problem which primarily deals with finding an optimal bankruptcy level V_B of a firm which keeps a constant level of its debt and chooses its bankruptcy level endogenously so that the value of its equity is maximized. The firm declares bankruptcy when the value of its asset goes below the level V_B. This problem has been investigated by Leland and Leland and Toft in a sequence of their papers in [77] and [76], respectively. The work of Leland and Toft was extended further from diffusion to a Lévy process which is the independent sum of a linear Brownian motion and a compound Poisson process with one-sided exponential jumps by Hilberink and Rogers [58]. As it was suggested by Leland and Toft [76] and later by Hilberink and Rogers [58] that, subject to the limited liability constraint[3] of the equity value, the smooth pasting condition is used for optimality criterion for choosing the bankruptcy level V_B. In other recent work, Chen and Kou [29] generalized the works of Leland and Toft [76] and Hilberink and Rogers [58] by adding a two-sided exponential jumps compound Poisson process to a linear Brownian motion. They succeeded in *proving* that the optimal bankruptcy level is obtained by using the smooth pasting condition for the case considered there.

The main purpose of this chapter is threefold. Firstly to revisit the previous works of Leland and Toft [76] and Hilberink and Rogers [58] and show that the issue of choosing an optimal endogenous bankruptcy level can be dealt with analytically and numerically when the underlying source of randomness for the value of the firm's asset is replaced by a general Lévy process with no positive jumps. Secondly, by working with the latter class of Lévy processes we bring to light a new phenomenon, namely that, depending on the nature of the small jumps, the optimal default level may be determined by a principle of *continuous pasting* as opposed to the usual *smooth pasting*. Thirdly, we are able to *prove* the optimality of the default level according to the appropriate choice of pasting. This improves on the results of Hilberink and Rogers [58] who were only able to give a numerical justification for the case of smooth pasting. Our calculations are greatly eased by the recent perspective on fluctuation theory of spectrally negative Lévy processes in which many new identities are expressed in terms of the so called *scale functions*. To finish this chapter, we study analytically and numerically the behaviour of the term structure of credit spreads for very short maturity bonds when we allow the firm's assets to be driven by a general Lévy process with no positive jumps. The study reveals the fact that the credit spreads have strictly positive values, a feature typically observed in the financial market.

Chapter 7 In this chapter we discuss a robust numerical method to numerically produce the q-scale function $\{W^{(q)}(x) : q \geq 0, x \in \mathbb{R}\}$ of a general spectrally negative Lévy process $(X, \mathbb{P})$. The method is based on the *Esscher transform* of measure $\mathbb{P}^\nu$ under which X is taken and the scale function is determined. This change of measure makes it possible for the scale function to be bounded and hence makes numerical computation easier, fast and stable. Working under the new measure $\mathbb{P}^\nu$ and using the method of Abate and Whitt [1] and Choudhury et al. [31], we give a fast stable numerical algorithm for the computation. The algorithm has been extensively used to give numerical verification of the main results presented in this thesis.

[3]Equity must worth non-negative for all values V of the firm's asset bigger than equal to the bankruptcy level V_B.

1.3 Publication details

The material presented in this thesis has resulted in the following research papers.

(i) Kyprianou, A. E. and Surya, B. A. A note on the change of variable formula with local time-space for bounded variation Lévy processes. To appear in *Séminaire de Probabilité XL*, Lecture Notes in Mathematics, Springer-Verlag.

(ii) Kyprianou, A. E. and Surya, B. A. On the Novikov-Shiryaev optimal stopping problems in continuous time. Appeared in *Electronic Communications in Probability*, Vol. **10** (2005), 146-154.

(iii) Kyprianou, A. E. and Surya, B. A. Principles of smooth and continuous fit in the determination of endogenous bankruptcy levels. To appear in *Finance and Stochastics*, Springer-Verlag.

(iv) Surya, B. A. An approach for solving perpetual optimal stopping problems driven by Lévy processes. To appear in *Stochastics*.

(v) Surya, B. A. Evaluating scale functions of spectrally negative Lévy processes. Submitted for publication to *Journal of Applied Probability*.