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The Logical Structure of Relations

De Logische Structuur van Relaties
(met een samenvatting in het Nederlands)

PROEFSCHRIFT

ter verkrijging van de graad van doctor aan de Universiteit Utrecht op gezag van de rector magnificus, prof. dr. J. C. Stoof, ingevolge het besluit van het college voor promoties in het openbaar te verdedigen op vrijdag 5 november 2010 des middags te 12.45 uur

door

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Promotoren: Prof. dr. A. Visser
Prof. dr. K. Fine

It is one of the reasons for the slow progress of philosophy that its fundamental questions are not, to most people, the most interesting, and therefore there is a tendency to hurry on before the foundations are secure.

Bertrand Russell, *On the nature of truth and falsehood*, 1910.

Preface

During the last four years I have thought about relations almost every day. My goal was to expose as clearly as possible the logical structure of ordinary relations, like the love relation. This thesis contains the results of my quest.

It all started when I read Kit Fine's paper *Neutral relations*. Before reading that paper, I never questioned the structure of relations. Relations to me were simply sets of ordered pairs (or more generally tuples) of objects. But Fine's paper made me realize that this was far too simple. In fact, his analysis was a staggering blow to my whole way of thinking.

Fine argued that the *standard view* on relations was fundamentally wrong. As an alternative he proposed an *antipositionalist view* on relations.[1] It not only took me quite some time to understand the antipositionalist view, but what troubled me most was that I could not gauge its impact. Did it mean that I had to give up my normal way of viewing anything at all in the world? And what exactly was the essence of relations?

I worked on these fascinating questions in two stages. First, I wrote a master's thesis in which I defined and compared mathematical models for different views on relations. A main result was a proof of a natural correspondence between my models for different views on relations. Subsequently, for my PhD project, I refined the analysis by taking the notion of occurrences of objects into account. Initially, this complicated things drastically. The notion of occurrences appeared less simple than I thought. In a few steps I lost myself in the higher infinite and obtained results that involved measurable cardinals.

At the beginning of 2009 I had a breakthrough in my thinking about relations. All the time I had implicitly identified relational states with relational

[1] These notions will be explained in detail in this thesis.

complexes. But by allowing a single state to have more than one corresponding complex within the same relation, the structure of the world suddenly seemed much simpler. I have elaborated this idea into a general theory of relations, which—in my opinion—looks very promising as a framework for understanding the nature of relations.

There is one more thing I wish to bring up. At a seminar shortly after the start of my PhD project, I was asked the following questions: In what sense were my investigations philosophical or even metaphysical? How could a formalization of relations help in obtaining metaphysical insights? Did my analysis really say something about relations or only about their representations? Had the problem of the structure of relations not already been solved long ago?

Since that seminar, these questions remained in the back of my mind. I would like to say a few words about them. First of all, I do not care too much whether or not my work is qualified as philosophical or as metaphysical. What matters is whether or not it helps in getting a better idea of what 'real' relations are. I strongly believe that my formal, mathematical approach is fruitful in this respect, and that the results obtained provide a solid foundation for a theory of the logical structure of relations. It is true that in my analysis I discuss at certain points representations of relations, but my primary focus is on the features of relations themselves. That the problem of relations has been solved long ago is extremely unlikely. Although historical research may reveal interesting similarities in ideas about the structure of relations, there will almost certainly be significant differences on a detailed level, simply because I make extensive use of logico-mathematical concepts developed in the 20th century. To make an admittedly immodest comparison, the ancient Greeks already proposed that the world consists of atoms, but sophisticated tools were indispensable for further developments of this idea.

I want to emphasize that I wrote my thesis not only for people with a mathematical background. It should be possible to get a good impression of the ideas also when the technical details are skipped. The results, however, are very abstract. I often use the love relation as an example, but anyone who hopes to find anything unique about this relation, will be disappointed. Globally substituting 'love' with 'hate' would not change the claims in any significant way.

Joop Leo, October 2010

Acknowledgements

I feel privileged to have had the opportunity to work on things that fascinate me. To start doing philosophical research after working for more than 10 years as a consultant and manager in the IT business, is a rather unusual move. I owe my gratitude to Albert Visser for offering me a PhD position at the Department of Philosophy of Utrecht University. As my promotor, Albert supported me in setting and pursuing my own goals, but he also shared a genuine interest in the problems I tried to solve. I learned a lot from the cheerful way in which he approaches abstract philosophical questions and from his brilliant insights in logic and mathematics. Playing in his room with cups and glasses was also very helpful in getting a grip on the essence of relations.

As I wrote in the preface, my research on relations started with reading a paper by Kit Fine about his new, intriguing view on relations. I am extremely grateful for his generous support in the last four years. In 2008 I spent three weeks at New York University as a visiting scholar, and I had the chance to talk a lot with him about my investigations. His relaxed and completely open way of looking at my problems has been most inspiring. I thank Kit also for all his insightful comments on my papers and for bringing me into contact with other people doing interesting research on related topics.

The Department of Philosophy in Utrecht has been a perfect working environment in several respects. Vincent van Oostrom helped me greatly with his questions, and his original and humorous analyses of mathematical and philosophically related issues. Special thanks also to my roommate Clemens Grabmayer for interesting discussions and for always being helpful with providing advice with respect to the style of my papers. I benefited from his thorough knowledge of LaTeX. I also thank my other roommates Nick Vaporis for nice conversations (first in English, but later completely in Dutch) and Sander Bruggink for explaining the basics of LaTeX and for his implementation of the sliding block puzzle I got from my father. I enjoyed the lunches and weekly lunch seminar with the other members of the Theoretical Philosophy group very much, in particular I like to mention Rosalie Iemhoff, Herman Hendriks, Menno Lievers, Janneke van Lith, Jesse Mulder, and Thomas Müller. Further, I thank the system group and the administration of the department for their assistance.

Attending the Dutch Research Seminar for Analytic Philosophy organized by Herman Philipse has been both disenchanting and enlightening. Comments on the manuscript of my first paper ("Modeling relations") made by Herman, Victor Gijsbers, Rosja Mastop, and Fred Muller made me realize more than ever that a mathematical treatment of a subject is no guarantee for sound metaphysical conclusions.

A number of other people have commented on my work. I benefited greatly from suggestions and comments made by Keimpe Algra, José Luis Bermúdez, Patrick Blackburn, Harm Boukema, Ben Caplan, Fabrice Correia, Shamik Dasgupta, Ali Enayat, Allen Hazen, Marcus Kracht, Benedikt Löwe, Ruth Millikan, Francesco Orilia, Peter Simons, Lars Tump, Jan Willem Wieland, and Paul Ziche. For improving the English of my thesis I thank Joel Anderson, Tony Booth, and Riëtte Leo very much.

The cover of the thesis is made by Loed Stolte. It depicts Narcissus and Echo, who play a special role in a number of examples in this thesis.

Finally, I like to mention Mo and Gitte. We know by experience how complex relations can be. I thank them both for making it possible to write this thesis.

Contents

Chapter 1

Introduction

> The question of relations is one of the most important that arise in philosophy, as most other issues turn on it: monism and pluralism; the question whether anything is wholly true except the whole of truth, or wholly real except the whole of reality; idealism and realism, in some of their forms; perhaps the very existence of philosophy as a subject distinct from science and possessing a method of its own.
>
> Bertrand Russell, *Logical Atomism*, 1924.

1.1 The question of relations

Many people see relations primarily as ways in which *people* stand to each other. Then they think of relations like *loving*, *hating*, *knowing*, etc. But there is also a more general conception of relations that involves all kinds of entities. We have, for example, the *smaller than* relation and the *betweenness* relation, which may be applied to books, countries, numbers, and so many other things.

The question I would like to answer in this PhD thesis is: *What is the logical structure of relations?* By this I mean, the totality of formal properties of relations that make no reference to the specific nature of their constituents. Many aspects of relations will therefore not be considered. In fact, most of what I say about the love relation equally applies to the hate relation. Although I immediately must add that it is certainly not excluded that the logical structure of these relations is quite different. For example, in order to love someone it may perhaps be necessary to love yourself, but to hate someone you may not have to hate yourself.

The question of relations is old. As with so many things, it goes at least back to Aristotle (± 350 BC). But what makes it very interesting to reconsider it *now* is that mathematical-logical developments in the 20th century offer radically new tools for approaching the problem. In particular, new modeling techniques may shed a whole new light on the structure of relations.

This research will be quite abstract, and its relevance may not immediately be evident to everyone. Let me briefly comment on this. In our ordinary understanding of the world, a leading role is played by noticing resemblances and differences between things and events. Analytically investigating what the essence of relations is, might deepen our understanding of fundamental aspects of reality. As such it is an important (metaphysical) topic. Furthermore, I think that this research may have applications, in particular in psychology and in artificial intelligence. It is, for example, not unlikely that it may be used to improve our understanding of the way humans and animals 'learn' relations, and to build better artificial intelligence systems.

In this introductory chapter, I first briefly sketch some historical views of relations. Then I discuss some basic notions of relations, like the distinction between external and internal relations, states of affairs, the degree of a relation, and symmetry in relations. In Section 1.4, views of relations as distinguished by Kit Fine are introduced. These views will be analyzed in detail in the subsequent chapters of this thesis. In Section 1.5, the methodological approach of this investigation is described. Because the modeling of relations fulfills a key role in the thesis, a preview is given in Section 1.6. Moreover in this section, a stepwise development of antipositional models is presented. The last section contains an outline of the thesis.

1.2 Historical views of relations

In this section, a kaleidoscopic overview is given of some historical views of relations that depart significantly from more contemporary views. The views discussed are from Aristotle, some medieval thinkers and Leibniz. It should be noted that many of the problems concerning relations that we find in the philosophical tradition will not reappear in this thesis. More modern views, like Russell's, will be considered later in this chapter.

1.2.1 Aristotle's view

Aristotle (384-322 BC) identifies in the *Categories* relatives as one of the ten main categories, the irreducible kinds of being. The other main categories are substance, quantity, quality, where, when, being-in-a-position, having, acting, and being-acted-upon.

Aristotle's name for relatives is *ta pros ti* (τὰ πρός τι), which literally means "things toward something". Interestingly, Aristotle never uses a noun for 'relation' [Ari62, p. 98] [Ari86, p. 292]. [1]

Chapter 7 of the *Categories*, which contains a detailed discussion of relatives, starts as follows:

> We call *relatives* all such things as are said to be just what they are, *of* or *than* other things, or in some other way *in relation to* something else. For example, what is larger is called what it is *than* something else (it is called larger than something); and what is double is called what it is *of* something else (it is called double of something); similarly with all other such cases. [Ari84a, 6a36-6b1]

For Aristotle, relatives seem to be things in so far as they stand in relation to something else. But Aristotle also argues that terms as 'head' and 'hand' signify parts of substances rather than relatives. Heads and hands are things standing in relations, but we can know what such things essentially are without knowing to what they are related. Therefore, later in Chapter 7, he refines his definition of relatives:

> . . . those things are relatives for which *being is the same as being somehow related to something* . . . [Ari84a, 8a31-32]

[1] Nevertheless, in some translations of the works of Aristotle occasionally the word "relation" is used where "relative" seems more appropriate. This is for example the case in the translation of the *Categories* by H. Cooke [Ari73] and in the translation of the *Metaphysics* by W.D. Ross [Ari84b]. Here, I use more literal renderings of J. L Ackrill, J. Annas, and C. Kirwan.

I find it difficult to really understand what Aristotle says about relatives in the *Categories*, but he seems to be saying that an object is only a relative in so far as it is conceived as a specific relative. As Ackrill says in his notes on the *Categories*: "[the slave] Callias is a relative in so far as he is *called a slave*" [Ari62, p. 98].

In the *Metaphysics*, Aristotle gives a shorter treatment of relatives. In Book Δ of the *Metaphysics*, a threefold classification of relations is given:

> Some things are called RELATIVE as double is relative to half and triple relative to a third, and in general multiple relative to submultiple and exceeding relative to exceeded; others as the able-to-heat is relative to the heatable and the able-to-cut relative to the cuttable, and in general the able-to-act relative to the affectible; and others as the measurable is relative to the measure, and knowable relative to knowledge, and perceptible relative to perception. [Ari93, 1020b26-33]

In Aristotle's discussion of this classification, some parts are hard to follow, like the claim that "the thought is not relative to that of which it is a thought" [Ari93, 1021a32-33]. In Book N of the *Metaphysics*, Aristotle again argues that a relative is not a substance or real object:

> ... the great and the small, and the like, must be relative. But relatives least of all are entities or real objects, coming as they do after both quality and quantity. A relative is a characteristic of quantity, as has been said, not matter, since there is something else ⟨serving as matter⟩ for both relative in general and its parts and forms. For there is nothing either great or small, many or few, or relative in general, which is not many or few or small or relative as being something else. An indication that a relative is least of all a kind of real object and existing thing is the fact that relatives alone do not come into being or pass away or change in the way that increase and diminution occur in quantity, alteration in quality, locomotion in place, sheer coming into being and passing away in the case of a real object. There is none of this with relatives. A thing will be greater or less or equal without itself changing if *another* thing changes in quantity. Also, the matter of each thing (and thus of a real

> object) must be that kind of thing potentially; but a relative is neither potentially nor actually a real object. It is absurd, or rather impossible, to make what is not a real object an element of and prior to a real object, for all the other categories come later. [Ari76, 1088a21-b4]

By saying that "a relative is a characteristic of quantity", it may seem that Aristotle adopts here a more restricted view on relatives than in the *Categories* where he also admits of relatives as depending directly on objects, like being a slave of a master. However, he makes the remark in the context of a discussion of the ontological status of *great* and *small*, and obviously he intends to refer only to a specific type of relatives. What I regard as the main point of this passage of Book N is Aristotle's claim that a relative is not a real object because a thing can become greater or less than another thing without itself changing.

Remark 1.2.1. According to Paul Studtmann [Stu07], it would be a mistake to think that the category of relatives contains what we would nowadays call 'relations'. Although he is right, I think there is a remarkable correspondence between Aristotle's view and a very modern view on relations, namely the *antipositionalist view*. According to the antipositionalist view, relations consist of a network of complexes interrelated by substitutions. Like the relatives of Aristotle, each complex is a particular instance, and no reference is made to relations as incomplete objects. The antipositionalist view is a central theme of this thesis.

1.2.2 Medieval views

Aristotle's view on relations was enormously influential during the Middle Ages. There were different interpretations and new ideas, but—according to Jeffrey Brower [Bro09]— medieval thinkers unanimously regarded polyadic properties as something that can only exist in the mind. Like Aristotle, they considered substances and accidents as the only extramental beings. Furthermore, the medievals thought that both are not polyadic in nature. Outside the mind, an instance of a property would belong to only one subject.

What could correspond to relational concepts could in the view of medievals only be individual substances or monadic accidents. Until the fourteenth century, the medievals followed Aristotle, by accepting that the corresponding items are always accidents. So, on this view, what makes a predication aRb true are individuals a and b and a pair of accidents F and G.

We see in the Middle Ages forms of realism as well as forms of anti-realism about relations. Peter Auriol is a representative of an extreme form of anti-realism. He not only claimes that relations exist only in the mind, but that nothing in extramental reality corresponds to our relational concepts. Although William of Ockham shares Auriol's view that relations only exist in the mind as concepts, he insists that things may be related independently of the mind.

We should distinguish a reductive from a non-reductive form of realism. According to the reductive form, the items corresponding to relational concepts are ordinary accidents, that is, accidents falling under Aristotelian categories other than relation, whereas according to the non-reductive form relatives are accidents of a *sui generis* type. The relevant accidents in both forms are regarded as 'grounds' or 'foundations' of relations. Representatives of reductive realism are Peter Abelard and William of Ockham; representatives of non-reductive realism are Albert the Great and John Duns Scotus. Abelard and Ockham regard their reductive theories as being in line with Aristotle, although, interestingly, Ockham first believed Aristotle had a non-reductive opinion [Ock74, pars I cap. 49, p. 154].

For medieval philosophers, the relational term 'taller' in *Simmias's being taller than Socrates* primarily signified Simmias's height and only indirectly Socrates's height. In *Simmias's thinking about Socrates*, they would even see no real accident of Socrates involved. Instead, they would consider Socrates as related merely by a *relation of reason*. Similarly, creatures are related to God by real relations, but, since God was thought to have no accidents, God is related to creatures only by a relation of reason.

As said before, Aristotle claimed that things are related by their accidents. But this view is problematic for the Christian doctrine of the Trinity. Thomas Aquinas says in the *Summa Theologica, question 28* that it would be a heresy to say that God is not really Father or Son. But at the same time he admits that God has no accidents. The solution was to admit that certain relational situations involve nothing more than individual substances. This started a shift in thinking about relations. In the fourteenth century, it became the common view that substances themselves are responsible for relating and that only in exceptional cases they are related by their accidents.

As an aside, I want to mention that for Aquinas the identity relation is a mere "being of reason" (ens rationis), since this relation requires two relata, but there are not two distinct things in extramental reality that can serve as relata. In *Summa Theologica, question 13, article 7*, he writes:

> Sometimes both what we say of x and what we say of y is true of them not because of any reality in them, but because they are being thought of in a particular way. When, for instance, we say that something is identical to itself, the two terms of the relation only exist because the mind takes one thing and thinks of it twice, thus treating it as though it has a relation to itself. [Aqu06, p. 153]

Another important shift in the fourteenth century was the characterization of the term 'relation'. As Brower [Bro09] notices, prior to the fourteenth century it signified in most cases "whatever it is in the relational situation that does the relating", but Ockham and John Buridan maintain that the term 'relation' always signifies beings of reason. In that sense, they maintain a form of anti-realism about relations, although a weaker form than Auriol's anti-realism, because they think that many relational judgements are true independently of the mind.

1.2.3 Leibniz's view

Gottfried Wilhelm Leibniz (1646-1716) is also an anti-realist with respect to relations. He asked himself where a relation which links two objects is located. He argues that it cannot be only in one of them and also not in a kind of void between them. He writes in 1714 in a letter to Bartholomaeus Des Bosses, a Jesuit theologian:

> I do not believe that you will admit an accident that is in two subjects at the same time. My judgement about relations is that paternity in David is one thing, sonship in Solomon another, but that the relation common to both is a merely mental thing whose basis is the modifications of the individuals. [Lei86, p. 609]

To express relational facts in terms of non-relational facts, Leibniz uses a special connective, namely *et eo ipso* (*and by that itself /and by that very fact*). For example, he gave of the sentence

> Paris amat Helenam (Paris loves Helen)

the following paraphrase

> Paris est amator et eo ipso Helena est amata (Paris is a lover and by that very fact Helen is beloved).

According to Leibniz's nominalistic metaphysics, there are in the real word only individual substances and their accidents. Each of these real accidents can be in no more than one substance. Therefore, when relations are conceived of as accidents of multiple substances, they necessarily are non-real. In a famous letter to Samuel Clarke (1716), Leibniz again declares certain relations to be "mere ideal things", but he calls them "nevertheless useful":

> I shall allege another example to show how the mind uses, upon occasion of accidents which are in subjects, to fancy to itself something answerable to those accidents out of the subjects. The ratio or proportion between two lines L and M may be conceived three several ways: as a ratio of the greater L to the lesser M, as a ratio of the lesser M to the greater L, and, lastly, as something abstracted from both, that is, as the ratio between L and M without considering which is the antecedent or which the consequent, which the subject and which the object. And thus it is that propositions are considered in music. In the first way of considering them, L the greater, in the second, M the lesser, is the subject of the accident which philosophers call "relation". But which of them will be the subject in the third way of considering them? It cannot be said that both of them, L and M together, are the subject of such an accident; for, if so, we should have an accident in two subjects, with one leg in one and the other in the other, which is contrary to the notion of accidents. Therefore, we must say that this relation, in this third way of considering it, is indeed out of the subjects; but being neither a substance nor an accident, it must be a mere ideal thing, the consideration of which is nevertheless useful. [Lei86, p. 704]

In this passage, Leibniz uses the word "relation" in different senses. In one sense, a relation is a real accident of a single subject, but if it is conceived of as an accident of more than one subject, he regards it as something that exists only in the mind.

There is much controversy about whether or not Leibniz thought that all relational propositions are reducible to nonrelational ones. According to Russell [Rus92, p. 15], Leibniz believed that propositions must, in the last analysis, have a subject and predicate. Yet for Russell it is clear that such a reduction is not always possible. For example, the proposition *there are three men* "cannot be regarded as a mere sum of subject-predicate propositions, since the number only results from the singleness of the proposition, and would be absent if three propositions, asserting each the presence of each man, were juxtaposed" [Rus92, p. 12].

In [Rus05, pp. 42–43] Russell considers the relation *earlier*. If you would like to reduce *A's being earlier than B* to nonrelational propositions, then you may attempt this by means of dates. But, then, according to Russell, you would have to say that the date of A is earlier than the date of B, and so you have not escaped from the relation.

Benson Mates [Mat86, pp. 215–218] has a completely different opinion about Leibnizian reductions. He regards it as a mistake to assume that a reduction of a proposition P to a proposition Q requires that these propositions are logically equivalent. He maintains that Leibniz certainly does not say that for every relation R there are predicates F and G such that for any x and y we have xRy iff (Fx and Gy), or more formally:

$$\forall x, y\, [\, xRy \Leftrightarrow (Fx \ \& \ Gy)\,].$$

For some relations this assertion is obviously false. Take, for example, for R the identity relation. Then the assertion would imply that all objects are identical. To see this, let a, b be arbitrary objects. Assume that for any x and y we have the equivalence $x = y$ iff (Fx and Gy). Because $a = a$ and $b = b$, the equivalence implies that Fa and Gb. So, by the same equivalence, we would also have $a = b$.

Mates remarks that Russell does not take Leibniz to demand this kind of reduction, but that Russell would not accept as a Leibnizian reduction of

Theaetetus is taller than Socrates,

the pair of propositions

Theaetetus is 6 feet tall,
Socrates is 5 feet tall.

In Russell's view, we need for an implication of the original proposition also the relational proposition

$$6 > 5.$$

So, according to Russell a reduction of a relation between things to properties of things failed. Mates, however, finds Russell's argument unconvincing. He argues that it is absolutely impossible for the original proposition to be false and for the two propositions of the lengths of Theaetetus and of Socrates to be true. He thinks that Leibniz would say that the original relational proposition is reducible to these two nonrelational propositions.

I can't say if Mates is right that this less demanding form of reduction is indeed Leibniz's own view. That would require a more detailed investigation of Leibniz's original texts. But apart from that, I think Mates's conception of a reduction is interesting, since its universal validity may not be easy to reject on logical grounds. At least I have no counterexample. On the other hand, postulating that such reductions are always possible may be regarded as speculative and without any foundation. It may, for example, be hard to find a satisfactory reduction for the cat's sitting on the mat, especially if there is no absolute space.

1.2.4 Later developments

According to Donald Mertz [Mer96, p. 43], up to the late nineteenth century relations were almost universally held to be reducible to properties. But maybe he should have mentioned Kant as an important exception. According to Rae Langton, Kant denied the reducibility of relations of substances to intrinsic properties of substances. For God could have created a world with different laws of nature but with substances having the same intrinsic properties as in our world. Moreover, she argues that Kant's distinction between phenomena and things in themselves is based on this irreducibility claim. [Lan98]

Relations are now often regarded as a separate subcategory of universals. But there are at present also many alternative systems of categories. Some are parsimonious in the sense that they have no universals and/or no relations. For example, nominalists hold that there are no universals, but *moderate* nominalists admit properties and relations, which they conceive

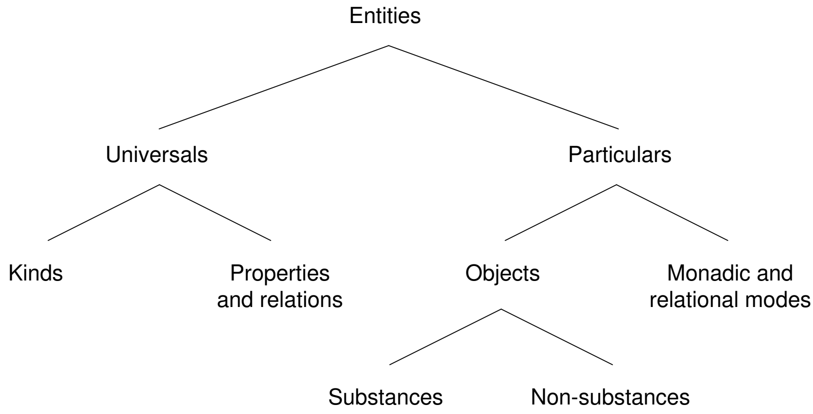


Figure 1.1: Jonathan Lowe's four-category ontology [Low06, p. 39].

of as particulars. These 'particularized qualities' are often called *tropes*. An example of a recent categorial hierarchy with relations and properties as a subcategory of universals is given in Figure 1.1.

Wittgenstein also had a rather unusual ontology. In the *Notebooks*, he counts relations and properties as objects: "Auch Relationen und Eigenschaften etc. sind *Gegenstände*" [Wit84, p. 16.6.15]. However, apparently there is no passage in the *Tractatus* that also says or implies this (cf. [Key63, p. 14, footnote 2]). According to Hans-Johann Glock, nominalist interpreters maintain that the *Tractatus* treats relations and properties as logical forms [Glo04, p. 104].

Of course, much more could be said about the history of relations. But my only goal here was to provide a minimal historical context for the systematic treatment of relations of this thesis. The overview suggests that in the distant past the primary focus was on the ontological status of relations. In my analysis, ontological questions play a less prominent role. Much can be said about the logical structure of relations independently of the question whether relations are really 'out there' or whether they are construction of our own mind.

1.3 Some basic notions

In the coming chapters, I will talk a lot about notions like the degree of a relation, symmetry, states of affairs and relational complexes. In this section, their meaning will be explained.

1.3.1 Internal and external relations

An important distinction that goes back to Hume, is the distinction between *internal* and *external* relations. Internal relations are relations whose existence supervenes upon the existence of its terms. For example, the *greater than* relation for numbers is an internal relation. External relations are simply those that are not internal. So, *being neighbors* is an external relation. External relations are ontologically the more interesting ones, since they might add something to the "fundamental furniture of the universe".

Bradley [Bra83, pp. 21–29] disputes the existence of external relations. According to him the assumption that external relations exist, leads to an infinite regress. Let me briefly sketch his argument. Assume that Romeo loves Juliet. Then this fact cannot come down to the existence of the constituents Romeo, Juliet, and 'loves', because these constituents need to be related in a certain way. Bradley calls this the 'relating relation'. This gives a new fact. Now for this new fact we can apply the same reasoning as above to show that we also need another 'relating relation', and so on. Bradley claims that in this way we get a vicious infinite regress.

Many philosophers have contested Bradley's regress. Frege, for example, avoided the regress by treating relations as objects that by themselves are incomplete ("unvollständig, ergänzungsbedürftig oder ungesättigt" [Fre75, p. 22]), and that can come together with arguments without further mediation. As a consequence, relations do not need additional relations to connect them to their arguments. Another way out may be offered by the antipositionalist view on relations discussed in this thesis. On this view, a relation consists of a network of complete complexes, and Bradley's regress cannot start, since a relation is simply not a constituent of a fact.

Original terms	Ogden translation	P & M translation
Tatsache	fact	fact
Sachlage	state of affairs	situation
Sachverhalt	atomic fact	state of affairs

Figure 1.2: Terminology of the *Tractatus*.

1.3.2 States of affairs

A relation has different *instantiations*. For example, the love relation has as instantiation the state of Paris's loving Helen, and the state of Tristan's loving Isolde. Instantiations of a relation are often conceived of as *states of affairs*. A state of affairs that actually obtains or holds is a *fact*. Examples are:

- Abraham's being older than Sarah.
- Cleopatra and Caesar's being the parents of Caesarion.
- Shanghai's being the largest city of the world and New York's being the largest city of the United States.

When the term 'state of affairs' is used, one often refers to Wittgenstein. Wittgenstein uses the following related terms in the *Tractatus*: *Tatsache*, *Sachlage*, and *Sachverhalt*. 'Tatsache' can be translated as 'fact', but the translation of the other two terms is somewhat problematic. In the Ogden translation of the Tractatus, which was approved by Wittgenstein, Sachlage is translated as 'state of affairs', and Sachverhalt as 'atomic fact'. However, in the also commonly used Pears and McGuinness translation of the Tractatus, 'Sachlage' is translated as 'situation', and Sachverhalt as 'state of affairs' (see Figure 1.2). According to John Nelson [Nel99] almost all commentators on Wittgenstein have accepted the Pears and McGuinness translation as the better one, but Nelson argues that the Ogden translation of *Sachlage* and *Sachverhalt* is the correct one.

As one can read in Appendix III of the *Notebooks 1914-1916* of Wittgenstein [Wit84, p. 130], Russell asked after reading the *Tractatus*: "What is the difference between Tatsache and Sachverhalt?". It is remarkable that Russell asked this question, because the notion of Tatsache and the notion of Sachverhalt play a key role in the book. Anyhow, Wittgenstein wrote back to Russell: "Sachverhalt is, what corresponds to an Elementarsatz if it is true. Tatsache is what corresponds to the logical product of elementary props when this product is true."

For Wittgenstein [Wit98, prop. 1.1] the world is the totality of facts, not of things: "Die Welt ist die Gesamtheit der Tatsachen, nicht der Dinge." David Armstrong [Arm97, p. 1] has a similar thesis. His hypothesis is that "the world, all there is, is a world of states of affairs". Armstrong is very explicit about the modality of states of affairs:

> All states of affairs are contingent. Their constituents, both particulars and universals, are likewise contingent existents. Since the world is a world of states of affairs, there are no other truthmakers for any truths except these contingent states of affairs and the contingently existent constituents. Modal truths, therefore, while not contingent truths, have nothing but these contingent beings as truthmakers. [Arm97, p. 150]

Facts are what make a proposition or statement true or false. For example, the fact that Mona Lisa is smiling makes the proposition that Mona Lisa is smiling true. Russell says in *The Philosophy of Logical Atomism*:

> If I say 'It is raining', what I say is true in a certain condition of weather and is false in other conditions of weather. The condition of weather that makes my statement true (or false as the case may be), is what I should call a 'fact'. [Rus86, p. 163]

For Frege, a fact is just a thought that is true. In *Der Gedanke* he says it as follows:

> "Tatsachen! Tatsachen! Tatsachen!" ruft der Naturforscher aus, wenn er die Notwendigkeit einer sicheren Grundlegung der Wissenschaft einschärfen will. Was ist eine Tatsache? Eine Tatsache ist ein Gedanke, der wahr ist. [Facts! Facts! Facts! cries the scientist if he wants to emphasize the necessity for a firm foundation for science. What is a fact? A fact is a thought that is true.] [Fre03a, p. 57]

Barwise & Perry [BP83, p. 49] use the term 'situations' as a general term for both states of affairs and for events. In their terminology, states of affairs are static situations and events are more dynamic situations.

Another interesting view is that of David Lewis. He considers states of affairs as sets of possible worlds. He illustrates his view in an appealing way:

> The set of all and only those worlds that include a talking donkey as a part, for instance, is the state of affairs *there being a talking donkey*. ... The concrete world—or rather, any concrete world—selects just those sets of worlds that have it as a member. That is what it is for a state of affairs to obtain ... [Lew02, p. 185].

In Figure 1.1, states of affairs did not occur as a category. According to Lowe [Low02, p. 385] the ontological status of facts is controversial and it is questionable whether they should be given a fundamental role in an ontological system. But if we would like to assign them a fundamental role in a system of ontology, then a question is where to put them? Are states of affairs abstract or concrete entities? According to one view, states of affairs are very similar to propositions and therefore abstract. But according to another view, states of affairs like that the cat is on the mat, are concrete entities.

An interesting question is whether states of affairs can be *molecular*. Are there conjunctive, disjunctive, conditional, and negative states of affairs? Armstrong accepts only conjunctive states of affairs [Arm97, p. 35]. Russell is somewhat more reserved in his judgement. He does not think it is plausible that in the actual objective world there is a single disjunctive fact corresponding to a proposition like 'p or q'. Furthermore, he says he is inclined to think there are negative facts, but he does "not say positively that there are, but that there may be" [Rus86, pp. 186–187].

A related question is whether states of affairs can be *atomic*. It is conceivable that states of affairs are composed of molecular states 'all the way down'. I will not discuss this issue here, but I would like to remark that Russell considers facts expressed in "This is white" and "This is left of that" as atomic [Rus86, p. 176].

In all the relational models that will be developed in this thesis, a general notion of *states* will be used. The exact identity of these states is not so important for the argument. If you like, you may substitute for them 'states of affairs', 'situations', 'circumstances', 'occasions', or 'events'.

A related notion is the notion of *complexes*, which can be conceived as structured instantiations of relations. Often the states and complexes of a relation are simply identified. Russell does this in [Rus84, pp. 79–80] and Fine does it in [Fin00, pp. 4–5]. This is also the approach taken in the coming three chapters. It is only in Chapter 5 that a sharp distinction is made between states and complexes.

1.3.3 Unigrade and multigrade relations

The *degree* or *adicity* of a relation can be defined as the least upper bound of the number of arguments (relata) in the different instantiations of the relation. For example, the love relation has degree 2 (when we ignore issues of time), the adjacency relation has degree 3, and the relation of *being surrounded*, viewed as a relation of an object and a finite number of surrounding objects, has degree $\aleph_0$.

Whether there really are (fundamental) relations of a particular adicity may well be an empirical question. One can imagine that all ontologically fundamental relations are dyadic. One can even imagine that all properties are constructed out of polyadic relations.

On the assumption that universals exist, properties are often regarded as monadic universals and relations as polyadic universals. Properties can be seen as a limiting case of a universal. In the context of this thesis, I will not treat properties and relations separately. I will simply conceive of properties as monadic relations.

Relations for which each of their instantiations have the same number of arguments are called *unigrade*; If instantiations may have differing number of arguments then they are called *multigrade*.

Not all relations are unigrade. For example, the relation of *being surrounded by* has a variable number of objects in its different instantiations. Armstrong [Arm97, p. 85] regards such relations as 'second-class' relations, since their different instantiations would differ in their "essential nature". He considers it a truism that a universal is strictly identical in its different instantiations. But, as Fraser MacBride [Mac05, p. 574] notices, Armstrong gives no reason to accept the assumption that universals have their adicities essentially.

1.3.4 Symmetry in relations

Consider the adjacency relation. Then for any objects a and b, if a is adjacent to b, we also have that b is adjacent to a. We therefore call the adjacency relation *symmetrical*. Furthermore, since a's being adjacent to b is the same state as b's being adjacent to a, we say that the relation is *strictly symmetrical*. (See [Fin00, p. 17] for a definition of strict symmetry for n-ary relations.)

It seems obvious that not all relations are symmetric: if I love you, then this does not mean that you also love me. Nevertheless, Cian Dorr [Dor04] argues that all relations are completely symmetric, that is, any permutation of the arguments does not change the state. If predicates do not necessarily correspond to relations, then it is not immediately clear that this claim has to be false. According to Dorr it is even an open question whether any of our current predicates corresponds to relations.

Dorr's argument for the claim that all relations are completely symmetric is rather complex and I will not discuss it here. I only mention three examples Dorr gives of how to interpret theories with non-symmetric predicates in theories with only symmetric predicates.

The first example is simple. Consider the non-symmetric predicate 'is part of' as it occurs in mereology. It is easy to see that this predicate can be defined in terms of the symmetric predicate 'overlaps': x is part of y iff whatever overlaps x overlaps y.

The next example is a tricky one. Let P_1xy be an arbitrary binary predicate. Then we can define a symmetric ternary predicate P_2xyz such that P_1xy iff P_2xyy. It is easy to generalize this to predicates of any adicity. To Dorr, this interpretation looks like cheating, and I agree with him.

The third example given by Dorr is a more complicated and more interesting one. It is originally from Allen Hazen [Haz99]. Hazen proved that (well-founded) set theory can be formulated in terms of the following two symmetric predicates:

$$x \mathsf{E} y \;:\Leftrightarrow\; x \in y \text{ or } y \in x,$$

$$x \mathsf{O} y \;:\Leftrightarrow\; x \text{ and } y \text{ have the same rank},$$

where the *rank* of a set is defined as the least ordinal number greater than the rank of any member of the set.

If there is at least one urelement, then it can be proved that:

$$a \in b \;\Leftrightarrow\; a \,\mathsf{E}\, b \;\&\; \exists x\,[\,x \,\mathsf{O}\, b \;\&\; \forall y\,[\,y \,\mathsf{O}\, a \Rightarrow y \,\mathsf{E}\, x\,]].$$

In words, this says that a is an element of b iff a is E-related to b and there is a set x of the same rank as b that is E-related to everything of the same rank as a. For x one may choose $\mathcal{P}^{\alpha}(A) \cup A$, where A is the set of urelements and α is the rank of b.

If there are no urelements, then this equivalence is not true for $a = \{\varnothing\}$ and $b = \varnothing$. Hazen shows that the assumption of an urelement can be avoided by a more complicated equivalence. As a consequence, for Zermelo-Fraenkel set theory an equivalent axiomatization can be given with as primitives only E and O.

In this thesis I will not assume that all relations or 'fundamental' relations have specific symmetry properties.

1.4 Modern views of relations

In his paper 'Neutral relations', Fine distinguishes three views of relations. In the next chapter, 'Modeling Relations', these views will be discussed in more detail, but here I give a short introduction.

Standard view

The standard view assumes that the arguments of a relation always come in a certain order. For example, suppose a block a is on top of a block b. Then, according to the standard view, we have *two* relations for this state, namely the relation *on top of* and the relation *beneath*. If we look at the given state from the perspective of the relation *on top of*, then the object a comes first, but from the perspective of the converse relation the object b comes first. However, in the state itself there is no order between the objects. This makes the standard view weak.

Positionalist view

In the positionalist view each relation has a set of argument-places or positions. For example for the state of a block a being on top of a block b, we would have a relation *Vertical Placement* with argument-places *Top* and *Bottom*. A great advantage of the positionalist view is that for the given state we have a single *neutral* relation. But also this view is not without problems, one of them is that we are not inclined to accept the existence of argument-places on a fundamental metaphysical level.

Antipositionalist view

In Fine's antipositionalist view there is no order between the arguments, nor are there argument-places. Instead the completion of a relation by certain objects is a *multi-valued* operation. For example, for the antipositionalist vertical placement relation, the completion by blocks a and b consists of *two* states. To distinguish states, the antipositionalist makes use of the notion of *substitution*. In a state like *a's being on top of b*, substituting c for a and d for b yields the state of *c's being on top of d*. In the antipositionalist view a relation consists of a network of states that are interrelated by the notion of substitution.

Interestingly, Russell already questioned in 'The Principles of Mathematics' [Rus72, § 218–219] whether converse relations are really distinct. For an asymmetrical relation $\mathfrak{R}$ and its converse $\breve{\mathfrak{R}}$ he asks: "Are $a\mathfrak{R}b$ and $b\breve{\mathfrak{R}}a$ really different propositions, or do they only differ linguistically?" He then argues in favor of the view that $\mathfrak{R}$ and $\breve{\mathfrak{R}}$ must be distinct. However, later, in 'The Theory of Knowledge' [Rus84, p. 87, note 3], he explicitly declares that he no longer agrees with this view. Furthermore, he says:

> It might perhaps be supposed that every relation has one proper sense, i.e. that it goes essentially *from* one term *to* another. In the case of timeline-relations, it might be thought that it is more proper to go from the earlier to the later term than from the later to the earlier. And in many relations it might be thought that one term is *active* while the other is *passive*; thus "*A* loves *B*" seems more natural than "*B* is loved by *A*". But this is a peculiarity of certain relations, of which others show no trace. Right and left, up and down, greater and less, for example, have obviously no peculiar 'natural' direction. And in the cases where there seemed to be a 'natural' direction, this will be found to have no logical foundation. In a dual complex, there is no essential order as between the terms. The order is

> introduced by the words or symbols used in naming the complex, and does not exist in the complex itself.
> ... [W]e cannot find a vestige of difference between x preceding y and y succeeding x. The two are merely different names for one and the same time-sequent. ... We must therefore explain the sense of a relation without assuming that a relation and its converse are different entities. [Rus84, p. 87]

Russell also offers a form of positionalism as a solution:

> To solve this problem, we require the notion of *position* in a complex with respect to the relating relation. With respect to time-sequence, for example, two terms which have the relation of sequence have recognizably two different positions, in the way that makes us call one of them *before* and the other *after*. [Rus84, p. 88]

He also gives a more general account:

> "Let γ be a complex whose constituents are $x_1, x_2, \ldots, x_n$ and a relating relation $\Re$. Then any of these constituents has a certain relation to the complex. We may omit the consideration of $\Re$, which obviously has a peculiar position. The relations of $x_1, x_2, \ldots, x_n$ to γ are their 'positions' in the complex; let us call them $C_1, C_2, \ldots, C_n$. [Rus84, p. 146]

Russell does not say whether different relations may have positions in common, or whether different constituents of a single complex may occupy the same position. Fine does not see any reasonable basis for identifying positions of different relations [Fin00, p. 12], nor does he accept that in a complex different relata may occupy the same position [Fin00, p. 17].

A critical discussion of the antipositional view is given by MacBride [Mac07]. In this paper he also provides more historical information on the standard and the positionalist view. In Chapter 2 of this thesis, objections against positionalism are discussed, and a comparison is made with antipositionalism. As we will see, the analysis provides support for antipositionalism, but it also provides a justification for using positional representations for relations.

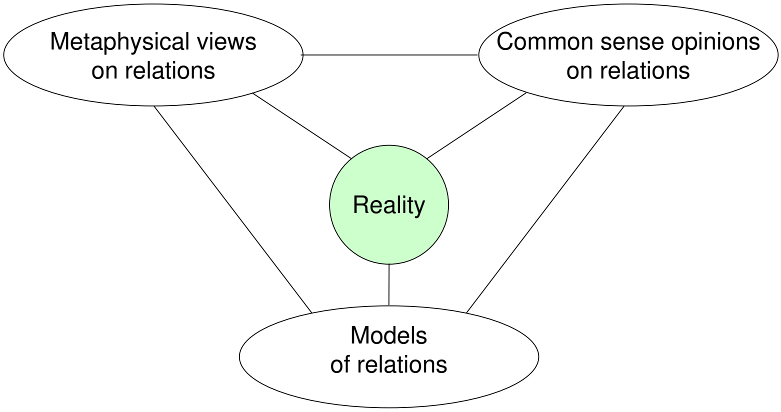


Figure 1.3: Methodological approach.

1.5 Methodological approach

The ultimate goal of this investigation is to determine the essence of the structure of relations. A number of subgoals can be formulated in the form of questions: (1) What is exactly the correspondence between the different views on relations? (2) How do the views relate to common sense opinions of relations? (3) How can the views be elaborated in the form of (mathematical-logical) theories of relations? (4) What do the theories tell us about the constituents of a relation and what makes a relation a genuine unity? (5) What is the most promising theory of relations?

To answer these questions, I will make extensive use of mathematical models for relations. The main reason for using such models is that they make a profound analysis of the different views possible. I do not start with models for individual relations, but I define types of models for classes of relations as seen from different views on relations, in particular, from the views as distinguished by Fine. Subsequently, these models are compared and analyzed, and the results are given a metaphysical interpretation. The underlying idea is that iterating these steps will lead to improved metaphysical views of relations and improved models. See Figure 1.3.

In general, models are powerful instruments. They do not necessarily mirror phenomena in reality, but they can also offer a partial representation that abstracts from the real nature. There are many types of models: heuristic models, explanatory models, idealized models, analogue models, mathematical models, didactic models, etc. Heuristic models, which are used in searching for relationships between various phenomena, are relatively innocuous, but other kind of models, like explanatory models, play a central role in scientific theorizing. In practice, models may also be of more than one type.

In modeling reality we try to establish an analogy between the model and certain aspects of reality. For this purpose various things can be used. For example, we can use tangible objects like spheres and rods to create chemistry models, or we can cut out wooden hearts as models for the love relation.

An important aspect of models is their faithfulness. In general, not every feature of a model has a representational function. For example, for chemistry the material of a model of spheres and rods is irrelevant. Similarly, the models I define for relations have irrelevant properties. They are defined as tuples, but the order of the components does not mirror anything of the underlying relations. Furthermore, because in my definitions I abstract from all kinds of properties relations might have, the defined types of models may contain models that do not correspond to any 'real' relation. We have to be very aware of this fact in translating properties of the models to the metaphysics of relations.

Limitations of a model should not automatically be interpreted as a weakness. On the contrary, deliberately abstracting from certain aspects helps to highlight aspects we like to concentrate on. It may, for example, be useful to consider also infinite models for things that are known to be finite.

The main reason for using mathematical models is their precision and their great flexibility to formulate all kinds of subtleties. The price, however, is that they are often harder to appreciate, because they often lack a simple visual form. In such cases it might be helpful to create in addition tangible models for explaining these abstract models.

An alternative for using models to learn more about the structure of relations, would be the use of an axiomatization of relations. I think this would certainly be a useful complementary approach in this case. However, in this thesis, I limit myself to models. The choice for models seems more straightforward, because some of the motivating questions are about representations for relations.

1.6 Introducing models for relations

The models developed in this thesis are all formulated in set theoretical terms. They all consist of tuples of the form $\langle S, O, \ldots \rangle$, where S is a set of states and O is a set of objects. But, since the states and objects of many relations are obviously not sets of Zermelo-Fraenkel set theory (without atoms), we must be careful in using principles like the axiom of choice.

The models for the standard view and the positionalist view are relatively straightforward, but to appreciate the models used in this thesis for the antipositionalist view some additional explanations may be helpful.

1.6.1 Models for the standard view

What I said in Section 1.4 about the standard view was that arguments of a relation always come in a certain order, but nothing was said about how many arguments a relation may have. In the models for this view I allow an infinite number of arguments. Furthermore, I assume that the order of the arguments is well-ordered. The resulting models have the following form:

$$\langle S, O, \alpha, \Gamma, H \rangle,$$

with S a nonempty set of states, O a nonempty set of objects, α an ordinal number, Γ a function from O^α to S, and $H \subseteq S$ a set representing the states that hold.

Assuming that the order of the arguments is well-ordered is perhaps a somewhat arbitrary choice. An alternative would be the weaker assumption that the arguments are totally (or partially) ordered, but it should be noticed that then the orders may have non-trivial automorphisms, and that as a result the arguments may have only a relative order. However, for now the issue is not really that important since no result in the thesis depends on my choice for the arguments being well-ordered.

1.6.2 Models for the positionalist view

The models for the positionalist view are very similar to the models for the standard view. The main difference is that the ordinal α is replaced by a set of positions P:

$$\langle S, O, P, \Gamma, H \rangle,$$

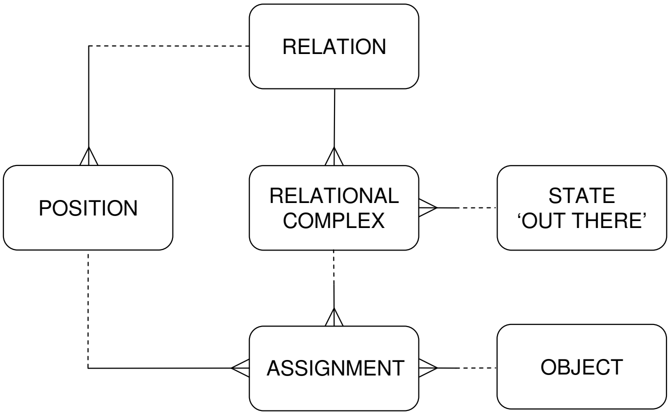


Figure 1.4: Entity-relationship diagram for a positional theory.

with S, O, H as before, P a set of positions, and Γ a function from O^P to S.

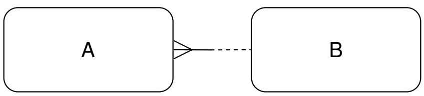
The set of positions may be finite or infinite. We also allow that the set of positions is empty. Then $O^P = \{\varnothing\}$.

The models say nothing about the question whether or not objects *occupy* positions in the states. In Fine's conception of the positionalist view, however, such an occupation was assumed.

It will not be excluded that a position belongs to more than one relation, but no specific meaning will be attached to a shared position.

For the positionalist view a graphical representation can be given in the form of an *entity-relationship diagram* (see Figure 1.4).[2]

In such diagrams,

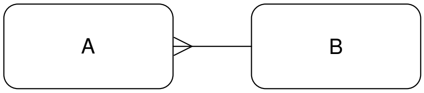


should be read as:

> Each instance of A relates to one instance of B, and
> each instance of B may relate to one or more instances of A.

[2]In this entity-relationship diagram I assumed that each position belongs to only one relation.

And



should be read as:

> Each instance of A relates to one instance of B, and
> each instance of B relates to one or more instances of A.

Note that in the definition of positional models the notion of relational complexes does not occur, but relational complexes can be identified with the elements of O^P.

1.6.3 Models for the antipositionalist view

Here, I develop in steps four types of models that reflect ideas of the antipositionalist view. Of these models, only model 4 will be used in the next chapters.

Model 1
As I said in Section 1.4, on the antipositionalist view the completion of a relation is a *multi-valued* operation. For example, the completion of the love relation by Simone and Jean-Paul contains both the state of Simone's loving Jean-Paul as well as the state of Jean-Paul's loving Simone. Furthermore, since we have no positions, there is no reason to assume that we always have a fixed number of arguments. These ideas lead us to define Model 1 as:

$$\langle S, O, A, \Upsilon, H \rangle,$$

with S, O, H as before, $A \subseteq \mathcal{P}(O)$ a set of argument sets, and $\Upsilon : A \to \mathcal{P}(S)$ a completion operation.

In this model, A is a set of subsets of O, but we could refine this to a set of multisets with elements in O. Then each element can stand for an occurrence of an object in a state. Such a multiset approach is also suggested by Fine [Fin00, p. 20].

Model 2
An obvious shortcoming of Model 1 is that it has no mechanism to distinguish between different completions of a set of constituents. We like to

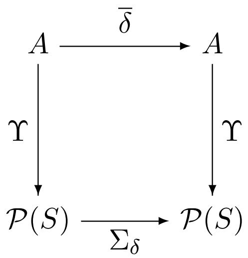
express in our models that objects of a state can be substituted by other objects, and that this results in another state. For example, we want to be able to express that in a state where Simone loves Jean-Paul, we can replace Simone and Jean-Paul by Hannah and Martin, respectively. By adding a substitution operation we get Model 2:

$$\langle S, O, A, \Upsilon, \Sigma, H\rangle,$$

with S, O, A, Υ, H as in Model 1, and with $\Sigma : S \times O^O \to S$ such that:

1. Σ is a *monoidal action* on states, that is,
   (a) $\Sigma(s, \mathsf{id}_O) = s$,
   (b) $\Sigma(s, \delta' \circ \delta) = \Sigma\left(\Sigma(s, \delta), \delta'\right)$,
2. $\overline{\Sigma}(\Upsilon(\alpha), \delta) = \Upsilon(\overline{\delta}(\alpha))$. [3]

The first condition agrees with how we understand substitution. The intended interpretation of $\Sigma(s, \delta)$ is the state we get when we simultaneously substitute for each object d in the state s the object $\delta(d)$. The second condition states that for any $\delta : O \to O$ the following diagram commutes:



with Σ_δ defined by $\Sigma_\delta(S') := \overline{\Sigma}(S', \delta)$.

MODEL 3

Model 2 could be criticized for not being very elegant. The interdependencies between Σ and Υ make the model rather complicated. We can simplify things if, instead of starting with a function Υ for the completion operation, we start with a function $\Delta : S \to \mathcal{P}(O)$ that gives for each state its objects. We may do so if each s in S belongs to exactly one set $\Upsilon(\alpha)$. This results in Model 3:

$$\langle S, O, \Delta, \Sigma, H\rangle,$$

[3] We denote lifted functions by overlining the function name. For example, $\overline{\Sigma}$ denotes the lifted function from $\mathcal{P}(S) \times O^O$ to $\mathcal{P}(S)$ defined by $\overline{\Sigma}(S', \delta) := \{\Sigma(s, \delta) \mid s \in S'\}$.

with S, O, Σ, H as above, Σ a monoidal action on states, and $\Delta : S \to \mathcal{P}(O)$ such that:

1. $\delta =_{\Delta(s)} \delta' \Rightarrow \Sigma(s, \delta) = \Sigma(s, \delta')$,
2. $\forall s \in S\, \forall d \in \Delta(s)\, \exists \delta, \delta'\, [\, \delta =_{\Delta(s)-\{d\}} \delta' \;\&\; \Sigma(s, \delta) \neq \Sigma(s, \delta')\,]$.

Let us look at the conditions. Condition 1 says that the result of a substitution is determined by what is substituted for the constituents of a state, and Condition 2 says that for each constituent d of a state there is a substitution for which it makes a difference for the result which object is substituted for d. So, these conditions seem to agree with our idea of constituents of a state.

Model 4
Model 3 also looks a bit complicated. But we can simplify things further. It is not difficult to see that for a given function Σ there can be at most one function Δ such that the Conditions 1 and 2 of Model 3 are fulfilled. For suppose Δ and Δ' both satisfy these conditions. Let δ, δ' be such that $\delta =_{\Delta(s) \cap \Delta'(s)} \delta'$. Define δ'' by

$$\delta''(a) = \begin{cases} \delta(a) & \text{if } a \in \Delta(s) - \Delta'(s), \\ \delta(a) = \delta'(a) & \text{if } a \in \Delta(s) \cap \Delta'(s), \\ \delta'(a) & \text{if } a \in \Delta'(s) - \Delta(s). \end{cases}$$

Then $\delta'' =_{\Delta(s)} \delta$ and $\delta'' =_{\Delta'(s)} \delta$. So, $\Sigma(s, \delta) = \Sigma(s, \delta'') = \Sigma(s, \delta')$. Thus, it follows that $\Delta = \Delta'$.

From this observation it follows that we could alternatively define a model $\langle S, O, \Sigma, H \rangle$ with the same strength as Model 3 by demanding that Σ is a monoidal action on states, and by additionally demanding that there exists a function Δ that fulfills Conditions 1 and 2 of Model 3. However, models that do not fulfill these additional conditions might also be interesting for further study. Therefore we define Model 4 as:

$$\langle S, O, \Sigma, H \rangle,$$

with S, O, Σ, H as above, and Σ a monoidal action on states.

This last model is the *substitution model* as defined in Chapter 2 of this thesis. An attractive feature of the model is its simplicity. It has, however,

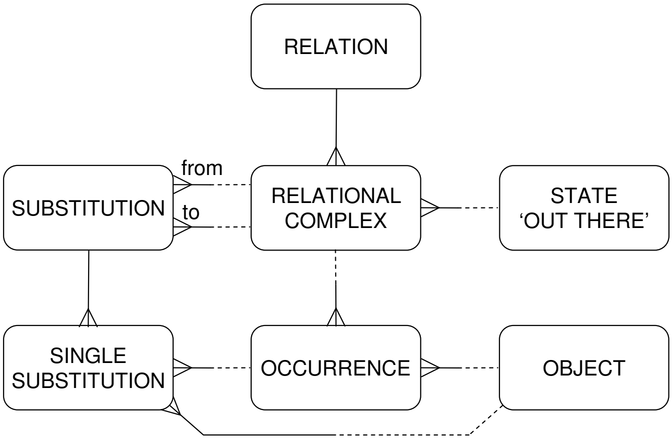


Figure 1.5: ER diagram for an antipositional theory.

also a serious limitation since substitution does not work on individual *occurrences* of objects. In Chapter 3 of this thesis, models with a more refined substitution mechanism are defined.

An entity-relationship (ER) diagram for an antipositional theory with occurrences of objects is given in Figure 1.5. Chapter 4 contains an alternative entity-relationship diagram for an antipositional theory of relations without any reference to states 'out there'. Note that also in the definition of the antipositional models given above the notion of relational complexes does not occur. The reason is that for these models I assume that for a given relation each state has at most one corresponding complex. For the theory developed in Chapter 4, this assumption will be dropped.

1.7 Outline of the thesis

The body of this thesis consists of four papers of which the first three have been published in a refereed journal and the fourth in a preprint series of Utrecht University. The fourth paper has also just been submitted to a refereed journal. The content of the papers in the thesis is the same as that of the original publications. Only the references have been moved to the bibliography at the end of the thesis.

In a sense, the papers of the thesis reflect the development of my thinking about relations. As an introduction, let me say some informal things about their realization.

At the start of the PhD project I had a global plan, but I did not foresee the specific topics of each paper. Rather, the results of the issues addressed in each paper suggested in a natural way the topics for the next papers. For example, initially, I had no intention to write about relational complexes, which is the topic of the fourth paper. It was only in writing the third paper about occurrences that I began to realize that distinguishing relational states from relational complexes might be a good idea. In my opinion this approach resulted in a thesis of which the different parts clearly form a coherent whole.

In the first paper of this thesis, 'Modeling relations', I define and compare models for the standard view, the positionalist view and the antipositionalist view of relations. My burning question was the following. If the positionalist view is wrong, what does this mean for our way of thinking? Are we still justified to use positional representations of relations? It was enlightening and stimulating to discover that a logico-mathematical approach could be so useful for solving metaphysical problems.

In the second paper, 'The identity of argument-places', I investigate—starting from an antipositionalist view—for what kind of relations a *neutral* (re)construction of argument-places is possible. This paper is the result of a struggle. Being convinced that a neutral reconstruction should be possible in ordinary set theory, I spent a lot of time searching for a solution in this direction. At one point, however, I got a flash of insight. I came to see that no such construction could be made, but the next problem was how to prove this rigorously. I succeeded by introducing a formal notion of *neutrality*—a notion which may have wider applications.

In the third paper, 'Modeling occurrences of objects in relations', I refine the models of the first paper for the antipositionalist view to obtain a better understanding of occurrences. Initially, I expected that the models would be straightforward generalizations of the antipositional models developed in the first paper, but there were some surprising complications. Almost any model turned out to have proper refinements—provided that arbitrary large measurable cardinals exist. Furthermore, certain desirable uniqueness properties for maximal refinements were quite hard to prove.

In the fourth paper, 'Relational complexes', I develop what I call a *polymorphic theory of relations.* In my previous analysis of relations some irritating problems had begun to surface, in particular, models of some ordinary relations had a quite complicated structure. It was a liberating idea to allow for a given relation a single state to have more than one corresponding complex. This idea resulted is a powerful theory of relations in which all kinds of operations can be defined in a crystal-clear way—at least that is what I argue in this last paper.

Each paper is more or less self-contained. Nevertheless, the first paper will probably be helpful for a full appreciation of the other three papers. Similarly, the third paper, and in particular its non-technical parts, provides some background for the fourth paper. Personally, I would read this thesis by initially skipping the proofs. Reading the introductions and conclusions of each paper first may be a good strategy.

In the epilogue of the thesis, I evaluate the main results, discuss the reliability of the results, and make some suggestions for future investigations. I will not only look at related opportunities in philosophy, but also at how the theoretical results of this thesis may be used for challenging research areas in psychology and artificial intelligence.

Chapter 2

Modeling relations*

Abstract

In the ordinary way of representing relations, the order of the relata plays a structural role, but in the states themselves such an order often does not seem to be intrinsically present. An alternative way to represent relations makes use of positions for the arguments. This is no problem for the love relation, but for relations like the adjacency relation and cyclic relations, different assignments of objects to the positions can give exactly the same states. This is a puzzling situation. The question is what is the internal structure of relations? Is the use of positions still justified, and if so, what is their ontological status? In this paper mathematical models for relations are developed that provide more insight into the structure of relations "out there" in the real world.

2.1 Introduction

When I say "Koos loves Marietje" do I signify the same state of affairs as I would if I said "Marietje is loved by Koos"? If you accept that states of affairs are "out there" in reality, you will probably say "yes". But then we have two ways to describe a single state of affairs. Which one is better? If we can't say, then we might ask if there is a way to express the state of affairs in a neutral way.

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This is the problem Kit Fine takes up in his paper "Neutral Relations" [Fin00]. Fine shows the inadequacy of what he calls the *standard view* on relations, according to which all relations hold of objects in a given order. In search of a better alternative, he first proposes a *positionalist view* on relations in which each relation comes with a set of positions. For example, for the amatory relation, the positions would be *Lover* and *Beloved*. Fine finds the positionalist view very natural and plausible, but he also regards it as problematic. One of Fine's objections is that on this view no relation can be strictly symmetric. A second alternative proposed by Fine is his *antipositionalist view*, which he claims combines the virtues of the standard and the positionalist view.

If Fine is right that the positionalist view is wrong, then this would be very disturbing, because it seems so natural and fundamental for our way of thinking. Therefore, I have three basic questions: (1) Can the positionalist view in some way be saved from the objections raised by Fine? (2) For what kind of relations are positional representations still adequate, and for what kind are they not? (3) What is the most natural way to look at relations?

In this paper I develop mathematical models for different views on relations that are not only of interest in themselves, but that also increase our insight into the adequacy of different conceptions of relations. In particular, I define *directional models* that agree with the standard view, *positional models* that agree with the positionalist view, and an elegant type of models, the *substitution models*, inspired by Fine's antipositionalist view. I prove that a natural subclass of the substitution models corresponds in a well-defined way to a subclass of the positional models. As a consequence, without any commitment to an ontology of positions, positional representations of relations are justified for a large class of relations, including all kinds of symmetrical relations.

The structure of the paper is as follows. Section 2.2 briefly discusses the views on relations as distinguished by Fine. Section 2.3 to 2.5 form the core of this paper. In Section 2.3 types of mathematical models for relations are developed, in Section 2.4 the relationship between the types is studied, and in Section 2.5 we take a closer look at the positional structure of models. In Section 2.6 we focus our attention on metaphysical aspects of the structure of relations. In Section 3.10 I finish with a recapitulation of the main argument, conclusions and suggestions for further inquiry.

A final note about the scope of this paper. The relations considered are relations "out there" in the real world. Occasionally I will use the term *"real"* *relations* to stress this point. If you like, you may regard the mathematical relations as a subclass of the "real" relations. However, in my arguments I will freely use what are called "relations" in set theory as tools.

2.2 Views on relations

In his paper "Neutral Relations", Kit Fine presents three views on relations. He calls them the *standard view*, the *positionalist view* and the *antipositionalist view*. I will briefly describe here the views as presented by Fine, and criticize his objections against positionalism. In the description I will stay close to Fine's original formulations.

2.2.1 Standard view

According to the standard view, objects in a relation always come in a given order. For example, we may say that the relation *loves* holds of a and b (in that order) just in case a loves b. A consequence of this view is that each binary relation has a *converse* [Fin00, p. 2]. For example, the converse of the relation *on top of* is the relation *beneath*. The consequence that each binary relation has a converse is a shortcoming of this view for the following reason.

Suppose a block a is on top of a another block b. Then we have a state of affairs s that may be described as the state of a's being on top of b, but that may also be described as the state of b's being beneath a. If s is a genuine relational complex, i.e. a state consisting of a relation in combination with its relata, then there must be a *single* relation that can be correctly said to figure in the complex in combination with the given relata. We have no reason to choose either *on top of* or *beneath* for this relation. Whatever this relation is, it cannot have a converse. Therefore the standard view is objectionable from a metaphysical perspective. [Fin00, pp. 3-4]

If we consider relations as belonging to reality rather than to our representation of it, then the order of the arguments is to be attributed to our representations, not to the relation itself. It is an artefact of our language that it leads us to suppose that relations themselves must apply to arguments in a given order. [Fin00, p. 6]

There is also a linguistic argument against the standard view. In a graphic language, the love predicate could be a heart-shaped body with a red and a black side. On the red side we write the name of the lover and on the black side the beloved one. The relation signified by the heart does not fit in with the standard view, since the sides of the heart are not ordered in a relevant sense. [Fin00, pp. 6-7]

2.2.2 Positionalist view

The positionalist view assumes that each relation has a fixed number of positions or argument-places, which are *specific entities* that belong to the relation. For example, the love relation has the positions *Lover* and *Beloved*. [Fin00, p. 10]

There is no intrinsic order to the positions. This makes the positionalist view a *neutral* or *unbiased* conception of relations. This does however not imply that on this view every relation is neutral in the sense that there is no meaningful notion of converse for it. For we may get biased relations like *loves* by imposing an order on positions from the "outside". [Fin00, p. 11]

Fine has two objections to the positionalist view. The first objection is an ontological one. The positionalist is obliged to reify argument-places, and may have to include them among the "fundamental furniture of the universe". But according to Fine we are strongly inclined to think that there should be an account of relational facts without any reference to argument-places. The second objection concerns *strictly symmetric* relations, i.e. relations for which different assignments of the objects to the positions give identical states. For example, for the adjacency relation the state of a's being adjacent to b is the same as the state of b's being adjacent to a. But if a and b occupy distinct positions within a state, then switching the positions of a and b cannot yield the same state. A proposed way out to let objects in symmetric relations occupy the same position does not work; for cyclic relations the positions occupied must be distinct to distinguish certain states. Therefore, on the positionalist view no strictly symmetric relations are possible. [Fin00, pp. 16-17]

2.2.3 Antipositionalist view

On the positionalist view, the *completion* of a relation is the state we get from assigning objects to the argument-places of the relation. In this case it is a single-valued operation. However, the antipositionalist has no argument-places to which objects can be assigned. He takes completion as a *multi-valued* operation, yielding a plurality of states for the different ways in which the relation might be completed by the objects. For example, the completion of the love relation by Don José and Carmen contains the state of Don José's loving Carmen and the state of Carmen's loving Don José. [Fin00, p. 19]

To distinguish different states, the antipositionalist makes use of the idea that a state can be a completion of a relation *in the same manner* as another state. We say that a state s is a completion of a given relation $\Re$ by constituents $a_1, a_2, \ldots, a_m$ *in the same manner* as a state t is a completion of $\Re$ by constituents $b_1, b_2, \ldots, b_m$, if s can be obtained by simultaneously substituting $a_1, a_2, \ldots, a_m$ for $b_1, b_2, \ldots, b_m$ in t (and vice versa). We assume that if the a_i's are the same, then the corresponding b_i's are also the same (and vice versa). [Fin00, p. 20]

So, for example, the state of Anthony's loving Cleopatra is a completion of the love relation by Anthony and Cleopatra in the same manner as the state of Abelard's loving Eloise is a completion of the love relation by Abelard and Eloise.

The antipositionalist view has certain advantages over the positionalist view: (1) It does not have the ontological problem of the positionalist view, for it has no argument-places, (2) it does not have problems with strictly symmetric relations, and (3) it can account for variably polyadic relations. [Fin00, pp. 21-22]

The notion of *co-mannered completion*, i.e. the notion of completion *in the same manner*, should not be taken as a primitive. We defined the relation *in the same manner* in terms of substitution. Thus, we should see the notion of *co-mannered completion* as a special case of the more general notion of substitution. [Fin00, pp. 25-28]

The antipositionalist can reconstruct the notion of position in terms of *co-positionality*. We say that a in s is *co-positional* to b in t if s results from t by a substitution in which b goes into a (and vice versa) [Fin00, p. 29]. If the antipositionalist accepts the existence of strictly symmetric relations, he cannot satisfactorily reconstruct the positionalist's account of position, because if constituents occupy different positions, then interchanging constituents will give a different state [Fin00, p. 32].

On the standard conception, a relation applies to its relata in an absolute manner; on the positionalist conception, a relation applies to its relata relative to the positions of the relation, but with an absolute notion of position; on the antipositionalist conception, we have the relative notion of co-positionality. The antipositionalist has stripped the concept of a relation to its core. [Fin00, p. 32]

2.2.4 Criticizing Fine's objections to positionalism

The first objection Fine raised against the positionalist view is that a "full-blooded commitment to an ontology of positions" does not match our inclination to think that a position-free account of relations is possible. For me it is not clear whether we would *a priori* be strongly inclined to think that such an account is possible. But in any event, Fine comes with an elegant alternative account, the antipositionalist view.

According to Fine the antipositionalist can reconstruct positions, but I find his solution not very satisfactory. Fine claims [Fin00, p. 29]: "Positions can then be taken to be the abstracts of constituents in relational complexes with respect to the relation *co-positionality*." The way I understand this is that we would get for certain states of cyclic relations just one position. Another peculiarity of Fine's reconstruction of positions is that certain relations can have more positions than the maximum number of arguments an instance can have. Consider the love relation and the state s of Narcissus's loving Narcissus. By definition of co-positionality, Narcissus in s is only co-positional to other objects in states where they love themselves. This would give us three positions for the love relation instead of two. So, positions reconstructed in this way are in general not very similar to the positions of the positionalist. I find it somewhat confusing to call the reconstructed entities *positions*. Perhaps it would be better to call them *roles*.

Fine's second objection against the positionalist view is that, for strictly symmetric relations it is contradictory to assume both (1) that distinct objects occupy different positions, and (2) that position is preserved under substitution [Fin00, pp. 31-32]. However, in my view, this does *not* mean that positional representations are dubious. I even think that a positionalist might concede the argument of Fine, but respond that the positions of a relation only have a *mediating* role. Assigning objects to positions *yields* states, but I see no reason to assume that the objects *occupy* these positions within the states.

Finally, Fine argues that if we accept strictly symmetric relations, then the antipositionalist cannot give a satisfactory reconstruction of the positionalist's account of positions [Fin00, p. 32, note 22]. In his argument for this claim, Fine uses the supposition that according to a positionalist objects must occupy positions within the states. But if you drop this supposition – and as I argued, we have good reasons for doing so – then a satisfactory reconstruction of "normal" positions, as seen by the positionalist, is possible for a large class of relations including all kind of symmetric relations, as I will show in Section 2.4.

In response to this criticism, Fine has said [private communication, October 30, 2005] that in his paper "Neutral Relations" he was, for simplicity, ignoring the fact that substitution is properly done on occurrences, as is made clear in [Fin89]. If we use the notion of what it is for one occurrence of an individual to be co-positional with another occurrence, then we can avoid the difficulty over there being too many positions. If positions are something to be occupied, then we cannot properly distinguish different positions within a cyclic relation.

Further, Fine has granted that he has no objection to a "thin" notion of position (one which is not occupied) as such. But he does not think it is basic; exemplification or completion through thin positions must be understood in other terms. He remarked that he thinks his paper "Neutral Relations" was not clear on how both of his objections to positionalism are to positionalism as a *basic* account of what relations are.

In conclusion, I would say that these considerations give hope for a non-basic form of positionalism, but they still leave us with questions with respect to the adequacy of positional representations.

2.3 Framing relations

We will define different types of *frames* to model the *logical space* of relations. The frames will all be of the form $\langle S, O, ... \rangle$, where S is a nonempty set of states, and O a nonempty set of objects. We may extend the frames to *models* of the form $\langle S, O, ..., H \rangle$, where H is a subset of S representing the states that obtain.

2.3.1 Directional frames

We start with frames in which the order of the relata is relevant. Predicates like '__ loves __' can perfectly be expressed in these frames.

Definition 2.3.1. A *directional frame* is a quadruple $\mathcal{F} = \langle S, O, \alpha, \Gamma \rangle$, where S is a nonempty set of states, O is a nonempty set of objects, α is an ordinal number, and Γ is a function from O^α to S.

We call the cardinality of α the *degree of* the frame. We denote it as $\mathsf{degree}_\mathcal{F}$.

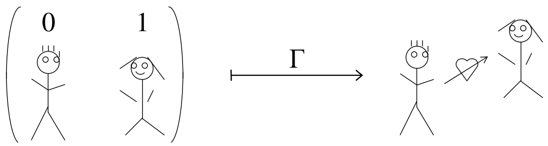
We call $\mathcal{F}$ a *full frame* if $\mathsf{im}\,\Gamma = S$.

Γ is the function that sends the sequence of objects $f \in O^\alpha$ to the state that is the completion of the modeled relation with f.

Note that we allow the degree of a frame to be infinite. We do not want to exclude upfront that some "real" relations might have an infinite number of relata per state. But also if all "real" relations are of finite degree, then it may still be useful to consider in our analysis frames of infinite degree, because it may highlight how certain properties depend on the degree.

Note further that we do not make use of typed domains for the objects. So, the models are not accurate for a relation like *drinks*, because "Mo drinks beer" corresponds in a natural way to a state, but "beer drinks Mo" does not. However, such refinements can easily be incorporated into the models.

Example 2.3.2. For the relation *loves* we can make a directional frame $\mathcal{F} = \langle S, O, \alpha, \Gamma \rangle$ with S states of loving, O people, $\alpha = \{0, 1\}$, and Γ depicted as:



⊣

Because the arguments in directional frames are ordered, binary directional frames have converses. More generally, all directional frames have *permutations*:

Definition 2.3.3. A directional frame $\mathcal{F} = \langle S, O, \alpha, \Gamma \rangle$ is a *permutation of* $\mathcal{F}' = \langle S', O', \alpha', \Gamma' \rangle$ if $S = S'$, $O = O'$, $\alpha = \alpha'$, and there is a bijection $\pi : \alpha \to \alpha$ such that for each $f \in O^\alpha$, $\Gamma(f) = \Gamma'(f \circ \pi)$.

We say that $\mathcal{F}$ has *strict symmetry* if there is a bijection $\pi : \alpha \to \alpha$ with $\pi \neq \mathsf{id}_\alpha$ such that for each $f \in O^\alpha$, $\Gamma(f) = \Gamma(f \circ \pi)$.

In our definition of directional frames we have chosen the arguments to be well-ordered. However, it is not obvious that this is the most appropriate choice for the infinite case. Perhaps we should also allow other linear orderings.

In Section 2.2.1 I mentioned Fine's objection against the standard view that, as a consequence of that view, each binary relation has a converse [Fin00, p. 2]. This objection may also be expressed as a shortcoming of directional frames. If there is a single underlying relation, then we would like to give a neutral representation for it. The directional frames obviously fail in this respect. We could of course take the class of permutations of a directional frame as a neutral representation, but, as we will show in this paper, there are simpler alternatives.

Remark 2.3.4. Fine thinks that there are both neutral and biased relations [Fin00, p. 1]. Therefore he might find for a certain class of relations directional models adequate. A different view on relations has been proposed by Timothy Williamson [Wil85]. For Williamson all relations are neutral, and he will probably see no reason (apart from conventional ones) to prefer for any relation any specific directional model as a representation.

2.3.2 Positional frames

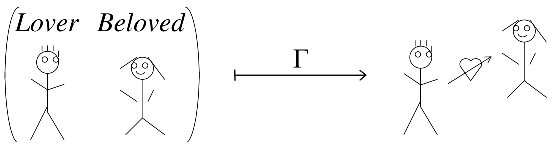
Instead of letting the order of the objects play a constitutive role for relations, we can assign objects to orderless *positions*:

Definition 2.3.5. A *positional frame* is a quadruple $\mathcal{F} = \langle S, O, P, \Gamma \rangle$, where S is a nonempty set of states, O is a nonempty set of objects, P is a set of positions, and Γ is a function from O^P to S.

We call the cardinality of P the *degree of* the frame. We denote it as $\mathsf{degree}_\mathcal{F}$.

We call $\mathcal{F}$ a *full frame* if $\mathsf{im}\,\Gamma = S$.

Example 2.3.6. For the love relation we can make a positional frame $\mathcal{F} = \langle S, O, P, \Gamma\rangle$ with S states of loving, O people, $P = \{\mathit{Lover}, \mathit{Beloved}\}$, and Γ depicted as:



⊣

Analogous to permutations of directional frames we can define *positional variants* of positional frames:

Definition 2.3.7. A positional frame $\mathcal{F} = \langle S, O, P, \Gamma\rangle$ is a *positional variant of* $\mathcal{F}' = \langle S', O', P', \Gamma'\rangle$ if $S = S'$, $O = O'$, and there is a bijection $\pi : P' \to P$ such that for each $f \in O^P$, $\Gamma(f) = \Gamma'(f \circ \pi)$.

For directional frames we discussed the problem with converses. This problem does not occur for positional frames, since there is no intrinsic order in the positions. In Section 2.2.2 I mentioned two objections of Fine against the positionalist view [Fin00, p. 16] whose impact for positional frames needs to be considered. The first objection is an ontological one, namely that positions do not belong to the "fundamental furniture of the universe". The second objection concerns strictly symmetric relations. Different assignments to positions may give identical completions.

The first objection has no force against positional frames as models for the logical space of relations if we do not have to take the identity of positions in the frames as basic. By presenting alternative frames and by analyzing their relationship with positional frames we will show that positions may be defined in other terms. Also the second objection I do not consider as a disqualification of positional frames. What Fine convincingly showed is that it would be wrong to assume that the positions correspond one-to-one to some kind of entities in the complexes of relations and that these entities are occupied by the constituents of the relation. But in the positional frames the positions are not part of the states nor is it said that objects occupy positions. The positions in the positional frames only have a kind of mediating function. Assigning objects to them *yields* states. In this paper I will defend that positional frames are most appropriate for representing a large class of relations.

It is possible that certain positions play individually absolutely no role for the states assigned by the function Γ. We call such positions *dummy positions*:

Definition 2.3.8. Let $\mathcal{F} = \langle S, O, P, \Gamma\rangle$ be a positional frame. We call $p \in P$ a *dummy position* if for each $f, g \in O^P$,[1]

$$f =_{P-\{p\}} g \Rightarrow \Gamma(f) = \Gamma(g).$$

Perhaps surprisingly, dummy positions cannot always be dropped all simultaneously as the next example shows.

Example 2.3.9. Let $\mathcal{F} = \langle S, O, P, \Gamma\rangle$ be a positional frame with O an infinite set of objects, S the set of subsets of O modulo a finite difference, i.e.

$$S = \{\widehat{A} \mid A \subseteq O\}$$

with $\widehat{A} = \{A' \subseteq O \mid A \mathbin{\Delta} A' \text{ is finite}\}$, P an infinite set of positions, and Γ defined by $\Gamma(f) = \widehat{\mathsf{im}\, f}$.[2] Then each $p \in P$ is a dummy position, but not for every f and g, $\Gamma(f) = \Gamma(g)$. $\dashv$

To handle *variadic* relations, i.e. relations with a variable number of relata, we could define a *variadic positional frame* as a quadruple $\langle S, O, P, \Gamma\rangle$, where Γ a function from V to S with V a set of partial functions from P to O. In this paper we will not discuss this type of frames.

2.3.3 Substitution frames

In this section, we present a type of frames, the *substitution frames*, that agrees with the antipositionalist view.[3]

Definition 2.3.10. A *substitution frame* is a triple $\mathcal{F} = \langle S, O, \Sigma\rangle$, where S is a nonempty set of states, O is a nonempty set of objects, and Σ is a function from $S \times O^O$ to S such that

[1] We say that $f =_X g$ if $f{\restriction}X = g{\restriction}X$, i.e. f restricted to X is equal to g restricted to X.

[2] $A \mathbin{\Delta} A' = (A - A') \cup (A' - A)$, the symmetric difference of A and A'.

[3] For a better appreciation of the definition of substitution frames it might be useful to look at [Leo05, pp. 23-25], where more frames are presented that reflect ideas of the antipositionalist view.

1. $\Sigma(s, \mathsf{id}_O) = s$,

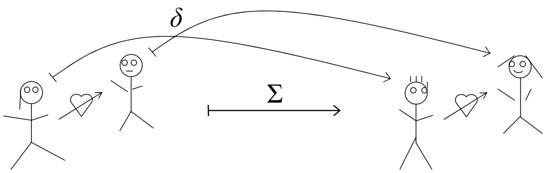
2. $\Sigma(s, \delta' \circ \delta) = \Sigma(\Sigma(s, \delta), \delta')$.

For convenience, we will often write $s \cdot_{\mathcal{F}} \delta$ or $s \cdot \delta$ for $\Sigma(s, \delta)$. Further, we will also often write $f \cdot g$ for $g \circ f$. With this notation, Σ is such that for all $s \in S$ and for all $\delta, \delta' \in O^O$, $s \cdot \mathsf{id}_O = s$, and $s \cdot (\delta \cdot \delta') = (s \cdot \delta) \cdot \delta'$.

The two conditions on Σ agree with how we understand substitution. The intended interpretation of $s \cdot \delta$ is the state we get when we simultaneously substitute in s for each object a the object $\delta(a)$.

Remark 2.3.11. The two conditions on Σ say that Σ is a *right action of the monoid O^O on S*. In terms of category theory we could alternatively have defined a substitution frame as a triple $\langle S, O, \Sigma^* \rangle$, where S is a nonempty set of states, O is a nonempty set of objects, and Σ^* is a functor from the monoid O^O to the monoid S^S.

Example 2.3.12. For the love relation we can make a substitution frame $\mathcal{F} = \langle S, O, \Sigma \rangle$ with S states of loving, O people, and Σ depicted as:



For the state s of Hans's loving Riëtte, and $\delta : O \to O$ with Hans $\mapsto$ Jan and Riëtte $\mapsto$ Jos, $s \cdot \delta$ is the state of Jan's loving Jos. With Σ defined in this way, the two conditions imposed on Σ in Definition 4.3.1 are clearly fulfilled. Take for example δ' with Jan $\mapsto$ Henk and Jos $\mapsto$ Lieke, then $s \cdot (\delta \cdot \delta')$ is the same state as $(s \cdot \delta) \cdot \delta'$, namely the state of Henk's loving Lieke. ⊣

Substitution frames can also accommodate variadic relations (i.e. variably polyadic relations), since states not connected by substitutions can be united into a single frame. For example, for the variadic relation *is surrounded by* we can make a substitution frame as follows:

Example 2.3.13. Define $\mathcal{F} = \langle S, O, \Sigma \rangle$ with O a set of objects, S states of objects being surrounded by a variable number of other objects, and Σ such that for the state of a_1's being surrounded by $a_2, a_3, ..., a_m$ a substitution may result in the state of b_1's being surrounded by $b_2, b_3, ..., b_m$, where we do not exclude that in a state certain objects may occur more than once. $\dashv$

We now define the objects or relata of a state. The idea is to define them as the objects for which it can make a difference for the resulting state which objects are substituted for them.[4] We have, however, to be a bit cautious in our formulation:

Definition 2.3.14. Let $\mathcal{F} = \langle S, O, \Sigma \rangle$ be a substitution frame. We call $A \subseteq O$ an *object-domain of* $s \in S$ if for every $\delta, \delta' : O \to O$,

$$\delta =_A \delta' \Rightarrow s \cdot \delta = s \cdot \delta'.$$

We define the *core of* s as:

$$\mathsf{Core}_{\mathcal{F}}(s) = \bigcap \{A \mid A \text{ is an object-domain of } s\}.$$

If $\mathsf{Core}_{\mathcal{F}}(s)$ is an object-domain, then we call this set the *objects of* s. We denote this set as $\mathsf{Ob}_{\mathcal{F}}(s)$. If $\mathsf{Core}_{\mathcal{F}}(s)$ is not an object-domain, then we leave $\mathsf{Ob}_{\mathcal{F}}(s)$ undefined.

We will often write $\mathsf{Core}(s)$ and $\mathsf{Ob}(s)$ for $\mathsf{Core}_{\mathcal{F}}(s)$ and $\mathsf{Ob}_{\mathcal{F}}(s)$.

Lemma 2.3.15. *Let $\mathcal{F} = \langle S, O, \Sigma \rangle$ be a substitution frame. For every $s \in S$, the object-domains of s form a (possibly non-proper) filter on O.*[5]

Proof. To prove that the object-domains of s are closed under finite intersection, let A and A' be object-domains of s. Let $\delta, \delta' : O \to O$ be such that $\delta =_{A \cap A'} \delta'$. Define

$$\delta''(a) = \begin{cases} \delta(a) & \text{if } a \in A - A', \\ \delta(a) = \delta'(a) & \text{if } a \in A \cap A', \\ \delta'(a) & \text{if } a \in A' - A. \end{cases}$$

Then $\delta'' =_A \delta$ and $\delta'' =_{A'} \delta'$. So, $s \cdot \delta = s \cdot \delta'' = s \cdot \delta'$. Thus, $A \cap A'$ is an object-domain of s.
It is trivial that the object-domains of s are upward closed.
Since an object-domain may be empty, we may have a non-proper filter. $\dashv$

[4] This approach to define the objects of a state corresponds to a remark Fine made in [Fin00, p. 26, note 15].

[5] As suggested by Albert Visser, it might be possible to give a general definition of the objects of s as the ultrafilters extending the collection of object-domains of s. The details of this approach are under development.

The next example, which is related to Example 2.3.9, shows that not every core is an object-domain.

Example 2.3.16. Let $\mathcal{F} = \langle S, O, \Sigma \rangle$ be a substitution frame with O an infinite set, S the set of subsets of O modulo a finite difference, i.e.

$$S = \{\widehat{A} \mid A \subseteq O\}$$

with $\widehat{A} = \{A' \subseteq O \mid A \,\Delta\, A' \text{ is finite}\}$, and Σ defined by[6]

$$\widehat{A} \cdot \delta = \widehat{\delta[A]}.$$

Σ is well-defined, since for any $A, B \subseteq O$, if $\widehat{A} = \widehat{B}$, then $\widehat{\delta[A]} = \widehat{\delta[B]}$. Further, $\mathcal{F}$ is a substitution frame, since

1. $\widehat{A} \cdot \mathsf{id}_O = \widehat{\mathsf{id}_O[A]} = \widehat{A}$,
2. $\widehat{A} \cdot (\delta \cdot \delta') = \widehat{(\delta \cdot \delta')[A]} = \widehat{\delta'[\delta[A]]} = \widehat{\delta[A]} \cdot \delta' = (\widehat{A} \cdot \delta) \cdot \delta'$.

It is not difficult to see that for any $A \subseteq O$ the core of $\widehat{A}$ is the empty set, but if A is infinite, then the empty set is not an object-domain of $\widehat{A}$. ⊣

The next lemma gives a characterization of the core of a state in terms of substitutions of one object at a time.

Lemma 2.3.17. *Let $\mathcal{F} = \langle S, O, \Sigma \rangle$ be a substitution frame. For every $s \in S$,*[7]

$$\mathsf{Core}(s) = \{a \mid \exists b \in O\, (s \cdot \mathsf{id}_O[a \mapsto b] \neq s)\}.$$

Proof. Consider $a \in \mathsf{Core}(s)$. Then $O - \{a\}$ is not an object-domain. Therefore, for some δ_1, δ_2 we have $\delta_1 =_{O-\{a\}} \delta_2$ and $s \cdot \delta_1 \neq s \cdot \delta_2$. We may choose $b \in O$ with $b \neq a$. Then, for $\delta_0 = \mathsf{id}_O[a \mapsto b]$ we have $\delta_0 \cdot \delta_1 = \delta_0 \cdot \delta_2$, and thus

$$(s \cdot \delta_0) \cdot \delta_1 = s \cdot (\delta_0 \cdot \delta_1) = s \cdot (\delta_0 \cdot \delta_2) = (s \cdot \delta_0) \cdot \delta_2.$$

So, because $s \cdot \delta_1 \neq s \cdot \delta_2$, we see that $s \cdot \delta_0 \neq s$.
The inclusion in the other direction is obvious. ⊣

In the next lemma and examples we investigate the relationship between object-domains, cores, and objects of different states.

[6] $f[X] = \{f(x) \mid x \in X\}$, the image of X under f.
[7] $f[a \mapsto b]$ denotes the function defined by $f[a \mapsto b](x) = b$ if $x = a$; $f(x)$ otherwise.

Lemma 2.3.18. *Let $\mathcal{F} = \langle S, O, \Sigma \rangle$ be a substitution frame. For every $s \in S$, if A is an object-domain of s, then $\delta[A]$ is an object-domain of $s \cdot \delta$.*

Proof. Let δ', δ'' be functions from O to O. If $\delta' =_{\delta[A]} \delta''$, then $\delta \cdot \delta' =_A \delta \cdot \delta''$. So, if A is an object-domain of s, then $s \cdot \delta \cdot \delta' = s \cdot \delta \cdot \delta''$. ⊣

Thus, by the lemma, if $\mathsf{Ob}(s)$ is defined, then $\mathsf{Core}(s \cdot \delta) \subseteq \delta[\mathsf{Ob}(s)]$. The next example shows that not always $\mathsf{Core}(s \cdot \delta) \subseteq \delta[\mathsf{Core}(s)]$.

Example 2.3.19. Let $\mathcal{F} = \langle S, O, \Sigma \rangle$ be a substitution frame with O an infinite set,

$$S = \{(A, \widehat{B}) \mid A, B \subseteq O\}$$

with $\widehat{B} = \{B' \subseteq O \mid B \mathbin{\Delta} B' \text{ is finite}\}$, and Σ defined by

$$(A, \widehat{B}) \cdot \delta = (\delta[A] \cup C, \widehat{\delta[B]})$$

with $C = \{a \mid \delta(b) = a \text{ for infinitely many } b \in B\}$.

It is not difficult to verify that Σ is well-defined, that $\mathcal{F}$ is indeed a substitution frame, and that $\mathsf{Core}(A, \widehat{B}) = A$. So, for $s_0 = (\varnothing, \widehat{O})$, $\mathsf{Core}(s_0) = \varnothing$, but for any constant function $c_a : D \to D$ with value a, $\mathsf{Core}(s_0 \cdot c_a) = a$. ⊣

Sometimes $\mathsf{Ob}(s \cdot \delta) \neq \delta[\mathsf{Ob}(s)]$, as is shown in the next example.

Example 2.3.20. Let $\mathcal{F} = \langle S, O, \Sigma \rangle$ be a substitution frame with $S = \mathcal{P}(O)$, and Σ defined by

$$s \cdot \delta = \begin{cases} \delta[s] & \text{if } \delta \text{ is injective on } s, \\ \varnothing & \text{otherwise.} \end{cases}$$

It is easy to verify that $\mathcal{F}$ is indeed a substitution frame, and that for all $s \in S$, we have $\mathsf{Ob}(s) = s$. But, we also see that for any function $\delta : O \to O$ that is not injective on s, $\mathsf{Ob}(s \cdot \delta) = \varnothing$. ⊣

If $\mathsf{Ob}(s)$ is undefined, then not necessarily $\mathsf{Ob}(s \cdot \delta)$ is undefined as well. This follows easily from Example 4.3.7. Also, if $\mathsf{Ob}(s)$ is defined, then not necessarily $\mathsf{Ob}(s \cdot \delta)$ is defined as well:

Example 2.3.21. Let $\mathcal{F} = \langle S, O, \Sigma \rangle$ be a substitution frame with O an infinite set,

$$S = \{(A, 0) \mid A \subseteq O\} \cup \{(\widehat{A}, 1) \mid A \subseteq O\}$$

with $\widehat{A} = \{A' \subseteq O \mid A \mathbin{\Delta} A' \text{ is finite}\}$, and Σ defined by

$$s \cdot \delta = \begin{cases} (\delta[A], 0) & \text{if } s = (A,0) \text{ and } \delta \text{ is injective on } s, \\ (\widehat{\delta[A]}, 1) & \text{if } s = (A,0) \text{ and } \delta \text{ is not injective on } s, \text{ or } s = (\widehat{A}, 1). \end{cases}$$

It is not difficult to verify that Σ is well-defined, and that $\mathcal{F}$ is indeed a substitution frame. Further, for all $s = (A, 0)$, $\mathsf{Ob}(s)$ is defined, but for any function $\delta : O \to O$ that is not injective on s, $\mathsf{Ob}(s \cdot \delta)$ is not defined. ⊣

For each substitution frame we can define its degree as a cardinal number:

Definition 2.3.22. Let $\mathcal{F} = \langle S, O, \Sigma \rangle$ be a substitution frame. For a state s in S, we define the *degree of* s as:

$$\mathsf{degree}(s) = \text{glb}\,\{|A| \mid A \text{ is an object-domain of } s\}.$$

The *degree of* $\mathcal{F}$ we define as:

$$\mathsf{degree}_{\mathcal{F}} = \text{lub}\,\{\mathsf{degree}(s) \mid s \in S\}.$$

Here $|A|$ denotes as usual the cardinality of A, "glb" denotes the greatest lower bound, and "lub" denotes the least upper bound. Note that the degree of s and the degree of $\mathcal{F}$ always exist and are indeed cardinal numbers.

If the degree of a frame is infinite, then either (1) for all states s the set $\mathsf{Ob}(s)$ is finite, but the size of the sets $\mathsf{Ob}(s)$ is unbounded, or (2) there is some state s for which $\mathsf{Ob}(s)$ is infinite or not defined. In the last case all object-domains of s are obviously infinite sets.

For any substitution frame $\mathcal{F} = \langle S, O, \Sigma \rangle$, $\mathsf{degree}_{\mathcal{F}} \leq |O|$. If $\mathcal{F}$ is of finite degree and $\mathsf{degree}_{\mathcal{F}} < |O|$, then the simultaneous substitution of many objects of a state can be defined in terms of substitutions of one object at a time. (cf. [Fin00, p. 26, note 15]).

We may ask ourselves whether substitution frames are perhaps not too limited. With substitution frames it is not possible to frame relations for which $\Re abc$ and $\Re cba$ represent the same state iff a and b are *not* equal. But could a "real" relation like this exist? Actually, I don't think this is very likely. However, if such relations exist, then it might be worth considering frames based on *injective* substitution. We will not discuss such frames in this paper.

2.4 Corresponding frames

In this section, we present the main results of this paper. We show how intimately related positional frames are with substitution frames. Metaphysically the results are of interest because they give a justification for using positional models for a large class of relations without any commitment to an ontology of positions. In other words, the use of positional frames for such relations does not force us to accept positions as fundamental entities, because we can treat positions as "light" products of our own mind. In Section 2.6 these metaphysical aspects will be discussed in more detail.

Definition 2.4.1. A substitution frame $\mathcal{F} = \langle S, O, \Sigma \rangle$ and a positional frame $\mathcal{G} = \langle S', O', P, \Gamma \rangle$ *correspond* if

1. $S = S' = \mathsf{im}\,\Gamma$,
2. $O = O'$,
3. $\Gamma(f) \cdot_{\mathcal{F}} \delta = \Gamma(f \cdot \delta)$.

Note that the first condition implies that $\mathcal{G}$ is a full frame. The last condition states that for any $\delta : O \to O$ the following diagram commutes:

$$\begin{array}{ccc} O^P & \xrightarrow{\tilde{\delta}} & O^P \\ \Big\downarrow \Gamma & & \Big\downarrow \Gamma \\ S & \xrightarrow[\Sigma_\delta]{} & S \end{array}$$

where $\tilde{\delta}$ is defined by $\tilde{\delta}(f) = f \cdot \delta$, and Σ_δ is defined by $\Sigma_\delta(s) = s \cdot_{\mathcal{F}} \delta$.

Further, we say that a substitution model $\mathcal{M} = \langle S, O, \Sigma, H \rangle$ and a positional model $\mathcal{N} = \langle S', O', P, \Gamma, H' \rangle$ *correspond* if their frames correspond and $H = H'$.

Not every substitution frame corresponds to a positional frame, but we will show in Theorems 3.7.2 and 2.4.6 that the *simple* substitution frames correspond to the *simple* positional frames (see Figure 2.1). Further, we will

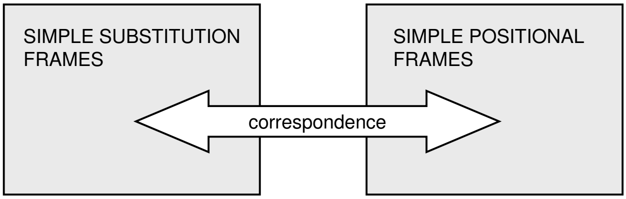


Figure 2.1: Relationship between simple frames.

show (1) that for each simple substitution frame of finite degree the corresponding positional frame of the same degree is unique, modulo positional variants, and (2) that for each simple positional frame the corresponding substitution frame is always unique.

Definition 2.4.2. Let $\mathcal{F} = \langle S, O, \Sigma \rangle$ be a substitution frame. We call $\mathcal{F}$ a *simple substitution frame* if there is a state s_0 such that

$$S = \{s_0 \cdot \delta \mid \delta : O \to O\}.$$

We call s_0 an *initial state.*

Theorem 2.4.3. *A substitution frame $\mathcal{F}$ corresponds to some positional frame $\mathcal{G}$ of the same degree iff $\mathcal{F}$ is a simple substitution frame.*

Further, if $\mathsf{degree}_{\mathcal{F}}$ is finite, then $\mathcal{G}$ is unique, modulo positional variants.

Proof. Let $\mathcal{F} = \langle S, O, \Sigma \rangle$ be a substitution frame, and let $\mathcal{G} = \langle S, O, P, \Gamma \rangle$ be a corresponding positional frame of the same degree. Let $f_0 : P \to O$ be an injection. Such a function exists, since $|P| = \mathsf{degree}_{\mathcal{G}} = \mathsf{degree}_{\mathcal{F}} \leq |O|$. By the injectivity of f_0, there is for each $f \in O^P$ a $\delta \in O^O$ such that $f = f_0 \cdot \delta$. Thus, by Condition 1 and 3 of the definition of corresponding frames,

$$S = \mathsf{im}\,\Gamma = \{\Gamma(f_0 \cdot \delta) \mid \delta : O \to O\} = \{\Gamma(f_0) \cdot_{\mathcal{F}} \delta \mid \delta : O \to O\}.$$

So, $\mathcal{F}$ is a simple substitution frame with $\Gamma(f_0)$ as an initial state.

Conversely, let $\mathcal{F} = \langle S, O, \Sigma \rangle$ be a simple substitution frame. We construct a corresponding positional frame $\mathcal{G} = \langle S, O, P, \Gamma \rangle$ as follows:

1. Choose an initial state $s_0 \in S$.

2. Choose an object-domain A of s_0 with $|A| = \mathsf{degree}_{\mathcal{F}}(s_0)$.

3. Define $P = A$.

4. Let f be an arbitrary element of O^P. Let f^+ extend f to $O \to O$. Define $\Gamma(f) = s_0 \cdot_{\mathcal{F}} f^+$.

Note that Γ is well-defined, since all extensions of f are identical on P and P is an object-domain of s_0.

We now show that the conditions of the definition of corresponding frames are fulfilled and that $\mathcal{G}$ has the same degree as $\mathcal{F}$.

Condition 1 follows from the following observations. Let s be an arbitrary state in S. Because s_0 is an initial state, for some δ we have $s = s_0 \cdot_{\mathcal{F}} \delta$. Let f in O^P be the restriction of δ to P. Then $\Gamma(f) = s_0 \cdot_{\mathcal{F}} \delta = s$, which proves that $S \subseteq \mathsf{im}\,\Gamma$. Conversely, from the definition of Γ it follows immediately that $\mathsf{im}\,\Gamma \subseteq S$.

Condition 2 is trivially fulfilled, and Condition 3 follows from

$$\Gamma(f) \cdot_{\mathcal{F}} \delta = (s_0 \cdot_{\mathcal{F}} f^+) \cdot_{\mathcal{F}} \delta = s_0 \cdot_{\mathcal{F}} (f \cdot \delta)^+ = \Gamma(f \cdot \delta).$$

This proves that $\mathcal{F}$ corresponds to $\mathcal{G}$.

Since s_0 is an initial state, for any s in S, $\mathsf{degree}(s) \leq \mathsf{degree}(s_0)$. Therefore, $\mathsf{degree}_{\mathcal{F}} = \mathsf{degree}(s_0) = |P| = \mathsf{degree}_{\mathcal{G}}$.

To prove the uniqueness claim of the theorem, assume $\mathsf{degree}_{\mathcal{F}}$ is finite. Let $\mathcal{G}' = \langle S, O, P', \Gamma' \rangle$ be another corresponding frame of the same degree. Consider again the injection $f_0 \in O^P$. Because the frames have the same, finite degree, there is an injection $f_0' \in O^{P'}$ such that $\Gamma'(f_0') = \Gamma(f_0)$. So there is a bijection $\pi : P' \to P$ such that $f_0' = \pi \cdot f_0$. Further, for each $f \in O^P$ there is a $\delta : O \to O$ such that $f = f_0 \cdot \delta$. Therefore,

$$\Gamma(f) = \Gamma(f_0 \cdot \delta) = \Gamma(f_0) \cdot_{\mathcal{F}} \delta = \Gamma'(f_0') \cdot_{\mathcal{F}} \delta = \Gamma'(f_0' \cdot \delta) = \Gamma'(\pi \cdot f_0 \cdot \delta) = \Gamma'(\pi \cdot f).$$

Thus, we showed that $\mathcal{G}$ and $\mathcal{G}'$ are positional variants. ⊣

If $\mathsf{degree}_{\mathcal{F}}$ is infinite, then the corresponding positional frames with the same degree are not always unique modulo positional variants. A trivial cause are dummy positions (see Definition 2.3.8), but dummy positions are not the only obstacle, as the next example shows.

Example 2.4.4. Let $\mathcal{F} = \langle S, O, \Sigma \rangle$ with $O = \omega$, the set of natural numbers, $S = \{s : \omega \to (\omega \cup \{\infty\}) \mid \exists i\, (s(i) = \infty)\}$, and Σ defined by

$$(s \cdot_{\mathcal{F}} \delta)(i) = \sum_{\delta(j)=i} s(j)$$

with $i + \infty = \infty + i = \infty + \infty = \infty$. Note that S can be regarded as a set of multisets. It is easy to check that $\mathcal{F}$ is a simple substitution frame. A peculiar property of $\mathcal{F}$ is that it has initial states s_0, s_0' such that for any $\delta : O \to O$ with $s_0 \cdot_{\mathcal{F}} \delta = s_0'$, δ is not injective on $\mathsf{Ob}(s_0)$, namely

$s_0 = [0^\infty, 1, 2, 3, ...]$, i.e. $s_0(0) = \infty$, and for $i \geq 1, s_0(i) = 1$;
$s_0' = [0^\infty, 1^\infty, 2, 3, ...]$, i.e. $s_0'(0) = s_0'(1) = \infty$, and for $i \geq 2, s_0'(i) = 1$.

We exploit this peculiarity to define two dissimilar positional frames:

$$\mathcal{G} = \langle S, O, \omega, \Gamma \rangle \text{ with } \Gamma(f) = s_0 \cdot_{\mathcal{F}} f;$$
$$\mathcal{G}' = \langle S, O, \omega, \Gamma' \rangle \text{ with } \Gamma'(f) = s_0' \cdot_{\mathcal{F}} f.$$

By the proof of Theorem 3.7.2 we see that $\mathcal{G}$ and $\mathcal{G}'$ correspond to $\mathcal{F}$. Clearly, $\mathcal{G}$ and $\mathcal{G}'$ have no dummy positions. To see that $\mathcal{G}$ and $\mathcal{G}'$ are not positional variants, let $\pi : \omega \to \omega$ be such that $\Gamma'(\mathsf{id}_\omega) = \Gamma(\mathsf{id}_\omega \cdot \pi)$. Then $s_0' = s_0 \cdot_{\mathcal{F}} \pi$, and so π cannot be bijective.

It is an open question what in this and similar cases could be regarded as the "most natural" corresponding frame. ⊣

Theorem 3.7.2 shows that for simple substitution frames of finite degree, we can reconstruct indirectly the notion of position in a satisfactory way. To characterize the positional frames that correspond to substitution frames, we need the following definition:

Definition 2.4.5. We say that a positional frame $\mathcal{G} = \langle S, O, P, \Gamma \rangle$ *respects substitution* if for every $\delta : O \to O$,

$$\Gamma(f) = \Gamma(g) \Rightarrow \Gamma(f \cdot \delta) = \Gamma(g \cdot \delta).$$

Now we state the counterpart of Theorem 3.7.2:

Theorem 2.4.6. *A positional frame $\mathcal{G}$ corresponds to some substitution frame iff $\mathcal{G}$ is a full frame that respects substitution.*

Further, the corresponding substitution frame is unique.

Proof. Let $\mathcal{G} = \langle S, O, P, \Gamma\rangle$ be a positional frame and let $\mathcal{F} = \langle S, O, \Sigma\rangle$ be a corresponding substitution frame. Assume $\Gamma(f) = \Gamma(g)$. Then for every $\delta : O \to O$,

$$\Gamma(f \cdot \delta) = \Gamma(f) \cdot_{\mathcal{F}} \delta = \Gamma(g) \cdot_{\mathcal{F}} \delta = \Gamma(g \cdot \delta).$$

So, $\mathcal{G}$ respects substitution. Further, by Condition 1 of the definition of corresponding frames, $\mathcal{G}$ is a full frame.

Conversely, let $\mathcal{G} = \langle S, O, P, \Gamma\rangle$ be a full, substitution-respecting positional frame. Let $\mathcal{F} = \langle S, O, \Sigma\rangle$ with Σ defined by

$$\Gamma(f) \cdot_{\mathcal{F}} \delta = \Gamma(f \cdot \delta).$$

Since $\mathcal{G}$ respects substitution, $\Gamma(f) = \Gamma(g) \Rightarrow \Gamma(f \cdot \delta) = \Gamma(g \cdot \delta)$. Therefore Σ is well-defined.

It is easy to see that $\mathcal{F}$ is a substitution frame:

1. $\Gamma(f) \cdot_{\mathcal{F}} \mathsf{id}_O = \Gamma(f \cdot \mathsf{id}_O) = \Gamma(f)$,
2. $\Gamma(f) \cdot_{\mathcal{F}} (\delta \cdot \delta') = \Gamma(f \cdot \delta \cdot \delta') = \Gamma(f \cdot \delta) \cdot_{\mathcal{F}} \delta' = (\Gamma(f) \cdot_{\mathcal{F}} \delta) \cdot_{\mathcal{F}} \delta'$.

The frames $\mathcal{F}$ and $\mathcal{G}$ correspond, since the conditions of the definition of corresponding frames are trivially fulfilled.

The uniqueness of $\mathcal{F}$ follows immediately from the fact that for any substitution frame $\mathcal{F}'$ that corresponds to $\mathcal{G}$ we must have $\Gamma(f) \cdot_{\mathcal{F}'} \delta = \Gamma(f \cdot \delta)$ by Condition 3 of the definition of corresponding frames. ⊣

Remark 2.4.7. For some substitution-respecting frames $\Gamma(f) = \Gamma(g)$ does not imply $\mathsf{im}\, f = \mathsf{im}\, g$, not even if the frames have no dummy positions. Take for example a frame $\mathcal{G} = \langle S, O, P, \Gamma\rangle$ with $|O| \geq 2$, $P = \{p_1, p_2\}$, and Γ such that

$$\Gamma(f) = \Gamma(g) \Leftrightarrow (f = g \text{ or } (f(p_1) = f(p_2) \text{ and } g(p_1) = g(p_2))).$$

Note that in the substitution frame corresponding to $\mathcal{G}$ the set of objects of one of the states is empty. So, in particular for some f, $\mathsf{Ob}(\Gamma(f)) \subsetneq \mathsf{im}\, f$. It is an open question whether some "real" relations have frames with such properties.

Definition 2.4.8. Let $\mathcal{G} = \langle S, O, P, \Gamma\rangle$ be a positional frame. We call $\mathcal{G}$ a *simple positional frame* if

1. $|O| \geq |P|$,
2. $\mathcal{G}$ is a full frame,
3. $\mathcal{G}$ respects substitution.

From Theorem 3.7.2 and Theorem 2.4.6 it follows immediately that:

Corollary 2.4.9. *Every simple substitution frame corresponds to a simple positional frame and vice versa.*

2.5 Positional structure

The frames developed in Section 2.3 might be too general or too limited for "real" relations. For example, it might be that relations can only have certain limited forms of strict symmetry. Because a positionalist view on relations is so natural for our way of thinking, we will investigate to what extent certain subtypes of the positional frames are adequate for "real" relations. In our analysis the notion of *positional structure* will play a central role.

Definition 2.5.1. Let $\mathcal{F} = \langle S, O, P, \Gamma \rangle$ be a positional frame. We define the *positional structure* $E_{\mathcal{F}}$ as:

$$E_{\mathcal{F}} = \{(f, g) \mid \Gamma(f) = \Gamma(g)\}.$$

Note that $E_{\mathcal{F}}$ is an equivalence relation. Insight in the positional structure for metaphysically meaningful relations might provide a better understanding of the essence of relations. We will define three subtypes of the positional frames in terms of structures that involve only their positions (see Figure 2.2). We follow a bottom-up approach, starting with the role-based frames.

2.5.1 Role-based frames

Positions can fulfill certain roles. In the positional frame for the amatory relation one position fulfills the role of *Lover* and the other position the role of *Beloved*. In this case positions and roles coincide. However, there is no compelling reason why there would always be a one-to-one correspondence between positions and roles. On the contrary, it is natural to say that in the positional frame for the adjacency relation the two positions fulfill the same role.

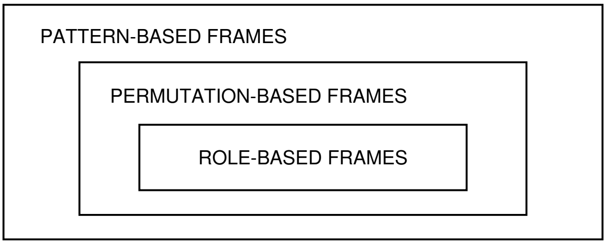


Figure 2.2: Subtypes of positional frames.

Definition 2.5.2. Let $\mathcal{F} = \langle S, O, P, \Gamma \rangle$ be a positional frame. Let p, p' be elements of P. We say that p' *fulfills the same role as* p if for some bijection $\pi : P \to P$,

$$p' = \pi(p) \ \& \ \forall f \in O^P \, (f \; E_{\mathcal{F}} \; (f \circ \pi)).$$

We define the *role of* p as:

$$\mathsf{Role}(p) = \{p' \in P \mid p' \text{ fulfills the same role as } p\},$$

and the *roles of* $\mathcal{F}$ as:

$$\mathsf{Roles}_{\mathcal{F}} = \{\mathsf{Role}(p) \mid p \in P\}.$$

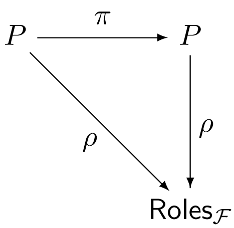
Note that the relation *fulfills the same role as* is an equivalence relation.

We now define a type of positional frames for which changing the positions of the objects does not change the corresponding state as long as the roles of objects are kept invariant.

Definition 2.5.3. Let $\mathcal{F} = \langle S, O, P, \Gamma \rangle$ be a positional frame, and let $E_{\mathcal{F}}$ be its positional structure. We call $\mathcal{F}$ a *role-based frame* if

$$E_{\mathcal{F}} = \{(f, f \circ \pi) \mid f \in O^P \ \& \ \pi \text{ is a role-preserving permutation}\},$$

where $\pi : P \to P$ *is a role-preserving permutation* if π is a bijection for which the following diagram commutes:



with $\rho : p \mapsto \mathsf{Role}(p)$.

Note that the role-preserving permutations form a group.

We defined roles within the context of individual frames. In this respect they differ from *thematic roles* which apply to arguments of different predicates. For example, thematic roles like *agent*, *patient*, and *location* are used to classify arguments of natural language predicates. We will not study such kind of global roles in this paper.

Certain strictly symmetric relations can be modeled by role-based models. For example, for the adjacency relation we can define a frame $\mathcal{F}$ with two positions, say *Next* and *Nixt*. If $\mathcal{F}$ is strictly symmetric, then $\mathcal{F}$ is clearly a role-based frame with one role. However, in general for circular relations no role-based frames are possible. We can see this by rephrasing an argument of Fine [Fin00, p. 17, note 10] in terms of roles:

Example 2.5.4. Let $\mathcal{F} = \langle S, O, P, \Gamma \rangle$ be a role-based frame with a, b, and c three different objects in O, $P = \{p_1, p_2, p_3\}$, and[8]

$$\Gamma abc = \Gamma bca = \Gamma cab.$$

Then the frame has just one role. But then also the state Γacb is necessarily identical to Γabc. Similar observations can of course be made for frames with more positions. Therefore, frames for "genuine" n-ary circular relations with $n \geq 3$ are not role-based. $\dashv$

2.5.2 Permutation-based frames

A natural generalization of the role-based frames are the *permutation-based frames*:

Definition 2.5.5. Let $\mathcal{F} = \langle S, O, P, \Gamma \rangle$ be a positional frame, and let $E_{\mathcal{F}}$ be its positional structure. We define the *permutation group of* $\mathcal{F}$ as:

$$\mathsf{Perm}_{\mathcal{F}} = \{\pi \in P^P \mid \pi \text{ is a bijection } \& \ \forall f \in O^P \ (f \ E_{\mathcal{F}} \ (f \circ \pi))\}.$$

We call $\mathcal{F}$ a *permutation-based frame* if

$$E_{\mathcal{F}} = \{(f, f \circ \pi) \mid f \in O^P \ \& \ \pi \in \mathsf{Perm}_{\mathcal{F}}\}.$$

[8] We use Γabc as an abbreviation for $\Gamma(\begin{pmatrix} p_1 & p_2 & p_3 \\ a & b & c \end{pmatrix})$.

Note that $\mathsf{Perm}_{\mathcal{F}}$ is indeed a group.

Circular relations can adequately be framed by permutation-based frames. For example, a ternary circular relation can be framed by a permutation-based frame $\mathcal{F}$ with $\mathsf{Perm}_{\mathcal{F}}$ generated by

$$\begin{pmatrix} p_1 & p_2 & p_3 \\ p_2 & p_3 & p_1 \end{pmatrix}.$$

Unfortunately, also this type of frames has shortcomings, in particular for certain relations of degree three and higher. Consider a relation $\mathfrak{R}$ in which $\mathfrak{R}abc$ represents the state that a loves b and b loves c. Then $\mathfrak{R}aba$ represents the same state as $\mathfrak{R}bab$, but aba is not a permutation of bab. This means that no permutation-based frame for this relation is possible. From this we can infer a more general conclusion, namely that the class of permutation-based frames is not closed under identification of positions.

2.5.3 Pattern-based frames

We now define a class of positional frames which turns out to be identical to the class of substitution-respecting frames. These frames are of special interest, since, as we showed in Theorem 2.4.6, the positional frames that correspond to a substitution frame are precisely the full, substitution-respecting frames.

Definition 2.5.6. Let $\mathcal{F} = \langle S, O, P, \Gamma \rangle$ be a positional frame, and let $E_{\mathcal{F}}$ be its positional structure. We define the *pattern of* $\mathcal{F}$ as:

$$\mathsf{Pattern}_{\mathcal{F}} = \{(\sigma, \sigma') \in Q^P \times Q^P \mid \forall h \in O^Q \, ((h \circ \sigma) \; E_{\mathcal{F}} \; (h \circ \sigma'))\}$$

with $Q = 2 \times P$.

We call $\mathcal{F}$ a *pattern-based frame* if

$$E_{\mathcal{F}} = \{(h \circ \sigma, h \circ \sigma') \mid h \in O^{2\times P} \;\; \& \;\; (\sigma, \sigma') \in \mathsf{Pattern}_{\mathcal{F}}\}.$$

Note that $\mathsf{Pattern}_{\mathcal{F}}$ is an equivalence relation.

Not every pattern-based frame is permutation-based, as the next example shows:

Example 2.5.7. Let $\mathcal{F} = \langle S, O, P, \Gamma \rangle$ be a positional frame with $P = \{p_1, p_2, p_3\}$, and Γabc being the state of a's loving b and b's loving c. $\mathcal{F}$ has a special symmetry, namely $\Gamma aba = \Gamma bab$. Hence, $\mathcal{F}$ is clearly not permutation-based. It is straightforward to verify that $\mathcal{F}$ is pattern-based with $\mathsf{Pattern}_\mathcal{F}$ consisting of all pairs (σ, σ) with $\sigma \in (2 \times P)^P$, and all pairs

$$(\begin{pmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_1 \end{pmatrix}, \begin{pmatrix} p_1 & p_2 & p_3 \\ q_2 & q_1 & q_2 \end{pmatrix})$$

with $q_1, q_2 \in 2 \times P$. ⊣

The next theorem is of metaphysical interest since it implies that the positional structure $E_\mathcal{F}$ of a substitution-respecting frame of finite degree is determined by a finite subset of $E_\mathcal{F}$. Also from an epistemological point of view this is of interest, since it means that in principle we can learn the positional structure of such frames by a finite number of substitutions.

Theorem 2.5.8. *A positional frame is pattern-based iff it respects substitution.*

Proof. Let $\mathcal{F} = \langle S, O, P, \Gamma \rangle$ be a positional frame with positional structure $E_\mathcal{F}$. Assume that $\mathcal{F}$ is pattern-based. Further, assume that $f \; E_\mathcal{F} \; g$. Then for some $(\sigma, \sigma') \in \mathsf{Pattern}_\mathcal{F}$ and some h we have $f = h \circ \sigma$ and $g = h \circ \sigma'$. So for every $\delta : O \to O$:

$$\delta \circ f = \delta \circ h \circ \sigma \;\; E_\mathcal{F} \;\; \delta \circ h \circ \sigma' = \delta \circ g.$$

So, $\mathcal{F}$ respects substitution.

Conversely, assume that $\mathcal{F}$ respects substitution. Obviously

$$E_\mathcal{F} \supseteq \{(h \circ \sigma, h \circ \sigma') \mid h \in O^{2 \times P} \;\; \& \;\; (\sigma, \sigma') \in \mathsf{Pattern}_\mathcal{F}\}.$$

To prove the reverse inclusion, assume that $f \; E_\mathcal{F} \; g$. We have an injection $j : \mathsf{im}\, f \cup \mathsf{im}\, g \to 2 \times P$. Choose a function $h : 2 \times P \to O$ such that $h \circ j$ is the identical embedding emb of $\mathsf{im}\, f \cup \mathsf{im}\, g$ in O. Define $\sigma = j \circ f$ and $\sigma' = j \circ g$. Then

$$h \circ \sigma = h \circ j \circ f = \mathsf{emb} \circ f = f.$$

Similarly, $h \circ \sigma' = g$. So, it is sufficient to show that $(\sigma, \sigma') \in \mathsf{Pattern}_\mathcal{F}$. But this follows from the fact that for every $h' : 2 \times P \to O$:

$$h' \circ \sigma = h' \circ j \circ f \;\; E_\mathcal{F} \;\; h' \circ j \circ g = h' \circ \sigma'.$$

⊣

Note that the permutation-based frames also respect substitution. Thus, it follows from the theorem and Example 2.5.7 that the class of permutation-based frames is a proper subclass of the pattern-based frames.

We are not going to discuss operations on frames and models, but I like to mention that a nice property of the class of pattern-based models is that they are closed under operations like identification of positions, conjunction, and disjunction. For more details about operations on models and relations, see [Leo05].

2.6 Metaphysical interpretations

In this section, we first look at metaphysical principles for relations. Then we translate technical results of the previous sections to metaphysical claims. In particular, we consider justification for positional representations and epistemological aspects of relations.

2.6.1 Metaphysical principles for relations

Let me start by formulating a number of metaphysical principles concerning states of relations. We postulate three sorts of entities: states, objects, and substitutions.

CONSTITUENTS PRINCIPLE:

CP-1 Every state of a relation has exactly one set of objects.

SUBSTITUTION PRINCIPLES:

SP-1 The objects of any state can simultaneously be substituted by other objects.

SP-2 Any substitution of objects in any state yields exactly one state of the same relation.

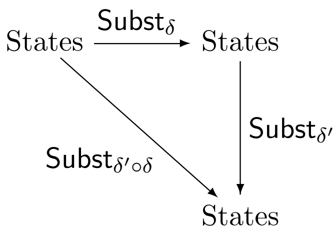
SP-3 An object a belongs to a state iff for some substitution it makes a difference for the resulting state which object is substituted for a.

SP-4 Substitution is a monoidal action on states, i.e.

(a) for any state s, the identity substitution yields the same state, i.e.

$$\mathsf{Subst}_{\mathsf{id}}(s) = s,$$

(b) the following diagram commutes:



with Subst_δ representing a substitution of objects.

Factuality principle:

FP-1 Every state either obtains or does not obtain.

What supporting arguments can be given for these principles? It seems hard to give any conclusive arguments, since I have not given any clear definition of what relations are. Nevertheless, for several views on relations these principles might be acceptable. Let me briefly comment on each principle separately.

The constituents principle by itself does not say much. Therefore I consider it in combination with the other principles. The first substitution principle is perhaps too restrictive. It might be argued that within the context of a relation a state can have two kinds of objects: those that can be substituted by other objects, and those that fulfill a kind of background role for the state. For example, for the relation $\mathfrak{R}$ in which $\mathfrak{R}ab$ is the state that there is a flight from a to b via Amsterdam, Amsterdam could be regarded as a fixed object for the states of this relation.

We might ask whether the second substitution principle is perhaps not too strong and that it would be better to replace it with a weaker principle, namely:

SP-2′ Any substitution of objects in any state yields *no more than* one state of the same relation.

This weaker principle might be a better choice if you consider the states of a relation as *possible* states of affairs. That Bin Laden loves Bush is possible, but that $1 = 2$ is clearly not possible. What is also impossible, I think, is that I am identical to Mo, my daughter. Such examples make clear that not every substitution yields a possible state of affairs. However, if you regard states of relations as propositions or if you are willing to accept *impossible* states of affairs, then the stronger principle seems preferable.

To a certain extent it is a matter of definition whether you accept substitution principle SP-3. One might entertain the view that constituents of constituents are constituents of the same state. So, on that view, for the state of Gitte's loving Mo the heart of Gitte is also a constituent of this state. A way to effectively deal with this view would be to refine the constituents principle and to enrich our models with a dependency ordering on O, the set of objects.

I think that our intuitive understanding of the identity substitution and of composing substitutions is fully in accordance with the fourth substitution principle.

Note that in the principles we only talk about substituting objects and not about substituting individual occurrences of an object by possibly different objects. The reason for not considering a more refined substitution mechanism is that otherwise we would have to make clear what exactly occurrences are. It might perhaps be possible to do this, but it would be an extra complication.

Another point is that we can ask ourselves if there is not a more primitive operation in which substitution could be expressed. Fine [Fin00, p. 27] discusses the question whether substitution should be understood in terms of a structural operation. However, he considers the notion of substitution of a lower logical type. But even if Fine would be wrong in this respect, this would not make the substitution principles less credible.

With respect to the factuality principle, note that the principle does not say that there are states that do not obtain. So, it is also true for hard actualists, who hold that the only states of affairs that exist are those that obtain.

The considerations above make it plausible that any relation can be modeled by a substitution model. However, this does not mean that such models are always completely satisfactory. In some cases the model might leave out essential aspects of the relation, like the (relative) order of relata in biased relations. Or consider the variadic relation *standing in a line*, where we allow an object to have multiple occurrences. Then how could we get from a line of length n to a line of length $n-1$? The state transition graph $\langle S, E\rangle$ with S the states of the relation, and E the set of pairs (s, s') such that s' can be obtained from s by substitution contains isolated islands of states. To get a more satisfactory model, we perhaps need to consider *subtraction* of objects from a state as a complementary basic operation. This will be a topic for further inquiry.

2.6.2 Justifying positional representations

I do not claim to have obtained a complete understanding of the essence of relations. I cannot even give a satisfactory definition. In this section, we limit ourselves to *simple* relations:

Definition 2.6.1. We call a relation $\Re$ a *simple relation* if

1. $\Re$ satisfies the constituents principle, the substitution principles, and the factuality principle;
2. $\Re$ has an initial state, i.e. a state from which any state of the relation can be obtained by substitution.

We define the *degree of* $\Re$ as the maximum number of objects per state.

Claim 1. Any simple relation can be modeled by a unique simple substitution model.

Note that we do not claim that the substitution model models all aspects of the relation.

Definition 2.6.2. Given a simple relation $\Re$. Then we call a positional model $\mathcal{M}$ a *natural model for* $\Re$ if $\mathcal{M}$ corresponds to the substitution model for $\Re$.

By Theorem 3.7.2 we have the following result:

Claim 2. Any simple relation of finite degree has a natural positional model of the same degree that is unique, modulo positional variants.

The last claim gives a justification for using positional representations for a large class of relations. I consider this claim as a main result of this paper. It is a modest result, but it is also very fundamental. Perhaps it looks completely trivial, but I do not think it is. The fact that we continually use positional representations for relations is in itself no evidence for their validity.

What is also worth noting is that for a simple relation of finite degree, a natural positional model (modulo positional variants) of the same degree has exactly the same information content as its substitution model.

I do not claim that the positional model for a simple relation is unique in an absolute sense. The model is defined in terms of our definition of correspondence between substitution models and positional models. Other definitions of correspondence might give other positional models. But the given definition strikes me as the most natural one. It seems plausible that if a relation can adequately be modeled by a substitution model and a positional model, then these models correspond.

For simple relations, substitution of objects in any state yields by definition exactly one state of the same relation. For the class of relations satisfying the weaker substitution principle that says that any substitution of objects in any state yields *no more than* one state of the same relation, another kind of models might be more appropriate, namely substitution models with a *partial* function Σ and positional models with a *partial* function Γ. For these kind of models, a theory very similar to the one given in this paper could be developed. In particular, we would get results similar to Claims 1 and 2.

Claim 2 says nothing about the ontological status of positions. But can we say something about the ontology of positions? Can we deny them a place in the "fundamental furniture of the universe"? I think we have no reason to grant them such an honorable place, but I also don't think that we have as yet a decisive argument why they cannot belong to this furniture. As long as you do not consider positions as things occupied by objects within the states, their fundamental existence seems hard to disprove. It is perhaps also not contradictory to claim that objects occupy a kind of *internal* positions within the states. For example, you could argue that the states of the adjacency relation have two internal positions, but that these positions have *no determinate identity.* The indeterminacy of the positions could be compared with the quantum-theoretic ontic indeterminacy of the electrons of a He atom (cf. [Low94]). For a cyclic relation the internal

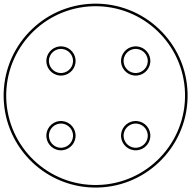


Figure 2.3: Could states have internal positions without identity?

positions might be argued to be partially indiscernible like the unlabeled vertices of a regular mathematical polygon are (see Figure 2.3).

Perhaps for arbitrary positional structures a similar defense can be given for a kind of internal (super)positions. But we do well to realize that the adequacy of positional models does not imply that the constituents of a state really need to occupy some kind of position. I think we can be perfectly happy with assigning to positions nothing more than the status of an innocent mental construction, *unless* we have reason to assume that there is a "real" relation that can only be adequately modeled by a positional model that does *not* respect substitution. Then we still have something to explain.

Although the two claims in this section are about relations, they could be generalized to any entity that satisfies the principles of the previous section. For an extreme nominalist who does not believe in relations, substitution might also still be a notion that makes sense, and he might find positional models useful. Substitution definitely has a wider scope of application than just states of affairs. I see no objection for applying substitution also to situations and propositions, and for using positional representations for them.

2.6.3 Epistemological aspects of relations

Substitution models seem to be more primitive than positional models. So why would we use positional models? I think there are good reasons for this. The strength of positional models (and of directional models) is that they provide us a well-organized framework for all states, and the possibility to refer efficiently to each individual state. Natural languages like English obviously take advantage of this kind of representation. As far as I can see, almost any linguistic relational statement in English makes explicitly or implicitly use of positions.

With respect to determining the structure of simple relations we have the following result:

Claim 3. For any natural positional model of finite degree, the positional structure can in principle be determined by a finite number of substitutions.

The claim is a direct consequence of Theorem 2.4.6, Theorem 2.5.8, and the presupposition that in principle one can determine for any pair of substitutions whether they result in the same state. I don't know if this is true in practice. Also, I do not know to what extent it is needed for a practical understanding of a relation to know explicitly or tacitly its positional structure.

How do we "learn" relations? I have as yet no answer to this question. I have not investigated what kind of empirical research has been done on this subject, but it would be very interesting to find answers to the following questions:

Do we learn relations by substitution, by abstraction, by positional representations or via processes with a completely different logic? Do we learn complex relations by applying operations like conjunction to simple relations? In processing perception, do we use neutral rather than biased relations? How do small children learn relations, and how animals and other organisms? What is the role of language in learning relations? Do all natural languages use directional and positional representations of relations?

Answers to these questions might deepen our insight in fundamental aspects of the way we understand and represent the world. In addition, they might suggest new learning programs or new ways for learning Artificial Intelligence systems to "discover" and handle relations. For example, if we can implement a general notion of substitution in an AI system, then it might perhaps be possible to learn the system a variety of relations by examples.

2.7 Conclusions

My aim in this paper was to develop models for relations that would give a better understanding of the essence of relations. To what extent did we accomplish this goal? Let me recapitulate the main results. We developed mathematical models for the views on relations that Fine described in his paper "Neutral Relations". We proved that the *simple substitution frames* correspond in a natural way with the *simple positional frames*, and that if the frames have the same finite degree, then the correspondence is one-to-one, modulo positional variants. Further, I argued that the simple substitution models adequately model a large class of relations, which we called the *simple relations*.

The results can be interpreted in two directions. They provide support for the antipositionalist view. They show – with the proviso that adequate substitution models and adequate positional models for a relation correspond – that for simple relations of finite degree the primitive notion of substitution has the same expressive power as the use of positions, modulo positional variants. On the other hand, the positionalist can claim for the same class of relations that the results show that his use of positions is innocent.

One of the objections Fine raised against the positionalist view was that it could not handle strictly symmetric relations. But, as I argued in Section 2.2.4, this is only an argument against a positionalist who would claim that objects *occupy* positions in relational states. If a positionalist only claims a *mediating* role for positions, then this argument of Fine does not apply.

Do we have arguments to prefer one view over the other? A strong argument for the antipositionalist view is that it is based on the very general notion of substitution, a primitive kind of operation. I don't think that for a positional approach a similar claim can be made. The positionalist could point out that there are positional models that do not correspond to substitution models, but the converse is also true. Moreover, for positional models that do not correspond to substitution models it seems highly unlikely that they could adequately model "real" relations. These models probably have no metaphysical significance at all for relations. In defense of a positional approach, it may be claimed that positional representations are very natural and practical. But that does not mean that they are very basic. In fact it is not an argument for a positionalist view, but only for the use of a certain representation. I conclude that the results of this paper give extra support to the antipositionalist view, but that they also give a justification for the use of positional representations for relations.

The models and theory developed in this paper can have more use than their contribution in reaching the conclusion I drew in the previous paragraph. The models can be useful tools for analyzing relations, and also for empirical research on how we "learn" relations. With respect to learnability, I showed in Section 2.6.3 that the positional structure of simple relations of finite degree can in principle be learned by a finite number of substitutions. Also further theoretical study of substitution models might be helpful to deepen our understanding of the structure of relations. We already mentioned a promising approach to defining objects of states in substitution models in terms of ultrafilters.

I want to conclude with a remark about the research approach I have taken in this paper. I started with questions about the structure of "real" relations. To find answers, I developed mathematical models that highlighted certain aspects of relations. By playing with the models and studying properties of them, ideas for new models emerged and connections between them became clear. Subsequently, these insights could be translated back to characteristic properties of "real" relations. I think that this type of approach might be fruitful also for other areas of metaphysics. Unfortunately, in metaphysics there is still hardly any consensus about almost anything. I would say that ontological claims about our world have value only if they are accompanied by very strong arguments. Developing and analyzing mathematical models or axiomatic systems might provide the right level of certainty in this respect. Doing physics without mathematics is hardly thinkable, but strangely enough, many seem to think that metaphysics can largely do without it. A reason to refrain from using rigorous mathematical methods for metaphysics might be that such methods are considered too difficult to handle for this discipline. This might be true in some cases, but if we want to take metaphysics seriously, I see no other way.

Acknowledgements The idea to develop mathematical models for the views on relations as presented by Kit Fine comes from Albert Visser. Comments from Kit Fine on my master's thesis about modeling relations gave impetus to writing this paper. Discussions with Rosja Mastop contributed to a better understanding of the subject. Questions from Vincent van Oostrom and his detailed reading of a semi-final version of this paper helped to get many parts in a better shape. I thank them very much for their support and contribution.

Chapter 3

The identity of argument-places*

Abstract

Argument-places play an important role in our dealing with relations. However, that does not mean that argument-places should be taken as primitive entities. It is possible to give an account of "real" relations in which argument-places play no role. But if argument-places are not basic, then what can we say about their identity? Can they e.g. be reconstructed in set theory with appropriate urelements? In this paper we show that for some relations, argument-places cannot be modeled in a *neutral* way in $\mathsf{V}[A]$, the cumulative hierarchy with basic ingredients of the relation as urelements. We argue that a natural way to conceive of argument-places is to identify them with abstract, structureless points of a derivative structure exemplified by positional frames. In case the relation has symmetry these points may be indiscernible.

3.1 Introduction

"Adam" occurs first in "Adam loves Eve", but Adam does not occur first in Adam's loving Eve. The order is a representational artifact, since there simply is no *intrinsic* order or direction between the arguments in the states

*Reprinted from *The Review of Symbolic Logic*, 1:335–354, 2008.

of a relation. A more faithful, neutral representation makes use of unordered argument-places like *lover* and *beloved*. It is often assumed that such argument-places are primitive entities. But as Kit Fine convincingly argued in [Fin00] there is a more basic neutral view on relations in which argument-places do not occur as primitives. In this so-called *antipositionalist* view on relations a key role is played by the general operation of substitution.

If we should not take argument-places as primitive, what can we say about their identity? Can we (re)construct them in a satisfactory way? These questions are particularly of interest, since argument-places play such a prominent role in the way we deal with relations in ordinary life.

In [Leo08a] we showed that for so-called *simple* relations of finite degree, we can construct argument-places or positions that are unique, modulo some equivalence relation. But this result does not seem completely satisfactory from a metaphysical point of view. Could we perhaps also construct them as unique in an absolute sense?

Whether this is possible may depend on the demands we want to impose upon such a construction. One demand seems obvious: we do not want to allow arbitrary choices within the construction.

According to Fine, we can transform biased relations, i.e. relations in which the arguments are ordered, into unbiased ones by taking a "permutation class" of biased relations and by identifying each argument-place of the unbiased relation with a function that takes each biased relation of the permutation class into a corresponding numerical position [Fin00, p. 15]. However, Fine also mentions – without further elaboration – that there are certain complications [Fin00, p. 15, footnote 9].

We will show that Fine's construction only works for relations without strict symmetry. Initially, I guessed it would be possible to develop a similar construction that would work for any relation. But after several attempts to find such a construction this turned out to be impossible. This suggest that we should look for a radically different approach to define argument-places in a neutral way. We will propose to define them as *abstractions* of the positions of positional frames for relations. This may be the most natural view on the identity of relations, although the ontological status of such abstract argument-places may still be a point of discussion.

The outline of the paper is as follows. In Section 3.2 we give an informal explanation of the different views on relations and of ways to reconstruct argument-places. Then in Section 3.3 we define mathematical models/frames corresponding with the views on relations. Most of our results will be formulated in terms of relational frames.

Fine's construction to transform biased relations into unbiased ones is discussed in Section 3.4. In Section 3.5 we introduce a formal notion of *neutrality* of a set with respect to another set, and in Section 3.6 we show that argument-places cannot always be constructed in a formally neutral sense with respect to the permutation class of biased relational frames. We use this result in Section 3.7 to prove that for certain *simple* relations no formally neutral reconstruction of argument-places is possible within the context of ordinary set theory.

In Section 3.8 we argue that the impossibility of the construction may be due to limitations of ordinary set theory as a modeling medium. In Section 3.9 we consider the possibility of conceiving argument-places as *abstractions* of the positions of positional frames for relations. We end in Section 3.10 with a consideration about the metaphysical relevance of the results.

3.2 Informal explanation

Fine [Fin00] presents three views on relations: the standard view, the positionalist view and the antipositionalist view. For readers not familiar with Fine's paper 'Neutral Relations' or with my paper 'Modeling Relations' I start with a very brief characterization of the views.

According to the standard view, the arguments of any relation are ordered. We have for example the biased relation *loves* and its converse *is loved by*. But as we remarked in the introduction, there is no order between the arguments of the state of Adam's loving Eve. So, if this state is a genuine relational complex, it should contain a relation in which the arguments are not ordered.

The positionalist view assumes that any relation comes with positions, as for example *Lover* and *Beloved*. A great advantage of this view is that we can identify a neutral relation for the state of Adam's loving Eve. But an ontological objection against this view is that it regards positions as part of the "fundamental furniture of the universe".

The antipositionalist view does not assume any ordering between arguments nor any positions. Instead, the states of a relation form a network of states interrelated by *substitutions.* For example, from the state of Adam's loving Eve we can obtain the state of Clark's loving Lois by substituting Clark for Adam and Lois for Eve.

A proponent of any of the three views may be interested in adequate positional representations for relations. The main question of this paper is how positions of a relations can be constructed in an *unbiased* or *neutral* way. Here we will consider this question by discussing a few examples.

Take as starting point the set Φ of the biased relations *loves* and *is loved by.* These relations are permutations of each other. The first (second) numerical position of *loves* corresponds to the second (first) position of *is loved by.* Now we get an unbiased positional representation of the amatory relation by identifying the position *lover* with the function from Φ to $\{0, 1\}$ that maps *loves* to 0 and *is loved by* to 1, and by identifying the position *beloved* with the function that maps *loves* to 1 and *is loved by* to 0.

Also for certain permutation classes of symmetric relations we may define positions in a neutral way. Take for example the adjacency relation. Then the permutation class of this relation consists just of nothing but the adjacency relation itself. Defining positions as the set $\{0, 1\}$ gives an unbiased representation.

Unfortunately, not for every relation with symmetry we can construct positions as a neutral set. Take the complex relation $\Re$ with $\Re abcd$ the state of *a's loving b, b's hating c, c's loving d, and d's hating a.* We will argue in Section 3.6 that such a relation puts us in a situation similar to that of Buridan's ass. It seems impossible to define positions for such relations as a set in $\mathsf{V}[S, O]$, the cumulative hierarchy with states S and objects O of the relation as urelements, in a completely non-arbitrary way.

But if positions cannot be defined in $\mathsf{V}[S, O]$ in a neutral way, then there is still a viable alternative. We propose to define positions as *abstractions*, i.e. as structureless places exemplified by representations in $\mathsf{V}[S, O]$. Assigning objects to such positions *yields* states. Despite the fact that such positions may be indiscernible, it would in general be wrong to assume that objects *occupy* positions within the states. In the next sections these ideas will be elaborated in more detail.

3.3 Relational models

In [Leo08a] we defined *frames* for relations to model the logical space of relations, the frames being all of the form $\langle S, O, ... \rangle$ where S is a nonempty set of states, and O a nonempty set of objects. We reserved the word *models* for extensions of the frames with a subset H of S representing the states that obtain. Here we briefly repeat the main definitions.

3.3.1 Relational frames

We present three types of frames corresponding with different views on relations as presented by Fine [Fin00]:

directional frames	$\sim$	*standard view*
positional frames	$\sim$	*positionalist view*
substitution frames	$\sim$	*antipositionalist view*

These type of frames are not uniquely defined by the different views. But for our purposes they adequately model the different views on relations.

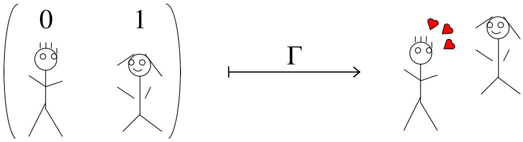
Directional frames

A *directional frame* models the logical space of relations in which the arguments are ordered in a specific way:

Definition 3.3.1. A *directional frame* is a quadruple $\mathcal{F} = \langle S, O, \alpha, \Gamma \rangle$, where S is a nonempty set of states, O is a nonempty set of objects, α is an ordinal number, and Γ is a function from O^α to S.

We call the cardinality of α the *degree of* the frame. We denote it as $\mathsf{degree}_{\mathcal{F}}$.

For the relation *loves* we can make a directional frame with Γ depicted as:



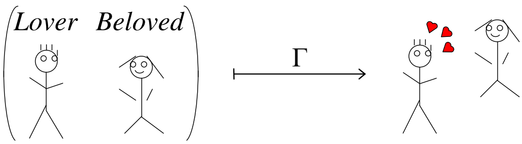
Positional frames

A *positional frame* models the logical space of *neutral* relations, i.e. relations for which the order of the arguments is irrelevant:

Definition 3.3.2. A *positional frame* is a quadruple $\mathcal{F} = \langle S, O, P, \Gamma \rangle$, where S is a nonempty set of states, O is a nonempty set of objects, P is a set of positions, and Γ is a function from O^P to S.

We call the cardinality of P the *degree of* the frame. We denote it as $\mathsf{degree}_{\mathcal{F}}$.

For the love relation we can make a positional frame with Γ depicted as:



Substitution frames

A *substitution frame* also models the logical space of neutral relations. This type of frame is more abstract and at first sight probably more difficult to appreciate than the two other types. It might be helpful to take a look at [Leo05, pp. 23–25] where substitution frames are developed in a number of steps.

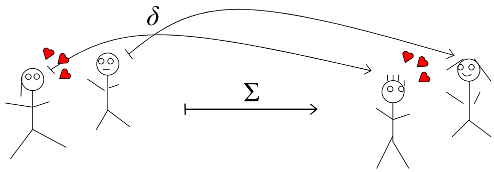
Definition 3.3.3. A *substitution frame* is a triple $\mathcal{F} = \langle S, O, \Sigma \rangle$, where S is a nonempty set of states, O is a nonempty set of objects, and Σ is a function from $S \times O^O$ to S such that

1. $\Sigma(s, \mathsf{id}_O) = s$,

2. $\Sigma(s, \delta' \circ \delta) = \Sigma(\Sigma(s, \delta), \delta')$.

For convenience, we will often write $s \cdot_{\mathcal{F}} \delta$ or $s \cdot \delta$ for $\Sigma(s, \delta)$, and $f \cdot g$ for $g \circ f$.

For the love relation we can make a substitution frame with Σ depicted as:



We define for a substitution frame the *objects* of its states and the *degree* of its states and of the frame itself as follows:

Definition 3.3.4. Let $\mathcal{F} = \langle S, O, \Sigma \rangle$ be a substitution frame. We call $A \subseteq O$ an *object-domain of* $s \in S$ if for every $\delta, \delta' : O \to O$,[1]

$$\delta =_A \delta' \Rightarrow s \cdot \delta = s \cdot \delta'.$$

We define the *core of* s as:

$$\mathsf{Core}_{\mathcal{F}}(s) = \bigcap \{A \mid A \text{ is an object-domain of } s\}.$$

If $\mathsf{Core}_{\mathcal{F}}(s)$ is an object-domain, then we call this set the *objects of* s. We denote this set as $\mathsf{Ob}_{\mathcal{F}}(s)$. If $\mathsf{Core}_{\mathcal{F}}(s)$ is not an object-domain, then we leave $\mathsf{Ob}_{\mathcal{F}}(s)$ undefined.

We will often write $\mathsf{Core}(s)$ and $\mathsf{Ob}(s)$ for $\mathsf{Core}_{\mathcal{F}}(s)$ and $\mathsf{Ob}_{\mathcal{F}}(s)$.

Definition 3.3.5. Let $\mathcal{F} = \langle S, O, \Sigma \rangle$ be a substitution frame. For a state s in S, we define the *degree of* s as:

$$\mathsf{degree}_{\mathcal{F}}(s) = \text{glb}\, \{|A| \mid A \text{ is an object-domain of } s\}.$$

The *degree of* $\mathcal{F}$ we define as:

$$\mathsf{degree}_{\mathcal{F}} = \text{lub}\, \{\mathsf{degree}_{\mathcal{F}}(s) \mid s \in S\}.$$

Here $|A|$ denotes as usual the cardinality of A, glb denotes the greatest lower bound, and lub denotes the least upper bound.

[1] We say that $f =_X g$ if $f{\upharpoonright}X = g{\upharpoonright}X$, i.e. f restricted to X is equal to g restricted to X.

3.3.2 Permutations and positional variants

Directional frames have *permutations*:

Definition 3.3.6. A directional frame $\mathcal{F} = \langle S, O, \alpha, \Gamma \rangle$ is a *permutation of* a frame $\mathcal{F}' = \langle S', O', \alpha', \Gamma' \rangle$ if $S = S'$, $O = O'$, $\alpha = \alpha'$, and there is a bijection $\pi : \alpha \to \alpha$ such that for each $f \in O^\alpha$, $\Gamma(f) = \Gamma'(f \circ \pi)$.

We denote $\mathcal{F}$ as $\pi(\mathcal{F}')$, and define the *permutation class of* $\mathcal{F}$ as

$$\Phi_{\mathcal{F}} = \{\pi(\mathcal{F}) \mid \pi \in \mathsf{Perm}(\alpha)\}$$

with $\mathsf{Perm}(\alpha)$ the bijections from α to α.

We say that $\mathcal{F}$ has *strict symmetry* if there is a bijection $\pi : \alpha \to \alpha$ with $\pi \neq \mathsf{id}_\alpha$ such that $\mathcal{F} = \pi(\mathcal{F})$.

Note that if $\mathcal{F}$ is a permutation of $\mathcal{F}'$, then $\Phi_{\mathcal{F}} = \Phi_{\mathcal{F}'}$.

In a similar way we define for positional frames the notion of *positional variants*:

Definition 3.3.7. A positional frame $\mathcal{F} = \langle S, O, P, \Gamma \rangle$ is a *positional variant of* a frame $\mathcal{F}' = \langle S', O', P', \Gamma' \rangle$ if $S = S'$, $O = O'$, and there is a bijection $\pi : P' \to P$ such that for each $f \in O^P$, $\Gamma(f) = \Gamma'(f \circ \pi)$.

We denote $\mathcal{F}$ as $\pi(\mathcal{F}')$.

3.3.3 Corresponding frames

Directional frames and positional frames may correspond in an obvious way:

Definition 3.3.8. A directional frame $\mathcal{F} = \langle S, O, \alpha, \Gamma \rangle$ and a positional frame $\mathcal{G} = \langle S', O', P, \Gamma' \rangle$ *correspond* if $S = S'$, $O = O'$, and there is a bijection $\mu : P \to \alpha$ such that for each $f \in O^\alpha$, $\Gamma(f) = \Gamma'(f \circ \mu)$.

We denote $\mathcal{F}$ as $\mu(\mathcal{G})$.

Note that directional frames corresponding to a positional frame $\mathcal{G}$, are not necessarily permutations of each other. This is a consequence of our somewhat arbitrary choice to demand in the definition of permutations of directional frames that $\alpha = \alpha'$.

For substitution frames and directional/positional frames we define correspondence as follows:

Definition 3.3.9. A substitution frame $\mathcal{F} = \langle S, O, \Sigma \rangle$ and a directional/ positional frame $\mathcal{G} = \langle S', O', X, \Gamma \rangle$ *correspond* if

1. $S = S' = \mathsf{im}\,\Gamma$,
2. $O = O'$,
3. $\Gamma(f) \cdot_{\mathcal{F}} \delta = \Gamma(f \cdot \delta)$.

As said at the beginning of Section 3.3, relational models can be defined by extending the frames with a subset H of S representing the states that obtain. But also more luxurious models can be considered by taking into account possible worlds where states can obtain. In our analysis, however, only the logical space of relations plays a role.

3.4 Fine's construction

In this section, we show that directional frames without strict symmetry, i.e. frames for which $\mathcal{F} = \pi(\mathcal{F})$ only if $\pi = \mathsf{id}_\alpha$, have corresponding positional frames that are uniquely determined by $\Phi_{\mathcal{F}}$, the permutation class of $\mathcal{F}$. The construction of the positional frames is essentially Fine's construction to transform biased relations into unbiased ones [Fin00, p. 15]:

Theorem 3.4.1. *Let $\mathcal{F} = \langle S, O, \alpha, \Gamma \rangle$ be a directional frame without strict symmetry, and let $\mathcal{G}_{\mathcal{F}} = \langle S, O, P, \overline{\Gamma} \rangle$ be defined by:*

$$P = \{p : \Phi_{\mathcal{F}} \to \alpha \mid \forall \pi \in \mathsf{Perm}(\alpha)\,(p(\pi(\mathcal{F})) = \pi(p(\mathcal{F})))\},$$

$$\overline{\Gamma}(f) = \Gamma(f \circ \mu) \text{ with } \mu : \alpha \to P \text{ such that } \forall i \in \alpha\,(\mu(i)(\mathcal{F}) = i).$$

Then $\mathcal{G}_{\mathcal{F}}$ is a positional frame corresponding to $\mathcal{F}$, and if $\mathcal{F}'$ is a permutation of $\mathcal{F}$, then $\mathcal{G}_{\mathcal{F}} = \mathcal{G}_{\mathcal{F}'}$.

Proof. We show that: (1) $\mathcal{G}_{\mathcal{F}}$ is a well-defined positional frame; (2) $\mathcal{G}_{\mathcal{F}}$ corresponds to $\mathcal{F}$; (3) if $\mathcal{F}' = \tau(\mathcal{F})$, then $\mathcal{G}_{\mathcal{F}} = \mathcal{G}_{\mathcal{F}'}$.

1. To prove that $\overline{\Gamma}$ is well-defined it is sufficient to show that there is exactly one function $\mu \in P^\alpha$ such that $\forall i \in \alpha\,(\mu(i)(\mathcal{F}) = i)$. We assumed that $\mathcal{F}$ has no strict symmetry, so $\pi(\mathcal{F}) = \pi'(\mathcal{F}) \Leftrightarrow \pi = \pi'$. Therefore, for every $i \in \alpha$ there is exactly one $p \in P$ with $p(\mathcal{F}) = i$.

2. To prove that $\mathcal{G}_{\mathcal{F}}$ corresponds to $\mathcal{F}$ it is sufficient to show that μ is bijective. Now μ is clearly injective, because for every $i \in \alpha$, $\mu(i)(\mathcal{F}) = i$. Furthermore, let p be an arbitrary element of P. Then $\mu(p(\mathcal{F}))(\mathcal{F}) = p(\mathcal{F})$, from which it follows that $\mu(p(\mathcal{F})) = p$. So, μ is also surjective.

3. Let $\mathcal{F}' = \langle S, O, \alpha, \Gamma' \rangle$ be another frame in $\Phi_{\mathcal{F}}$, say $\mathcal{F}' = \tau(\mathcal{F})$, and let $\mathcal{G}_{\mathcal{F}'} = \langle S, O, P', \overline{\Gamma'} \rangle$. To prove that $P' = P$, let p be an element of P. Then

$$\begin{aligned} p(\pi(\mathcal{F}')) &= p(\pi(\tau(\mathcal{F}))) \\ &= p((\pi \circ \tau)(\mathcal{F})) \\ &= (\pi \circ \tau)(p(\mathcal{F})) \\ &= \pi(\tau(p(\mathcal{F}))) \\ &= \pi(p(\tau(\mathcal{F}))) \\ &= \pi(p(\mathcal{F}')). \end{aligned}$$

Thus, p also belongs to P'. So, $P \subseteq P'$, and, mutatis mutandis, $P' \subseteq P$.

To prove that $\overline{\Gamma'} = \overline{\Gamma}$, let $\mu' \in P^\alpha$ be such that $\forall i \in \alpha\, (\mu'(i)(\mathcal{F}') = i)$. We have to prove that for any $f \in O^P$, $\Gamma'(f \circ \mu') = \Gamma(f \circ \mu)$. Because $\mathcal{F}' = \tau(\mathcal{F})$, we have $\Gamma'(f) = \Gamma(f \circ \tau)$. So, in particular, $\Gamma'(f \circ \mu') = \Gamma(f \circ \mu' \circ \tau)$. So, it is sufficient to prove that $\mu' \circ \tau = \mu$.

$$\begin{aligned} (\mu' \circ \tau)(i)(\mathcal{F}) &= \mu'(\tau(i))(\mathcal{F}) \\ &= \mu'(\tau(i))(\tau^{-1}(\mathcal{F}')) \\ &= \tau^{-1}(\mu'(\tau(i))(\mathcal{F}')) \\ &= \tau^{-1}(\tau(i)) \\ &= i. \end{aligned}$$

Because there is exactly one $\overline{\mu} \in P^\alpha$ such that $\forall i \in \alpha\, (\overline{\mu}(i)(\mathcal{F}) = i)$, we see that $\mu' \circ \tau = \mu$. ⊣

Unfortunately, if $\mathcal{F}$ has strict symmetry, then the construction of $\mathcal{G}_{\mathcal{F}}$ in Theorem 3.4.1 fails because then not for every $i \in \alpha$ there is a $p \in P$ with $p(\mathcal{F}) = i$. In such cases P may contain less than $\mathsf{card}(\alpha)$ elements. For example, if $\mathcal{F}$ has complete strict symmetry, then P is empty. And if $\mathcal{F}$ is a ternary frame with $\mathcal{F} = \pi(\mathcal{F})$ iff $\pi(1) = 1$, then P does not contain 3 functions – as we would like to have – but only 1.

3.5 Neutrality

The positional frame constructed in Theorem 3.4.1 is in an intuitive sense *neutral* with respect to the permutation class of $\mathcal{F}$. In this section, we give a formal definition of neutrality for set theory with atoms or urelements. We will use this notion in the next section to show that the permutation classes of some directional frames have no neutral corresponding positional frame in the cumulative hierarchy with the elements of S and O as urelements.

Let $\mathsf{V}[A]$ be the cumulative hierarchy with atoms A. Any function $u : A \to A$ can be lifted to a function $\widetilde{u} : \mathsf{V}[A] \to \mathsf{V}[A]$ in an obvious way:

- $\widetilde{u}(a) = u(a)$ for any $a \in A$,
- $\widetilde{u}(X) = \{\widetilde{u}(x) \mid x \in X\}$.

We may regard $\widetilde{u}(X)$ as the result of a transformation where for each $a \in A$ all its occurrences in X are substituted by $u(a)$.

We will treat any function f as a set of ordered pairs. Thus we may speak about $\widetilde{u}(f)$ as the image of this set.

We have the following elementary properties for $\widetilde{u}$:

Lemma 3.5.1. *Let $u : A \to A$ be lifted to $\widetilde{u} : \mathsf{V}[A] \to \mathsf{V}[A]$. Then:*

1. $\widetilde{u \circ v} = \widetilde{u} \circ \widetilde{v}$,
2. *if* $u = \mathsf{id}_A$, *then* $\widetilde{u} = \mathsf{id}_{\mathsf{V}[A]}$,
3. u *is injective iff* $\widetilde{u}$ *is injective,*
4. *if* u *is bijective, then* $\widetilde{u^{-1}} = \widetilde{u}^{-1}$,
5. *if* u *is injective, then* $\widetilde{u}$ *maps any function* $f : X \to Y$ *with* $X, Y \in \mathsf{V}[A]$ *to a function* $\widetilde{u}(f) : \widetilde{u}(X) \to \widetilde{u}(Y)$ *with*

$$(\widetilde{u}(f))(\widetilde{u}(x)) = \widetilde{u}(f(x)),$$

6. *if* u *is injective, then* $\widetilde{u}$ *is an endo-functor on* $\mathsf{V}[A]$, *i.e.*
   (a) *for any* $X \in \mathsf{V}[A]$, $\widetilde{u}(\mathsf{id}_X) = \mathsf{id}_{\widetilde{u}(X)}$,

(b) for any function $f : X \to Y$ and $g : Y \to Z$ with $X, Y, Z \in \mathsf{V}[A]$,

$$\widetilde{u}(f \cdot g) = \widetilde{u}(f) \cdot \widetilde{u}(g).$$

Proof. We prove Property 1 by $\in$-induction: (i) If $x \in A$, then $\widetilde{u \circ v}(x) = u \circ v(x) = u \circ \widetilde{v}(x) = \widetilde{u} \circ \widetilde{v}(x)$. (ii) Let $x \in \mathsf{V}[A]$ and assume $\widetilde{u \circ v}(z) = \widetilde{u} \circ \widetilde{v}(z)$ for every $z \in x$. Then $\widetilde{u \circ v}(x) = \{\widetilde{u \circ v}(z) \mid z \in x\} = \{\widetilde{u} \circ \widetilde{v}(z) \mid z \in x\} = \widetilde{u}(\{\widetilde{v}(z) \mid z \in x\}) = \widetilde{u} \circ \widetilde{v}(x)$. So, by $\in$-induction, $\widetilde{u \circ v} = \widetilde{u} \circ \widetilde{v}$.

Properties 2 to 4 can be proved in a similar way by $\in$-induction.

To prove Property 5, assume $u : A \to A$ is injective. Then by Property 3, $\widetilde{u}$ is also injective. So, if f is a function, then $\widetilde{u}(f)$ is a function as well. Furthermore, if $f : x \mapsto y$, then $\widetilde{u}(f) : \widetilde{u}(x) \mapsto \widetilde{u}(y)$.

Property 6a is trivial and Property 6b follows from Property 5:

$$\begin{aligned}
(\widetilde{u}(f \cdot g))(\widetilde{u}(x)) &= \widetilde{u}((f \cdot g)(x)) \\
&= \widetilde{u}(g(f(x)) \\
&= (\widetilde{u}(g))(\widetilde{u}(f(x))) \\
&= (\widetilde{u}(g))((\widetilde{u}(f))(\widetilde{u}(x))) \\
&= (\widetilde{u}(f) \cdot \widetilde{u}(g))(\widetilde{u}(x)).
\end{aligned}$$

⊣

Using this lemma it is easy to prove that if $\mathcal{F}$ and $\mathcal{G}$ are corresponding frames in $\mathsf{V}[S, O]$ and $u : S \cup O \to S \cup O$ is injective, then $\widetilde{u}(\mathcal{F})$ and $\widetilde{u}(\mathcal{G})$ correspond as well.

We now define what it means for a set in $\mathsf{V}[A]$ to be neutral with respect to another set in $\mathsf{V}[A]$.

Definition 3.5.2. For $X, Y \in \mathsf{V}[A]$ we say that Y is *neutral with respect to* X if for any bijection $u : A \to A$,

$$\widetilde{u}(X) = X \quad \Rightarrow \quad \widetilde{u}(Y) = Y.$$

Note that if Y is neutral with respect to X, and Z is neutral with respect to Y, then Z is also neutral with respect to X. However, if Y and Z are both neutral with respect to X, then Z is not necessarily neutral with respect to Y.

Example 3.5.3. Let $A = \{a, b\}$ be a set of atoms. Then $\{a, b\}$ and every set in V are neutral with respect to $\{a, b\}$, but $\{a\}$ and $\{a, \{b\}\}$ are not. However, every set in $\mathsf{V}[A]$ is neutral with respect to $\{a, \{b\}\}$. $\dashv$

To see the relevance of this formal notion of neutrality for modeling "real" structures, consider a set A of specific entities. Suppose we are given a set $X \in \mathsf{V}[A]$ that models certain connections between the entities in A. Then for any deterministic construction of a set Y purely on the basis of the structure of X (treating the elements of A as atoms), the set Y must be neutral with respect to X. Thus, if we can show that no member of a certain class of models is neutral with respect to a given model, then this may give us valuable information about the impossibility of certain constructions.

3.6 Neutrality w.r.t. permutation classes

In this section, we treat the states S and objects O as urelements. Unless mentioned otherwise, we do not assume that S and O are disjoint.

Let us now face the case in which a directional frame $\mathcal{F} = \langle S, O, \alpha, \Gamma \rangle$ may have strict symmetry. Then it is simple to construct a corresponding positional frame $\mathcal{G} = \langle S, O, P, \overline{\Gamma} \rangle$ in which all frames in $\Phi_{\mathcal{F}}$, the permutation class of $\mathcal{F}$, are more or less equally well represented:

1. Choose $\Pi \subseteq \mathsf{Perm}(\alpha)$ with $\mathsf{id}_\alpha \in \Pi$ and $\forall \mathcal{F}' \in \Phi_{\mathcal{F}}\, \exists!\, \pi \in \Pi\, (\pi(\mathcal{F}) = \mathcal{F}')$.

2. Define $P = \{p : \Phi_{\mathcal{F}} \to \alpha \mid \forall \pi \in \Pi\, (p(\pi(\mathcal{F})) = \pi(p(\mathcal{F})))\}$.

3. Define $\overline{\Gamma}(f) = \Gamma(f \circ \mu)$ with μ such that $\forall i \in \alpha\, (\mu(i)(\mathcal{F}) = i)$.

We can prove in a similar way as we did for the frames without strict symmetry, that $\mathcal{G}$ corresponds to $\mathcal{F}$. But if $\mathcal{F}$ belongs to $\mathsf{V}[S, O]$, then $\mathcal{G}$ is not necessarily neutral with respect to the permutation class of $\mathcal{F}$, as a consequence of Theorem 3.6.7 later in this section.

The next lemma shows that if $u : S \cup O \to S \cup O$ is injective, then $\widetilde{u}{\restriction}\Phi_{\mathcal{F}}$ is a structure preserving mapping.

Lemma 3.6.1. *Let $\mathcal{F} = \langle S, O, \alpha, \Gamma \rangle \in \mathsf{V}[S, O]$ be a directional frame. If $u : S \cup O \to S \cup O$ is injective and $\pi \in \mathsf{Perm}(\alpha)$, then $\widetilde{u}{\restriction}\Phi_{\mathcal{F}} : \Phi_{\mathcal{F}} \to \Phi_{\widetilde{u}(\mathcal{F})}$ is a bijection for which the following diagram commutes:*

$$\begin{array}{ccc} \Phi_{\mathcal{F}} & \xrightarrow{\widetilde{u}{\restriction}\Phi_{\mathcal{F}}} & \Phi_{\widetilde{u}(\mathcal{F})} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \Phi_{\mathcal{F}} & \xrightarrow[\widetilde{u}{\restriction}\Phi_{\mathcal{F}}]{} & \Phi_{\widetilde{u}(\mathcal{F})} \end{array}$$

Proof. Let $\pi(\mathcal{F}) = \langle S, O, \alpha, \Gamma' \rangle$. Then, by Lemma 3.5.1,

$$(\widetilde{u}(\Gamma'))(\widetilde{u}(f)) = \widetilde{u}(\Gamma'(f)) = \widetilde{u}(\Gamma(f \circ \pi)) = (\widetilde{u}(\Gamma))(\widetilde{u}(f) \circ \pi).$$

So, $\widetilde{u}(\pi(\mathcal{F})) = \pi(\widetilde{u}(\mathcal{F}))$, from which it follows that the diagram of the lemma commutes and that $\widetilde{u}{\restriction}\Phi_{\mathcal{F}} : \Phi_{\mathcal{F}} \to \Phi_{\widetilde{u}(\mathcal{F})}$ is surjective. By Property 3 of Lemma 3.5.1 we also see that $\widetilde{u}{\restriction}\Phi_{\mathcal{F}}$ is injective. ⊣

As we might have expected, if $\mathcal{F}$ has no strict symmetry, then the corresponding positional frame $\mathcal{G}_{\mathcal{F}}$ defined in Theorem 3.4.1 is neutral with respect to $\Phi_{\mathcal{F}}$:

Theorem 3.6.2. *Let $\mathcal{F} = \langle S, O, \alpha, \Gamma \rangle \in \mathsf{V}[S, O]$ be a directional frame without strict symmetry, and let $\mathcal{G}_{\mathcal{F}} = \langle S, O, P, \overline{\Gamma} \rangle$ be defined by:*

$$P = \{p : \Phi_{\mathcal{F}} \to \alpha \mid \forall \pi \in \mathsf{Perm}(\alpha)\, (p(\pi(\mathcal{F})) = \pi(p(\mathcal{F})))\},$$

$$\overline{\Gamma}(f) = \Gamma(f \circ \mu) \text{ with } \mu \text{ such that } \forall i \in \alpha\, (\mu(i)(\mathcal{F}) = i).$$

Then $\mathcal{G}_{\mathcal{F}}$ is neutral with respect to $\Phi_{\mathcal{F}}$.

Proof. Let $u : S \cup O \to S \cup O$ be a bijection such that $\widetilde{u}(\Phi_{\mathcal{F}}) = \Phi_{\mathcal{F}}$. We will show that $\widetilde{u}(P) = P$ and that $\widetilde{u}(\Gamma) = \Gamma$.

With the use of Lemma 3.5.1 and Lemma 3.6.1 it is easy to see that:

$$\widetilde{u}(P) = \{p : \widetilde{u}(\Phi_{\mathcal{F}}) \to \alpha \mid \forall \pi \in \mathsf{Perm}(\alpha)\, (p(\pi(\widetilde{u}(\mathcal{F}))) = \pi(p(\widetilde{u}(\mathcal{F}))))\}.$$

Because $\Phi_{\mathcal{F}} = \widetilde{u}(\Phi_{\mathcal{F}})$, and because, by Theorem 3.4.1, $\mathcal{G}_{\mathcal{F}} = \mathcal{G}_{\widetilde{u}(\mathcal{F})}$, it follows that $\widetilde{u}(P) = P$. Furthermore, by Lemma 3.5.1,

$$(\widetilde{u}(\overline{\Gamma}))(\widetilde{u}(f)) = (\widetilde{u}(\Gamma))((\widetilde{u}(f)) \circ \widetilde{u}(\mu))$$

with $\widetilde{u}(\mu)$ such that $\forall i \in \alpha\, ((\widetilde{u}(\mu))(i)(\widetilde{u}(\mathcal{F})) = i)$.

Because $\widetilde{u}(P) = P$, and because $\mathcal{G}_{\mathcal{F}} = \mathcal{G}_{\widetilde{u}(\mathcal{F})}$, it also follows that $\widetilde{u}(\overline{\Gamma}) = \overline{\Gamma}$. ⊣

For the next results we need the notion of *permutation-based frames*:

Definition 3.6.3. Let $\mathcal{F} = \langle S, O, \alpha, \Gamma\rangle$ be a directional frame. We define the *permutation group of* $\mathcal{F}$ as:

$$\mathsf{Perm}_{\mathcal{F}} = \{\pi \in \alpha^\alpha \mid \pi \text{ is a bijection } \& \ \forall f \in O^\alpha\, (\Gamma(f \circ \pi) = \Gamma(f))\}.$$

We call $\mathcal{F}$ a *permutation-based frame* if for every $f, f' \in \alpha^\alpha$,

$$\Gamma(f') = \Gamma(f) \ \Rightarrow\ f' = f \circ \pi \text{ for some } \pi \in \mathsf{Perm}_{\mathcal{F}}.$$

Every directional frame $\mathcal{F}$ with complete strict symmetry has a corresponding positional frame that is neutral with respect to $\Phi_{\mathcal{F}}$:

Example 3.6.4. Let $\mathcal{F} = \langle S, O, \alpha, \Gamma\rangle \in \mathsf{V}[S, O]$ be a directional frame with complete strict symmetry, i.e. $\mathsf{Perm}_{\mathcal{F}} = \mathsf{Perm}(\alpha)$. Then obviously $\langle S, O, \alpha, \Gamma\rangle$ considered as a positional frame is neutral with respect $\Phi_{\mathcal{F}}$, since $\mathcal{F}$ is the only frame in $\Phi_{\mathcal{F}}$. ⊣

Also for certain cyclic frames we have a positive result:

Example 3.6.5. Let $\mathcal{F} = \langle S, O, 4, \Gamma\rangle \in \mathsf{V}[S]$ be a permutation-based frame with permutation group $\mathsf{Perm}_{\mathcal{F}}$ generated by

$$\pi_0 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 0 & 1 & 2 \end{pmatrix}.$$

Let $\pi_1 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \end{pmatrix}$. Define $\mathcal{G}_{\mathcal{F}} = \langle S, O, P, \overline{\Gamma}\rangle$ with

$$P = \{p : \{\mathcal{F}, \pi_1(\mathcal{F})\} \to \{0, 1, 2, 3\} \mid p(\pi_1(\mathcal{F})) = \pi_1(p(\mathcal{F}))\},$$

$$\overline{\Gamma}(f) = \Gamma(f \circ \mu) \text{ with } \mu \text{ such that } \forall i \in \alpha\, (\mu(i)(\mathcal{F}) = i).$$

Then $\mathcal{G}_{\mathcal{F}}$ corresponds to $\mathcal{F}$ and is neutral with respect to $\Phi_{\mathcal{F}}$.

To see this, first note that by Lemma 3.6.1, if $u : S \to S$ is a bijection such that $\widetilde{u}(\Phi_{\mathcal{F}}) = \Phi_{\mathcal{F}}$, then $\widetilde{u}(\mathcal{F}) = \mathcal{F}$ or $\widetilde{u}(\mathcal{F}) = \pi_1(\mathcal{F})$. It follows by an analysis similar to the one in Theorem 3.6.2 that $\widetilde{u}(P) = P$ and $\widetilde{u}(\overline{\Gamma}) = \overline{\Gamma}$. ⊣

In Section 3.4 we saw how to create a neutral positional frame for the love relation. But now consider relation $\Re$ in which $\Re abcd$ represents the state of *a's loving b and c's loving d.* Let $\mathcal{F} \in \mathsf{V}[S, O]$ be a directional frame for this relation, and let $\mathcal{G} \in \mathsf{V}[S, O]$ be a corresponding positional frame. We claim that $\mathcal{G}$ cannot be neutral with respect to the permutation class $\Phi_{\mathcal{F}}$. We sketch a proof in the next example. Then, in Theorem 3.6.7, we prove a generalization of the claim.

Example 3.6.6. Let $\mathcal{F} = \langle S, O, \alpha, \Gamma\rangle \in \mathsf{V}[S, O]$ be a directional frame with for any $a, b, c, d \in O$

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ a & b & c & d \end{pmatrix} \quad \stackrel{\Gamma}{\longmapsto} \quad a \xrightarrow{\heartsuit} b \ \ \& \ \ c \xrightarrow{\heartsuit} d.$$

Let $\mathcal{G} = \langle S, O, P, \overline{\Gamma}\rangle \in \mathsf{V}[S, O]$ be a corresponding positional frame with $P = \{p_0, p_1, p_2, p_3\}$, and

$$\begin{pmatrix} p_0 & p_1 & p_2 & p_3 \\ a & b & c & d \end{pmatrix} \quad \stackrel{\overline{\Gamma}}{\longmapsto} \quad a \xrightarrow{\heartsuit} b \ \ \& \ \ c \xrightarrow{\heartsuit} d.$$

Let $u : S \cup O \to S \cup O$ be such that

$$u(a \xrightarrow{\heartsuit} b \ \ \& \ \ c \xrightarrow{\heartsuit} d) \ = \ b \xrightarrow{\heartsuit} c \ \ \& \ \ d \xrightarrow{\heartsuit} a,$$

and $u \restriction O = \mathsf{id}_O$. Obviously, $u \circ u = \mathsf{id}_{\mathsf{S} \cup \mathsf{O}}$, and so, by Lemma 3.5.1, $\widetilde{u} \circ \widetilde{u} = \mathsf{id}_{\mathsf{V}[S,O]}$. Furthermore, it is not difficult to see that $\widetilde{u}(\Phi_{\mathcal{F}}) = \Phi_{\mathcal{F}}$.

Now suppose $\mathcal{G}$ is neutral with respect to $\Phi_{\mathcal{F}}$. Then

$$\begin{pmatrix} \widetilde{u}(p_0) & \widetilde{u}(p_1) & \widetilde{u}(p_2) & \widetilde{u}(p_3) \\ a & b & c & d \end{pmatrix} \quad \stackrel{\overline{\Gamma}}{\longmapsto} \quad b \xrightarrow{\heartsuit} c \ \ \& \ \ d \xrightarrow{\heartsuit} a.$$

Thus,

$$\widetilde{u}(p_0, p_1, p_2, p_3) = (p_1, p_2, p_3, p_0) \ \text{ or } \ \widetilde{u}(p_0, p_1, p_2, p_3) = (p_3, p_0, p_1, p_2).$$

But then $\widetilde{u} \restriction P \circ \widetilde{u} \restriction P \neq \mathsf{id}_P$, contradicting $\widetilde{u} \circ \widetilde{u} = \mathsf{id}_{\mathsf{V}[S,O]}$. Therefore, $\mathcal{G}$ cannot be neutral with respect to $\Phi_{\mathcal{F}}$. ⊣

How should we interpret this example? It surely does not say that it is impossible to find a natural positional frame for the disjoint conjunction of two love relations. A quite natural positional frame for it is the frame with positions $Lover_1$, $Beloved_1$, $Lover_2$, and $Beloved_2$. Interestingly, this frame is neutral with respect to the permutation class of $\mathcal{F}$, if $\mathcal{F}$ would have been defined in $\mathsf{V}[S_0, O]$ with S_0 being the states of the ordinary love relation, and the conjunction of states s_1 and s_2 would have been modeled as $\{s_1, s_2\}$.

In the example, we could depict each state as four objects equally spaced on a circle such that rotating them by 180° always gives the same state, but rotating them by 90° gives a different state when the objects are not all the same. In the next theorem we prove that any positional frame $\mathcal{F} \in \mathsf{V}[S, O]$ with this property and S, O disjoint has no corresponding positional frame that is neutral with respect to the permutation class of $\mathcal{F}$. We will use this theorem to argue that it is very unlikely that for any "real" relation we can always make a neutral choice for a corresponding positional frame.

Theorem 3.6.7. *Let $\mathcal{F} = \langle S, O, 4, \Gamma \rangle \in \mathsf{V}[S, O]$ be a permutation-based frame with $S \cap O = \varnothing$ and permutation group $\mathsf{Perm}_{\mathcal{F}}$ generated by*

$$\pi_0 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & 3 & 0 & 1 \end{pmatrix}.$$

Then no corresponding positional frame is neutral with respect to $\Phi_{\mathcal{F}}$.

Proof. Let $\pi_1 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 0 & 1 & 2 \end{pmatrix}$. Define $u : S \cup O \to S \cup O$ by:

$$u(x) = \begin{cases} \Gamma(f \circ \pi_1) & \text{if } x = \Gamma(f), \\ x & \text{otherwise.} \end{cases}$$

We will show that:

1. u is a well-defined bijective function with $u \circ u = \mathsf{id}_{S \cup O}$.
2. $\widetilde{u}(\Phi_{\mathcal{F}}) = \Phi_{\mathcal{F}}$.
3. $\widetilde{u}(\mathcal{G}) \neq \mathcal{G}$ for any positional frame $\mathcal{G}$ corresponding to $\mathcal{F}$.

1. Suppose $\Gamma(f') = \Gamma(f)$. Then, because $\mathcal{F}$ is permutation-based, $f' = f$ or $f' = f \circ \pi_0$. Assume $f' = f \circ \pi_0$. Then $f' \circ \pi_1 = f \circ \pi_1 \circ \pi_0$ because

$\pi_0 = \pi_1 \circ \pi_1$. So, $\Gamma(f' \circ \pi_1) = \Gamma(f \circ \pi_1)$ because $\pi_0 \in \mathsf{Perm}_{\mathcal{F}}$. It follows that u is well-defined.

Because $(u \circ u)(\Gamma(f)) = \Gamma(f \circ \pi_1 \circ \pi_1) = \Gamma(f \circ \pi_0) = \Gamma(f)$, we see that $u \circ u = \mathsf{id}_{S \cup O}$. So, u is bijective, and, by Lemma 3.5.1, $\widetilde{u} \circ \widetilde{u} = \mathsf{id}_{\mathsf{V}[S,O]}$.

2. Because $S \cap O = \varnothing$, we have $u \restriction O = \mathsf{id}_O$. Thus $\widetilde{u}(\mathcal{F}) = \pi_1(\mathcal{F})$, and so, because $\Phi_{\mathcal{F}} = \Phi_{\pi_1(\mathcal{F})}$, we have by Lemma 3.6.1, $\widetilde{u}(\Phi_{\mathcal{F}}) = \Phi_{\mathcal{F}}$.

3. Let $\mathcal{G} = \langle S, O, P, \overline{\Gamma} \rangle$ be a positional frame corresponding to $\mathcal{F}$. Say $\mathcal{G} = \mu_0(\mathcal{F})$ with

$$\mu_0 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ p_0 & p_1 & p_2 & p_3 \end{pmatrix}.$$

Now suppose $\widetilde{u}(\mathcal{G}) = \mathcal{G}$. Then because $\widetilde{u}(P) = P$, for any $f \in O^P$ and any $p \in P$,

$$\begin{aligned} f(\widetilde{u}(p)) &= (f \circ \widetilde{u} \restriction P)(p) \\ &= \widetilde{u}((f \circ \widetilde{u} \restriction P)(p)) && \text{because } u \restriction O = \mathsf{id}_O \\ &= (\widetilde{u}(f \circ \widetilde{u} \restriction P))(\widetilde{u}(p)) && \text{by Lemma 3.5.1, prop. 5.} \end{aligned}$$

So, $f = \widetilde{u}(f \circ \widetilde{u} \restriction P)$. Therefore

$$\begin{aligned} \overline{\Gamma}(f) &= (\widetilde{u}(\overline{\Gamma}))(f) && \text{because } \widetilde{u}(\overline{\Gamma}) = \overline{\Gamma} \\ &= (\widetilde{u}(\overline{\Gamma}))(\widetilde{u}(f \circ \widetilde{u} \restriction P)) \\ &= \widetilde{u}(\overline{\Gamma}(f \circ \widetilde{u} \restriction P)) && \text{by Lemma 3.5.1, prop. 5.} \end{aligned}$$

Because $\overline{\Gamma}(f) = \Gamma(f \circ \mu_0)$ and $\widetilde{u}(\Gamma(f)) = u(\Gamma(f)) = \Gamma(f \circ \pi_1)$, we get

$$\Gamma(f \circ \mu_0) = \Gamma(f \circ \widetilde{u} \restriction P \circ \mu_0 \circ \pi_1).$$

Because $\mathsf{Perm}_{\mathcal{F}}$ is generated by π_0, we have $|O| \geq 2$, and so, because $\mathcal{F}$ is permutation-based, $\mu_0 = \widetilde{u} \restriction P \circ \mu_0 \circ \pi_1$ or $\mu_0 = \widetilde{u} \restriction P \circ \mu_0 \circ \pi_1 \circ \pi_0$. It follows that

$$\widetilde{u} \restriction P = \begin{pmatrix} p_0 & p_1 & p_2 & p_3 \\ p_1 & p_2 & p_3 & p_0 \end{pmatrix} \quad \text{or} \quad \widetilde{u} \restriction P = \begin{pmatrix} p_0 & p_1 & p_2 & p_3 \\ p_3 & p_0 & p_1 & p_2 \end{pmatrix}.$$

But this contradicts that $u \circ u = \mathsf{id}_{S \cup O}$. So, $\widetilde{u}(\mathcal{G}) \neq \mathcal{G}$. ⊣

Note that for any O with at least two objects, a frame can be defined that fulfills the conditions of the theorem. Also note that if $\mathcal{F}$ would be in $\mathsf{V}[S]$ and $O \in \mathsf{V}$, then we would get a similar result as in the theorem.

What conclusions can we draw from this theorem? Does it follow that some "real" relations do not have a positional frame that is neutral with respect to the permutation class of any reasonable frame for it? Unfortunately, we have not found an example of an atomic "real" relation that could be modeled by frame $\mathcal{F}$ of the theorem. So, let us look at molecular relations.

Suppose we have two binary relations $\Re_1$ and $\Re_2$. Let $\Re$ be the quaternary relation with $\Re abcd$ being the state of

$$\Re_1 ab \ \ \& \ \ \Re_2 bc \ \ \& \ \ \Re_1 cd \ \ \& \ \ \Re_2 da.$$

Then, depending on properties of $\Re_1$ and $\Re_2$, the relation $\Re$ may have a frame $\mathcal{F}$ as in Theorem 3.6.7. It may even be the case that if $\mathcal{F}$ is defined in $\mathsf{V}[S_1, S_2, O]$, with S_1 being the states of $\Re_1$ and S_2 being the states of $\Re_2$, then there is no corresponding positional frame that is neutral with respect to $\Phi_{\mathcal{F}}$. This is for example the case when $\Re_1 ab$ is the state of $a's$ loving b and $\Re_2 ab$ the state of $a's$ hating b. Examples like this put us more or less in a similar position as Buridan's ass. Like the ass cannot choose between two piles of hay, we seem to be unable to make a deliberate choice of sets for the positions of the relation $\Re$.

3.7 Neutrality w.r.t. substitution frames

In the previous section we started with permutation classes of biased frames to create positional frames that are neutral with respect to these classes. But it is not clear that such a class of biased frames itself always gives an unbiased account of the underlying relation. It may be better to start instead with more primitive means, like substitution frames. We investigate that in this section.

We restrict our discussion to *simple substitution frames*:

Definition 3.7.1. Let $\mathcal{F} = \langle S, O, \Sigma \rangle$ be a substitution frame. We call $\mathcal{F}$ a *simple substitution frame* if there is a state s_0 such that

$$S = \{s_0 \cdot \delta \mid \delta : O \to O\}.$$

We call s_0 an *initial state.*

In [Leo08a] we defined the notion of a *simple relation* in terms of metaphysical principles satisfied by the relation. In this paper we will use the term in a more loose sense by calling a relation simple if it can *adequately* be modeled by a simple substitution frame. Furthermore, we will say that a simple relation has a neutral positional frame if the positional frame corresponds to a substitution frame for the relation, and it is neutral with respect to this substitution frame.

In [Leo08a] we proved the following theorem about the relationship between substitution frames and positional frames:

Theorem 3.7.2. *A substitution frame $\mathcal{F}$ corresponds to some positional frame $\mathcal{G}$ of the same degree iff $\mathcal{F}$ is a simple substitution frame.*

Furthermore, if $\mathsf{degree}_{\mathcal{F}}$ is finite, then $\mathcal{G}$ is unique, modulo positional variants.

In the proof of the theorem we constructed $\mathcal{G}$ as follows:

1. Choose an initial state $s_0 \in S$.
2. Choose an object-domain A of s_0 with $|A| = \mathsf{degree}_{\mathcal{F}}(s_0)$.
3. Define $P = A$.
4. Let f be an arbitrary element of O^P. Let f^+ extend f to $O \to O$. Define $\Gamma(f) = s_0 \cdot_{\mathcal{F}} f^+$.

Obviously, we may get in this way many positional frames that are each not neutral with respect to $\mathcal{F}$.

Definition 3.7.3. Let $\mathcal{F} = \langle S, O, \Sigma \rangle$ be a substitution frame of finite degree. We say that $\mathcal{F}$ has *strict symmetry* if there is a state $s \in S$ and a $\delta \neq \mathsf{id}_{\mathsf{Ob}(s)}$ such that $s \cdot \delta = s$.

For any substitution frame of finite degree without strict symmetry, we can construct a corresponding positional frame that is neutral with respect to it:

Example 3.7.4. Let $\mathcal{F} = \langle S, O, \Sigma \rangle \in \mathsf{V}[S, O]$ be a simple substitution frame of finite degree without strict symmetry. Let S_0 be the set of initial states of $\mathcal{F}$. Define $\mathcal{G} = \langle S, O, P, \Gamma \rangle$ with

$$P = \{p : S_0 \to O \mid \forall s \in S_0\, (p(s) \in \mathsf{Ob}(s)\ \& \\ \forall \text{ bijection } \pi : O \to O\, (p(s \cdot_{\mathcal{F}} \pi) = \pi(p(s))))\},$$

$$\Gamma(f) = s_0 \cdot \delta \text{ with } s_0 \in S_0 \text{ and } \forall p \in P\,(\delta(p(s_0)) = f(p)).$$

It is not difficult to show that $\mathcal{G}$ is neutral with respect to $\mathcal{F}$. ⊣

As an aside, we note that the positions defined in the example can be identified with the *roles* of the substitution frame:

Definition 3.7.5. Let $\mathcal{F} = \langle S, O, \Sigma \rangle$ be a simple substitution frame of finite object-degree. Then for any initial state $s_0 \in S$ and $a_0 \in \mathsf{Ob}(s_0)$, we define the *role of* a_0 *in* s_0 as:

$$\mathsf{Role}(s_0, a_0) = \{(s, a) \mid \exists \delta\,(s \cdot \delta = s_0 \;\;\&\;\; a \in \mathsf{Ob}(s) \;\;\&\;\; \delta(a) = a_0)\}.$$

Furthermore, we define the *roles of* $\mathcal{F}$ as:

$$\mathsf{Roles}_{\mathcal{F}} = \{\mathsf{Role}(s_0, a_0) \mid s_0 \in S \text{ is an initial state } \;\&\;\; a_0 \in \mathsf{Ob}(s_0)\}.$$

The next theorem in combination with the results of the previous section shows that there is little hope for a neutral reconstruction of argument-places in ordinary set theory for every relation:

Theorem 3.7.6. *Let substitution frame $\mathcal{F} = \langle S, O, \Sigma \rangle \in \mathsf{V}[S, O]$ and directional frame $\mathcal{G} = \langle S, O, \alpha, \Gamma \rangle$ be corresponding frames of the same finite degree. Then $\mathcal{F}$ and $\Phi_{\mathcal{G}}$, the permutation class of $\mathcal{G}$, are neutral with respect to each other.*

Proof. An informal argument to see the correctness of this theorem is that we can construct $\mathcal{F}$ deterministically purely based on the structure of $\Phi_{\mathcal{G}}$, and vice versa. Our formal argument requires a few more steps:

Let $u : S \cup O \to S \cup O$ be a bijection such that $\widetilde{u}(\Phi_{\mathcal{G}}) = \Phi_{\mathcal{G}}$. Then there is a bijection $\pi : \alpha \to \alpha$ such that for each $f \in O^{\alpha}$, $(\widetilde{u}(\Gamma))(f) = \Gamma(\pi \cdot f)$. Furthermore, because $\mathcal{F}$ and $\mathcal{G}$ correspond, there is for each $s \in S$ an f

such that $s = \Gamma(f)$. So,

$$\begin{aligned}
\widetilde{u}(s) \cdot_{\widetilde{u}(\mathcal{F})} \widetilde{u}(\delta) &= \widetilde{u}(s \cdot_{\mathcal{F}} \delta) && \text{by Lemma 3.5.1, prop. 5} \\
&= \widetilde{u}(\Gamma(f) \cdot_{\mathcal{F}} \delta) \\
&= \widetilde{u}(\Gamma(f \cdot \delta)) && \text{because } \mathcal{F} \text{ and } \mathcal{G} \text{ correspond} \\
&= (\widetilde{u}(\Gamma))\widetilde{u}(f \cdot \delta) && \text{by Lemma 3.5.1, prop. 5} \\
&= \Gamma(\pi \cdot \widetilde{u}(f \cdot \delta)) \\
&= \Gamma(\pi \cdot \widetilde{u}(f) \cdot \widetilde{u}(\delta)) && \text{by Lemma 3.5.1, prop. 6} \\
&= \Gamma(\pi \cdot \widetilde{u}(f)) \cdot_{\mathcal{F}} \widetilde{u}(\delta) && \text{because } \mathcal{F} \text{ and } \mathcal{G} \text{ correspond} \\
&= \widetilde{u}(\Gamma(f)) \cdot_{\mathcal{F}} \ \widetilde{u}(\delta) \\
&= \widetilde{u}(s) \cdot_{\mathcal{F}} \widetilde{u}(\delta).
\end{aligned}$$

It follows that $\widetilde{u}(\mathcal{F}) = \mathcal{F}$, and thus that $\mathcal{F}$ is neutral with respect to $\Phi_{\mathcal{G}}$.

Conversely, let $u : S \cup O \to S \cup O$ be a bijection such that $\widetilde{u}(\mathcal{F}) = \mathcal{F}$. Then for each $s \in S$ and $\delta \in O^O$, $u(s) \cdot_{\mathcal{F}} \widetilde{u}(\delta) = u(s \cdot_{\mathcal{F}} \delta)$. Let s_0 be an initial state of $\mathcal{F}$. Then $u(s_0)$ is also an initial state. Because $\mathcal{F}$ and $\mathcal{G}$ are of the same finite degree, we have for some injection f_0, $s_0 = \Gamma(f_0)$, and for some injection f_1, $u(s_0) = \Gamma(f_1)$. So, because $\mathsf{im}\, f_1 = \mathsf{Ob}(u(s_0)) = \widetilde{u}(\mathsf{Ob}(s_0)) = \widetilde{u}(\mathsf{im}\, f_0)$, there is a bijection $\pi : \alpha \to \alpha$ such that $f_1 = \pi \cdot \widetilde{u}(f_0)$. Furthermore, for each $f \in O^\alpha$ there is a δ such that $f = f_0 \cdot \delta$. So,

$$\begin{aligned}
(\widetilde{u}(\Gamma))(\widetilde{u}(f)) &= \widetilde{u}(\Gamma(f)) && \text{by Lemma 3.5.1, prop. 5} \\
&= \widetilde{u}(\Gamma(f_0) \cdot_{\mathcal{F}} \delta) && \text{because } \mathcal{F} \text{ and } \mathcal{G} \text{ correspond} \\
&= \Gamma(f_1) \cdot_{\widetilde{u}(\mathcal{F})} \widetilde{u}(\delta) && \text{by Lemma 3.5.1, prop. 5} \\
&= \Gamma(f_1) \cdot_{\mathcal{F}} \widetilde{u}(\delta) && \text{because } \widetilde{u}(\mathcal{F}) = \mathcal{F} \\
&= \Gamma(\pi \cdot \widetilde{u}(f_0)) \cdot_{\mathcal{F}} \widetilde{u}(\delta) \\
&= \Gamma(\pi \cdot \widetilde{u}(f_0) \cdot \widetilde{u}(\delta)) && \text{because } \mathcal{F} \text{ and } \mathcal{G} \text{ correspond} \\
&= \Gamma(\pi \cdot \widetilde{u}(f_0 \cdot \delta)) && \text{by Lemma 3.5.1, prop. 5} \\
&= \Gamma(\pi \cdot \widetilde{u}(f)).
\end{aligned}$$

It follows that $\widetilde{u}(\Phi_{\mathcal{G}}) = \Phi_{\mathcal{G}}$, and thus that $\Phi_{\mathcal{G}}$ is neutral with respect to $\mathcal{F}$. ⊣

So, if the degree of the frames is finite, then it makes for a neutral reconstruction of argument-places no real difference whether we start with a substitution frame or with a permutation class of a corresponding directional frame of the same degree. But what if the degrees are infinite?

In the first part of the proof of the theorem we did not use any restriction on the degrees of the frames. So, if $\mathcal{F}$ and $\mathcal{G}$ correspond, then $\mathcal{F}$ is always neutral with respect to the permutation class of $\mathcal{G}$. Interestingly, we need the restriction for the converse:

Example 3.7.7. Let $\mathcal{F}_1 = \langle S_1, O, \Sigma_1 \rangle$ with $O = \omega$, the set of natural numbers, $S_1 = \{s : \omega \to (\omega \cup \{\infty\}) \mid \exists i\,(s(i) = \infty)\}$, and Σ_1 defined by

$$(s \cdot_{\mathcal{F}_1} \delta)(i) = \sum_{\delta(j)=i} s(j)$$

with $i + \infty = \infty + i = \infty + \infty = \infty$.

Let

$s_0 = [0^\infty, 1, 2, 3, ...]$, i.e. $s_0(0) = \infty$, and for $i \geq 1, s_0(i) = 1$;
$s_0' = [0^\infty, 1^\infty, 2, 3, ...]$, i.e. $s_0'(0) = s_0'(1) = \infty$, and for $i \geq 2, s_0'(i) = 1$.

Define

$$\mathcal{G}_1 = \langle S_1, O, \omega, \Gamma_1 \rangle \text{ with } \Gamma_1(f) = s_0 \cdot_{\mathcal{F}_1} f;$$
$$\mathcal{G}_1' = \langle S_1, O, \omega, \Gamma_1' \rangle \text{ with } \Gamma_1'(f) = s_0' \cdot_{\mathcal{F}_1} f.$$

In [Leo08a] we showed that $\mathcal{G}_1$ and $\mathcal{G}_1'$ correspond to $\mathcal{F}_1$, and that $\mathcal{G}_1$ and $\mathcal{G}_1'$ are not permutations of each other.

Now define $\mathcal{F} = \langle S, O, \Sigma \rangle$ with $S = S_1 \times S_1$, and

$$(s, s') \cdot_{\mathcal{F}} \delta = (s \cdot_{\mathcal{F}_1} \delta, s' \cdot_{\mathcal{F}_1} \delta).$$

Furthermore, define $\mathcal{G} = \langle S, O, \omega + \omega, \Gamma \rangle$ with

$$\Gamma(f) = (\Gamma_1(f_1), \Gamma_1'(f_1')) \text{ with } f_1(\alpha) = f(\alpha) \text{ and } f_1'(\alpha) = f(\omega + \alpha).$$

Obviously $\mathcal{F}$ is a substitution frame and $\mathcal{G}$ is a corresponding directional frame. Now conceive $\mathcal{F}$ and $\mathcal{G}$ as elements of $\mathsf{V}[S]$. Define $u : S \to S$ with $u(s, s') = (s', s)$. It is not difficult to verify that $\widetilde{u}(\mathcal{F}) = \mathcal{F}$, but $\widetilde{u}(\Phi_{\mathcal{G}}) \neq \Phi_{\mathcal{G}}$. ⊣

From Theorem 3.6.7 and 3.7.6, and the fact that every simple substitution frame has a corresponding positional frame, it follows that:

Corollary 3.7.8. *Not every simple substitution frame has a corresponding positional frame that is neutral with respect to it.*

Note that by Theorem 3.6.7 the corollary already applies to frames of degree 4.

3.8 Alternative constructions

What do the results in the previous sections tell us? Does it follow from the fact that for certain relations positions cannot be modeled in a neutral way as sets that these relations cannot have argument-places as derived entities?

I think such a conclusion would be too hasty. First of all, we should ask ourselves if we did not perhaps use too limited a notion of neutrality. Maybe we could well-order the objects or states of a relation and exploit this to deterministically construct or select a positional frame for it. The next example shows that for a given well-ordering of the objects such a selection can be made for simple relations of finite degree.

Example 3.8.1. Let $\mathcal{F} = \langle S, O, \alpha, \Gamma \rangle$ be a frame of finite degree for a given relation. Suppose we are given a well-ordering $<$ of the objects O. Then this induces a lexicographical ordering $<$ of O^α. Because α is finite, this ordering of O^α is clearly a well-ordering. So, for any $\mathcal{F}' = \langle S, O, \alpha, \Gamma' \rangle$ and $\mathcal{F}'' = \langle S, O, \alpha, \Gamma'' \rangle$, we may define

$$\mathcal{F}' < \mathcal{F}'' \quad \text{:if} \quad \Gamma'(f_0) < \Gamma''(f_0) \text{ with } f_0 \text{ the least element of } \{f : \alpha \to O \mid \Gamma'(f) \neq \Gamma''(f)\}.$$

It is easy to see that this last relation is a linear order. Because α is finite, $\Phi_{\mathcal{F}}$ is finite. So, the permutation class $\Phi_{\mathcal{F}}$ contains a least element, which we can select as positional frame for the relation by ignoring the order of α. ⊣

If α is infinite, then the lexicographical ordering of O^α is not always a well-ordering. So, in that case the construction of the previous example does not work. If O and α are both $\mathbb{N}$, the set of natural numbers, then no method for how to well-order O^α is known. Perhaps other constructions might work to uniquely select one frame in $\Phi_{\mathcal{F}}$. For example, if O and S are representable in V, the cumulative hierarchy, and if $\mathsf{V} = \mathsf{L}$, with L the constructible universe, then we can use a definable well-ordering of L to uniquely select an element of $\Phi(\mathcal{F})$.

But how realistic is it to assume that O and S can always be well-ordered in a deterministic way or that they are representable by sets in any model of ZFC, that is, Zermelo-Fraenkel set theory (ZF) with the axiom of choice (AC)? Recall that O and S are *objects* and *states*. We may even have no

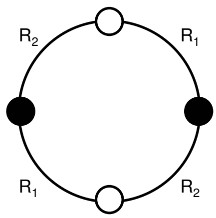


Figure 3.1: A neutral graphical representation of argument-places.

reason to assume *a priori* that the elements of O and S are always all discernible. But even if we would restrict ourselves to relations with each a finite set of discernible objects, then because of the diversity of objects, it is still questionable whether a single method can be found to deterministically order each set O. Also it is not very mathematically elegant to require an extrinsic ordering of the objects. In summary, I doubt that looking for specific orderings of the sets of objects and states of a relation is very promising as a general solution.

There is a stronger argument why the conclusion that relations cannot have argument-places as derived entities is too hasty. There is up front no reason why argument-places would have to be sets in V or $\mathsf{V}[S, O]$. *It could be the case that ordinary set theory as a modeling medium is too limited.* Other modeling media might be more adequate for some relations. We could, for example, model the quaternary relation $\mathfrak{R}$ at the end of Section 3.6 graphically in a neutral way, as shown in Figure 3.1.

But I doubt that for every simple relation of finite degree a non-arbitrary graphical representation is possible. For example, if we have an *atomic* relation with a directional model as in Theorem 3.6.7, then I do not see how to label the positions in a neutral way. There is, however, a related and more promising approach to define argument-places in a neutral way. The approach will probably appeal to at least the structuralists among us. We discuss it in the next section.

3.9 Argument-places as abstractions

Let $\mathfrak{R}$ be a simple relation of finite degree, and $\mathcal{F}$ a substitution frame for it. As we remarked earlier and proved in [Leo08a] a corresponding positional frame $\mathcal{G}$ exists, which is unique modulo positional variants. I propose to

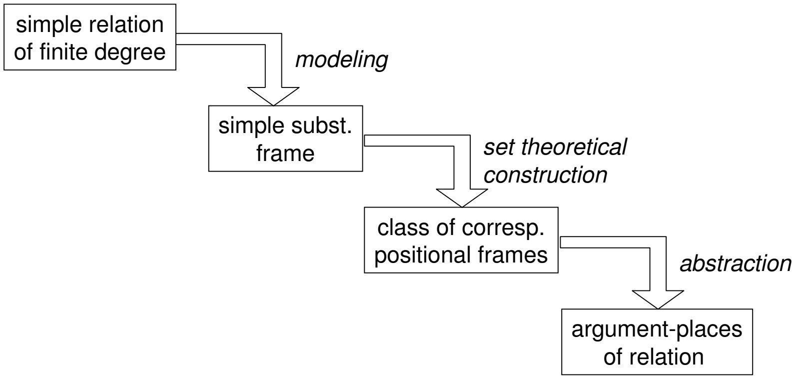


Figure 3.2: Construction of abstract argument-places.

define the argument-places of $\Re$ as structureless *places* exemplified by the positions of $\mathcal{G}$. See Figure 3.2.

This approach gives us argument-places as entities of a structure obtained by abstraction. I have no strong opinion about the question whether we should take them literally as genuine *objects* or that we should regard talk of argument-places as abstractions, as nothing but a convenient way of saying something about the underlying positional frames. I regard this issue as part of a much more comprehensive ontological debate that I want to avoid here. See e.g. [Par04] and [Sha97].

Consider the adjacency relation. With the view of argument-places as abstractions, the two argument-places for this relation fulfill exactly the same role. Since the argument-places have no internal structure, the two argument-places are completely indiscernible from each other. More generally, the argument-places of a relation fulfilling the same role are indiscernible. For simple relations of finite degree the concept of roles as we use it here matches with the roles as defined in Section 3.7.

In [Fin00, p. 17] Fine gave the following objection against the positionalist view that a relation has argument-places. On this view, the adjacency relation has two argument-places, say *Next* and *Nixt*. Assigning a to *Next* and b to *Nixt* will intuitively give the same state as assigning a to *Nixt* and b to *Next*. Yet they must be distinct since the argument-places occupied by a and b are distinct. As a consequence, strictly symmetric relations cannot have argument-places.

In [Leo08a, p. 358] I countered Fine's objection by arguing that we could also assume that assigning objects to argument-places *yields* states and that I see no reason to assume that objects *occupy* argument-places within the states. I think the view of argument-places as abstractions gives us additional ammunition against Fine's objection, since if the arguments would occupy argument-places, then switching arguments would not give a distinct state if the argument-places are indiscernible.

Nevertheless, we should not conclude that on the view of argument-places as abstractions for any relation, the arguments may occupy argument-places in the states. For consider the relation $\Re$ with $\Re abc$ being the state of *a's loving b and b's loving c*. Then the relation has three discernible argument-places, say p_0, p_1, and p_2. If assigning a to p_0, b to p_1 and a to p_2 gives the state of *a's loving b and b's loving a*, then we get the same state if we assign b to p_0, a to p_1 and b to p_2. Since in the first case a is assigned to two argument-places, and in the second case to only one argument-place, it is obviously impossible for the arguments to occupy the argument-places.

On the view of argument-places as abstractions, the internal structure of positions of a positional model is not particularly interesting, since on this view argument-places have by definition no internal structure. The main value that a specific positional model may have from this perspective is that in some cases it gives a nice or canonical representation for the argument-places of a relation.

We should not, in general, regard the positions of a positional frame as *names* for the argument-places as abstractions. They could only be names if each would uniquely refer to an entity, but this is impossible if the argument-places are indiscernible. What we can do, however, is regard the positions of a positional frame collectively as a *representation* of the abstracted argument-places.

The indiscernibility of argument-places as abstractions might be a problem in the following respect: how can we assign different objects to them when they are indiscernible? For a relation with complete strict symmetry, it might be enough just to say which objects are to be assigned to the argument-places, but for cyclical relations somehow an order has to be specified. A solution might be to use a representative positional frame: First, we assign objects to its positions, and then we abstract to get the required assignment.

The construction of argument-places as abstractions is not applicable for every substitution frame. Some substitution frames of infinite degree have corresponding positional frames that are at first sight equally natural, but that are nevertheless not positional variants of each other (e.g. $\mathcal{F}_1$ in Example 3.7.7). Consequently, for relations with such substitution frames a construction of argument-places as abstractions seems impossible. Maybe we should not interpret this as a weakness of the construction, but rather as a peculiarity of certain exotic relations (if they exist) that they do not have argument-places.

In [Sha97] a theory of structures is sketched and in [Fin02] a framework for a general theory of abstraction is given. It would be interesting to investigate whether argument-places as abstractions fit into these theories.

With argument-places as abstractions we obviously take a step outside the realm of ordinary set theory. However, we may define them in an extension of ZF along the lines sketched in [Fin98]. By taking as urelements not only the states in S and the objects in O, but also what Fine calls *variable objects*, positions could be defined as a *system* of variable objects.

3.10 Conclusions

We started our analysis with the presupposition that argument-places are not primitive entities. I think that argument-places are indeed not primitive since we can give a position-free account of relations based on the notion of substitution, which I regard as more primitive than the notion of argument-places. But to vindicate a position-free account either a satisfactory construction of argument-places is needed or an argument why the notion of argument-places is problematic.

For a (re)construction of argument-places we made use of mathematical models for relations. By using a new, formal notion of neutrality, we showed that every simple relation without strict symmetry and some other simple relations, like those with complete strict symmetry, have neutral positional frames. But we also showed that it is impossible to construct for every simple relation, even if its degree is finite, a neutral positional frame. As a consequence it is highly unlikely that for every relation, argument-places can be defined in a canonical way in ordinary set theory.

I consider the positive and the negative results about the existence of neutral positional frames to be primarily of interest from a representational perspective. I do not immediately see what conclusions we should draw about the ontological character of the argument-places for relations that do have a natural and neutral positional frame. Also with interpreting the negative results we should be careful. I certainly do not want to conclude from them that the identity of argument-places is problematic. But the negative results do show that for a neutral construction of argument-places of certain relations we have to go beyond the limits of ordinary set theory.

We showed that for the class of simple relations of finite degree a unique construction of argument-places as places in a structure exemplified by positional frames is possible. I personally find this view of argument-places as abstractions very natural. However, I do not know whether we should consider argument-places as genuine objects or that we should only talk about them as such. Regarding them as genuine objects is obviously in conflict with Leibniz's Principle of the Identity of Indiscernibles. Since we are dealing with constructed entities, I don't expect that a further analysis of argument-places will itself provide a ground for giving up Leibniz's Principle. We can probably paraphrase all references to argument-places as objects in terms of references to positions of positional frames. But if Leibniz's Principle is denied for whatever reason, then perhaps the assignment of an ontological status to argument-places may be just a matter of choice motivated by what appears to us as the most convenient perspective.

One may question whether the results obtained have any metaphysical significance, for we did not discuss any new metaphysical principles, nor did we reveal any relationship between ontologically fundamental entities. However, by giving a clarification and justification of the notion of argument-places, we contributed in an indirect way to forming and elaborating ideas about the essence of relations. Our ordinary way of using argument-places apparently is pretty much in agreement with treating argument-places as abstractions. I consider this not only to be support for common sense, but also to be confirmation of the antipositionalist view on relations. As such, the results do have metaphysical significance.

Acknowledgements The formulation and proof of the main negative result about the existence of neutral positional frames (Theorem 3.6.7) became much clearer thanks to Albert Visser's suggestion to reformulate it in terms of set theory with urelements. Furthermore, I am grateful to Kit Fine for helpful comments and to Clemens Grabmayer for many stylistic suggestions.

Chapter 4

Modeling occurrences of objects in relations*

Abstract

We study the logical structure of 'real' relations, and in particular the notion of occurrences of objects in a state. We start with formulating a number of principles for occurrences and defining corresponding mathematical models. These models are analyzed to get more insight in the formal properties of occurrences. In particular, we prove uniqueness results that tell us more about the possible logical structures relations might have.

4.1 Introduction

Relations are sometimes simply identified with sets of tuples of certain objects. For mathematical relations this is obviously appropriate since they *are* sets of tuples. But what about other relations, like, for example, the love relation? The idea that the state of Albert's loving Karen would be nothing but a tuple seems perverse.

According to Kit Fine [Fin00], the standard view of relations is that the constituents of a relation always come in a certain order. But then any relation also has a converse relation. For example, the relation *is older than* has the converse relation *is younger than*. Now consider the state of

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Thomas's being older than Nick. If—in a certain context—you regard this as exactly the same state as Nick's being younger than Thomas, and you regard this state as a relational complex of a single underlying relation, then the question is, of *which* relation?

On an alternative view of relations any relation comes with orderless positions or argument-places to which objects can be assigned. For example, for the love relation we have the positions *lover* and *beloved*. Unfortunately, this alternative view is also problematic. Fine [Fin00, pp. 16–17] raises two objections. First, positions seem ontologically excessive, and second, he regards positions unsuitable for strictly symmetric relations, like the adjacency relation, since switching positions of arguments would give a different state. (See [Leo08b] for a more detailed discussion of argument-places).

A very promising view of relations introduced by Fine is based on the notion of substitution. According to this *antipositionalist view* the states of a relation form a network in which substituting objects of a state by other objects yields another state. For example, substituting Roos for Albert and Bas for Karen in the state of Albert's loving Karen, results in the state of Roos's loving Bas.

In [Leo08a] we defined models that represent different views of relations. In particular we defined *substitution models*. We argued that substitution models adequately model a large class of 'real' relations. However, we also noted that these substitution models have certain limitations. For example, they have no typed domains for the objects. As a consequence, they are not accurate for a relation like 'drinks', because "Mo drinks tea" corresponds in a natural way to a state, but "tea drinks Mo" does not. I do not consider this limitation to be a serious one, since typed domains can be incorporated into the models in a straightforward way.

Substitution models as they were defined might have a more serious shortcoming. It could be argued that the substitution mechanism of the models is perhaps not refined enough, since objects are not explicitly substituted for individual *occurrences of* objects, but only for objects in a global sense. For example, for the state of Narcissus's loving Narcissus, the model did not allow for a substitution resulting in the state of Echo's loving Narcissus. Now, if occurrences of objects are a *basic* notion for relations, then this is a serious limitation of our substitution models.

But what exactly are occurrences? The notion of occurrences does not have the reputation of being crystal-clear. Benson Mates [Mat72, p. 49] even called it a "woolly notion". Occurrences can be considered in different contexts, e.g. in expressions [Wet93; JV04; Kra07]. In this paper we try to get a better grip on the nature of occurrences in the logical space of 'real' relations by developing mathematical models for relations.

Our approach is as follows. First, we formulate an initial set of principles for occurrences. Then, in light of these principles, we define mathematical models in which occurrences have a constitutive function. We perform a technical investigation of these models to get a better understanding of the formal properties of occurrences. Section 4.6 is the more philosophical part of the paper. There we try to say more about the nature of occurrences and consider the question whether, for 'real' relations, models with a refined substitution mechanism for occurrences of objects are in a relevant sense more complete or adequate than models with an undifferentiated substitution mechanism. In Section 4.6 we also briefly discuss the idea of explicitly distinguishing between states 'out there' and relational complexes, and to allow single states to figure as relational complexes in more than one relation. On a first reading of this paper, it might not be a bad idea to peek ahead to this section.

4.2 Basic principles for occurrences

In his paper 'The Problem of *De Re* Modality' [Fin89] Kit Fine proposes the development of a *general theory of constituent structure*, where the basic structure of the entities involved is given by the operation of substitution. Fine considers the following syntactic notions to be basic: *occurrence of*, *occurrence in*, and *substitution*. He gives the following example of a basic principle for these notions:

> One basic principle, for example, is that if F' is the result of substituting E' for the occurrence e of E within F, then there is an occurrence e' of E' within F' such that the result of substituting any expression E'' for e' within F' is identical to the result of substituting E'' directly for e in F.

The objective of this paper is not to develop a general theory of constituent structure. We restrict ourselves to the structure of relations. We postulate the following basic principles for occurrences of objects in relations:

P-1 Each relation has a nonempty set of states.

P-2 Each state of a relation has exactly one set of occurrences of objects.

P-3 Each occurrence is an occurrence of exactly one object.

P-4 Each occurrence has a type that corresponds to a domain of objects.

P-5 Objects of the same type can simultaneously be substituted for occurrences of objects in a state.

P-6 Any substitution of objects for occurrences in a state results in exactly one state of the same relation.

P-7 For any occurrence of an object in a state it makes for some substitution a difference for the resulting state which object is substituted for it.

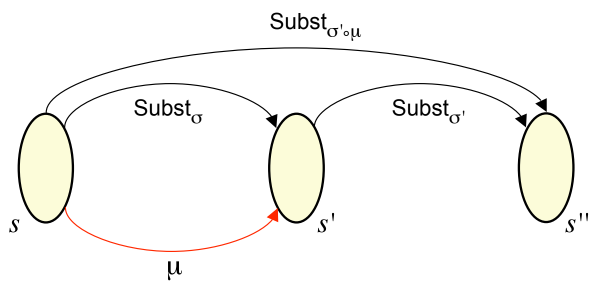
P-8 Composition principle: if a substitution in state s results in s', then there is a mapping μ from the occurrences in s to the occurrences in s' such that

(a) μ maps each occurrence α in s to an occurrence of the object substituted for α, and

(b) any substitution in s' gives the same result as when we substitute in s for each occurrence α the object that is substituted in s' for $\mu(\alpha)$.

It should be noted that what is at stake here is substitution of occurrences of objects in states of relations and not of occurrences of terms in expressions denoting states.

We may depict the composition principle P-8 as follows:



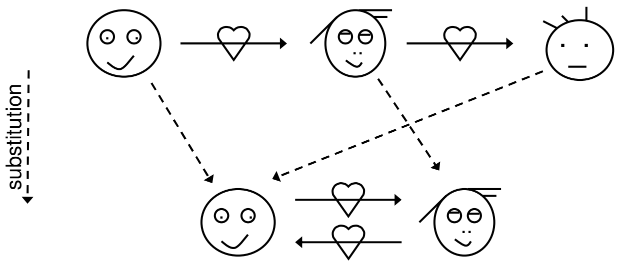


Figure 4.1: Coalescence of occurrences.

I do not claim that the principles given above are in any sense complete. For example, one might perhaps like to add that different states have no occurrence in common. Also from these principles it cannot always be deduced uniquely how many occurrences a state has. They do not tell us whether the state of Narcissus' loving Narcissus has one or two occurrences.

Principle P-8 may seem too weak. The mapping μ is surjective by P-7 and P-8, but could we not also demand that μ is injective? Unfortunately, for certain relations this seems problematic, since substitution might perhaps result in a *coalescence* of occurrences. This is illustrated by the following three exemplary situations.

1. Consider the variably polyadic relation of *waiting for the bus*. Take the state of Janneke and Vincent's waiting. If we substitute Janneke for Vincent, we expect to get—if anything—a state with only one occurrence.

2. Consider the ternary relation $\Re$ where $\Re abc$ is the state that *a loves b and b loves c*. Assume that the order of the conjuncts is irrelevant. Now suppose that a, b, c are three different objects, and that each has one occurrence in the state $\Re abc$. Substituting in this state a for the occurrence of c gives the state $\Re aba$. Simultaneously substituting in the state $\Re abc$, the objects b, a, b for the occurrences of a, b, c gives the state $\Re bab$, which appears to be the same state as the state $\Re aba$. It follows by principle P-8 that the state $\Re aba$ has only one occurrence of a and only one occurrence of b. So, μ cannot be injective (see Figure 4.1).

3. As a last example of a possible coalescence consider the conjunction $\Re \;\&\; \Re$ of a relation $\Re$ with itself. We may envision that the occurrences of each state $s \;\&\; s$ in the conjunction relation correspond one-to-one to the occurrences of s itself in $\Re$.

There may be arguments that can be used to reject a possible coalescence of occurrences. Against a coalescence for the ternary love relation it may for example be objected that substitution is a more subtle operation than we just seemed to suppose, and that substituting a for the occurrence of c in the state $\Re abc$ does *not* give the same result as substituting b, a, b for a, b, c in the state $\Re abc$. In Section 4.6 we will discuss such an alternative view in more detail.

In the next sections we will study the principles from a technical perspective by defining and analyzing mathematical models based on them.

4.3 Modeling occurrences

We define two types of *substitution frames* to model the *logical space* of relations. In the first type occurrences play no role and substitution is defined for objects. In the second type we have a more refined substitution mechanism working on occurrences of objects.

4.3.1 Undifferentiated substitution frames

We call frames with substitution defined for objects *undifferentiated substitution frames*. Since this type of frames is extensively discussed in [Leo08a], we give the definition without further explanation:

Definition 4.3.1. An *undifferentiated substitution frame* is a triple $\mathcal{F} = \langle S, O, \Sigma \rangle$, where S is a nonempty set of states, O is a nonempty set of objects, and Σ is a function from $S \times O^O$ to S such that

1. for all $s \in S$, $\Sigma(s, \mathsf{id}_O) = s$,
2. for all $s \in S$ and $\delta, \delta' \in O^O$, $\Sigma(\Sigma(s, \delta), \delta') = \Sigma(s, \delta' \circ \delta)$.

For convenience, we will often write $s \cdot_{\mathcal{F}} \delta$ or $s \cdot \delta$ for $\Sigma(s, \delta)$. Further, we will also often write $f \cdot g$ for $g \circ f$. With this notation, Σ is such that for all $s \in S$ and for all $\delta, \delta' \in O^O$, $s \cdot \mathsf{id}_O = s$, and $(s \cdot \delta) \cdot \delta' = s \cdot (\delta \cdot \delta')$.

We now give a definition of the objects of a state. Roughly put, they are the objects for which it makes a difference for the resulting state which objects are substituted for them.

Definition 4.3.2. Let $\mathcal{F} = \langle S, O, \Sigma \rangle$ be an undifferentiated substitution frame. We call $A \subseteq O$ an *object-domain of* $s \in S$ if for every $\delta, \delta' : O \to O$,

$$\delta =_A \delta' \;\Rightarrow\; s \cdot \delta = s \cdot \delta'.$$ [1]

We define the *core objects of* s as:

$$\mathsf{Core\text{-}ob}_{\mathcal{F}}(s) = \bigcap \{A \mid A \text{ is an object-domain of } s\}.$$

If $\mathsf{Core\text{-}ob}_{\mathcal{F}}(s)$ is an object-domain, then we call this set the *objects of* s. We denote this set as $\mathsf{Ob}_{\mathcal{F}}(s)$. If $\mathsf{Core\text{-}ob}_{\mathcal{F}}(s)$ is not an object-domain, then we leave $\mathsf{Ob}_{\mathcal{F}}(s)$ undefined.

For each substitution frame we can define its degree as a cardinal number:

Definition 4.3.3. Let $\mathcal{F} = \langle S, O, \Sigma \rangle$ be an undifferentiated substitution frame. For a state s in S, we define the *object-degree of* s as:

$$\mathsf{ob\text{-}degree}_{\mathcal{F}}(s) = \mathrm{glb}\,\{|A| \mid A \text{ is an object-domain of } s\}.$$

The *Object-degree of* $\mathcal{F}$ we define as:

$$\mathsf{ob\text{-}degree}_{\mathcal{F}} = \mathrm{lub}\,\{\mathsf{ob\text{-}degree}_{\mathcal{F}}(s) \mid s \in S\}.$$

Here $|A|$ denotes as usual the cardinality of A, "glb" denotes the greatest lower bound, and "lub" denotes the least upper bound. Note that the degree of s and the degree of $\mathcal{F}$ always exist and are indeed cardinal numbers.

4.3.2 Differentiated substitution frames

In this subsection we define differentiated substitution frames, discuss their adequacy, and present some basic properties.

Definition

We give a definition of *differentiated substitution frames* that is based on the principles for occurrences given in Section 4.2:

[1] For functions f, g we say that $f =_X g$ if $f{\restriction}X = g{\restriction}X$, that is, f restricted to X is equal to g restricted to X

Definition 4.3.4. A *differentiated substitution frame* is a tuple $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta\rangle$, where S is a nonempty set of states, O is a nonempty set of objects, Oc is a set of occurrences, Π is a mapping from Oc to O, and Θ is a function from $S \times O^{\mathsf{Oc}}$ to S such that

1. for all $s \in S$, $\Theta(s, \Pi) = s$,
2. for all $s \in S$ and $\sigma : \mathsf{Oc} \to O$ there is a mapping $\mu : \mathsf{Oc} \to \mathsf{Oc}$ such that
   (a) $\Pi \circ \mu = \sigma$,
   (b) for all $\sigma' : \mathsf{Oc} \to O$, $\Theta\left(\Theta(s, \sigma), \sigma'\right) = \Theta(s, \sigma' \circ \mu)$.

We say that μ *corresponds to* $(s, \sigma) \mapsto \Theta(s, \sigma)$. For $\alpha \in \mathsf{Oc}$, we say that α is an *occurrence of* $\Pi(\alpha)$.

Note that 2(a) implies that Π is surjective.

We will often write $s \cdot_{\mathcal{G}} \sigma$ or $s \cdot \sigma$ for $\Theta(s, \sigma)$. With this notation, Θ is such that (1) $s \cdot \Pi = s$, and (2) $\mu \cdot \Pi = \sigma$ and $(s \cdot \sigma) \cdot \sigma' = s \cdot (\mu \cdot \sigma')$.

If $s \cdot \sigma = s'$, then we denote the corresponding *transition* as $s \xrightarrow{\sigma} s'$.

Since the only type of frames we will discuss in this paper are differentiated and undifferentiated substitution frames, we will often just call them *frames* when it is clear from the context what type of frame we are talking about.

With any differentiated substitution frame we can associate a category with objects all states in S and with morphisms all triples $\langle s, \mu, s'\rangle$ with $\mu : \mathsf{Oc} \to \mathsf{Oc}$ satisfying conditions 2(a) and 2(b) of the definition of a differentiated substitution frame for some $\sigma : \mathsf{Oc} \to O$ with $s \cdot \sigma = s'$.

Note that differentiated and undifferentiated substitution frames are both defined in the language of second-order logic. In Appendix C we will consider some alternative definitions of frames that also fulfill the principles in the previous section.

Adequacy of the definition

For reasons of simplicity we deliberately chose to let our definition of a differentiated substitution frame deviate slightly from the principles in Section 4.2. The main differences are:

1. Substitution frames do not have typed domains for the objects.
2. The definition does not imply that every state has exactly one set of occurrences.
3. The definition is not completely consistent with the composition principle P-8.

Ad 1. As we remarked in the introduction, typed domains can easily be incorporated into our models. For our purposes, however, they do not really play a role.

Ad 2. In the next subsection we will define occurrences of states in accordance with principle P-7. However, as Example 4.3.7 will show, our definition of differentiated substitution frames is too liberal, since occurrences are not always defined for every state. So, principle P-2 is not always fulfilled. But I think this should not be considered to be a serious weakness of our models. Allowing certain borderline cases might be illuminating. It might even be that principle P-2 turns out to be just too strict for certain 'real' relations.

Ad 3. The function μ corresponding to a transition $s \xrightarrow{\sigma} s'$ might map the occurrences of s to a proper superset of the occurrences of s'. Although I do not have an example of a 'real' relation where this is the case, I see no compelling reasons to exclude the possibility.

A further point to note is that in our models states may share the same occurrences. In our analysis we will take care to explicitly mention if any of our results depend on this.

Basic properties

We define occurrences of states as follows:

Definition 4.3.5. Let $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta \rangle$ be a differentiated substitution frame. Then we call $X \subseteq \mathsf{Oc}$ an *occurrence-domain of* $s \in S$ if for all $\sigma, \sigma' : \mathsf{Oc} \to O$,

$$\sigma =_X \sigma' \;\Rightarrow\; s \cdot \sigma = s \cdot \sigma'.$$

We define the *core occurrences of* s as:

$$\mathsf{Core\text{-}oc}_{\mathcal{G}}(s) = \bigcap \{X \mid X \text{ is an occurrence-domain of } s\}.$$

If $\mathsf{Core\text{-}oc}_{\mathcal{G}}(s)$ is an occurrence-domain, then we call this set the *occurrences of* s, and we denote it as $\mathsf{Oc}_{\mathcal{G}}(s)$. If $\mathsf{Core\text{-}oc}_{\mathcal{G}}(s)$ is not an occurrence-domain, then we leave $\mathsf{Oc}_{\mathcal{G}}(s)$ undefined.

Definition 4.3.6. We say that a transition $s \xrightarrow{\sigma''} s''$ is a *composition of* $s \xrightarrow{\sigma} s'$ *and* $s' \xrightarrow{\sigma'} s''$ if $\sigma'' =_X \mu \cdot \sigma'$ for some occurrence-domain X of s and some mapping μ corresponding to $s \xrightarrow{\sigma} s'$.

It may happen that $\mathsf{Core\text{-}oc}(s)$ is not an occurrence-domain:

Example 4.3.7. Let $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta \rangle$ be a differentiated substitution frame with O an infinite set, $\mathsf{Oc} = O$, $\Pi = \mathsf{id}_O$, S the set of subsets of O modulo a finite difference, that is,

$$S = \{\widehat{A} \mid A \subseteq O\}$$

with $\widehat{A} = \{A' \subseteq O \mid A \,\Delta\, A' \text{ is finite}\}$, and Θ defined by

$$\widehat{A} \cdot \sigma = \widehat{\sigma[A]}.^2$$

Θ is well-defined, since for any $A, B \subseteq O$, if $\widehat{A} = \widehat{B}$, then $\widehat{\sigma[A]} = \widehat{\sigma[B]}$. Further, $\mathcal{G}$ is a substitution frame, since

1. $\widehat{A} \cdot \Pi = \widehat{\mathsf{id}_O[A]} = \widehat{A}$, and
2. $(\widehat{A} \cdot \sigma) \cdot \sigma' = \widehat{\sigma[A]} \cdot \sigma' = \widehat{\sigma'[\sigma[A]]} = \widehat{(\sigma \cdot \sigma')[A]} = \widehat{A} \cdot (\sigma \cdot \sigma')$.

It is not difficult to see that for any $A \subseteq O$, $\mathsf{Core\text{-}oc}(\widehat{A})$ is the empty set, but if A is infinite, then the empty set is not an occurrence-domain of $\widehat{A}$. $\dashv$

For undifferentiated substitution frames, a similar example can be given that shows $\mathsf{Core\text{-}ob}(s)$ is not always an object-domain [Leo08a, Example 3.16].

Lemma 4.3.8. *Let* $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta \rangle$ *be a differentiated substitution frame. For every* $s \in S$*, the occurrence-domains of* s *form a (possibly nonproper) filter on* Oc*.*

[2] $f[X] = \{f(x) \mid x \in X\}$, the image of X under f.

Proof. To prove that the occurrence-domains of s are closed under finite intersection, let X and X' be occurrence-domains of s. Let $\sigma, \sigma' : \mathsf{Oc} \to O$ be such that $\sigma =_{X \cap X'} \sigma'$. Define

$$\sigma''(\alpha) = \begin{cases} \sigma(\alpha) & \text{if } \alpha \in X - X', \\ \sigma(\alpha) = \sigma'(\alpha) & \text{if } \alpha \in X \cap X', \\ \sigma'(\alpha) & \text{if } \alpha \in X' - X. \end{cases}$$

Then $\sigma'' =_X \sigma$ and $\sigma'' =_{X'} \sigma'$. So, $s \cdot \sigma = s \cdot \sigma'' = s \cdot \sigma'$. Thus $X \cap X'$ is an occurrence-domain of s.
It is trivial that the occurrence-domains of s are upward closed.
Since an occurrence-domain may be empty, we may have a nonproper filter. ⊣

Definition 4.3.9. For a differentiated substitution frame $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta \rangle$, we call $A \subseteq O$ an *object-domain of* $s \in S$ if for some occurrence-domain X of s,

$$A = \Pi[X].$$

We define the *core objects of* s as:

$$\mathsf{Core\text{-}ob}_{\mathcal{G}}(s) = \bigcap \{A \mid A \text{ is an object-domain of } s\}.$$

If $\mathsf{Core\text{-}ob}(s)$ is an object-domain, then we call this set the *objects of* s, and we denote it as $\mathsf{Ob}(s)$. If $\mathsf{Core\text{-}ob}(s)$ is not an object-domain, then we leave $\mathsf{Ob}(s)$ undefined.

Lemma 4.3.10. *Let $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta \rangle$ be a differentiated substitution frame. For every $s \in S$, the object-domains of s form a (possibly nonproper) filter on O.*

Proof. To prove that the object-domains of s are closed under finite intersection, let A and A' be object-domains of s. Then $A = \Pi[X]$ and $A' = \Pi[X']$ for some occurrence-domains X, X'. So, $A \cap A' = \Pi[X] \cap \Pi[X'] \supseteq \Pi[X \cap X']$. By Lemma 4.3.8, $X \cap X'$ is an occurrence-domain. It follows that $A \cap A'$ is an object-domain of s.
By the surjectivity of Π and the upward closedness of occurrence-domains, the object-domains of s are also upward closed.
Since an object-domain may be empty, we may have a nonproper filter. ⊣

In the next lemma we show that (i) the objects of the core-occurrences of a state form a subset of its core-objects, and (ii) if the occurrences of a state exist, then the objects of the state also exist, and are exactly the objects of the occurrences of the state.

Lemma 4.3.11. *Let $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta \rangle$ be a differentiated substitution frame. For every state $s \in S$,*

(i) $\Pi[\mathsf{Core\text{-}oc}(s)] \subseteq \mathsf{Core\text{-}ob}(s)$.

(ii) *If $\mathsf{Oc}(s)$ exists, then $\mathsf{Ob}(s)$ exists and $\Pi[\mathsf{Oc}(s)] = \mathsf{Ob}(s)$.*

Proof. By Definition 4.3.5 and Definition 4.3.9,

$$\begin{aligned} \Pi(\mathsf{Core\text{-}oc}(s)) &= \Pi[\bigcap\{X \mid X \text{ is an occurrence-domain of } s\}] \\ &\subseteq \bigcap\{\Pi[X] \mid X \text{ is an occurrence-domain of } s\} \\ &= \mathsf{Core\text{-}ob}(s). \end{aligned}$$

To prove the second claim, assume $\mathsf{Oc}(s)$ exists. Then $\mathsf{Oc}(s)$ is an occurrence-domain of s. So, $\Pi[\mathsf{Oc}(s)]$ is an object-domain of s. Thus $\mathsf{Core\text{-}ob}(s) \subseteq \Pi[\mathsf{Oc}(s)]$. By the first claim of the lemma, we also have the reverse inclusion. So, $\mathsf{Core\text{-}ob}(s) = \Pi[\mathsf{Oc}(s)]$. Because $\Pi[\mathsf{Oc}(s)]$ is an object-domain, the claim follows. ⊣

The inclusion of Lemma 4.3.11 may be a proper inclusion, as the next example shows.

Example 4.3.12. Let $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta \rangle$ be a differentiated substitution frame with $O = \{a, b\}$, $S = \{s_1, s_2\}$, $\mathsf{Oc} = O \times \omega$, $\Pi(x, n) = x$, and Θ defined by

$$s \cdot \sigma = \begin{cases} s_1 & \text{if } s = s_1, \text{ and both } \sigma^{-1}(a) \text{ and } \sigma^{-1}(b) \text{ are infinite,} \\ s_2 & \text{otherwise.} \end{cases}$$

We see that $\mathsf{Ob}(s_1) = \{a, b\}$, but $\mathsf{Core\text{-}oc}(s_1) = \varnothing$ and $\mathsf{Oc}(s_1)$ is undefined. ⊣

The next lemma expresses the core-occurrences of a state in terms of single substitutions:

Lemma 4.3.13. *If in* $\mathsf{Core\text{-}oc}(s)$ *the number of occurrences of each object is finite, then*

$$\mathsf{Core\text{-}oc}(s) = \{\alpha \mid \exists b \in O\ [\, s \cdot \Pi[\alpha \mapsto b] \neq s\,]\}.^{3}$$

Proof. If O has just one element, then it is obviously true. So, assume that O has at least two elements. Consider $\alpha_0 \in \mathsf{Core\text{-}oc}(s)$. Let $a = \Pi(\alpha_0)$ and b be some object in O with $b \neq a$. Define $\sigma_0 = \Pi[\alpha_0 \mapsto b]$. By the definition of differentiated substitution frames, there is a mapping $\mu_0 : \mathsf{Oc} \to \mathsf{Oc}$ such that

$$\mu_0 \cdot \Pi = \sigma_0 \text{ and for all } \sigma : \mathsf{Oc} \to O,\ \ (s \cdot \sigma_0) \cdot \sigma = s \cdot (\mu_0 \cdot \sigma).$$

Assume that $A = \{\alpha \in \mathsf{Core\text{-}oc}(s) \mid \Pi(\alpha) = a\}$ is finite. Then there is an $\alpha_1 \in A$ such that $\alpha_1 \notin \mu_0[\mathsf{Core\text{-}oc}(s)]$. It follows that $s \cdot \sigma_0 \neq s$.
The inclusion in the other direction is obvious. ⊣

In the lemma we cannot drop the finiteness condition, as the next example shows.

Example 4.3.14. Let $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta \rangle$ be a differentiated substitution frame with $O = \{a, b\}$, $S = \{s_0, s_1, s_2, ..., s_\infty\}$, $\mathsf{Oc} = O \times \omega$, $\Pi(x, n) = x$, and Θ defined by

$$s \cdot \sigma = \begin{cases} s_0 & \text{if } s = s_n \text{ with } n \in \omega \text{ and } \sigma(a,0) = \sigma(b,0),\\ s_n & \text{if } s = s_n \text{ with } n \in \omega \text{ and } \sigma(a,0) \neq \sigma(b,0),\\ s_{|\sigma^{-1}(x)|} & \text{if } s = s_\infty \text{ and } \sigma^{-1}(x) \text{ is finite},\\ s_\infty & \text{otherwise.}^{4} \end{cases}$$

Then $\mathsf{Oc}(s_\infty) = O \times \omega$, but for any $b \in O$ and $\alpha \in \mathsf{Oc}$, $s_\infty \cdot \Pi[\alpha \mapsto b] = s_\infty$. ⊣

The notion of *occurrence-degree* and *object-degree* are defined as follows:

Definition 4.3.15. Let $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta \rangle$ be a differentiated substitution frame. For a state s in S, we define the *occurrence-degree of* s as:

$$\mathsf{oc\text{-}degree}_{\mathcal{G}}(s) = \text{glb}\ \{|X| \mid X \text{ is an occurrence-domain of } s\}.$$

[3] $f[a \mapsto b]$ denotes the function defined by $f[a \mapsto b](x) = b$ if $x = a$; $f(x)$ otherwise.
[4] $|X|$ denotes as usual the cardinality of X.

The *occurrence-degree of* $\mathcal{G}$ we define as:

$$\textsf{oc-degree}_{\mathcal{G}} = \text{lub}\,\{\textsf{oc-degree}_{\mathcal{G}}(s) \mid s \in S\}.$$

The *object-degrees* $\textsf{ob-degree}_{\mathcal{G}}(s)$ and $\textsf{ob-degree}_{\mathcal{G}}$ are defined in a similar way by starting with the object-domains of s.

In Section 4.4 we will see that the occurrence-degree of a frame can be much higher than its object-degree.

4.3.3 Underlying frames

There is a natural embedding of the undifferentiated substitution frames in the class of differentiated substitution frames, and, conversely, each differentiated substitution frame also has an underlying undifferentiated substitution frame:

Definition 4.3.16. Let $\mathcal{F} = \langle S, O, \Sigma \rangle$ be an undifferentiated substitution frame and let $\mathcal{G} = \langle S', O', \textsf{Oc}, \Pi, \Theta \rangle$ be a differentiated substitution frame. We say that $\mathcal{F}$ *underlies* $\mathcal{G}$ if $S = S'$, $O = O'$, and for every $s \in S$ and $\delta : O \to O$,

$$s \cdot_{\mathcal{G}} (\Pi \cdot \delta) = s \cdot_{\mathcal{F}} \delta.$$

We call $\mathcal{G}$ a *basic refinement of* $\mathcal{F}$ if $\mathcal{F}$ underlies $\mathcal{G}$, $\textsf{Oc} = O$ and $\Pi = \textsf{id}_O$.

Note that in a basic refinement, different states may have occurrences in common. To prevent this, we could alternatively have defined for a basic refinement the set of occurrences as $S \times O$.

Theorem 4.3.17. *(i) Each undifferentiated substitution frame has a unique basic refinement. (ii) Each differentiated substitution frame has a unique underlying undifferentiated substitution frame. (iii) If $\mathcal{F}$ is an undifferentiated substitution frame, then the underlying undifferentiated substitution frame of the basic refinement of $\mathcal{F}$ is $\mathcal{F}$ itself.*

Proof. (i) This follows immediately from the definition of a basic refinement and the definition of a differentiated substitution frame.

(ii) Let $\mathcal{G} = \langle S, O, \textsf{Oc}, \Pi, \Theta \rangle$ be a differentiated substitution frame. Let $\mathcal{F} = \langle S, O, \Sigma \rangle$ be defined by $s \cdot_{\mathcal{F}} \delta = s \cdot_{\mathcal{G}} (\Pi \cdot \delta)$. Then by Condition 1 of the definition of a differentiated substitution frame $s \cdot_{\mathcal{F}} \mathsf{id}_O = s \cdot_{\mathcal{G}} (\Pi \cdot \mathsf{id}_O) = s$. By Condition 2 of the definition of a differentiated substitution frame:

$$\begin{aligned}(s \cdot_{\mathcal{F}} \delta) \cdot_{\mathcal{F}} \delta' &= (s \cdot_{\mathcal{G}} (\Pi \cdot \delta)) \cdot_{\mathcal{G}} (\Pi \cdot \delta') \\ &= s \cdot_{\mathcal{G}} (\mu \cdot \Pi \cdot \delta') \qquad \text{for some } \mu \text{ with } \mu \cdot \Pi = \Pi \cdot \delta \\ &= s \cdot_{\mathcal{G}} (\Pi \cdot \delta \cdot \delta') \\ &= s \cdot_{\mathcal{F}} (\delta \cdot \delta').\end{aligned}$$

So, $\mathcal{F}$ is an undifferentiated substitution frame. Its uniqueness follows immediately from the definition of an underlying undifferentiated substitution frame.

(iii) This also follows immediately from the definitions. ⊣

In Appendix B we will define categories for undifferentiated and differentiated substitution frames and show that there is an adjunction between them.

Lemma 4.3.18. *Let $\mathcal{G}$ be a basic refinement of $\mathcal{F} = \langle S, O, \Sigma \rangle$. Then for every $s \in S$, the occurrence-domains of s in $\mathcal{G}$ are the same as the object-domains of s in $\mathcal{G}$ and the same as the object-domains of s in $\mathcal{F}$.*

Proof. The lemma follows immediately from the definitions. ⊣

We may use this lemma to deduce certain results about undifferentiated substitution frames from results about differentiated substitution frames. For example, from Lemma 4.3.10 it follows that also for undifferentiated substitution frames the object-domains of s form a filter on O, and from Lemma 4.3.13 we get for undifferentiated substitution frames a characterization of the core of s. But of course, these results could also be obtained more directly.

The next lemma shows that refining a frame does not change the object-domains of its states.

Lemma 4.3.19. *If $\mathcal{F}$ underlies $\mathcal{G}$, then any state has in $\mathcal{F}$ the same object-domains as in $\mathcal{G}$.*

Proof. Let $\mathcal{F} = \langle S, O, \Sigma \rangle$ underly $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta \rangle$. Let A be an object-domain of s in $\mathcal{F}$. If O consists of one object, then $\varnothing$ is an object-domain of

s in $\mathcal{G}$, and so is A. Therefore we may assume that O contains (at least) two different objects a and b. Let $\sigma_1, \sigma_1' : \mathsf{Oc} \to O$ be such that $\sigma_1 =_{\Pi^{-1}[A]} \sigma_1'$. Define

$$\sigma_2(\alpha) = \begin{cases} \sigma_1(\alpha) = \sigma_1'(\alpha) & \text{if } \Pi(\alpha) \in A, \\ \sigma_1(\alpha) & \text{if } \Pi(\alpha) = a, \\ \sigma_1'(\alpha) & \text{if } \Pi(\alpha) = b, \\ \Pi(\alpha) & \text{otherwise.} \end{cases}$$

We will show that $s \cdot_{\mathcal{G}} \sigma_2 = s \cdot_{\mathcal{G}} \sigma_1$. Then, by symmetry of the definition of σ_2, we also have $s \cdot_{\mathcal{G}} \sigma_2 = s \cdot_{\mathcal{G}} \sigma_1'$, and thus A is also an object-domain of s in $\mathcal{G}$.

Define $\delta_0 : O \to O$ by

$$\delta_0(d) = \begin{cases} d & \text{if } d \in A, \\ a & \text{otherwise.} \end{cases}$$

Define $\sigma_0 = \Pi \cdot \delta_0$. Then $s \cdot_{\mathcal{G}} \sigma_0 = s \cdot_{\mathcal{F}} \delta_0 = s$. Because $\mathcal{G}$ is a differentiated substitution frame, there is a function $\mu_0 : \mathsf{Oc} \to \mathsf{Oc}$ such that $\mu_0 \cdot \Pi = \sigma_0$, and for any $\sigma : \mathsf{Oc} \to O$, $(s \cdot_{\mathcal{G}} \sigma_0) \cdot_{\mathcal{G}} \sigma = s \cdot_{\mathcal{G}} (\mu_0 \cdot \sigma)$. So,

$$s \cdot_{\mathcal{G}} \sigma_2 = (s \cdot_{\mathcal{G}} \sigma_0) \cdot_{\mathcal{G}} \sigma_2 = s \cdot_{\mathcal{G}} (\mu_0 \cdot \sigma_2) = s \cdot_{\mathcal{G}} (\mu_0 \cdot \sigma_1) = (s \cdot_{\mathcal{G}} \sigma_0) \cdot_{\mathcal{G}} \sigma_1 = s \cdot_{\mathcal{G}} \sigma_1.$$

Conversely, let A be an object-domain of s in $\mathcal{G}$. Then there is an occurrence-domain X of s such that $A = \Pi[X]$. So, for any $\delta, \delta' : O \to O$ with $\delta =_A \delta'$, we have $\Pi \cdot \delta =_X \Pi \cdot \delta'$, and thus $s \cdot_{\mathcal{F}} \delta = s \cdot_{\mathcal{G}} (\Pi \cdot \delta) = s \cdot_{\mathcal{G}} (\Pi \cdot \delta') = s \cdot_{\mathcal{F}} \delta'$. It follows that A is also an object-domain of s in $\mathcal{F}$. ⊣

As a direct consequence of the lemma, if $\mathcal{F}$ underlies $\mathcal{G}$, then for any state s, $\mathsf{Core\text{-}ob}_{\mathcal{F}}(s) = \mathsf{Core\text{-}ob}_{\mathcal{G}}(s)$, and $\mathsf{ob\text{-}degree}_{\mathcal{F}}(s) = \mathsf{ob\text{-}degree}_{\mathcal{G}}(s)$.

4.4 Refining occurrences

An undifferentiated substitution frame may underly various differentiated substitution frames. In establishing for a given relation the most adequate one, our principles in Section 4.2 seem to leave us with a lot of choices. In this section we investigate a natural (pre)ordering of frames. A consequence of the main result to be presented here, is that for a large class of relations *unique* maximally differentiated substitution frames exist.

We define refinements of differentiated substitution frames as follows:

Definition 4.4.1. Let $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta\rangle$ and $\mathcal{G}' = \langle S', O', \mathsf{Oc}', \Pi', \Theta'\rangle$ be differentiated substitution frames. We say that $\mathcal{G}$ is a *refinement of* $\mathcal{G}'$ if $S = S'$, $O = O'$, and for every $s \in S$ there is a function $\tau : \mathsf{Oc} \to \mathsf{Oc}'$ such that

1. $\tau \cdot \Pi' = \Pi$,
2. for every $\sigma : \mathsf{Oc}' \to O$, $s \cdot_{\mathcal{G}} (\tau \cdot \sigma) = s \cdot_{\mathcal{G}'} \sigma$.

We say that $\mathcal{G}$ and $\mathcal{G}'$ are *equally refined* if for every $s \in S$ the function τ is injective on some occurrence-domain X of s in $\mathcal{G}$.

We call a refinement a *proper* refinement, if the frames are not equally refined.

We also call $\mathcal{G}$ a refinement of an undifferentiated frame $\mathcal{F}$ if $\mathcal{G}$ is a refinement of the basic refinement of $\mathcal{F}$.

Note that $\mathcal{G}$ is a refinement of an undifferentiated substitution frame $\mathcal{F}$ iff $\mathcal{F}$ underlies $\mathcal{G}$. Also note that the relation *is equally refined* is an equivalence relation.

For an undifferentiated substitution frame $\mathcal{F}$ we will call $\mathcal{G}$ a proper refinement, if $\mathcal{G}$ is a proper refinement of the basic refinement of $\mathcal{F}$.

Lemma 4.4.2. *Differentiated substitution frames with all states of finite occurrence-degree are refinements of each other iff they are equally refined.*

Proof. The lemma follows immediately from the definitions. ⊣

I am not sure whether the claim of the lemma is also true for all frames of infinite occurrence-degree. But refinements—modulo equal refinedness—obviously form a preorder. In the next subsections we investigate the structure of this preorder in more detail.

4.4.1 Spurious refinements

Perhaps contrary to what one would expect, an undifferentiated substitution frame for a relation like the love relation has (many) proper refinements:

Theorem 4.4.3. *Let $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta\rangle$ be a differentiated substitution frame with O finite, and $\mathsf{oc\text{-}degree}_{\mathcal{G}}$ also finite, but unequal zero. Then for any infinite cardinal κ, there is a refinement $\mathcal{G}'$ of $\mathcal{G}$ with $\mathsf{oc\text{-}degree}_{\mathcal{G}'} = \kappa$.*

Proof. Let κ be an infinite cardinal. Define the filter F on κ by $A \in F$ iff $|\kappa - A| < \kappa$. Let U be an ultrafilter on κ extending F. (Such an ultrafilter exists by the Axiom of Choice.) Then for any $A \in U$, $|A| = \kappa$. Further, since O is finite, for any $\tau \in O^\kappa$, there is exactly one $a \in O$ such that $\{\lambda \mid \tau(\lambda) = a\} \in U$.

It follows that we can make a proper refinement of $\mathcal{G}$ by replacing each occurrence by κ occurrences of the same object. More precisely, define $\mathcal{G}' = \langle S, O, \mathsf{Oc}', \Pi', \Theta'\rangle$ with

1. $\mathsf{Oc}' = \mathsf{Oc} \times \kappa$,
2. $\Pi' = \mathsf{proj}_1|_{\mathsf{Oc}'} \cdot \Pi$,
3. $s \cdot_{\mathcal{G}'} \sigma = s \cdot_{\mathcal{G}} \overline{\sigma}$ with $\overline{\sigma}$ such that $\forall \alpha \in \mathsf{Oc},\ \{\lambda \mid \sigma(\alpha, \lambda) = \overline{\sigma}(\alpha)\} \in U$.

We will show that (i) $\mathcal{G}'$ is a differentiated substitution frame, (ii) $\mathcal{G}'$ is a refinement of $\mathcal{G}$, and (iii) $\mathsf{oc\text{-}degree}_{\mathcal{G}'} = \kappa$.

(i) We show that $\mathcal{G}'$ satisfies Conditions 1 and 2 of the definition of a differentiated substitution frame.

1. $s \cdot_{\mathcal{G}'} \Pi' = s$ because $\{\lambda \mid \Pi'(\alpha, \lambda) = \Pi(\alpha)\} = \kappa \in U$.
2. Consider any $s \in S$, and $\sigma : \mathsf{Oc}' \to O$. Let $\overline{\sigma}$ be as in the definition of $\mathcal{G}'$, let $\overline{\mu} : \mathsf{Oc} \to \mathsf{Oc}$ correspond to $s \xrightarrow{\overline{\sigma}} s \cdot \overline{\sigma}$, and let $\mu : \mathsf{Oc}' \to \mathsf{Oc}'$ be such that
   (a) $\mu \cdot_{\mathcal{G}'} \Pi' = \sigma$,
   (b) $\mu(\alpha, \lambda) = (\overline{\mu}(\alpha), \lambda)$ if $\sigma(\alpha, \lambda) = \overline{\sigma}(\alpha)$.

For any $\sigma' : \mathsf{Oc}' \to O$, let $\overline{\sigma}'$ be such that $\forall \alpha \in \mathsf{Oc},\ \ \{\lambda \mid \sigma'(\alpha, \lambda) = \overline{\sigma}'(\alpha)\} \in U$. Then

$$\begin{aligned}(s \cdot_{\mathcal{G}'} \sigma) \cdot_{\mathcal{G}'} \sigma' &= (s \cdot_{\mathcal{G}} \overline{\sigma}) \cdot_{\mathcal{G}} \overline{\sigma}' \\ &= s \cdot_{\mathcal{G}} (\overline{\mu} \cdot \overline{\sigma}') \\ &= s \cdot_{\mathcal{G}'} (\mu \cdot \sigma'),\end{aligned}$$

where the last equation can be proved as follows. Define for each $\alpha \in \mathsf{Oc}$ the following sets:

$$A_\alpha = \{\lambda \mid \mu(\alpha, \lambda) = (\overline{\mu}(\alpha), \lambda)\},$$
$$B_\alpha = \{\lambda \mid \sigma'(\alpha, \lambda) = \overline{\sigma}'(\alpha)\},$$
$$C_\alpha = \{\lambda \mid \sigma'(\mu(\alpha, \lambda)) = \overline{\sigma}'(\overline{\mu}(\alpha))\}.$$

Then $A_\alpha \cap B_{\overline{\mu}(\alpha)} \subseteq C_\alpha$. Because $A_\alpha, B_{\overline{\mu}(\alpha)} \in U$, and U is a filter, we have $C_\alpha \in U$, from which it follows that $s \cdot_{\mathcal{G}'} (\mu \cdot \sigma') = s \cdot_{\mathcal{G}} (\overline{\mu} \cdot \overline{\sigma}')$. Thus $\mathcal{G}'$ is a differentiated substitution frame.

(ii) To prove that $\mathcal{G}'$ is a refinement of $\mathcal{G}$, define $\tau : \mathsf{Oc}' \to \mathsf{Oc}$ by $\tau(\alpha, \lambda) = \alpha$. Then clearly τ is surjective and $\Pi' = \tau \cdot \Pi$. Further, $s \cdot_{\mathcal{G}} \sigma = s \cdot_{\mathcal{G}'} (\tau \cdot \sigma)$, because for each $\alpha \in \mathsf{Oc}$, $\{\lambda \mid \sigma(\tau(\alpha, \lambda)) = \sigma(\alpha)\} = \kappa \in U$.

(iii) Since $\mathsf{oc\text{-}degree}_{\mathcal{G}}$ is finite, and κ is infinite, we have $\mathsf{oc\text{-}degree}_{\mathcal{G}'} \leq \kappa$. Since $\mathsf{oc\text{-}degree}_{\mathcal{G}} \neq 0$, there is a $s_0 \in S$ and $\sigma_0 : \mathsf{Oc} \to O$ such that $s_0 \cdot_{\mathcal{G}} \sigma_0 \neq s_0$. Let X be an occurrence-domain of s_0 in $\mathcal{G}'$. Define $\sigma'_0 : \mathsf{Oc}' \to O$ by

$$\sigma'_0(\alpha, \lambda) = \begin{cases} \Pi(\alpha) & \text{if } (\alpha, \lambda) \in X, \\ \sigma_0(\alpha) & \text{otherwise.} \end{cases}$$

Then $\sigma'_0 =_X \Pi'$. So, $s_0 \cdot_{\mathcal{G}'} \sigma'_0 = s_0$. Now suppose $|X| < \kappa$. Then, by the definition of U, for any $\alpha \in \mathsf{Oc}$, $\{\lambda \mid (\alpha, \lambda) \in \mathsf{Oc}' - X\} \in U$, and thus $s_0 \cdot_{\mathcal{G}'} \sigma'_0 = s_0 \cdot_{\mathcal{G}} \sigma_0 \neq s_0$. So, we have a contradiction. It follows that $\mathsf{oc\text{-}degree}_{\mathcal{G}'} = \kappa$. ⊣

What about frames with an infinite number of objects? Do they also always have proper refinements? By a rather straightforward modification of the proof of the previous theorem we can show that *any* frame with at least one transition from a state to a different one has a proper refinement—provided that there are arbitrary large measurable cardinals:

Theorem 4.4.4. *Let $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta \rangle$ be a differentiated substitution frame with $\mathsf{oc\text{-}degree}_{\mathcal{G}} \neq 0$. Then for any measurable cardinal number $\kappa > \max(|O|, \mathsf{oc\text{-}degree}_{\mathcal{G}})$, there is a refinement $\mathcal{G}'$ of $\mathcal{G}$ with $\mathsf{oc\text{-}degree}_{\mathcal{G}'} = \kappa$.*

Proof. By definition, an uncountable cardinal κ is measurable iff there is a nonprincipal κ-complete ultrafilter on κ, where a filter is called κ-complete if it is closed under intersection of less than κ sets. See e.g. [Jec06, p. 127] or [Kan05, p. 26].

Now let κ be a measurable cardinal greater than $\mathsf{max}(|O|, \mathsf{oc\text{-}degree}_{\mathcal{G}})$, and let U be a nonprincipal κ-complete ultrafilter over κ. Then for any $A \in U$, $|A| = \kappa$. Further, since $\kappa > |O|$, for any $\tau \in O^\kappa$, there is exactly one $a \in O$ such that $\{\lambda \mid \tau(\lambda) = a\} \in U$.

Define $\mathcal{G}' = \langle S, O, \mathsf{Oc}', \Pi', \Theta' \rangle$ exactly as in the proof of previous theorem. We may prove that $\mathcal{G}'$ is a differentiated substitution frame and that $\mathcal{G}'$ is a refinement of $\mathcal{G}$ in the same way as we did in (i) and (ii) of the proof of the previous theorem.

To prove that $\mathsf{oc\text{-}degree}_{\mathcal{G}'} = \kappa$, we first note that since $\mathsf{oc\text{-}degree}_{\mathcal{G}} \leq \kappa$, and κ is infinite, we have $\mathsf{oc\text{-}degree}_{\mathcal{G}'} \leq \kappa$. Then we may follow (iii) of the proof of the previous theorem to complete the proof. ⊣

For 'real' relations the ultra-refinements constructed in the proof of the last two theorems do not seem to be adequate, since the states do not have a well-defined set of occurrences. Probably no metaphysical significance should be given to the existence of these spurious refinements.

4.4.2 Normal refinements

In this section we focus on frames of finite occurrence-degree, which, as a consequence of Lemma 4.4.2, form—modulo equal refinedness—a partial order. In particular we want to investigate whether two given frames have a supremum. We begin with a negative result:

Example 4.4.5. Let $\mathcal{F} = \langle S, O, \Sigma \rangle$ be an undifferentiated substitution frame with $O = \{a, b\}$, $S = \{\{a\}, \{b\}, \{a, b\}\}$, and Σ defined by

$$s \cdot \delta = \delta[s].$$

Then $\mathcal{F}$ underlies a frame $\mathcal{G}$ with for each $s \in S$, $\mathsf{oc\text{-}degree}_{\mathcal{G}}(s) = 2$. But for any natural number $k > 2$, $\mathcal{F}$ also underlies a frame $\mathcal{G}'$ in which the state $\{a, b\}$ has k occurrences for a and k occurrences for b. Since in $\mathcal{G}'$ the other states necessarily have only one occurrence, it follows immediately that $\mathcal{G}$ and $\mathcal{G}'$ have no refinement of finite occurrence-degree in common. Some extra thought shows that they also have no refinement of infinite occurrence-degree in common. ⊣

In the example, the frame $\mathcal{G}'$ has the property of having an occurrence-degree that is greater than its object-degree. Although we can imagine that this might be the case for certain 'real' relations, we expect that in 'normal' cases the occurrence-degree and object-degree will be the same:

Definition 4.4.6. We call a frame or refinement *normal* if its occurrence-degree is the same as its object-degree. We call a normal frame or refinement *maximal* if it has no proper, normal refinement.

The next theorem will be very useful in evaluating possible structures of relations:

Theorem 4.4.7. *Let Ψ be a nonempty collection of normal refinements of a common substitution frame of finite object-degree. Then there is a least common refinement of the frames in Ψ, which is also normal and unique, modulo equal refinedness.*

Proof. We will construct a normal frame $\mathcal{G}^* = \langle S, O, \mathsf{Oc}^*, \Pi^*, \Theta^* \rangle$ of which we will show that it is a unique least common refinement of the frames in Ψ.

(i) *Construction of $\mathcal{G}^*$*:
Choose a frame $\mathcal{G}_0 = \langle S, O, \mathsf{Oc}_0, \Pi_0, \Theta_0 \rangle$ in Ψ.
Define $\mathsf{Oc}^* = \mathsf{Oc}_0 \times O$, and $\Pi^*(\alpha, a) = a$. For the definition of Θ^* we need some preparations:

Let $\xrightarrow{*}$ be the transitive closure of the relation $\rightarrow$ defined by:

$$s \rightarrow s' \text{ if } s \cdot_{\mathcal{G}} \sigma = s' \text{ for some } \mathcal{G} \in \Psi \text{ and some } \sigma.$$

For each $s \in S$ define

$$\Delta_s = \{ s' \in S \mid s \xrightarrow{*} s' \ \& \ s' \xrightarrow{*} s \}.$$

In each Δ_s choose s_0 with $\mathsf{ob\text{-}degree}_{\mathcal{G}_0}(s_0)$ maximal. Because the frames in Ψ are normal refinements of a common frame, we have for each $\mathcal{G} \in \Psi$, $\mathsf{oc\text{-}degree}_{\mathcal{G}}(s_0) = \mathsf{ob\text{-}degree}_{\mathcal{G}_0}(s_0)$. It follows that for any $\mathcal{G} \in \Psi$ and any $s \in S$ there is a transition from s_0 to s in $\mathcal{G}$ iff there is a transition from s_0 to s in $\mathcal{G}_0$. Thus, for every $s_1 \in \Delta_s$, we may choose σ_1 such that $s_0 \cdot_{\mathcal{G}_0} \sigma_1 = s_1$. Now define for an arbitrary $\sigma : \mathsf{Oc}^* \rightarrow O$,

$$s_1 \cdot_{\mathcal{G}^*} \sigma = s_0 \cdot_{\mathcal{G}_0} (\tau_1 \cdot \sigma)$$

with $\tau_1 : \mathsf{Oc}_0 \rightarrow \mathsf{Oc}^*$ defined by $\tau_1(\alpha) = (\alpha, \sigma_1(\alpha))$. This completes the definition of $\mathcal{G}^*$.

(ii) *Proof that $\mathcal{G}^*$ is a differentiated substitution frame*:
We will show that $\mathcal{G}^*$ fulfills the two conditions of Definition 4.3.4. Let $\mathcal{G}_0$, s_0, s_1, σ_1, τ_1 be as in the definition of $\mathcal{G}^*$. Then:

1. $s_1 \cdot_{\mathcal{G}^*} \Pi^* = s_0 \cdot_{\mathcal{G}_0} (\tau_1 \cdot \Pi^*) = s_0 \cdot_{\mathcal{G}_0} \sigma_1 = s_1$.
2. Consider any $\sigma : \mathsf{Oc}^* \to O$. We have to show that there is a μ such that
   (a) $\mu \cdot \Pi^* = \sigma$,
   (b) for all $\sigma' : \mathsf{Oc}^* \to O$, $(s_1 \cdot_{\mathcal{G}^*} \sigma) \cdot_{\mathcal{G}^*} \sigma' = s_1 \cdot_{\mathcal{G}^*} (\mu \cdot \sigma')$.

Let $s_1' = s_1 \cdot_{\mathcal{G}^*} \sigma$, and let s_0', σ_1', τ_1' be related entities of s_1' in the construction of $\mathcal{G}^*$. As we noted before, for any $\mathcal{G} \in \Psi$ and any $s \in S$ there is a transition from s_0 to s in $\mathcal{G}$ iff there is a transition from s_0 to s in $\mathcal{G}_0$. So, because there is a transition from s_0 to s_1' in $\mathcal{G}_0$, and because there is a path from s_1' to s_0' of transitions in frames of Ψ, there is a transition σ_0 from s_0 to s_0' in $\mathcal{G}_0$. Then for any corresponding mapping $\mu_0 : \mathsf{Oc} \to \mathsf{Oc}$ we have:

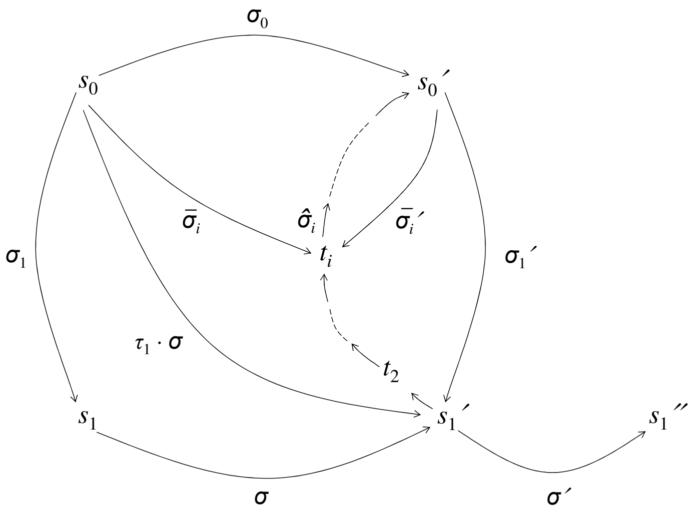
$$\begin{aligned}(s_1 \cdot_{\mathcal{G}^*} \sigma) \cdot_{\mathcal{G}^*} \sigma' &= s_0' \cdot_{\mathcal{G}_0} (\tau_1' \cdot \sigma') \\ &= (s_0 \cdot_{\mathcal{G}_0} \sigma_0) \cdot_{\mathcal{G}_0} (\tau_1' \cdot \sigma') \\ &= s_0 \cdot_{\mathcal{G}_0} (\mu_0 \cdot \tau_1' \cdot \sigma') \\ &= s_0 \cdot_{\mathcal{G}_0} (\tau_1 \cdot \tau_1^* \cdot \mu_0 \cdot \tau_1' \cdot \sigma') \qquad \text{with } \tau_1 \cdot \tau_1^* = \mathsf{id}_{\mathsf{Oc}_0} \\ &= s_1 \cdot_{\mathcal{G}^*} (\tau_1^* \cdot \mu_0 \cdot \tau_1' \cdot \sigma').\end{aligned}$$

This proves condition 2b.

Condition 2a requires some extra work. Let $X = \{(\alpha, \sigma_1(\alpha)) \mid \alpha \in \mathsf{Oc}_{\mathcal{G}_0}(s_0)\}$. Showing that $\tau_1^* \cdot \mu_0 \cdot \tau_1' \cdot \Pi^* =_X \sigma$ for some appropriate μ_0, will prove 2a. Because $\tau_1(\alpha) = (\alpha, \sigma_1(\alpha))$, $(\tau_1 \cdot \tau_1^*)(\alpha) = \alpha$ and $\tau_1' \cdot \Pi^* = \sigma_1'$, this is equivalent to showing that $\mu_0 \cdot \sigma_1' =_{\mathsf{Oc}_{\mathcal{G}_0}(s_0)} \tau_1 \cdot \sigma$ for some appropriate μ_0.

Choose a path $s_1' = t_1 \xrightarrow{\widehat{\sigma}_1} t_2 \xrightarrow{\widehat{\sigma}_2} \dots \xrightarrow{\widehat{\sigma}_{n-1}} t_n = s_0'$ of transitions in frames of Ψ such that for each transition $t_i \xrightarrow{\widehat{\sigma}_i} t_{i+1}$ in $\mathcal{G}$ the mapping $\widehat{\sigma}_i \restriction \mathsf{Oc}_{\mathcal{G}}(t_i)$ is injective.

For any $\mathcal{G} \in \Psi$, there is a natural one-to-one correspondence π between the occurrences of s_0 in $\mathcal{G}$ and in $\mathcal{G}_0$, namely $\pi(\alpha) = \alpha'$ iff $\Pi(\alpha) = \Pi_0(\alpha')$. For convenience, we will simply identify the occurrences of s_0 in $\mathcal{G}$ and $\mathcal{G}_0$. For s_0' we will do similarly. Now define transitions $s_0 \xrightarrow{\overline{\sigma}_i} t_i$ as follows:



1. $s_0 \xrightarrow{\overline{\sigma}_1} t_1$ is the transition $s_0 \xrightarrow{\tau_1 \cdot \sigma} t_1$,

2. $s_0 \xrightarrow{\overline{\sigma}_{i+1}} t_{i+1}$ is a composition of $s_0 \xrightarrow{\overline{\sigma}_i} t_i$ and $t_i \xrightarrow{\widehat{\sigma}_i} t_{i+1}$.

Also define transitions $s_0' \xrightarrow{\overline{\sigma}_i'} t_i$:

1. $s_0' \xrightarrow{\overline{\sigma}_1'} t_1$ is the transition $s_0' \xrightarrow{\sigma_1'} t_1$,

2. $s_0' \xrightarrow{\overline{\sigma}_{i+1}'} t_{i+1}$ is a composition of $s_0' \xrightarrow{\overline{\sigma}_i'} t_i$ and $t_i \xrightarrow{\widehat{\sigma}_i'} t_{i+1}$.

Because for each transition $t_i \xrightarrow{\widehat{\sigma}_i} t_{i+1}$ in $\mathcal{G}$ the mapping $\widehat{\sigma}_i \restriction \mathsf{Oc}_{\mathcal{G}}(t_i)$ is injective, it follows that if $s_0 \xrightarrow{\overline{\sigma}_{i+1}} t_{i+1}$ is a composition of $s_0 \xrightarrow{\sigma_0} s_0'$ and $s_0' \xrightarrow{\overline{\sigma}_{i+1}'} t_{i+1}$, then $s_0 \xrightarrow{\overline{\sigma}_i} t_i$ is a composition of $s_0 \xrightarrow{\sigma_0} s_0'$ and $s_0' \xrightarrow{\overline{\sigma}_i'} t_i$. So, choosing $s_0 \xrightarrow{\sigma_0} s_0'$ as a composition of $s_0 \xrightarrow{\overline{\sigma}_n} t_n$ and an inverse of $s_0' \xrightarrow{\overline{\sigma}_n'} t_n$ gives by induction that $s_0 \xrightarrow{\overline{\sigma}_1} t_1$ is a composition of $s_0 \xrightarrow{\sigma_0} s_0'$ and $s_0' \xrightarrow{\overline{\sigma}_1'} t_1$. Thus, $\mu_0 \cdot \sigma_1' =_{\mathsf{Oc}_{\mathcal{G}}(s_0)} \tau_1 \cdot \sigma$.

(iii) *Proof that $\mathcal{G}^*$ is a refinement of each frame in Ψ:*
Let s_0, s_1, σ_1 be as in the construction of $\mathcal{G}^*$. Let $\mu_1 : \mathsf{Oc}_0 \to \mathsf{Oc}_0$ correspond to $s_0 \xrightarrow{\sigma_1} s_1$ in $\mathcal{G}_0$. Let $\tau : \mathsf{Oc}^* \to \mathsf{Oc}_0$ be a function with

$$\tau(\alpha, a) = \begin{cases} \mu_1(\alpha) & \text{if } a = \sigma_1(\alpha), \\ \alpha' \text{ with } \Pi_0(\alpha') = a & \text{otherwise.} \end{cases}$$

Then

1. $\tau \cdot \Pi_0 = \Pi^*$, because

$$\Pi_0(\tau(\alpha, a)) = \begin{cases} \Pi_0(\mu_1(\alpha)) = \sigma_1(\alpha) = \Pi^*(\alpha, a) & \text{if } a = \sigma_1(\alpha), \\ a = \Pi^*(\alpha, a) & \text{otherwise.} \end{cases}$$

2. for every $\sigma : \mathsf{Oc}_0 \to O$, $s_1 \cdot_{\mathcal{G}^*} (\tau \cdot \sigma) = s_0 \cdot_{\mathcal{G}_0} (\tau_1 \cdot \tau \cdot \sigma) = s_0 \cdot_{\mathcal{G}_0} (\mu_1 \cdot \sigma) = s_1 \cdot_{\mathcal{G}_0} \sigma$.

So, $\mathcal{G}^*$ is a refinement of $\mathcal{G}_0$.

To see that $\mathcal{G}^*$ is also a refinement of any other frame $\mathcal{G}$ in Ψ, observe that if in the construction of $\mathcal{G}^*$ we would have chosen $\mathcal{G}$ instead of $\mathcal{G}_0$, then we would have obtained a frame equally refined as $\mathcal{G}^*$. Thus it follows that $\mathcal{G}^*$ is a refinement of each frame in Ψ.

(iv) *Proof that $\mathcal{G}^*$ is a unique least common refinement of the frames in Ψ*: Let $\mathcal{G}^\diamond = \langle S, O, \mathsf{Oc}^\diamond, \Pi^\diamond, \Theta^\diamond \rangle$ be an arbitrary common refinement of the frames in Ψ. Let s_0, s_1, σ_1 be as in the construction of $\mathcal{G}^*$. Then there is a mapping $\tau^\diamond : \mathsf{Oc}^\diamond \to \mathsf{Oc}^*$ with $\tau^\diamond \cdot \Pi^* = \Pi^\diamond$ and $\tau^\diamond(\alpha) \in \mathsf{Oc}_{\mathcal{G}^*}(s_0)$ for any $\alpha \in \mathsf{Oc}_{\mathcal{G}^\diamond}(s_0)$. Let $\mu_1 : \mathsf{Oc}^\diamond \to \mathsf{Oc}^\diamond$ be a bijection corresponding to $s_0 \xrightarrow{\tau^\diamond \cdot \sigma_1} s_1$ in $\mathcal{G}^\diamond$, and let $\mu_2 : \mathsf{Oc}^* \to \mathsf{Oc}^*$ correspond to $s_0 \xrightarrow{\sigma_1} s_1$ in $\mathcal{G}^*$. Further, let $\tau : \mathsf{Oc}^\diamond \to \mathsf{Oc}^*$ be a function with

$$\tau(\alpha) = \begin{cases} (\mu_1^{-1} \cdot \tau^\diamond \cdot \mu_2)(\alpha) & \text{if } \alpha \in \mathsf{Oc}_{\mathcal{G}^\diamond}(s_1), \\ \alpha' \text{ with } \Pi^*(\alpha') = \Pi^\diamond(\alpha) & \text{otherwise.} \end{cases}$$

Then

1. $\tau \cdot \Pi^* = \Pi^\diamond$, because $(\tau^\diamond \cdot \mu_2 \cdot \Pi^*)(\alpha) = (\tau^\diamond \cdot \sigma_1)(\alpha) = (\mu_1 \cdot \Pi^\diamond)(\alpha)$ if $\alpha \in \mathsf{Oc}_{\mathcal{G}^\diamond}(s_0)$.

2. $s_1 \cdot_{\mathcal{G}^\diamond} (\tau \cdot \sigma) = s_0 \cdot_{\mathcal{G}^\diamond} (\tau^\diamond \cdot \mu_2 \cdot \sigma) = s_0 \cdot_{\mathcal{G}^*} (\mu_2 \cdot \sigma) = s_1 \cdot_{\mathcal{G}^*} \sigma$.

So, $\mathcal{G}^\diamond$ is a refinement of $\mathcal{G}^*$, and thus, because we assumed $\mathcal{G}^\diamond$ to be an arbitrary common refinement of the frames in Ψ, we see that $\mathcal{G}^*$ is not only a least common refinement, but also unique, modulo equal refinedness. ⊣

In the theorem we assumed that the object-degree of the frames is finite. I am not sure whether this condition could be dropped.

Remark 4.4.8. We can also prove that for any nonempty collection of normal refinements of a common substitution frame of finite object-degree, there is a greatest common subrefinement, which is also normal and unique, modulo equal refinedness. The construction of this subrefinement is analogous to the construction of $\mathcal{G}^*$ in Theorem 4.4.7, only here we use, instead of $\xrightarrow{*}$, the transitive closure of the relation $\rightarrow_1$ defined by:

$$s \rightarrow_1 s' \;\text{ if } s \cdot_{\mathcal{G}} \sigma = s' \text{ for all } \mathcal{G} \in \Psi \text{ and some } \sigma.$$

The proof that $\mathcal{G}^*$ is indeed the greatest common subrefinement and unique is relatively simple.

It follows that for a substitution frame of finite object-degree, the normal refinements—modulo equal refinedness—form a complete lattice.

A direct consequence of Theorem 4.4.7 is the following uniqueness result:

Corollary 4.4.9. *A normal substitution frame of finite object-degree has—modulo equal refinedness—a unique maximal normal refinement.*

4.4.3 Coalescence-free refinements

Of special interest are frames in which no coalescence of occurrences takes place:

Definition 4.4.10. We call a frame $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta \rangle$ *coalescence-free* if each transition $s \xrightarrow{\sigma} s'$ has a corresponding $\mu : \mathsf{Oc} \rightarrow \mathsf{Oc}$ that is injective on an occurrence-domain of s.

Note that coalescence-free normal frames are maximally refined.

We will characterize coalescence-free frames in terms of their underlying undifferentiated frames. We will restrict ourselves to cases where the underlying frame is of finite object-degree and *simple*:

Definition 4.4.11. We call a frame $\mathcal{F} = \langle S, O, \Sigma \rangle$ *simple* if there is a state s_0 such that

$$S = \{s_0 \cdot \delta \mid \delta : O \rightarrow O\}.$$

We call s_0 an *initial state.*

Similarly, we call a frame $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta \rangle$ *simple* if there is a state s_0 such that

$$S = \{s_0 \cdot \sigma \mid \sigma : \mathsf{Oc} \rightarrow O\}.$$

We define a class of frames whose transitions from an initial state s_0 are unique, modulo loops of s_0:

Definition 4.4.12. We call a simple frame $\mathcal{F} = \langle S, O, \Sigma \rangle$ of finite object-degree a *loop-initial frame* if for any initial state s_0, and any $\delta_1, \delta_2 \in O^O$,

$$s_0 \cdot \delta_1 = s_0 \cdot \delta_2 \;\Rightarrow\; \exists \delta_0 \, [\, \delta_2 =_{\mathsf{Ob}(s_0)} \delta_0 \cdot \delta_1 \text{ and } s_0 \cdot \delta_0 = s_0 \,].$$

Theorem 4.4.13. *Let $\mathcal{F}$ be a simple undifferentiated substitution frame of finite object-degree. Then $\mathcal{F}$ has a coalescence-free normal refinement iff $\mathcal{F}$ is a loop-initial frame.*

Proof. Assume $\mathcal{F}$ has a coalescence-free normal refinement $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta \rangle$. Let s_0 be an initial state of $\mathcal{F}$, and let δ_1, δ_2 be such that $s_0 \cdot_{\mathcal{F}} \delta_1 = s_0 \cdot_{\mathcal{F}} \delta_2$. Then, because $\mathcal{G}$ is a refinement of $\mathcal{F}$, $s_0 \cdot_{\mathcal{G}} (\Pi \cdot \delta_1) = s_0 \cdot_{\mathcal{G}} (\Pi \cdot \delta_2)$.

For $i = 1, 2$ let $\mu_i : \mathsf{Oc} \to \mathsf{Oc}$ correspond to the transition $\Pi \cdot \delta_i$ from s_0. Since all states have the same, finite occurrence-degree, we may assume that the mappings μ_i are bijective. Clearly, there is a mapping $\delta_0 : O \to O$ such that $\Pi \cdot \delta_0 =_{\mathsf{Oc}(s_0)} \mu_2 \cdot \mu_1^{-1} \cdot \Pi$. So,

$$\begin{aligned}
s_0 &= s_0 \cdot_{\mathcal{G}} (\mu_1 \cdot \mu_1^{-1} \cdot \Pi) \\
&= (s_0 \cdot_{\mathcal{G}} (\Pi \cdot \delta_1)) \cdot_{\mathcal{G}} (\mu_1^{-1} \cdot \Pi) \\
&= (s_0 \cdot_{\mathcal{G}} (\Pi \cdot \delta_2)) \cdot_{\mathcal{G}} (\mu_1^{-1} \cdot \Pi) \\
&= s_0 \cdot_{\mathcal{G}} (\mu_2 \cdot \mu_1^{-1} \cdot \Pi) \\
&= s_0 \cdot_{\mathcal{G}} (\Pi \cdot \delta_0) \\
&= s_0 \cdot_{\mathcal{F}} \delta_0.
\end{aligned}$$

Further,

$$\begin{aligned}
\Pi \cdot \delta_0 \cdot \delta_1 &=_{\mathsf{Oc}(s_0)} \mu_2 \cdot \mu_1^{-1} \cdot \Pi \cdot \delta_1 \\
&=_{\mathsf{Oc}(s_0)} \mu_2 \cdot \mu_1^{-1} \cdot \mu_1 \cdot \Pi \\
&=_{\mathsf{Oc}(s_0)} \mu_2 \cdot \Pi \\
&=_{\mathsf{Oc}(s_0)} \Pi \cdot \delta_2.
\end{aligned}$$

So, $\delta_0 \cdot \delta_1 =_{\mathsf{Ob}(s_0)} \delta_2$. We have proved that $\mathcal{F}$ is a loop-initial frame.

Conversely, assume $\mathcal{F} = \langle S, O, \Sigma \rangle$ is a loop-initial frame. Define $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta \rangle$ as follows:

1. Define $\mathsf{Oc} = S \times O$.
2. Choose an initial state s_0 of $\mathcal{F}$.
3. For each $s \in S$ choose one $\delta_s \in O^O$ such that $s_0 \cdot_{\mathcal{F}} \delta_s = s$.
4. Define $\Pi(s, d) = \delta_s(d)$.
5. Define $s \cdot_{\mathcal{G}} \sigma = s_0 \cdot_{\mathcal{F}} \delta$ with for all $d \in O$, $\delta(d) = \sigma(s, d)$.

We prove that $\mathcal{G}$ is a differentiated substitution frame by showing that $\mathcal{G}$ fulfills the two conditions of Definition 4.3.4.

(1) $s \cdot_{\mathcal{G}} \Pi = s_0 \cdot_{\mathcal{F}} \delta_s = s$.

(2) Consider any $s \in S$ and $\sigma : \mathsf{Oc} \to O$. Then $s \cdot_{\mathcal{G}} \sigma = s_0 \cdot_{\mathcal{F}} \delta_{s \cdot_{\mathcal{G}} \sigma}$. Also we have $s \cdot_{\mathcal{G}} \sigma = s_0 \cdot_{\mathcal{F}} \delta$ with for all $d \in O$, $\delta(d) = \sigma(s, d)$. Because $\mathcal{F}$ is a loop-initial frame, $\delta =_{\mathsf{Ob}(s_0)} \delta_0 \cdot \delta_{s \cdot_{\mathcal{G}} \sigma}$ and $s_0 \cdot_{\mathcal{F}} \delta_0 = s_0$ for some δ_0. Define a function $\mu : \mathsf{Oc} \to \mathsf{Oc}$ with $\mu(s, d) = (s \cdot_{\mathcal{G}} \sigma, \delta_0(d))$, and for any other state $s' \in S$, $\Pi(\mu(s', d)) = \sigma(s', d)$. Then

$$\begin{aligned} \Pi(\mu(s, d)) &= \Pi(s \cdot_{\mathcal{G}} \sigma, \delta_0(d)) \\ &= \delta_{s \cdot_{\mathcal{G}} \sigma}(\delta_0(d)) \\ &= \sigma(s, d). \end{aligned}$$

So, $\mu \cdot \Pi = \sigma$.

We have $(s \cdot_{\mathcal{G}} \sigma) \cdot_{\mathcal{G}} \sigma' = s_0 \cdot_{\mathcal{F}} \delta'$ with for all $d \in O$, $\delta'(d) = \sigma'(s \cdot_{\mathcal{G}} \sigma, d)$. Further, $s \cdot_{\mathcal{G}} (\mu \cdot \sigma') = s_0 \cdot_{\mathcal{F}} \delta''$ with for all $d \in O$, $\delta''(d) = (\mu \cdot \sigma')(s, d)$. So, because $\mu(s, d) = (s \cdot_{\mathcal{G}} \sigma, \delta_0(d))$, we see that $\delta'' = \delta_0 \cdot \delta'$. Thus, because $s_0 \cdot_{\mathcal{F}} \delta_0 = s_0$, we have $(s \cdot_{\mathcal{G}} \sigma) \cdot_{\mathcal{G}} \sigma' = s \cdot_{\mathcal{G}} (\mu \cdot \sigma')$. This completes the proof that $\mathcal{G}$ is a differentiated substitution frame.

To show that $\mathcal{G}$ is a refinement of $\mathcal{F}$, we note that $s \cdot_{\mathcal{G}} (\Pi \cdot \delta) = s_0 \cdot_{\mathcal{F}} \delta'$ with for all $d \in O$, $\delta'(d) = (\Pi \cdot \delta)(s, d)$. So, $s_0 \cdot_{\mathcal{F}} \delta' = (s_0 \cdot_{\mathcal{F}} \delta_s) \cdot_{\mathcal{F}} \delta = s \cdot_{\mathcal{F}} \delta$.

Further, it follows from item 5 of the definition of $\mathcal{G}$ that $\mathsf{oc\text{-}degree}_{\mathcal{G}}(s) = \mathsf{ob\text{-}degree}_{\mathcal{F}}(s_0)$, and thus that $\mathcal{G}$ is a coalescence-free normal frame. $\dashv$

Note that in the second part of the proof we *choose* certain functions from O to O. But since O consists of *objects*, we may perhaps not assume that O is representable in ZFC. So, perhaps it would be more accurate to add an additional condition in the theorem. Because the object-degree of $\mathcal{F}$ is finite, it would be enough to assume that O can be totally ordered.

4.5 Restricting frames

We can define operations like conjunction, disjunction and negation for undifferentiated substitution frames in a rather straightforward way. For example, the conjunction $\mathcal{F}\ \&\ \mathcal{F}'$ will have states $s\ \&\ s'$ with s a state of $\mathcal{F}$ and s' a state of $\mathcal{F}'$, and substitution defined by $(s\ \&\ s')\cdot\delta = s\cdot\delta\ \&\ s'\cdot\delta$. Note that the definition only requires that the following condition is satisfied:

$$s\ \&\ s' = t\ \&\ t' \ \Rightarrow\ s\cdot\delta\ \&\ s'\cdot\delta = t\cdot\delta\ \&\ t'\cdot\delta.$$

For differentiated substitution frames the situation is a bit more complicated. For example, in defining conjunction it is not immediately clear whether or not we should let the occurrences of a state $s\ \&\ s$ correspond one-to-one to the occurrences of s itself. Also, we should maybe be content with results that are unique modulo equal refinedness.

A really controversial issue is how to interpret such operations metaphysically. For example, in *The Philosophy of Logical Atomism* Bertrand Russell said that when he argued that there were negative facts, it nearly produced a riot [Rus56, p. 211]. Russell says that on the whole he is inclined to believe that there are negative facts, but no disjunctive facts [Rus56, p. 215]. David Armstrong rejects both negative and disjunctive facts, but he does accept conjunctive facts and totality facts [Arm97]. We will not pursue this issue here.

What I would like to discuss here in more detail is the notion of *restriction* for relations. Consider a frame for the love relation with states x *loving* y. If we restrict the states to those of x *loving Mo*, then we get a new frame for this restricted set of states. We get another type of restriction if we take as states only x *loves* x. More generally, we define:

Definition 4.5.1. Let $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta\rangle$ and $\mathcal{G}' = \langle S', O', \mathsf{Oc}', \Pi', \Theta'\rangle$ be substitution frames. We say that $\mathcal{G}$ is a *restriction of* $\mathcal{G}'$ if $S \subseteq S'$, $O = O'$, and for every $s \in S$ there is an $X \subseteq \mathsf{Oc}'$ and a function $\tau : X \to \mathsf{Oc}$ such that

1. $\tau \cdot \Pi =_X \Pi'$,
2. for every $\sigma : \mathsf{Oc} \to O$, $\quad s \cdot_{\mathcal{G}} \sigma = s \cdot_{\mathcal{G}'} \sigma'$ with $\sigma' =_X \tau \cdot \sigma$ and $\sigma' =_{\mathsf{Oc}'-X} \Pi'$.

We call a restriction $\mathcal{G}$ of $\mathcal{G}'$ *simple* if $\mathcal{G}$ is a simple substitution frame.

We call a restriction $\mathcal{G}$ of $\mathcal{G}'$ *maximal for* S if for any $\mathcal{G}_1$ with state set S, and such that $\mathcal{G}$ is a restriction of $\mathcal{G}_1$ and $\mathcal{G}_1$ is a restriction of $\mathcal{G}'$, the frames $\mathcal{G}$ and $\mathcal{G}_1$ are equally refined.

Note that if $\mathcal{G}'$ is a refinement of $\mathcal{G}$, then $\mathcal{G}$ is a restriction of $\mathcal{G}'$.

The relation *restriction of* is transitive:

Lemma 4.5.2. *If $\mathcal{G}_1$ is a restriction of $\mathcal{G}_2$ and $\mathcal{G}_2$ is a restriction of $\mathcal{G}_3$, then $\mathcal{G}_1$ is a restriction of $\mathcal{G}_3$.*

Proof. The proof follows straightforwardly from the definition of restriction. $\dashv$

The relation *maximal restriction of* is not transitive:

Example 4.5.3. Let $\Re xyuv$ be the state $x \mathrel{-\!\heartsuit\!\!\rightarrow} y \;\;\&\;\; u \mathrel{-\!\heartsuit\!\!\rightarrow} v$. Let $\mathcal{G}_1$ be a frame with states $\Re xyuv$ and such that each state has four occurrences. Let $\mathcal{G}_2$ be a maximal restriction of $\mathcal{G}_1$ with states $\Re xyyv$, and such that the states $\Re xyyx$ have one occurrence of x and one occurrence of y. Now for a given b let $\mathcal{G}_3$ be a maximal simple restriction of $\mathcal{G}_2$ with states $\Re xbbv$. Then, $\Re xbbx$ obviously has one occurrence of x in $\mathcal{G}_2$, but this state has two occurrences of x in some refinement of $\mathcal{G}_3$. Since this refinement is also a restriction of $\mathcal{G}_1$, it follows that $\mathcal{G}_3$ is not a maximal restriction of $\mathcal{G}_1$. $\dashv$

Note that the example also shows that coalescence-free frames are not closed under the restriction relation.

Also note that for any frame $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta\rangle$ and any nonempty $S' \subseteq S$, there is a restriction of $\mathcal{G}$ with state set S', namely $\mathcal{G}' = \langle S', O, \mathsf{Oc}, \Pi, \Theta'\rangle$ with $s \cdot_{\mathcal{G}'} \sigma = s$ for every $\sigma : \mathsf{Oc} \rightarrow O$. Because of such (degenerated) restrictions, maximal restrictions will obviously in general not be unique, modulo equal refinedness. However, if we restrict ourselves to simple restrictions, we have the following uniqueness result:

Theorem 4.5.4. *Let $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta\rangle$ be a normal frame of finite object-degree. Then for any $S' \subseteq S$, there is—modulo equal refinedness—at most one simple maximal restriction of $\mathcal{G}$ for S'.*

Proof. Let $\mathcal{G}_1 = \langle S', O, \mathsf{Oc}_1, \Pi_1, \Theta_1\rangle$ and $\mathcal{G}_2 = \langle S', O, \mathsf{Oc}_2, \Pi_2, \Theta_2\rangle$ be simple restrictions of $\mathcal{G}$. First we will show that $\mathcal{G}_1$ and $\mathcal{G}_2$ have the same underlying undifferentiated substitution frame. Then by Theorem 4.4.7, $\mathcal{G}_1$ and $\mathcal{G}_2$ have a least common refinement, which is normal and unique, modulo equal refinedness. We will show that this refinement is also a restriction of $\mathcal{G}$. Then the theorem follows since there are—modulo equal refinedness—only a finite number of normal refinements of $\mathcal{G}_1$.

(i) Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be the underlying undifferentiated substitution frames of $\mathcal{G}_1$ and $\mathcal{G}_2$. Choose a state $s_0 \in S'$ with $\mathsf{oc\text{-}degree}_{\mathcal{G}}(s_0)$ maximal. Then s_0 is an initial state of $\mathcal{G}_1$ and $\mathcal{G}_2$, because otherwise there would be a state $t_0 \in S'$ with $\mathsf{oc\text{-}degree}_{\mathcal{G}}(t_0) > \mathsf{oc\text{-}degree}_{\mathcal{G}}(s_0)$. It is not difficult to see that $\mathsf{Ob}_{\mathcal{G}_1}(s_0) = \mathsf{Ob}_{\mathcal{G}_2}(s_0)$, and that for any $\delta : O \to O$, $s_0 \cdot_{\mathcal{F}_1} \delta = s_0 \cdot_{\mathcal{F}_2} \delta$. So, for any $s \in S'$, there is a δ' such that $s \cdot_{\mathcal{F}_1} \delta = (s_0 \cdot_{\mathcal{F}_1} \delta') \cdot_{\mathcal{F}_1} \delta = s_0 \cdot_{\mathcal{F}_1} (\delta' \cdot \delta) = s_0 \cdot_{\mathcal{F}_2} (\delta' \cdot \delta) = (s_0 \cdot_{\mathcal{F}_2} \delta') \cdot_{\mathcal{F}_2} \delta = s \cdot_{\mathcal{F}_2} \delta$. Thus, $\mathcal{F}_1 = \mathcal{F}_2$.

(ii) Consider the construction of $\mathcal{G}^*$ in the proof of Theorem 4.4.7 starting with $\mathcal{G}_1$ and $\mathcal{G}_2$. Let $s_0, s_1, \sigma_1, \tau_1$ be as in the construction of $\mathcal{G}^*$.

Further, let $X_0 \in \mathsf{Oc}_{\mathcal{G}}(s_0)$ and $\widetilde{\tau}_0 : X_0 \to \mathsf{Oc}_1$ be such that $\widetilde{\tau}_0 \cdot \Pi_1 =_{X_0} \Pi$, and for every $\sigma : \mathsf{Oc}_1 \to O$, $\; s_0 \cdot_{\mathcal{G}_1} \sigma = s_0 \cdot_{\mathcal{G}} \sigma'$ with $\sigma' =_{X_0} \widetilde{\tau}_0 \cdot \sigma$ and $\sigma' =_{\mathsf{Oc}' - X_0} \Pi$.

Define $\widetilde{\sigma}_1 : \mathsf{Oc} \to O$ by

$$\widetilde{\sigma}_1(\alpha) = \begin{cases} \sigma_1(\widetilde{\tau}_0(\alpha)) & \text{if } \alpha \in X_0 \\ \Pi(\alpha) & \text{otherwise.} \end{cases}$$

Let $\widetilde{\mu}_1$ correspond to $s_0 \xrightarrow{\widetilde{\sigma}_1} s_1$. Then $\widetilde{\mu}_1$ is injective on X_0, because there is in $\mathcal{G}$ also a transition from s_1 to s_0. Now define $X_1 = \widetilde{\mu}_1[X_0]$ and $\widetilde{\tau}_1 = (\widetilde{\mu}_1 \restriction X_0)^{-1} \cdot \widetilde{\tau}_0 \cdot \tau_1$. It is straightforward to check that $\widetilde{\tau}_1 \cdot \Pi^* =_{X_1} \Pi$, and for every $\sigma : \mathsf{Oc}^* \to O$, $\; s \cdot_{\mathcal{G}^*} \sigma = s \cdot_{\mathcal{G}} \sigma'$ with $\sigma' =_{X_1} \widetilde{\tau}_1 \cdot \sigma$ and $\sigma' =_{\mathsf{Oc} - X_1} \Pi$. So, it follows that $\mathcal{G}^*$ is a restriction of $\mathcal{G}$. ⊣

Other forms of restriction that never introduce a coalescence of occurrences are conceivable. But for such restrictions more than one state may have the same underlying state in the original relation. For example, if we start with the 'double' love relation of Example 4.5.3 then the state $x \overset{\heartsuit}{\to} y \;\&\; y \overset{\heartsuit}{\to} x$ may underly two states of a restriction, namely a state with two occurrences of x and one of y and another state with one occurrence of x and two of y. Also one state of a restriction may perhaps have more than one underlying state. This would for example be the case if we can make a further restriction to states $x \mathrel{-\!\heartsuit\!\!\rightarrow} y \;\; \& \;\; y \mathrel{-\!\heartsuit\!\!\rightarrow} x$.

4.6 Back to reality

A key question here is how to determine, for any state of a relation, what its occurrences of objects are. For the love relation the state of Echo's loving Narcissus obviously has two occurrences, but what about the state of Narcissus's loving Narcissus? If this state has one occurrence, then substituting Narcissus for Echo in Echo's loving Narcissus gives a coalescence of occurrences.

If in the 'real' world no coalescence of occurrences takes place, then for many relations the logical space is straightforwardly determined. But if we may not exclude coalescence of occurrences, then the principles in Section 4.2 often leave us with many choices.

We are inclined to think that there is just one love relation, one adjacency relation, one similarity relation, etc. But this might perhaps be disputed. Why would we exclude the possibility that there is a love relation where the state of Narcissus's loving Narcissus has only one occurrence of Narcissus, and another relation where Narcissus's loving Narcissus has two occurrences of him? Maybe we should not even exclude the possibility of a love relation with both states, one with two occurrences of him and another with one. Could it be that there simply *is* no deeper metaphysical fact that determines the right choice?

These considerations might engender the uncomfortable feeling that we could easily get stuck with an abundance of relations for the same states of affairs 'out there' in reality. However, what we want is a clear, uncomplicated canonical view on the logical structure of relations.

In our mathematical analysis we have arrived at a few results that provide more insight with regard to the possibilities:

(i) By Corollary 4.4.9, any normal substitution frame of finite object-degree has a unique maximal normal refinement.[5]

[5] More precisely we should say: unique up to equal refinedness.

(ii) By Theorem 4.5.4, any normal substitution frame of finite object-degree has for each subset of states at most one simple maximal restriction.

We might formulate the first result in terms of relations as follows. Suppose we modeled the logical space of a given relation by a substitution frame for which the maximum number of occurrences of its states is finite and equal to the maximum number of objects of its states. Suppose further that the 'real' occurrences of the relation are a maximal refinement of the occurrences of the frame. Then the logical space of the relation is in fact uniquely determined by the original frame.

It seems reasonable to postulate that the logical space for restrictions of relations is maximal, *if unique* in an appropriate sense. By Theorem 4.5.4, we know that this applies to relations whose logical space corresponds to a simple restriction of a normal frame of finite object-degree. What makes this result of metaphysical interest is that they probably form a large class of 'real' relations. Encouraged by the two previously mentioned uniqueness results we might go a step further and postulate the following *maximality principle*:

P-x The occurrences of every relation are always maximally refined.

If this principle is true, then that would be very nice, since it would make the notion of occurrences more accessible. Let us test its viability by examining some objections that could be made:

Objection 1: Example 4.4.5 does not support the maximality principle. The example shows a substitution frame whose refinements have no refinement in common. So, in this case there is no unique maximum. Furthermore, it is an open question whether all normal substitution frame of *infinite* object-degree have a unique maximal normal refinement.

Objection 2: A conjunction of a relation with itself will introduce a coalescence of occurrences, if the occurrences of a state $s \ \& \ s$ correspond one-to-one with the occurrences of s. If so, then this would obviously contradict the maximality principle.

Objection 3: Relations whose states have a *set-like* character are straightforward counterexamples to the maximality principle. Take the relation of *waiting for the bus*. If we may substitute Janneke for Vincent in the state

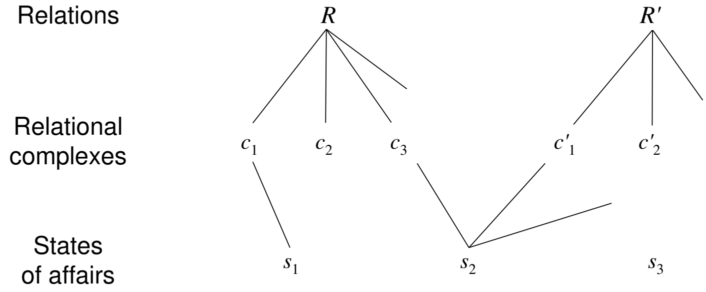


Figure 4.2: Should we distinguish relational complexes from states of affairs?

of Janneke and Vincent's waiting for the bus, the resulting state will have only one occurrence, and therefore will not be maximally refined.

These objections seem strong, but there may be escape routes available:

Against Objection 1 it might be argued that the frame of the example is farfetched and probably does not correspond to a 'real' relation. As a safer alternative, we could restrict the maximality principle to relations whose logical space corresponds to normal frames of finite degree.

Some people will reject Objection 2, because they will deny that there are conjunctive relations. But let us suppose that conjunctive relations exist. Then why would the occurrences of s & s correspond one-to-one to the occurrences of s? I don't see a convincing argument. Even if s & s and s are identical states, then there could still be a way out, namely by arguing that not states, but *relational complexes* have occurrences. A relational complex could be conceived of as a structured perspective on a state 'out there', where we give up the idea that there is a one-to-one correspondence or even identity between states of affairs and relational complexes.[6] (See Figure 4.2.) With this approach, there seems to be no compelling reason why there should be a one-to-one correspondence between occurrences of different relational complexes corresponding to the same state.

Maintaining both states 'out there' and relational complexes, however, has as potential drawback that it might give rise to an inflation of ontology. A more detailed analysis is needed to address this issue. Note, by the way, that in our formal analysis we also allowed states to belong to more than

[6] Such a one-to-one correspondence is often taken for granted. For example, Russell [Rus84, p. 80] says: "there is certainly a one-one correspondence of complexes and facts".

one frame: in the definition of a restriction of a frame (Definition 4.5.1) the states of a restriction of a frame $\mathcal{G}$ are a subset of the states of $\mathcal{G}$ itself. If this is not acceptable, then an alternative definition of a restriction might be given that takes *hiding* and *merging* of occurrences as primitive operations.

With respect to Objection 3 two different replies are possible. First, one could argue that the objects of states with a set-like character are sets themselves, not the members of the sets. For the given example, this would mean that in fact we should substitute the set consisting of Janneke for the set consisting of Janneke and Vincent. Alternatively, one could argue that certain substitutions in states with a set-like character are highly dubious, since it is in some cases unclear what exactly the result should be. In the example, it may be more natural to leave substituting Janneke for Vincent undefined. The interrelatedness of the states may more adequately be expressed by the operation of *subtraction*. It seems natural to say that *subtracting* Vincent from the state of Janneke and Vincent's waiting for the bus results in the state of Janneke's waiting for the bus. If we follow this road, then we need a weaker version of principle P-6:

P-6′ Any substitution of objects for occurrences in a state results in *at most* one state of the same relation.

In addition, we should formulate a principle for subtracting occurrences similar to P-6′ and a composition principle for subtractions.

As an aside, note that principle P-6′ also has the advantage that it allows us to keep meaningless or (conceptually) impossible states out of our ontology. Substituting, for example, a for b in *a's being adjacent to b* might be problematic. If you do not accept the existence of conceptually impossible states, then it could be better to leave the result of this substitution undefined.

To conclude this discussion, I think that a maximality principle like principle P-x might be the right choice for getting close to capturing the essence of occurrences. But to make such a principle really plausible, we need to conduct a more profound study of the notion of states, of substitution and probably of other operations, like subtraction. Also a follow-up of our formal analysis will be needed where partial substitution and subtraction functions are taken into account.

A view on relations without coalescence of occurrences still seems an attractive alternative, because of its simplicity. We already countered two arguments against it, namely a coalescence in relations with set-like states, and coalescence introduced by a conjunction of relations. In Section 4.2 we gave a third objection against a coalescence-free account. We considered the ternary relation $\Re$ where $\Re abc$ is the state of *a's loving b & b's loving c*. This relation has the peculiar property that $\Re bab$ and $\Re aba$ are identical states. As a consequence, this state can have only one occurrence of a and one occurrence of b, and thus in a transition from $\Re abc$ to this state we get coalescence of occurrences. How can we counter this objection?

Again relational complexes may be the solution. We could argue that there are *two* relational complexes corresponding to the state of *a's loving b & b's loving a*, namely one relational complex with two occurrences of a and one with two occurrences of b. Only in this case we would have to accept that one state 'out there' can have more than one relational complex within the *same* relation. This approach looks quite natural if we regard the relation $\Re$ as a restriction of a quaternary relation with states like *a's loving b & c's loving d*, because then one of the relational complexes corresponding to *a's loving b & b's loving a* in $\Re$ could be taken as the result of merging two occurrences of a and the other relational complex as the result of merging two occurrences of b. For any relation with a comparable symmetry, a similar 'solution' could be given. We also regard this issue as something that requires further investigation.

There is one final issue of metaphysical importance that I would like to discuss, namely the question: Do we really *need* occurrences?

Occurrences are definitely a very useful part of our representation of reality, but that in some cases the best way to express certain properties of states is in terms of occurrences does not force us to any ontological commitment to them. So, do we have a compelling reason to assume that *for relations* occurrences are ontologically basic?

Let us assume, for the moment, that the maximality principle is valid, and in addition that for every 'real' relation the underlying undifferentiated substitution frame has a unique maximal refinement. Would this not imply that a complete account of relations could be given in terms of undifferentiated substitutions? If so, then a parsimonious undifferentiated substitution mechanism might be enough for the ontology of relations.

I only see one strong argument in favor of occurrences being basic for relations, namely that some relations might just not have enough objects for an occurrence-free account of relations. Consider a relation whose occurrence-degree is larger than its object-degree. Then undifferentiated substitution might not provide enough information about the interconnection of the states. We might have such a situation for example with a conjunction $\Re$ & $\Re$ & .... & $\Re$, where $\Re$ is a binary relation with a finite number of objects. Probably such situations do not only occur in complex relations, but also in what we would regard as elementary relations. An example of such an elementary relation could maybe be something like a relation with states of *a's loving b more than c* in a mini-world with two inhabitants.

My conclusion is that we have good reason to consider occurrences of relations to be a primitive notion. Further, I think our analysis has brought us closer to revealing the essence of the logical structure of relations. Although some major issues are still open, the way we have articulated them may very well contribute to their solution.

Acknowledgements I am grateful to Kit Fine for illuminating discussions about coalescence of occurrences. Further, I thank Albert Visser and the referee for valuable comments on the manuscript.

A Occurrences and roles

An object can fulfill one or more roles in a state. For example in the amatory relation an object can play the role of *lover* or the role of *beloved.* In this appendix we investigate how roles are related to occurrences.

We only define roles for simple substitution frames of finite degree. The definition will be such that

1. any object of any initial state fulfills exactly one role,

2. if s_0 is an initial state, $a_0 \in \mathsf{Ob}(s_0)$, and $a \in \mathsf{Ob}(s)$, then a fulfills in s the role of a_0 in s_0 iff there is a δ such that $s = s_0 \cdot \delta$ and $\delta(a_0) = a$.

It is not clear how to generalize the definition for arbitrary undifferentiated substitution frames.

Definition A.1. Let $\mathcal{F} = \langle S, O, \Sigma \rangle$ be a simple undifferentiated substitution frame of finite object-degree. Then for any initial state $s_0 \in S$ and $a_0 \in \mathsf{Ob}(s_0)$, we define the *role of* a_0 *in* s_0 as

$$\mathsf{Role}(s_0, a_0) = \{(s, a) \mid \exists\delta\, [\, s \cdot \delta = s_0 \;\;\&\;\; a \in \mathsf{Ob}(s) \;\;\&\;\; \delta(a) = a_0\,]\}.$$

Further, we define the *roles of* $\mathcal{F}$ as

$$\mathsf{Roles}_{\mathcal{F}} = \{\mathsf{Role}(s_0, a_0) \mid s_0 \in S \text{ is an initial state } \;\;\&\;\; a_0 \in \mathsf{Ob}(s_0)\}.$$

For arbitrary $s \in S$, $a \in \mathsf{Ob}(s)$, we say that *a in s fulfills role* $\rho \in \mathsf{Roles}_{\mathcal{F}}$ if for some s_0, a_0 with $\rho = \mathsf{Role}(s_0, a_0)$ there is a mapping $\delta : O \to O$ such that

$$s = s_0 \cdot \delta \;\text{ and }\; a = \delta(a_0).$$

It is easy to see that if $\mathcal{F}$ is an n-ary simple substitution frame, then $\mathcal{F}$ has at most n roles. Objects sometimes fulfill more than one role in certain states. For example, if $\mathcal{F}$ models the amatory relation, then in the state where Narcissus loves himself, he fulfills both roles of the model. It is also possible that an n-ary model with $n > 1$, has only one role. This is for example the case for cyclic models.

For differentiated substitution frames of finite occurrence-degree we can define roles in a similar way:

Definition A.2. Let $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta \rangle$ be a simple differentiated substitution frame of finite occurrence-degree. Then for any initial state $s_0 \in S$ and $\alpha_0 \in \mathsf{Oc}(s_0)$, we define the *role of* α_0 *in* s_0 as

$$\begin{aligned}\mathsf{Role}(s_0, \alpha_0) = \{(s, \alpha) \mid \exists\sigma\, [\, s \cdot \sigma = s_0 \;\;\&\;\; \alpha \in \mathsf{Oc}(s) \\ \&\;\; \mu(\alpha) = \alpha_0 \text{ with } \mu \text{ corresponding to } s \xrightarrow{\sigma} s_0\,]\}.\end{aligned}$$

Further, we define the *roles of* $\mathcal{G}$ as

$$\mathsf{Roles}_{\mathcal{G}} = \{\mathsf{Role}(s_0, \alpha_0) \mid s_0 \in S \text{ is an initial state } \;\;\&\;\; \alpha_0 \in \mathsf{Oc}(s_0)\}.$$

For arbitrary $s \in S$, $\alpha \in \mathsf{Oc}(s)$, we say that *a in s fulfills role* $\rho \in \mathsf{Roles}_{\mathcal{G}}$ if for some s_0, α_0 with $\rho = \mathsf{Role}(s_0, \alpha_0)$ there is a mapping $\sigma : \mathsf{Oc} \to O$ such that

$$s = s_0 \cdot \sigma \;\text{ and }\; \mu(\alpha_0) = \alpha \text{ for a corresponding } \mu.$$

Consider again the relation $\Re$ where $\Re abc$ is the state that *a loves b & b loves c*. Let $\mathcal{G}$ be a differentiated substitution frame for it with $\mathsf{oc\text{-}degree}_{\mathcal{G}} = 3$. Then because the state $\Re aba$ is identical with the state $\Re bab$, a and b each fulfill three roles, but a and b each have only one occurrence in $\Re bab$.

Theorem A.3. *Let $\mathcal{G}$ be a simple differentiated substitution frame of finite occurrence-degree. Then each occurrence of a state of $\mathcal{G}$ fulfills exactly one role iff $\mathcal{G}$ is coalescence-free.*

Proof. The theorem follows immediately from the definitions. ⊣

B Categories of substitution frames

For a better understanding of the relation between undifferentiated and differentiated substitution frames, it is useful to consider them in terms of category theory.

There are various ways in which categories of substitution frames can be defined. We give for both types of substitution frames a definition that seems to be a rather natural choice.

Definition B.1. We define **USF** as the category with objects all undifferentiated substitution frames, and with morphisms from $\mathcal{F} = \langle S, O, \Sigma \rangle$ to $\mathcal{F}' = \langle S', O', \Sigma' \rangle$ all pairs $\langle f_S, f_O \rangle$ of functions $f_S : S \to S'$ and $f_O : O \to O'$ such that

1. f_O is injective,
2. $f_S(s \cdot_{\mathcal{F}} \delta) = f_S(s) \cdot_{\mathcal{F}'} \delta'$ with $\delta'(d') = \begin{cases} f_O(\delta(d)) & \text{if } d' = f_O(d), \\ d' & \text{otherwise.} \end{cases}$

Note that Condition 1 guarantees that the function δ' in Condition 2 is uniquely defined by the function δ.

Our category for differentiated substitution frames is somewhat more complicated:

Definition B.2. We define **DSF** as the category with objects all differentiated substitution frames and with morphisms from $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta \rangle$ to $\mathcal{G}' = \langle S', O', \mathsf{Oc}', \Pi', \Theta' \rangle$ all triples $\langle f_S, f_O, f_{\mathsf{Oc}} \rangle$ of functions $f_S : S \to S'$,

$f_O : O \to O'$ and $f_{\mathsf{Oc}} : S \times X \to \mathsf{Oc}$ with $X = (\Pi')^{-1}[\mathsf{im}\, f_O]$, the inverse image of $\mathsf{im}\, f_O$ by Π', such that

1. f_O is injective,
2. for each $s \in S$ and $\alpha' \in X$, $f_O(\Pi(f_{\mathsf{Oc}}(s, \alpha'))) = \Pi'(\alpha')$,
3. $f_S(s \cdot_{\mathcal{G}} \sigma) = f_S(s) \cdot_{\mathcal{G}'} \sigma'$ with $\sigma'(\alpha') = \begin{cases} \sigma(f_{\mathsf{Oc}}(s, \alpha')) & \text{if } \alpha' \in X, \\ \Pi'(\alpha') & \text{otherwise.} \end{cases}$

The next theorem states that there is an adjunction from USF to DSF.

Theorem B.3. *For the categories* USF *and* DSF *the functor that assigns to an undifferentiated substitution frame its basic refinement is a left-adjoint for the functor that assigns to a differentiated substitution frame its underlying undifferentiated substitution frame.*

Proof. Let $R : \mathsf{USF} \to \mathsf{DSF}$ be the functor that assigns to an undifferentiated substitution frame its basic refinement, and $U : \mathsf{DSF} \to \mathsf{USF}$ be the forgetful functor that assigns to a differentiated substitution frame its underlying undifferentiated substitution frame.

For each $\mathcal{F} \in \mathsf{USF}$ and $\mathcal{G} \in \mathsf{DSF}$, we get a bijection

$$\Phi_{\mathcal{F},\mathcal{G}} : \mathsf{DSF}(R(\mathcal{F}), \mathcal{G}) \to \mathsf{USF}(\mathcal{F}, U(\mathcal{G}))$$

by assigning $\langle f_S, f_O \rangle$ to the morphism $\langle f_S, f_O, f_{\mathsf{Oc}} \rangle$. It is not difficult to see that the family of bijections

$$\Phi : \mathsf{DSF}(R(\mathcal{F}), \mathcal{G}) \cong \mathsf{USF}(\mathcal{F}, U(\mathcal{G}))$$

is natural in $\mathcal{F}$ and $\mathcal{G}$. So, $\langle R, U, \Phi \rangle$ is an adjunction from USF to DSF. ⊣

The theorem can also be proved by characterizing an adjunction in terms of the unit of adjunction (see e.g. Theorem IV.1.2(i) of [ML98, p. 83]). The unit of adjunction is in this case simply the identity natural transformation $\eta : I_{\mathsf{USF}} \stackrel{\cdot}{\to} I_{\mathsf{USF}}$.

We should be careful when drawing conclusions from this theorem. In particular, I would not directly regard it as strong evidence of a natural relationship between undifferentiated and differentiated substitution frames. For some slightly different, but equally natural, definitions of a category of differentiated substitution frames, we do not get an adjunction.

C Alternative substitution frames

In this appendix we present two alternative types of frames. In the first type the states have disjoint occurrence-domains, and in the second type occurrences are explicitly mapped to occurrences. Further, we discuss two reasons why our attempts to simplify our definition of differentiated substitution frames failed.

C.1 Frames with disjoint occurrence-domains

In the definition of differentiated substitution frames we did not demand that different states have disjoint occurrence-domains. One may perhaps regard this as a shortcoming, but it can rather easily be fixed by defining a differentiated substitution frame as a sextuple $\mathcal{F} = \langle S, O, \mathsf{Oc}, \Pi, \Pi_S, \Theta \rangle$ with Π_S a mapping from Oc to S that provides the states in S with disjoint occurrence-domains $\mathsf{Oc}_s = \Pi_S^{-1}(s)$. Further, Θ is a mapping from $\{(s, \sigma) \mid s \in S \;\&\; \sigma \in O^{\mathsf{Oc}_s}\}$ to S such that:

1. $\Theta(s, \Pi \restriction \mathsf{Oc}_s) = s$,

2. for all $s \in S$ and $\sigma : \mathsf{Oc}_s \to O$ there is a mapping $\mu : \mathsf{Oc}_s \to \mathsf{Oc}_{\Theta(s,\sigma)}$ such that

   (a) $\Pi \circ \mu = \sigma$,

   (b) for all $\sigma' : \mathsf{Oc}_{\Theta(s,\sigma)} \to O$, $\Theta\left(\Theta(s, \sigma), \sigma'\right) = \Theta(s, \sigma' \circ \mu)$.

Because of the extra complexity of this definition we prefer to work with the initial definition of differentiated substitution frames, being on the alert for results that only applies to them, but not to those with disjoint occurrence-frames.

C.2 Frames explicitly mapping occurrences to occurrences

In our refined frames the correspondence between the occurrences of different states is implicit in the definition of Θ. We could make this correspondence explicit by defining frames as tuples $\langle S, O, \mathsf{Oc}, \Pi, \Psi \rangle$, where for any $s, s' \in S$, $\Psi(s, s')$ is a set of functions from Oc to Oc such that

1. $\mathsf{id}_{\mathsf{Oc}} \in \Psi(s, s)$,
2. if $\mu \in \Psi(s, s')$ and $\mu' \in \Psi(s', s'')$, then $\mu \cdot \mu' \in \Psi(s, s'')$,
3. if $\mu \in \Psi(s, s')$ and μ is a bijection, then $\mu^{-1} \in \Psi(s', s)$,
4. for all $s \in S$ and $\sigma : \mathsf{Oc} \to O$ there is exactly one $s' \in S$ and a mapping $\mu \in \Psi(s, s')$ such that $\mu \cdot \Pi = \sigma$.

There are several variants possible. For example, instead of assuming that the correspondence between the occurrences of one state and the occurrences of another one is *functional*, we could (perhaps more adequately) assume only a *relational* correspondence between the occurrences.

C.3 Failures to simplify our definition

The definition of a differentiated substitution frame perhaps looks more complex than needed. Our attempts to simplify the definition failed, because differentiated substitution frames lack certain nice properties, as we show in the next two observations.

Observation C.1. *Not for every differentiated substitution frame and every* $\mu : \mathsf{Oc} \to \mathsf{Oc}$ *we have* $(s \cdot (\mu \cdot \Pi)) \cdot \sigma = s \cdot (\mu \cdot \sigma)$.

Proof. Let $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta\rangle$ be a differentiated substitution frame without any symmetry and with a state s for which $\mathsf{Oc}(s) = \{\alpha_0, \alpha_1\}$ with $\alpha_0 \neq \alpha_1$ and $\Pi(\alpha_0) = \Pi(\alpha_1) = a$. Let $\mu : \mathsf{Oc} \to \mathsf{Oc}$ be such that $\mu(\alpha_0) = \alpha_1$ and $\mu(\alpha_1) = \alpha_0$. Then $s \cdot (\mu \cdot \Pi) = s$, but for any $b \in O$ with $b \neq a$, if $\sigma(\alpha_0) = a$ and $\sigma(\alpha_1) = b$, then $s{\cdot}\sigma \neq s{\cdot}(\mu{\cdot}\sigma)$. So, $(s{\cdot}(\mu{\cdot}\Pi)){\cdot}\sigma \neq s{\cdot}(\mu{\cdot}\sigma)$. $\dashv$

To a transition $s \xrightarrow{\sigma} s'$ more than one mapping $\mu : \mathsf{Oc} \to \mathsf{Oc}$ may correspond. The next observation shows that it is unlikely that we can always select one of them as a canonical mapping.

Observation C.2. *Not for every differentiated substitution frame, there is a representative function* $\overline{\mu} : S \times O^{\mathsf{Oc}} \to \mathsf{Oc}^{\mathsf{Oc}}$ *preserving composition, that is,*

(representative) $\overline{\mu}(s, \sigma)$ *corresponds to a transition* $s \xrightarrow{\sigma} s'$,

(preservation) $\overline{\mu}(s, \overline{\mu}(s, \sigma) \cdot \sigma') = \overline{\mu}(s, \sigma) \cdot \overline{\mu}(s \cdot \sigma, \sigma')$.

Proof. Let $\mathcal{F} = \langle S, O, \Sigma \rangle$ be an undifferentiated substitution frame with an initial state s_0 for which $\mathsf{Ob}(s_0) = \{a, b, c, d\}$, and such that $s_0 \cdot \delta = s_0 \cdot \delta'$ iff $\delta' =_{\mathsf{Ob}(s_0)} (\delta_0)^i \cdot \delta$ with $i \in \{0, 1, 2, 3\}$ and

$$\delta_0 \restriction \mathsf{Ob}(s_0) = \begin{pmatrix} a & b & c & d \\ d & a & b & c \end{pmatrix}.$$

Let $\mathcal{G} = \langle S, O, \mathsf{Oc}, \Pi, \Theta \rangle$ be a refinement of $\mathcal{F}$ with for each state four occurrences.

Now suppose there is a function $\overline{\mu} : S \times O^{\mathsf{Oc}} \to \mathsf{Oc}^{\mathsf{Oc}}$ such that:

1. $\overline{\mu}(s, \sigma) \cdot \Pi = \sigma$,
2. $s \cdot (\overline{\mu}(s, \sigma) \cdot \sigma') = (s \cdot \sigma) \cdot \sigma'$,
3. $\overline{\mu}(s, \overline{\mu}(s, \sigma) \cdot \sigma') = \overline{\mu}(s, \sigma) \cdot \overline{\mu}(s \cdot \sigma, \sigma')$.

Let $s_1 = s_0 \cdot_{\mathcal{F}} \delta_1$ with $\delta_1 \restriction \mathsf{Ob}(s_0) = \begin{pmatrix} a & b & c & d \\ a & b & a & b \end{pmatrix}$.

Further, let $\mathsf{Oc}(s_1) = \{0, 1, 2, 3\}$ with $\Pi(0) = \Pi(2) = a$, and let

$$\sigma_1 \restriction \mathsf{Oc}(s_1) = \begin{pmatrix} 0 & 1 & 2 & 3 \\ b & a & b & a \end{pmatrix}.$$

Then $s_1 \cdot \sigma_1 = s_1$. It is not difficult to see that by Properties (1) and (2) of $\overline{\mu}$:

$$\overline{\mu}(s_1, \sigma_1) \restriction \mathsf{Oc}(s_1) = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 0 & 1 & 2 \end{pmatrix} \text{ or } \overline{\mu}(s_1, \sigma_1) \restriction \mathsf{Oc}(s_1) = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \end{pmatrix}.$$

So, $\overline{\mu}(s_1, \sigma_1) \cdot \overline{\mu}(s_1, \sigma_1) \neq_{\mathsf{Oc}(s_1)} \mathsf{id}_{\mathsf{Oc}}$.

By Properties (1) and (3) of $\overline{\mu}$ we have for any $s \in S$:

$$\begin{aligned} \overline{\mu}(s, \Pi) &= \overline{\mu}(s, \overline{\mu}(s, \Pi) \cdot \Pi) \\ &= \overline{\mu}(s, \Pi) \cdot \overline{\mu}(s \cdot \Pi, \Pi) \\ &= \overline{\mu}(s, \Pi) \cdot \overline{\mu}(s, \Pi). \end{aligned}$$

So, because for any $s \in S$, $\overline{\mu}(s, \Pi)[\mathsf{Oc}(s)] = \mathsf{Oc}(s)$, we have $\overline{\mu}(s, \Pi) =_{\mathsf{Oc}(s)} \mathsf{id}_{\mathsf{Oc}}$, and so, because $\overline{\mu}(s_1, \sigma_1) \cdot \sigma_1 =_{\mathsf{Oc}(s_1)} \Pi$, we have $\overline{\mu}(s_1, \overline{\mu}(s_1, \sigma_1) \cdot \sigma_1) =_{\mathsf{Oc}(s_1)} \mathsf{id}_{\mathsf{Oc}}$. It follows that

$$\overline{\mu}(s_1, \overline{\mu}(s_1, \sigma_1) \cdot \sigma_1) \neq \overline{\mu}(s_1, \sigma_1) \cdot \overline{\mu}(s_1, \sigma_1) = \overline{\mu}(s_1, \sigma_1) \cdot \overline{\mu}(s_1 \cdot \sigma_1, \sigma_1),$$

contradicting Property (3) of $\overline{\mu}$. ⊣

Chapter 5

Relational complexes

Abstract

A theory of relations is presented that provides a detailed account of the logical structure of relational complexes. The theory draws a sharp distinction between relational complexes and relational states. A salient difference is that relational complexes belong to exactly one relation, whereas relational states may be shared by different relations. Relational complexes are conceived as a structured perspective on states 'out there' in reality. It is argued that only relational complexes have occurrences of objects, and that different complexes of the same relation may correspond to the same state.

5.1 Introduction

Many of us will consider *Oedipus's loving Jocasta* as a single state 'out there'. However, if we analyze this state by breaking it up into constituents, then we are confronted with a fundamental question: Does the state have an underlying relation with both Oedipus and Jocasta as relata, or with only one of them as a relatum?

In 'Theory of Knowledge', Bertrand Russell writes:

> A 'complex' is anything analyzable, anything which has *constituents*. [...] It may be questioned whether a complex is or is not the same as a 'fact', where a 'fact' may be described as what there is when a judgement is true, but not when it is false.

> [...] However this may be, there is certainly a one-one correspondence of complexes and facts, and for our present purposes we shall assume that they are identical. [Rus84, pp. 79–80]

In his paper 'Neutral Relations', Kit Fine expresses a similar view. He refers to states and facts as *complexes*, and writes: "We wish to adopt a conception of relations and their completions for which Uniqueness holds", where *Uniqueness* is the assumption that "no complex is the completion of two distinct relations" [Fin00, pp. 4–5].

What makes a view that identifies states 'out there' with relational complexes attractive is its apparent simplicity. Nevertheless, I have a serious problem with this view. I think it treats states 'out there' as more detailed than they in fact are. It seems natural to regard states 'out there' as *empirical entities*, that is, as empirically distinguishable entities, where empirical distinctions are ones that make a possible difference to the world. (See [Fin82, p. 58].) But if we do that, then identifying states with complexes is inconsistent with Uniqueness.

In order to retain a one-to-one correspondence between states and complexes, suppose we would give up Uniqueness. Now let us assume that the state of *Oedipus's loving Jocasta* has a single corresponding complex with one occurrence of Oedipus and one of Jocasta. Then this complex would not only be the completion of a binary relation, but of two unary relations as well. This, however, is not a desirable situation, since it obscures the interrelatedness of the complexes of a relation.[1] This suggests that giving up Uniqueness should not be the first choice.

It may perhaps be argued that an identification of states with complexes should be taken in a more restricted sense, namely that it applies only to the class of atomic or elementary relations. It could be maintained that the states of such relations are mutually disjoint and that each state has a unique corresponding complex or is even identical to it. This view, however, is also problematic since it makes a strong metaphysical claim that is far from evident. No one has ever given a convincing example of an atomic relation. Furthermore, an account of other relations would still be needed. What Russell said about this issue in 'My Philosophical Development' is interesting:

[1] In Section 5.2.2, other examples will be given that illustrate this point more dramatically.

> I have come to think, however, that, although many things can be known to be complex, nothing can be *known* to be simple, and, moreover, that statements in which complexes are named can be completely accurate, in spite of the fact that the complexes are not recognized as complex. [...] It follows that the whole question whether there are simples to be reached by analysis is unnecessary. [Rus05, p. 123]

Another attempt to 'save' a one-to-one correspondence between states and complexes, is to argue that each state 'out there' has a unique *canonical* corresponding complex. This could possibly be a complex with exactly one occurrence for each essentially involved object of the state.[2] So, for the state of *Narcissus's loving Narcissus* we could get a canonical complex of the monadic self-love relation, and for *Apollo's loving Daphne* we could get a canonical complex of the dyadic love relation (unless Eros is essentially involved, as well). It may, however, not always be clear what the essentially involved objects of a state are. For example, if we accept the existence of disjunctive states, then the notion of essentially involved objects requires further clarification.

A related issue concerns the structure of thoughts.[a] In 'Begriffsschrift' [Fre00, Section 9], Frege discusses the notion of functions and considers the proposition that *Cato killed Cato*. Frege notes that if we think of "Cato" as replaceable at its first occurrence, then "to kill Cato" is the function, but we also have a function "to be killed by Cato" and a function "to kill oneself". There is, however, controversy about the question whether the different decompositions correspond to different thoughts.

[a] Or maybe it is the same issue. In 'Der Gedanke', Frege said: "Was ist eine Tatsache? Eine Tatsache ist ein Gedanke, der wahr ist."[Fre03a, pp. 57–58] ("What is a fact? A fact is a thought that is true.")

[2] This approach may be advanced in particular by someone who regards (first-class) relations as universals, involving no particular particulars. (See [Arm97, pp. 92–93].)

José Luis Bermúdez [Ber01, pp. 94–95] claims that "“Cato killed Cato” can express four different thoughts", and that this also *has* to be Frege's position. On the other hand, Harold Hodes [Hod82, p. 162] argues that "Frege thought thoughts to be compositionally polymorphous". In the same vein, David Bell [Bel96, p. 596] talks about "the mistake of taking function/argument analysis to reveal intrinsic structure", and admits that he is more and more convinced that thoughts do not have a determinate, intrinsic structure.

By the way, Hodes [Hod82, p. 176] leaves open the possibility that something is wrong with the notion of Fregean thought itself: "Perhaps the Fregean notion of a thought is a hybrid, born of confusions created by divided reference." With 'divided reference' he means, on the one hand, reference in terms of possible worlds, and, on the other hand, reference in terms of something like Carnap's notion of intensional isomorphism. I think the confusion pointed at might also be formulated as a confusion between the notions of relational states and relational complexes.

Wittgenstein struggled with a similar issue and talked about "Das alte Problem von Komplex und Tatsache" [Wit84, entry 14.5.15]. Peter Simons [Sim92, p. 319] argues that "the *Sachverhalte* of the *Tractatus* are best seen not as atomic *facts*, but as atomic *complexes*". Furthermore, Simons [Sim92, p. 335] claims that "Wittgenstein thought the *Tractatus* embodied a confusion between complex and fact". In the note 'Complex and fact' [Wit75, pp. 301–303], Wittgenstein talks about a "muddle" and alludes to remarks in the *Tractatus*.

I started this short analysis of the correspondence between states and complexes with the assumption that states 'out there' are empirical entities. There might be good reasons to concede that many states 'out there' are non-empirical, but I think it is hard to deny that there is an important class of states 'out there' that are empirical. Therefore, my conclusion is that for a general account of relations we had better give up a one-to-one correspondence between states and complexes, and take another course.

In this paper, I present a theory of relations that explicitly distinguishes relational complexes from relational states. More specifically, *relational complexes are conceived as a structured perspective on relational states.* The theory is polymorphic in a strong sense, since it not only allows that a state 'out there' may belong to more than one relation, but also that for a fixed relation more than one relational complex can correspond to the same state.

I will not presuppose any particular view on the nature of relational states, or more generally, of states 'out there'. I will not even make any assumption about the existence of atomic complexes or states. The theory presented does not exclude the possibility that all states 'out there' are infinitely complex in the sense that all states are composed of simpler ones. Furthermore, it is not assumed (but not excluded) that relational states themselves have an intrinsic structure. The theory can, for example, be combined with viewing states as David Lewis's *sets of possible worlds* [Lew02], or as being akin to Kit Fine's *worldly facts* [Fin82], or David Armstrong's *states of affairs* [Arm97].

I call the theory the *polymorphic theory of relations.* It is presented in the form of a number of principles postulated throughout the paper (P-1 to P-22). In Appendix A the principles are also listed for easy access.

5.2 Polymorphic view on relations

In the polymorphic view on relations, a key role is played by the notions *relational states*, *relational complexes*, and *occurrences of objects.* I will not try to give an exact definition of these notions. My objective is to characterize them in terms of a system of metaphysical principles.

5.2.1 Structural principles

I postulate the following structural principles:

P-1 Each relation 'has' one or more relational complexes and each relational complex belongs to one and only one relation.

P-2 Each relational complex corresponds to one state 'out there' and each state 'out there' may correspond to one or more relational complexes.

P-3 Each relational complex may contain one or more occurrences of objects and each occurrence belongs to one and only one relational complex.

P-4 Each occurrence is the occurrence of one object and each object may be the content of one or more occurrences.

I call a state 'out there' that corresponds to a relational complex ξ of a relation $\Re$ a *relational state* of $\Re$, and denote it as $\mathsf{S}(\xi)$. I denote the occurrences of objects in a complex ξ as $\mathsf{Oc}(\xi)$, and the object of an occurrence α as $\mathsf{Ob}(\alpha)$. Furthermore, I call the objects of the occurrences in a relational complex the *objects of the complex*, and the objects of the complexes of a relation the *objects of the relation* itself.

The number of occurrences in a relational complex ξ is called the *degree* or *adicity of* ξ. The least upper bound of the degrees of the relational complexes of a relation $\Re$ is called the *degree* or *adicity of* $\Re$. If all relational complexes of a relation have the same degree, then the relation has a *fixed degree*, otherwise the relation has a *variable degree*. Relations of fixed degree are also called *unigrade relations*, and relations of variable degree are also called *multigrade relations*. An example of a multigrade relation is *being surrounded by*.

I consider a *property* as nothing but a monadic relation.

Remark 5.2.1. In order to keep the presentation simple, I ignore the possibility that each occurrence may be taken to have a type that corresponds to a domain of objects. The required changes here and in the principles further on are straightforward.

The structural principles are graphically represented in Figure 5.1, which is a so-called *entity-relationship diagram.*[3]

[3]Entity-Relationship Modeling is an established technique to model the information needs of organizations. (I used it myself frequently as a consultant for Oracle Corporation.) More information about this technique can be found in [Bar92] and [Tha00].

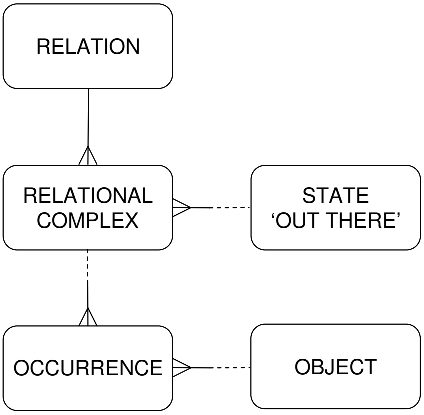
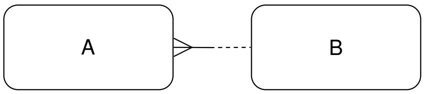
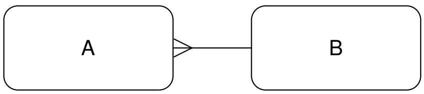


Figure 5.1: Entity-relationship diagram for the polymorphic theory.

In such diagrams, a relationship between entities of type A and B, where for each instance of A there is exactly one instance of B, and for each instance of B there may be one or more instances of A, is represented by:



If for each instance of B there is at least one related instance of A, then the right part of the line between the entities is also solid:



I don't expect that the structural principles will raise serious concerns, except perhaps Principle P-2. This principle will be evident for those who regard any set of possible worlds as a state 'out there', but those who think of states 'out there' as sparse, may perhaps consider it more adequate to make this correspondence optional. The exact consequences of this alternative choice are not immediately clear; we would have to find a satisfactory interpretation of complexes without corresponding states. Another possibility might be to let several states correspond to a given complex. For example, the complex of *x's being red or round* might correspond both to the state of *x's being red* and the state of *x's being round*.

The structural principles leave open whether a relation may have multiple relational complexes for the same state 'out there'. However, the principles in the next subsection will make this inevitable for certain relations.

5.2.2 Substitution principles

I do not assume that the occurrences in relational complexes are ordered, nor do I assume that relations have argument-places. The basic means by which relational complexes of a relation are thought to form a genuine unity is through substitution of objects for occurrences.

The notion of substitution is taken by Fine [Fin00] as a primitive operation for his so-called *antipositionalist view* on relations. In models for relations I developed in [Leo08a; Leo10], substitution of objects also play a central role. There I assumed that substitution takes place directly in relational states, but here I take a different approach. The next principles concern substitution of objects for occurrences in relational complexes:

P-5 Any substitution of objects for occurrences in a relational complex results in exactly one relational complex of the same relation.

P-6 For any complex, the identity substitution results in the same complex.

P-7 *Composition principle*: If a substitution σ in a relational complex ξ results in ξ', then there is a bijection μ from the occurrences in ξ to the occurrences in ξ', such that

(a) μ maps each occurrence α in ξ to an occurrence of the object substituted by σ for α,

(b) any substitution σ' in ξ' gives the same result as substituting in ξ for each occurrence α the object that σ' substitutes for $\mu(\alpha)$.

A mapping μ satisfying (a) and (b) of the composition principle is said to *correspond to* the substitution σ. Because of the injectivity of μ, substitution is said to be *coalescence-free*.

Denotation. Let $\Re$ be a relation, ξ a relational complex of $\Re$, and σ a mapping from the occurrences in ξ to the objects of $\Re$. I denote the result of substituting in ξ according to σ as $\xi \cdot_{\Re} \sigma$ or $\xi \cdot \sigma$. Furthermore, I denote the composition of a mapping f followed by a mapping g as $f \cdot g$.

With this denotation, part (b) of Principle P-7 says that

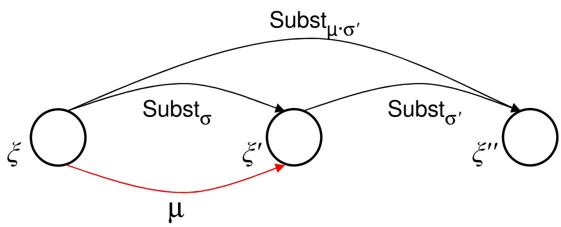
$$(\xi \cdot \sigma) \cdot \sigma' = \xi \cdot (\mu \cdot \sigma').$$

Let me make a few comments on the substitution principles.

Ad P-5. Certain substitutions result in complexes corresponding to impossible states. This happens, for example, when substituting a for the occurrence of b in the complex *a's being next to b*. If you want to exclude impossible states, then you could of course postulate a weaker principle than P-5, namely one stating that any substitution results in at most one relational complex.

Instead of assuming that any occurrence is an occurrence for which objects can be substituted, we might also consider introducing 'fixed' or 'hidden' occurrences of objects. In Section 5.3.2 the operation of *hiding* occurrences will be discussed.

Ad P-7. The composition principle forms the heart of the theory developed in this paper. It expresses the way different relational complexes form some kind of unity or resemble each other. This may be illustrated as follows:



It should be noticed that for a given substitution σ, a corresponding mapping μ is not necessarily unique; for symmetric relations the uniqueness may fail.

An important consequence of the principles P-1 to P-7 is that —for an empirical account of states 'out there'—certain relations are inevitably polymorphic in the sense that some states need to have more than one corresponding complex within the same relation:

Example 5.2.2. Suppose that states 'out there' are sets of possible worlds. Let $\mathfrak{R}$ be the parthood relation. The state that my nose is a part of myself has a corresponding complex with one occurrence of my nose and one of myself. It seems reasonable to assume that in no possible world I am a part of my nose and neither are you. But this means that the relation $\mathfrak{R}$ has a state—to wit, the empty state—that necessarily has more than one corresponding complex within $\mathfrak{R}$. ⊣

It could be objected that in the example we are dealing with a borderline case, since the state with multiple complexes is a vacuous state. So, let me also give another example.

Example 5.2.3. Let $\mathfrak{R}$ be the binary love relation with states $x \mathrel{-\!\heartsuit\!\!\to} y$, and let $\mathfrak{R}'$ be the unary relation with states $x \mathrel{-\!\heartsuit\!\!\to} d$ with d a fixed object. Now consider the conjunction of $\mathfrak{R}$ and $\mathfrak{R}'$.[4] Let a, c be distinct objects. Then the conjunctive state $a \mathrel{-\!\heartsuit\!\!\to} d \owedge c \mathrel{-\!\heartsuit\!\!\to} d$ has in this relation *two* corresponding complexes.

To see this, let b be another object and let ξ be a complex corresponding to the state $a \mathrel{-\!\heartsuit\!\!\to} b \owedge c \mathrel{-\!\heartsuit\!\!\to} d$. This complex has three occurrences. We might say that the occurrence of c is connected to a hidden occurrence of d. If we substitute in ξ the object d for the occurrence of b, then in the resulting complex the occurrence of c is still connected to a hidden occurrence of d. But if we would have started with a complex for the state $c \mathrel{-\!\heartsuit\!\!\to} b \owedge a \mathrel{-\!\heartsuit\!\!\to} d$, then substituting d for b, would have given a complex with the occurrence of a connected to a hidden occurrence of d. Thus, this complex is not identical to ξ, but they obviously have the same corresponding state 'out there'. ⊣

One could adopt an extremely refined notion of states 'out there' and argue that in the example the relations $\mathfrak{R}$ and $\mathfrak{R}'$ have no states in common. But if one regards states 'out there' as really being 'out there', this view is highly implausible. Alternatively, one could disallow 'object-involving' relations like $\mathfrak{R}'$, but I see no reason to do so for a general theory of relations.

In Principle P-7 the mapping μ is required to be bijective. As a result, complexes that can be obtained one from the other by substitution have a very uniform structure. I consider this as an advantage, although this may also give rise to multiple complexes for the same state. For example, consider

[4] In Section 5.3, conjunction of relations and hiding occurrences of objects will be discussed in more detail.

a *ternary* relation $\Re$ with all states being (commutative) conjunctions of the form $x \xrightarrow{\heartsuit} y \owedge y \xrightarrow{\heartsuit} z$. Then for distinct objects a, b the state $a \xrightarrow{\heartsuit} b \owedge b \xrightarrow{\heartsuit} a$ has two corresponding complexes in $\Re$, namely a complex with one occurrence of a and two of b, and another complex with two occurrences of a and one of b.

In [Leo10] I questioned the validity of a coalescence-free account. My expectation was that by allowing coalescence we could develop a satisfactory theory without multiple complexes for the same state within a single relation. But, as we have seen in Examples 5.2.2 and 5.2.3, then not all relational states can be conceived of as empirical entities. Of course, there may be other arguments for allowing coalescence for a general theory of relations. One of them could be the way substitution works for set-like relations. I discuss this issue in the next subsection.

Understanding substitution

The notion of substitution may need further explanation. Postulating a notion as primitive does not imply that its meaning is immediately clear. As I said before, I regard a relational complex as a structured perspective on a state 'out there'. But what does this mean? It might be tempting to entertain a picture-like representation for a relational complex, and to consider substitution simply as literally replacing in such a picture objects by other objects. But how accurate is such a representation?

Consider the adjacency relation. Suppose you have in front of you a cup adjacent to a glass. Then switching these objects obviously changes the scene. But, if the objects of a corresponding relational complex are switched, then it seems reasonable to assume that the result should be exactly the same relational complex. However, in working with a naive picture-like representation this may not be so easy to accomplish. Or suppose you want to substitute a chair for the glass. Then in such a representation you may have trouble deciding where exactly to put the chair. A naive picture-like representation may also be problematic for other relations. Take, for example, the relation of an *object x being wedged between objects y and z*. Then substituting for x an object of a different size, may require that the other objects are moved in the representation as well. What these examples show is that a picture-like representation for relational complexes may only be adequate if it is sufficiently abstract to accommodate these situations.

Another issue is how to understand substitution for a relation like *forming a group*. Suppose the group consisting of Groucho, Harpo and Chico has a corresponding complex with one occurrence for each person. Now what do we get if we substitute Groucho for the occurrence of Harpo? According to the substitution principles we get a complex with *two* occurrences of Groucho and one of Chico. But the state of Groucho and Chico forming a group also has a corresponding complex with one occurrence of each. Apparently the states with a set-like character have corresponding complexes with a multiset-like character. This might look a bit odd or artificial, but I don't see this as an objection to a coalescence-free account of substitution. Rather, I would say that each complex reflects precisely the way it can be obtained by substitution. In addition, we can of course assign to each state a canonical complex with one occurrence for each person. Alternatively, we could consider leaving substitution undefined in this case and postulate that substitution is only a partial function. (See also the discussion above of P-5.) In the next subsection, a complementary operation will be discussed that is relevant for relations like *forming a group*, namely subtracting occurrences from complexes.

5.2.3 Connectivity principles

A relation is given as a unity. But what exactly constitutes this unity; how do the complexes of a relation form a coherent whole?

For most ordinary relations, the relational complexes all have the same finite number of occurrences of objects. For this class of relations, the next principle characterizes the interrelatedness between the relational complexes.

P-8 For any relation $\Re$ of fixed finite degree, any relational complex of $\Re$ can be obtained from any other relational complex of $\Re$ via substitution.

With this principle it becomes possible to give positional representations for any relation of fixed finite degree. (See [Leo08a] and [Leo08b].)

The principle is limited to relations of *finite* degree because certain relations of fixed infinite degree may have for a subset of its occurrences a kind of variable degree. Take for example the relation of $x_1, x_2, \ldots, x_n$ *being a finite segment of* $y_1, y_2, \ldots$, with n variable and $y_1, y_2, \ldots$ an infinite list. Then P-8 obviously does not hold.

Note that by Principle P-8, a structure with complexes of the form $x_1, \ldots, x_n$ *playing against* $y_1, \ldots, y_m$ with n, m variable, but $n + m$ fixed, is not a relation. However, I do not consider this limitation of particular metaphysical significance. The polymorphic theory presented in this paper could be relaxed so that structures like this would be relations as well. A possibility could be to accept as a relation any substructure of a relation as long as it is closed under substitution.

For an arbitrary relation, I do not have a satisfactory solution for characterizing the unity of its complexes. However, for multigrade relations with a set-like character, like *forming a team*, *collectively supporting*, and *being surrounded by* the notion of *subtracting occurrences* may be helpful. We simply remove certain occurrences from a relational complex and get another complex.

For the operation of subtraction of occurrences additional principles can be proposed, in particular, a generalized composition principle that involves substitutions and/or subtractions. Care should be taken, however, not to confuse the subtraction of an occurrence with substituting the object of the occurrence with 'nothing', resulting in some kind of 'empty' occurrence.

For the multigrade relation of $x_1, \ldots, x_n$ *forming a team*, there is for any pair of relational complexes a substitution and/or subtraction from one of the complexes to the other. But for some other multigrade relations, we may need a third relational complex to make a connection. This might even be the case when the first two relational complexes have the same finite degree. We have such a situation for the relation of $x_1, \ldots, x_n$ *playing against* $y_1, \ldots, y_m$ with n, m variable.

As a connectivity principle for multigrade relations with a (multi)set-like character, it could be proposed that the relational complexes (with substitutions and/or subtractions as arrows) form a weakly connected graph.[5]

An important question is whether subtraction is a *basic* operation for multigrade relations. Kit Fine remarked [private communication, November 24, 2008] that he is inclined to regard the possibility of subtraction as a *symptom* of variable adicity and not as an *explanation*. I agree that subtraction is maybe not an explanation of variable adicity, but it may be a way to *characterize* the unity of the relational complexes of a multigrade relation.

[5]A directed graph is called *weakly connected* if its underlying undirected graph is connected.

Alternatively, could we not simply deny that there are multigrade relations? Could we not say that a relation like *being surrounded by* has only *two* occurrences: one for the surrounded entity and one for the set of surrounders?

By taking sets, wholes or aggregates as compound objects of relations, relations that at first sight appear to have variable adicity may turn out to have in fact a fixed adicity. The strategy, which goes back to Bolzano [Bol75, pp. 2–3] and to Frege [Fre83, p. 246], looks attractive, but does it work?

A difficulty with the strategy has been brought forward by Fraser MacBride [Mac05, pp. 583–584]. He points out that rendering a multigrade relation unigrade appears to presuppose the existence of at least one apparently multigrade relation, namely the *constitution relation*, to manufacture compound objects. To remove the multigrade appearance of the constitution relation would involve another multigrade relation, and so on. In other words, a complete removal of multigrade relations leads to an infinite regress. But, as Macbride notes, the regress can be avoided by admitting a multigrade relation of constitution.

There may, however, be an additional complication with rendering multigrade relations to unigrade relations if we start with a multigrade relation where the n objects have a relative order, but no object can in any genuine sense be said to be the first one. This is, for example, the case for a 'linked' love relation with states $x_1 \xrightarrow{\heartsuit} x_2 \xrightarrow{\heartsuit} \ldots \xrightarrow{\heartsuit} x_n$. If we want to render this relation as a unigrade relation, then the question is whether we have a *neutral* choice for the structure of the compound objects. Choosing sequences seems a bad idea, because then the compound objects of the resulting relation would have a first element, and this would conflict with taking the love relation as directionless.[6] Unfortunately, I do not see an acceptable artifact-free alternative. We could consider to introduce a *new* sort of compound object with the required structure, but it is not very clear to me what the ontology of such compound objects would be.

For characterizing arbitrary multigrade relations, the notion of subtracting occurrences is too limited. But, as I said, I have as yet no satisfactory general solution.

[6] This argument is related to Fine's objection against what he calls the *standard view* on relations [Fin00].

5.2.4 Identity criteria

Because more than one relational complex for a given relation may correspond to the same relational state, identity criteria for relations and relational complexes may be useful to prevent an unjustifiable abundance of relations with exactly the same relational states. The basic idea behind the criteria proposed is that a relational complex is a purely extensional notion.

Identity of relations

Relations with fully matching complexes—with respect to substitution—are called *qualitatively the same*. They are formally defined as follows.

Definition 5.2.4. Relations $\Re, \Re'$ are called *qualitatively the same* if there is a bijection π from the complexes of $\Re$ to the complexes of $\Re'$ such that for every complex ξ of $\Re$,[7]

1. $\mathsf{S}(\pi(\xi)) = \mathsf{S}(\xi)$,
2. there is a bijection $\tau : \mathsf{Oc}(\xi) \to \mathsf{Oc}(\pi(\xi))$ such that
   (a) for each occurrence α in ξ, $\mathsf{Ob}(\tau(\alpha)) = \mathsf{Ob}(\alpha)$,
   (b) for each substitution σ in ξ, $\pi(\xi \cdot \sigma) = \pi(\xi) \cdot (\tau^{-1} \cdot \sigma)$.

The love relation and the hate relation are obviously qualitatively not the same, because they have different states. But let $\Re$ be a unary relation with $\Re x$ being the complex *x's loves himself or x does not love himself*, and let $\Re'$ be a unary relation with $\Re' x$ being the complex *x's hates himself or x does not hate himself*, and suppose that for any x the states corresponding to the complexes $\Re x$ and $\Re' x$ are the same. Then $\Re$ and $\Re'$ are qualitatively the same.

The next lemma shows that there is a completely structure-preserving correspondence between the occurrences of objects in matching complexes of qualitatively the same relations.

[7] Recall that $\mathsf{S}(\xi)$ denotes the state corresponding to the complex ξ, that $\mathsf{Oc}(\xi)$ denotes the occurrences of objects in ξ, and that $\mathsf{Ob}(\alpha)$ denotes the object of the occurrence α.

Lemma 5.2.5. *Suppose $\Re$ and $\Re'$ are qualitatively the same. Then in the following commuting diagram*

$$\begin{array}{ccc} \mathsf{Oc}(\xi) & \xrightarrow{\ \tau\ } & \mathsf{Oc}(\pi(\xi)) \\ \mu \big\downarrow & & \big\downarrow \mu' \\ \mathsf{Oc}(\xi \cdot \sigma) & \xrightarrow{\ \widetilde{\tau}\ } & \mathsf{Oc}(\pi(\xi \cdot \sigma)) \end{array}$$

with $\pi, \tau, \widetilde{\tau}$ in line with Definition 5.2.4, with μ a mapping corresponding to substitution σ in ξ, and with $\mu' = \tau^{-1} \cdot \mu \cdot \widetilde{\tau}$, the mapping μ' corresponds to substitution $\tau^{-1} \cdot \sigma$ in $\pi(\xi)$.

Proof. Because μ' maps each occurrence α in $\pi(\xi)$ to an occurrence of the object $(\tau^{-1} \cdot \sigma)(\alpha)$, Condition (a) of the composition principle P-7 holds for μ'. To prove Condition (b) of P-7, let σ' be an arbitrary substitution in $\pi(\xi \cdot \sigma)$. Then

$$\begin{aligned} (\pi(\xi) \cdot (\tau^{-1} \cdot \sigma)) \cdot \sigma' &= \pi(\xi \cdot \sigma) \cdot \sigma' \\ &= \pi((\xi \cdot \sigma) \cdot (\widetilde{\tau} \cdot \sigma')) \\ &= \pi(\xi \cdot (\mu \cdot \widetilde{\tau} \cdot \sigma')) \\ &= \pi(\xi) \cdot (\mu' \cdot \sigma'). \end{aligned}$$

Thus, because μ' is bijective, μ' corresponds to substitution $\tau^{-1} \cdot \sigma$ in $\pi(\xi)$. ⊣

I postulate the following *identity criterion for relations*:

P-9 Relations are identical if and only if they are qualitatively the same.

The criterion expresses that a relation is completely determined by the network of interconnections of substitution in which the complexes with their corresponding states stand to each other. We might say that the complexes themselves have no internal complexity.

A bolder identity criterion could be considered, namely that relations are identical if and only if they have the same states. This could not be right, however, for some accounts of states ‘out there’. For suppose that states 'out there' are sets of possible worlds. Then the relation $\Re$ with complexes *x and y are the same number* and the relation $\Re'$ with complexes *x and y are not the same number* have the same states, namely the set of all possible worlds and the empty set. But $\Re$ and $\Re'$ are obviously not identical.

The identity criterion P-9 is stated for arbitrary relations. To play on the safe side, we might consider to limit it to relations where all complexes are connected to each other via substitution. I consider this as an issue for future investigation.

Identity of relational complexes

For relational complexes I also define the notion of *qualitative sameness.* The idea is that two complexes of the same relation are qualitative the same if substitutions in them are fully matching.

Definition 5.2.6. Relational complexes ξ of $\Re$ and ξ' of $\Re'$ are called *qualitatively the same* if there is a bijection τ from the occurrences in ξ to the occurrences in ξ' such that

1. for each occurrence α in ξ, $\mathsf{Ob}(\tau(\alpha)) = \mathsf{Ob}(\alpha)$,
2. a substitution σ in ξ is defined iff the τ-corresponding substitution in ξ' is defined, and $\mathsf{S}(\xi \cdot \sigma) = \mathsf{S}(\xi' \cdot (\tau^{-1} \cdot \sigma))$.

A non-trivial (but somewhat artificial) example of complexes that are qualitatively the same can be obtained as follows. Consider the ternary relation $\Re$ with $\Re xyz$ the complex *x is not identical to itself and y loves z*. Then in a possible worlds conception of states 'out there', the complexes $\Re xyz$ and $\Re xzy$ have the same corresponding state. It follows that the complexes are qualitatively the same.

Note that qualitative sameness of relational complexes is an equivalence relation. Moreover, the equivalence is preserved under substitution:

Lemma 5.2.7. *Suppose ξ and ξ' are qualitatively the same relational complexes, and let τ be as in Definition 5.2.6. Then for every substitution σ in ξ, the complexes $\xi \cdot \sigma$ and $\xi' \cdot (\tau^{-1} \cdot \sigma)$ are qualitatively the same as well.*

Proof. Let $\mu : \mathsf{Oc}(\xi) \to \mathsf{Oc}(\xi \cdot \sigma)$ correspond to substitution σ in ξ, and let μ' correspond to substitution $\tau^{-1} \cdot \sigma$ in ξ'. Then, by Principle P-7, the mapping $\widetilde{\tau} = \mu^{-1} \cdot \tau \cdot \mu'$ is a bijection. So, we have the following commuting diagram:

$$\begin{array}{ccc}
\mathsf{Oc}(\xi) & \xrightarrow{\quad\tau\quad} & \mathsf{Oc}(\xi') \\
\Big\downarrow \mu & & \Big\downarrow \mu' \\
\mathsf{Oc}(\xi \cdot \sigma) & \xrightarrow{\quad\widetilde{\tau}\quad} & \mathsf{Oc}(\xi' \cdot (\tau^{-1} \cdot \sigma))
\end{array}$$

The mapping $\widetilde{\tau}$ obviously fulfills Condition 1 of Definition 5.2.6. Furthermore,

$$\begin{aligned}
\mathsf{S}((\xi \cdot \sigma) \cdot \sigma') &= \mathsf{S}(\xi \cdot (\mu \cdot \sigma')) \\
&= \mathsf{S}(\xi' \cdot (\tau^{-1} \cdot \mu \cdot \sigma')) \\
&= \mathsf{S}(\xi' \cdot (\mu' \cdot \widetilde{\tau}^{-1} \cdot \sigma')) \\
&= \mathsf{S}(\xi' \cdot (\tau^{-1} \cdot \sigma) \cdot (\widetilde{\tau}^{-1} \cdot \sigma')).
\end{aligned}$$

Thus, $\widetilde{\tau}$ also fulfills Condition 2 of Definition 5.2.6. ⊣

For many relations, we neither want nor need multiple relational complexes for the same state. For example, for the adjacency relation we want that the state of *a's being next to b* has only one relational complex. Substituting a for the occurrence of b and b for the occurrence of a should give the same relational complex. To take care of such situations, I propose the following *identity criterion for relational complexes*:

P-10 Relational complexes that belong to the same relation are identical if and only if they are qualitatively the same.

Principle P-10 has the following nice property. If $\mathfrak{R}$ were to be a relation that does not fulfill P-10, then $\mathfrak{R}$ can be stripped down to a structure $\breve{\mathfrak{R}}$ that fulfills this principle, and in which everything that seems essential to $\mathfrak{R}$ is preserved. In particular, $\breve{\mathfrak{R}}$ also fulfills the composition principle P-7.

To see this, define $\breve{\mathfrak{R}}$ as a relation-like structure whose complexes are arbitrarily chosen representatives of qualitatively the same complexes in $\mathfrak{R}$, and with substitution defined by

$$\xi \cdot_{\breve{\mathfrak{R}}} \sigma = \xi' \quad \text{with } \xi' \text{ qualitatively the same as } \xi \cdot_{\mathfrak{R}} \sigma.$$

For $\xi \cdot_{\mathfrak{R}} \sigma$ and $\xi \cdot_{\breve{\mathfrak{R}}} \sigma$ let τ be a bijection as in Definition 5.2.6. Let μ be a mapping that corresponds—in accordance with the principle P-7—to the substitution σ applied to ξ. Then $(\xi \cdot_{\breve{\mathfrak{R}}} \sigma) \cdot_{\breve{\mathfrak{R}}} \sigma' = \xi \cdot_{\breve{\mathfrak{R}}} (\mu \cdot \tau \cdot \sigma')$. It follows that $\breve{\mathfrak{R}}$ also fulfills the composition principle.

Note that to obtain a relation as a result of this stripping process, it is essential that the stripping is done simultaneously for all relational complexes of $\mathfrak{R}$.

The next lemma shows that the identity of a relation can be expressed in terms of qualitatively the same complexes.

Lemma 5.2.8. *Let $\mathfrak{R}$ and $\mathfrak{R}'$ be relations. If for every complex in $\mathfrak{R}$ there is a qualitatively the same complex in $\mathfrak{R}'$, and vice versa, then $\mathfrak{R}$ and $\mathfrak{R}'$ are identical.*

Proof. Assume that for every complex $\mathfrak{R}$ there is a qualitatively the same complex in $\mathfrak{R}'$, and vice versa. Then, by P-10, there is a bijection π that maps the complexes of $\mathfrak{R}$ to their qualitatively the same counterparts in $\mathfrak{R}'$. This bijection π makes $\mathfrak{R}$ and $\mathfrak{R}'$ qualitatively the same relations. Thus, by P-9, they are identical. ⊣

It is interesting to note that, as a consequence of Principle P-10, the identity mapping is the only bijection π in Definition 5.2.4 that makes a relation qualitatively the same to itself.

Justifying the identity criteria

The identity criteria for relations and for relational complexes have some consequences that at first sight may strike us as odd:

Example 5.2.9. Let $\mathfrak{R}_0$ be a nullary relation with as its only state the state of *everyone's loving everyone and trusting everyone*. If the conjunction of this relation with the binary love relation is again a relation, say $\mathfrak{R}_1$, then how does $\mathfrak{R}_1$ look like? It seems reasonable to assume that $\mathfrak{R}_1$ is a binary relation. Furthermore, if relational states are conceived of as sets of possible worlds, then $\mathfrak{R}_1$ would obviously also have just one state. Now by the identity criterion for relational complexes, it would also have for any (unordered) pair of objects a, b just one complex with one occurrence of a and one b (and two of a if a and b are identical). So, if α, β are the occurrences in a complex in $\Re_1$, then we cannot say which of them fulfills the role of lover and which the role of beloved.

In a similar way, we can define a relation $\Re_2$ as the conjunction of $\Re_0$ with the binary trust relation. According to the identity criterion of relations, $\Re_1$ and $\Re_2$ would be identical. Thus, in the resulting relation no traces are left of either a 'love' or a 'trust' origin. $\dashv$

One might wonder whether a more refined account of relations that retains more of the origin of the underlying complexes and occurrences would be more adequate. Although I do not exclude the possibility of such an account, I think there are strong arguments in favor of the proposed identity criteria.

An important feature of the proposed criteria is their *definiteness*: the question whether two relations or relational complexes are identical has a clear, definite answer. Another strong feature of the proposed criteria is, as we will see in Section 5.3, that they allow us to define various operations on relations in a very natural way in terms of operations on relational states.

The proposed identity criteria are not the only ones conceivable with these features. We might, for example, think of defining identity criteria in terms of formal properties, like commutativity and associativity, of certain operations on relations. If we would have conjunction, disjunction and negation as operations, then probably we would also like to have De Morgan's laws for relations. There are, however, a few complications with this alternative approach: (1) When new operations are introduced, additional identity criteria may have to be defined. (2) A complete specification of a relation may require a complete specification of its composition.

A point of concern may be the question whether entertaining relations and complexes with more refined distinctions would provide a better understanding of the structure of the world. I think this is not very likely. A more refined account of relations would *in toto* probably not contain more information. For example, when we are interested in the possible origins of certain relations or their complexes, we could simply direct our attention to the generating relations. More generally, what really matters is to have an account of relations that provides access to the details, but the form in which they are presented is not essential.

5.3 Operations on relations

I discuss operations on relations that I consider as very basic, namely conjunction, disjunction, negation, and restriction. With these operations all kinds of complex relations can be constructed. Of course, other basic operations are conceivable as well. Besides, it is not excluded that certain relations supervene on others without any possible reduction. By this I mean that the existence of certain relations might be entailed by other relations, but that they are not explicitly definable in terms of these other relations.

5.3.1 Logical operations

Suppose we want to define the conjunction of relations with sets of complexes C_1 and C_2. Then for the complexes of the conjunction we cannot simply take $C_1 \times C_2$, because this may yield too many complexes, and because the ordering in the elements is undesirable. For the occurrences in the conjunction of complexes ξ and ξ' we also cannot simply take the union of the occurrences in ξ and ξ', because ξ and ξ' may be identical, and because other complexes may have the same conjunction. We must be more cautious, but we can define logical operations on relations in a clear and uniform way in terms of logical operations on states:

Definition 5.3.1. Let $\Re_1, \ldots, \Re_n$ be relations with sets of states $S_1, \ldots, S_n$ and sets of complexes $C_1, \ldots, C_n$. Let f be a mapping from $S_1 \times \ldots \times S_n$ to some set of states S.
We say that a relation $\Re$ with set of complexes C is *induced by f* if there is a surjection $\rho : C_1 \times \ldots \times C_n \to C$, and for every $\xi_1 \in C_1, \ldots, \xi_n \in C_n$, a bijection $\tau : X \to \mathsf{Oc}(\rho(\xi_1, \ldots, \xi_n))$, with $X = \bigcup_{i=1}^{n}(\mathsf{Oc}(\xi_i) \times \{i\})$, the disjoint union of $\mathsf{Oc}(\xi_1), \ldots, \mathsf{Oc}(\xi_n)$, such that

1. for each $(\alpha, i) \in X$, $\mathsf{Ob}(\tau(\alpha, i)) = \mathsf{Ob}(\alpha)$,
2. for each substitution σ in $\rho(\xi_1, \ldots, \xi_n)$ and τ-corresponding substitutions, $\mathsf{S}(\rho(\xi_1, \ldots, \xi_n) \cdot \sigma) = f(\mathsf{S}(\xi_1 \cdot \sigma_1), \ldots, \mathsf{S}(\xi_n \cdot \sigma_n))$.

The identity criteria of relations and relational complexes guarantee that $\Re$ is thus uniquely defined:

Lemma 5.3.2. *For the given collection of relations $\Re_1, \ldots, \Re_n$, there is at most one relation $\Re$ induced by f.*

Proof. Suppose that $\Re$ and $\Re'$ are both induced by f. By the identity criterion for relational complexes P-10, it follows that $\Re$ and $\Re'$ are qualitatively the same. Thus, by the identity criterion for relations P-9, it follows that $\Re$ and $\Re'$ are identical. $\dashv$

If for given relations $\Re_1, \ldots, \Re_n$ there is a relation induced by f, then it is denoted as $f(\Re_1, \ldots \Re_n)$ and its complexes are denoted as $f(\xi_1, \ldots \xi_n)$.

Note that the degree of $f(\Re_1, \ldots \Re_n)$ is equal to the sum of the degrees of the relations $\Re_1, \ldots \Re_n$. Also note that if f is the identity mapping from S_1 to S_1, then $f(\Re_1) = \Re_1$.

Suppose f is a binary mapping with for all states $f(s_1, s_2) = f(s_2, s_1)$. Then clearly also for all complexes $f(\xi_1, \xi_2) = f(\xi_2, \xi_1)$, if defined. The next lemma establishes a more general similarity between identities of compositions of states and identities of compositions of complexes.

Lemma 5.3.3. *Let $\Re_1, \ldots, \Re_n$ be relations with sets of states $S_1, \ldots, S_n$ and sets of complexes $C_1, \ldots, C_n$. Let π be a permutation of $\{1, \ldots, n\}$, and let $f_1, \ldots, f_m, g, h$ be mappings such that for all $s_1 \in S_1, \ldots, s_n \in S_n$,*

$$g(f_1(s_{\pi(1)}, \ldots, s_{\pi(k)}), f_2(s_{\pi(k+1)}, \ldots), \ldots, f_m(\ldots, s_{\pi(n)})) = h(s_1, \ldots, s_n).$$

Then for all $\xi_1 \in C_1, \ldots, \xi_n \in C_n$,

$$g(f_1(\xi_{\pi(1)}, \ldots, \xi_{\pi(k)}), f_2(\xi_{\pi(k+1)}, \ldots), \ldots, f_m(\ldots, \xi_{\pi(n)})) = h(\xi_1, \ldots, \xi_n),$$

if defined.

Proof. Suppose $g(f_1(\Re_{\pi(1)}, \ldots), \ldots, f_m(\ldots, \Re_{\pi(n)}))$ and $h(\Re_1, \ldots, \Re_n)$ are relations. Let us call them $\Re$ and $\Re'$, respectively. Let ρ be the mapping from the complexes of $\Re$ to the complexes of $\Re'$ defined by

$$g(f_1(\xi_{\pi(1)}, \ldots,), \ldots, f_m(\ldots, \xi_{\pi(n)})) \;\mapsto\; h(\xi_1, \ldots, \xi_n).$$

From the given equations for the states, it follows that ρ is well-defined. Now let σ be an arbitrary substitution in a complex ξ of $\Re$, and let σ' be the corresponding substitution in $\rho(\xi)$. It is easy to see that $\mathsf{S}(\xi{\cdot}\sigma) = \mathsf{S}(\rho(\xi){\cdot}\sigma')$. It follows that $\Re$ and $\Re'$ are qualitatively the same relations, and thus by P-9 they are identical. The equations for the complexes follow directly from this. $\dashv$

In the next subsections the lemma will be applied to standard logical operations, but let me already mention a few applications: (1) If the conjunction of any two states is commutative, then the conjunction of complexes is commutative as well. (2) If the double negation of any state is the state itself, then the double negation of any complex is the complex itself. (3) If we have De Morgan's laws for all states, like $-(s \ovee s') = -s \owedge -s'$, then we also have identities like $-(\xi \ovee \xi') = -\xi \owedge -\xi'$. But note that if for all s, $s \owedge s = s$, then this does not imply that $\xi \owedge \xi = \xi$.

Conjunction of relations

Russell [Rus56, pp. 209–211] sees no reason to accept conjunctive facts, for the truth of a proposition $p \owedge q$ depends only on the fact corresponding to p and the fact corresponding to q. Wittgenstein is also said to consider conjunctive facts as unnecessary (for example in [Lin03, p. 382]), but he does not deny their existence, since he writes in response to a question of Russell [Wit84, p. 130]: "Sachverhalt is, what corresponds to an Elementarsatz if it is true. Tatsache is what corresponds to the logical product of elementary props when this product is true." A different opinion is held by Armstrong. He accepts conjunctive facts and points out that there could perhaps be conjunctive universals 'all the way down' [Arm97, p. 32].

For the theory developed in this paper, I assume that for all states 'out there' s, s' the conjunction $s \owedge s'$ is a well-defined state 'out there', although certain conjunctive states might be more natural than others.

Which properties you attribute to the conjunctive states 'out there' will depend on your specific view, but commutativity and associativity seem to be evident. Frege, for example, remarks about commutativity for compound thoughts [Fre03b, p. 89]: "Dass "B und A" denselben Sinn hat wie "A und B" sieht man ein ohne Beweis nur dadurch, dass man sich des Sinnes bewusst wird.", and he adds in a footnote [note 5]: "Ein anderen Fall dieser Art ist der, dass "A und A" denselben Sinn hat wie "A"."

I define the conjunction of relations in terms of the conjunction of their states, and by making use of Definition 5.3.1:

Definition 5.3.4. Let $\mathfrak{R}$ and $\mathfrak{R}'$ be relations. Then, if it exists, the *conjunction of* $\mathfrak{R}$ *and* $\mathfrak{R}'$ is defined as the relation induced by the conjunctive operation on states. It is denoted as $\mathfrak{R} \owedge \mathfrak{R}'$, and its complexes as $\xi \owedge \xi'$.

I postulate that the conjunction of any pair of relations is also a relation:

P-11 Relation $\Re \owedge \Re'$ exists for any relations $\Re, \Re'$.

As we have seen in Example 5.2.3, if for each relation each relational state may only have one corresponding relational complex, then the conjunction of relations would not always exist. Also the next example shows that a conjunction of relations might introduce multiple relational complexes of the same relational state.

Example 5.3.5. Consider the conjunctive relation $\Re \owedge (\Re \owedge \Re)$. Suppose conjunction of states is commutative, associative and idempotent. Then the complexes $\xi \owedge (\xi \owedge \xi')$ and $\xi' \owedge (\xi \owedge \xi')$ correspond to the same relational state, but, in many cases, the complexes will be distinct. Note that if in a relation no multiple relational complexes were allowed for a single relational state, then substitution would in these cases inevitable lead to a coalescence of occurrences. ⊣

The next theorem follows directly from Lemma 5.3.3.

Theorem 5.3.6. *If conjunction of states is commutative (associative), then conjunction of complexes is commutative (associative) as well.*

So, it also follows that if conjunction of states is commutative (associative), then conjunction of relations is commutative (associative) as well.

Note that the theorem is not true for every theory of relations. For a theory where the arguments of the relations come in a certain order, conjunction of complexes and of relations is obviously not commutative. Also for a theory where the relations come with argument-places, conjunction of complexes is not commutative, but conjunction of relations can be commutative, provided that there are suitable identity criteria for sets of argument-places.

Besides a conjunction of two relations, we also might define a conjunction of an infinite number of relations, provided that we accept an infinite conjunction of states 'out there'.

Disjunction of relations

Is *the Statue of Liberty's being made of metal* a fact? This should be denied by those who deny that there are any disjunctive facts.[8] A problem with such an austere view is that it may make it practically impossible to determine of anything whether it is indeed a fact. As the quote from Russell I mentioned in the introduction says, "nothing can be *known* to be simple". It might be just as hard to determine of something that it is not disjunctive as to determine that it is not conjunctive.

Nevertheless, many philosophers have mixed feelings about disjunctive facts. Russell [Rus56, p. 215] is inclined to think that there are no disjunctive facts, but he also notes that the denial of disjunctive facts lead to certain difficulties. According to Fine [Fin82, p. 55] there are disjunctive *propositional* facts, but no disjunctive *worldly* facts. Armstrong [Arm97, pp. 43–46] denies the existence of disjunctive *first-class* facts, but he has no problem with disjunctive *second-class* and *third-class* facts, where these notions are defined in terms of classes of relations: first-class relations are universals; second-class relations have the necessary and sufficient condition that when truly predicated of particulars, the resultant truth is a contingent one; third-class relations are relations that are not contingent (for example, *being identical with a*). Armstrong explicitly notes that he has no objection to admitting disjunctive and negative relations, provided they are at best second-class relations.

Although I do not use Armstrong's classification for states and relations, I assume that for all states 'out there' s, s' there exists a disjunctive state $s \otimes s'$, and thus I define a disjunction $\mathfrak{R} \otimes \mathfrak{R}'$ of relations analogously to the conjunction of relations. As an aside, I want to remark that, if desired, Armstrong's classification can be superimposed on the theory developed here.

The disjunction of relations is defined in a similar way as the conjunction of relations:

Definition 5.3.7. Let $\mathfrak{R}$ and $\mathfrak{R}'$ be relations. Then, if it exists, the *disjunction of* $\mathfrak{R}$ *and* $\mathfrak{R}'$ is defined as the relation induced by the disjunctive operation on states. It is denoted as $\mathfrak{R} \otimes \mathfrak{R}'$, and its complexes as $\xi \otimes \xi'$.

I postulate that the disjunction of any pair of relations is a relation as well:

[8] At least if *being of metal* means being of iron or being of copper or being of copper, steel and gold, etc.

P-12 Relation $\Re \circledvee \Re'$ exists for any relations $\Re, \Re'$.

The next theorem also follows directly from Lemma 5.3.3.

Theorem 5.3.8. *If disjunction of states is commutative (associative), then disjunction of complexes is commutative (associative) as well.*

Negation of relations

Russell once remarked [Rus56, p. 211]: "I argued that there were negative facts, and it nearly produced a riot: the class would not hear of there being negative facts at all." I want to sidestep the discussion of whether there really are negative facts, but nevertheless admit for any state 'out there' s the existence of the *negation of* s, denoted as $-s$. In general, $-s$ should not be conceived as a negative state, since s itself may also seen as the negation of $-s$.

The definition of the negation of a relation is straightforward.

Definition 5.3.9. Let $\Re$ and $\Re'$ be relations. Then, if it exists, the *negation of* $\Re$ is defined as the relation induced by the negation operation on states. It is denoted as $-\Re$, and its complexes as $-\xi$.

I postulate that the negation of any relation is a relation as well:

P-13 Relation $-\Re$ exists for any relation $\Re$.

The next theorem about the double negation of states and complexes also follows directly from Lemma 5.3.3.

Theorem 5.3.10. *If for all states s in $\Re$ we have $--s = s$, then for all complexes ξ in $\Re$ we also have $--\xi = \xi$.*

Note that, by the theorem 5.3.10, atomic relations may in a certain sense also be regarded as complex.

When we combine the logical operations, we get again by Lemma 5.3.3 other interesting identities, like De Morgan laws:

Theorem 5.3.11.

1. *If for all states s in $\mathfrak{R}$, s' in $\mathfrak{R}'$ we have* $-(s \ovee s') = -s \owedge -s'$, *then for all complexes ξ in $\mathfrak{R}$, ξ' in $\mathfrak{R}'$ we also have* $-(\xi \ovee \xi') = -\xi \owedge -\xi'$.

2. *If for all states s in $\mathfrak{R}$, s' in $\mathfrak{R}'$ we have* $-(s \owedge s') = -s \ovee -s'$, *then for all complexes ξ in $\mathfrak{R}$, ξ' in $\mathfrak{R}'$ we also have* $-(\xi \owedge \xi') = -\xi \ovee -\xi'$.

5.3.2 Restriction of relations

Consider the dyadic love relation with relational complexes of the form $x \xrightarrow{\heartsuit} y$. Starting with this relation, we may construct a *self-love* relation as follows. (1) Select a relational complex of the original love relation with two occurrences of the same object, say Venus. (2) *Merge* these two occurrences to obtain a single one. (3) Define as relational complexes of the self-love relation the complexes that can be obtained via substitutions for this single occurrence.

In a similar way, we may define the relation of *loving Venus.* First, select a relational complex of the original love relation corresponding to a state of someone loving Venus. Then *hide* the occurrence of Venus, and define relational complexes with only one occurrence, namely an occurrence in the role of *lover.*

I call relations constructed in this way *restrictions.* For the love relation we have the following (classes of) restrictions:

1. For any object a, a monadic relation with relational complexes $a \xrightarrow{\heartsuit} y$.

2. For any object b, a monadic relation with relational complexes $x \xrightarrow{\heartsuit} b$.

3. A monadic relation with relational complexes $x \xrightarrow{\heartsuit} x$.

4. For any objects a and b, a medadic relation with just one relational complex $a \xrightarrow{\heartsuit} b$.[9]

Note that from the conjunction of the love relation with itself it is possible to regain the love relation by merging corresponding occurrences in a complex

[9] It is perhaps questionable whether we should regard relations of arity 0 as 'real' relations, but I see no compelling reason to exclude them.

$a \xrightarrow{\heartsuit} b \circledcirc a \xrightarrow{\heartsuit} b$. (I presuppose here that for all love-states $s \circledcirc s = s$, and use the identity criterion P-9.)

The approach for restricting the love relation may be generalized in a straightforward way to other relations:

Definition 5.3.12. Let $\mathfrak{R}$ be a relation, and ξ be one of its complexes. Let $\mathcal{P}$ be a partition of a subset of $\mathsf{Oc}(\xi)$ such that each element of $\mathcal{P}$ contains only occurrences of the same object.
We say that a relation $\mathfrak{R}'$ is a *restriction induced by* $\mathcal{P}$ if there is a complex ξ' in $\mathfrak{R}'$ and a bijection $\tau : \mathcal{P} \to \mathsf{Oc}(\xi')$ such that

1. each $\alpha' \in \mathsf{Oc}(\xi')$ and $\alpha \in \tau^{-1}(\alpha')$ are occurrences of the same object,
2. for each complex ξ'' of $\mathfrak{R}'$ there is a σ' such that $\xi' \cdot \sigma' = \xi''$,
3. a substitution σ' in ξ' is defined iff the substitution σ in ξ is defined with

$$\sigma(\alpha) = \begin{cases} \sigma'(\alpha') \text{ with } \alpha \in \tau^{-1}(\alpha') & \text{if such an } \alpha' \text{ exists,} \\ \text{the object of } \alpha & \text{otherwise,} \end{cases}$$

and $\mathsf{S}(\xi' \cdot \sigma') = \mathsf{S}(\xi \cdot \sigma)$.

Note that this operation does two things at once. It *hides* the occurrences not in $\bigcup \mathcal{P}$, and it *merges* the occurrences in each element of $\mathcal{P}$.

The identity criteria of relations and relational complexes guarantee that restrictions are thus uniquely defined:

Lemma 5.3.13. *There is at most one relation a restriction induced by* $\mathcal{P}$.

Proof. Suppose that $\mathfrak{R}$ and $\mathfrak{R}'$ are both restrictions induced by $\mathcal{P}$. By the identity criterion for relational complexes P-10, it follows that $\mathfrak{R}$ and $\mathfrak{R}'$ are qualitatively the same. Thus, by the identity criterion for relations P-9, it follows that $\mathfrak{R}$ and $\mathfrak{R}'$ are identical. ⊣

I postulate for restrictions the following principle:

P-14 Any restriction of a relation is also a relation.

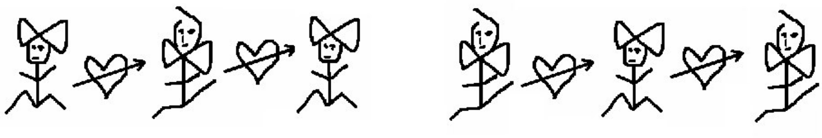


Figure 5.2: Two complexes for the same state.

Note that the definition of a restriction of relations allows us to choose $\mathcal{P} = \varnothing$. This gives a medadic relation with only one complex. Also note that if in a relation all complexes can be obtained from each other by a substitution, then choosing $\mathcal{P} = \{\{\alpha\} \mid \alpha \in \xi\}$ results in the original relation.

Although the definition of restrictions is simple, there are certain subtleties that need to be discussed.

A state may have *more* corresponding relational complexes in a restriction than in the original relation:

Example 5.3.14. Let $\Re$ be the 'double' love relation with relational complexes of the form $x \xrightarrow{\heartsuit} y \circledcirc u \xrightarrow{\heartsuit} v$. Then each state of this relation has one corresponding relational complex. But in the restriction $\Re'$ obtained by merging the occurrences of b in the relational complex $a \xrightarrow{\heartsuit} b \circledcirc b \xrightarrow{\heartsuit} c$, the state corresponding with $a \xrightarrow{\heartsuit} b \circledcirc b \xrightarrow{\heartsuit} a$ has *two* corresponding relational complexes: one complex with one occurrence of a and two of b and another complex with two occurrences of a and one of b. (See Figure 5.2.) ⊣

A state may also have *fewer* corresponding relational complexes in a restriction than in the original relation:

Example 5.3.15. For the relation $\Re'$ of Example 5.3.14, we can define a further restriction $\Re''$ by also merging the occurrences of a in the relational complex corresponding to the state of $a \xrightarrow{\heartsuit} b \circledcirc b \xrightarrow{\heartsuit} a$. This state has two relational complexes in $\Re'$, but only one in $\Re''$, as follows from the definition of restrictions. ⊣

As we would expect of any robust theory of relations, restriction is a transitive operation:

Theorem 5.3.16. *If $\Re''$ is a restriction of $\Re'$ and $\Re'$ is a restriction of $\Re$, then $\Re''$ is also a restriction of $\Re$.*

Furthermore, if $\mathfrak{R}'_1$ *is a restriction of* $\mathfrak{R}_1$ *and* $\mathfrak{R}'_2$ *is a restriction of* $\mathfrak{R}_2$*, then* $\mathfrak{R}'_1 \oslash \mathfrak{R}'_2$ *,* $\mathfrak{R}'_1 \varnothing \mathfrak{R}'_2$*,* $-\mathfrak{R}'_1$ *are restrictions of* $\mathfrak{R}_1 \oslash \mathfrak{R}_2$*,* $\mathfrak{R}_1 \varnothing \mathfrak{R}_2$*,* $\mathfrak{R}_1$*, respectively.*

Proof. Let $\mathfrak{R}'$ be the restriction of $\mathfrak{R}$ induced by $\mathcal{P}$, and let $\mathfrak{R}''$ be the restriction of $\mathfrak{R}'$ induced by $\mathcal{P}'$. Let $\tau : \mathcal{P} \to \mathsf{Oc}(\underset{\sim}{\xi'})$ be a bijection fulfilling the conditions of Definition 5.3.12. Now define $\widetilde{\mathfrak{R}}$ as the restriction of $\mathfrak{R}$ induced by

$$\{\bigcup \tau^{-1}[x] \mid x \in \mathcal{P}'\}.$$

Since substitution is coalescence-free, it follows that $\widetilde{\mathfrak{R}}$ is qualitatively the same as $\mathfrak{R}''$. So, by P-9, the identity criterion for relations, it follows that they are identical.

The second part of the theorem can be proved in a similar way. ⊣

By the identity criterion for relations (Principle P-9), it follows that we can regain the relation $\mathfrak{R}$ by a restriction of $\mathfrak{R} \oslash \mathfrak{R}$ if each state of $\mathfrak{R}$ can be obtained from any other state via substitution.

Coordinating occurrences. By merging occurrences, the original occurrences are no longer present in the resulting relation. This would be different if we had a mechanism for *linking* occurrences. In his book 'Semantic Relationism' [Fin07], Fine introduces the notion of *coordination* among certain entities as the strongest relation of synonymy. One of the forms of coordination concerns constituents of thought. In line with this, we might consider introducing coordination among occurrences of the same object in relational complexes. This would allow us to distinguish, for example, between (1) the relational complex of Cicero loving Cicero with two uncoordinated occurrences of Cicero, (2) the relational complex of Cicero loving Cicero with two coordinated occurrences of Cicero, and (3) the relational complex of Cicero loving himself with only one occurrence of Cicero. However, the applicability of coordination for elements of thought does not automatically mean that coordination is also needed on the level of relational complexes. I regard this as a topic for future investigation.

5.4 Alternative theories

The polymorphic theory is of course not the only possible theory for relations. In this section we consider some alternative theories. In particular, we will take a look at theories that do not posit both states 'out there' and complexes.

5.4.1 A theory without complexes

Can we build a theory with only states 'out there' and in which the states themselves have occurrences? The answer may depend on how refined the states are, but let us suppose that the states 'out there' are empirical entities. Then, as I will show, such a theory has unacceptable limitations.

There are basically two choices for a theory with only states:

(i) a theory in which each state belongs to only one relation;

(ii) a theory in which a state may belong to more than one relation.

The first option would accommodate only a very small class of relations. It would, for example, not be possible to define the conjunction and disjunction of a relation with itself, and also restrictions of a relation could not be defined, because these operations would yield states that were already in the original relations.

So, let us consider the more liberal kind of relational theory, where a state 'out there' may belong to more than one relation, and in which the states have occurrences relative to a relation. Equivalently, we may assume that relations do have complexes with occurrences, but no two complexes of the same relation correspond to the same state. This option gives considerably more freedom than the first one, but unfortunately, also this kind of theory has severe problems.

As follows from Example 5.2.2, a theory without complexes cannot accommodate a relation like the parthood relation if states are conceived of as possible worlds. Furthermore, for some relations the conjunction cannot be defined in such a theory. This is not only the case for conjunctions where $s \oslash \neg s$ and $s' \oslash \neg s'$ are taken as identical states, but also for some conjunctions without 'negative' components. We have already seen this in Example

5.2.3. There we considered the conjunction of the binary love relation with states $x \xrightarrow{\heartsuit} y$, and the unary relation with states $x \xrightarrow{\heartsuit} d$ with d a fixed object. We showed that the composition principle for substitution (P-7) cannot be fulfilled if each state has only one corresponding complex in the conjunctive relation. The problem is that for the state $a \xrightarrow{\heartsuit} d \owedge c \xrightarrow{\heartsuit} d$ the non-hidden occurrence of d should on the one hand be 'part of' the conjunct $a \xrightarrow{\heartsuit} d$, and on the other of the conjunct $c \xrightarrow{\heartsuit} d$. This shortcoming shows that a theory with only (empirical) states 'out there' cannot give a fully adequate account of the logical structure of relations.

5.4.2 A theory without states

It might be argued that in the polymorphic theory, the states 'out there' play a rather modest role. They may be looked at as a means for classifying relational complexes. If we could develop a plausible theory of relations without states, then that theory might be the first choice.

There are two ways to go. We can try to develop a theory that makes use of an equivalence relation on complexes or we can try to develop a completely free-standing theory of relations without such an equivalence relation. Let us consider both options.

A theory with empirically indistinguishable complexes

We can define an equivalence relation $\approx$ on complexes, where $\xi \approx \xi'$ means that ξ and ξ' are *empirically indistinguishable* from each other, that is, necessarily ξ obtains just in case ξ' obtains. With this equivalence relation, we can build a relational theory that makes no reference to states at all. Figure 5.3 depicts an entity-relationship (ER) diagram for this theory.

We can formulate principles for this state-less theory analogous to those of the polymorphic theory, with instead of P-2 a principle that says that each complex is empirically indistinguishable from one or more other complexes. Furthermore, some definitions need modifications, but most of them are straightforward. In particular, where $\mathsf{S}(\xi) = \mathsf{S}(\xi')$ is used, this should be replaced by $\xi \approx \xi'$. Only for the logical operations the modifications are a bit less trivial. For example, in the definition of conjunction we should stipulate that necessarily $\xi \owedge \xi'$ obtains iff ξ obtains and ξ' obtains.

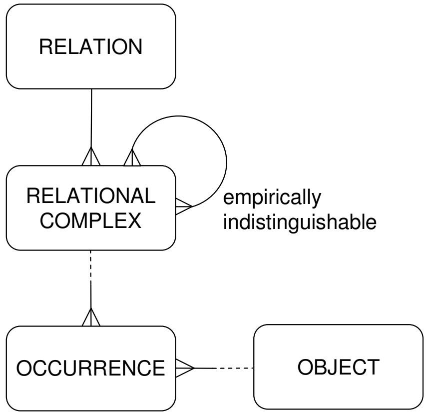


Figure 5.3: ER diagram for a relational theory without states.

Apparently, we can develop a relational theory that is very similar to the polymorphic theory. The question is: which theory is to be preferred? Unfortunately, I have no definite answer; there are strong arguments for both positions.

If states 'out there' are nothing but abstractions from complexes and there is no need for referencing them, then by the principle of parsimony, the state-less theory may be more appealing. However, if states are really 'out there' and complexes are a structured perspective on them, then the polymorphic theory seems more adequate. Or if the relational states are a proper subset of the states 'out there', then this would also be a consideration in favor of the polymorphic theory. The polymorphic theory is also more refined than the state-less theory if not all relational states are empirical entities. Furthermore, it may seem arbitrary to accept complexes in one's ontology, but not states.

A free-standing theory

Suppose our objective is to develop a completely free-standing relational theory that neither refers to states nor makes use of the notion of empirical indistinguishability. Then we cannot postulate identity criteria for relations and relational complexes similar to the principles P-9 and P-10, because these principles make essential use of an equivalence between complexes. For the same reason, the operations on relations need to be redefined. Let us consider how this could be done.

Logical operations. Any definition of conjunction of relations should provide clear identity criteria for conjunctive complexes. Probably we want identities like $\xi \owedge \xi' = \xi' \owedge \xi$ and $(\xi \owedge \xi') \owedge \xi'' = \xi \owedge (\xi' \owedge \xi'')$, and possibly other identities as well (like $\xi \owedge \xi = \xi$ under certain conditions). We could perhaps give an axiomatic specification, but what do we have for its justification, and in what sense would such a specification be complete? If disjunction and negation are also taken into account, the challenge becomes even bigger. Then we have, for example, also to justify—again without referencing states or the like—whether or not $\xi \owedge \xi' = -(-\xi \ovee -\xi')$.

Restriction of relations. For restrictions we have to answer the following kind of questions:

1. Is the binary relation with complexes $\Re xy \owedge \Re xy$ the same as the binary relation with complexes $\Re xy$?

2. If $\Re$ is a transitive, binary relation, then is the ternary relation with complexes $\Re xy \owedge \Re yz \owedge \Re xz$ the same as the ternary relation with complexes $\Re xy \owedge \Re yz$?

3. For a restriction with complexes $\Re xxy$, when do we have $\Re xxy = \Re yyx$?

4. Is restriction of relations a transitive operation?

These questions do not necessarily have unique answers, because more than one restriction operation could perhaps be defined with different results. An important requirement for any definition, however, is that all resulting structures can be conceived of as genuine (complex) relations. Furthermore, we do not want a multiplicity of relations for essentially the same entity 'out there'. From this perspective, the preferred answer to each of the questions above would be "yes". But I do not see, what a general identity criterion for restrictions in a free-standing theory might be.

In conclusion, I don't think that a free-standing theory is a good alternative. What I regard as a key problem is how to understand the notion of complexes without the notion of states 'out there' or the notion of empirical indistinguishability. Furthermore, as the previous discussion suggests, we may end up with a complex and hard to justify set of identity rules.

5.4.3 A theory without occurrences of objects

For a theory without occurrences of objects, there are at least three options: (i) a theory with only states, (ii) a theory with only complexes, and (iii) a theory with states and complexes.

The first two options are not really worth considering here, since they have at least the same drawbacks as the theories discussed in Sections 5.4.1 and 5.4.2. The third option, however, seems more interesting.

We could define a simplified polymorphic theory with comparable principles as of the original theory, but with substitution working directly on objects. Equivalently, we could accept occurrences, but then all occurrences in a complex should be occurrences of different objects.

For the simplified polymorphic theory, we could define notions like the *objects of a complex*, the *object-degree of a complex* and the *object-degree of* $\mathcal{R}$ itself. Roughly stated, the objects of a complex are the objects for which it makes a difference for the resulting complex which objects are substituted for them. See [Leo08a; Leo10] for more detailed definitions.

To every relation-like structure $\breve{\Re}$ of the polymorphic theory there corresponds a unique structure $\mathcal{R}$ of the simplified polymorphic theory. The idea is to define for each complex ξ in $\breve{\Re}$ and function $\delta : O \to O$, with O the objects of $\Re$,

$$\xi \odot \delta$$

as the complex obtained by substituting in ξ simultaneously $\delta(a)$ for each occurrence of each object a. The complexes of $\mathcal{R}$ may then be defined as equivalence classes of complexes of $\breve{\Re}$, where ξ and ξ' are considered equivalent if $\mathsf{S}(\xi \odot \delta) = \mathsf{S}(\xi' \odot \delta)$ for any $\delta : O \to O$.

An interesting question is under what conditions a relation-like structure $\breve{\Re}$ of the polymorphic theory can be uniquely reconstructed from a structure $\mathcal{R}$ of the occurrence-free theory. Let us look at a few simple cases.

1. Suppose that $\mathcal{R}$ has object-degree n, with n finite. Then $\mathcal{R}$ has exactly one corresponding structure $\breve{\Re}$ of the same degree. Furthermore, $\breve{\Re}$ is the only corresponding structure of degree $\leq$ the number of objects of $\mathcal{R}$ with no *dummy occurrences*, where α is a dummy occurrence if $\mathsf{S}(\xi \cdot \sigma) = \mathsf{S}(\xi \cdot \sigma')$ for all substitutions σ, σ' with $\sigma =_{\mathsf{Oc}(\xi)-\{\alpha\}} \sigma'$.

2. Suppose again that $\mathcal{R}$ has object-degree n, with n finite. Then $\mathcal{R}$ may have a corresponding structure $\breve{\Re}$ of degree $>$ the number of objects of $\mathcal{R}$,

and without dummy occurrences, as shown in the following examples:

(i) Let $\mathcal{R}_0$ be a structure with two objects and four complexes in the occurrence-free theory, and let $\mathcal{R}$ be the conjunction of $\mathcal{R}_0$ with itself. Then $\mathcal{R}$ may have more than one corresponding structure $\breve{\mathfrak{R}}$ of degree 4.

(ii) Let $\mathcal{R}$ have objects a, b, and complexes $\langle a, b\rangle$, $\langle b, a\rangle$, $\langle a\rangle$, $\langle b\rangle$ for states s_0, s_1, s_2, s_3, respectively. Then $\mathcal{R}$ has an *initial complex*, that is, a complex from which all other complexes can be obtained by substitution, but nevertheless $\mathcal{R}$ has a corresponding structure $\breve{\mathfrak{R}}$ without dummy occurrences of fixed degree 3, namely a structure with two complexes for state s_0, two complexes for s_1, one complex for s_2, and one complex for s_3.

These examples show that the polymorphic theory is more powerful than the less refined theory without occurrences. The problem with the less refined theory is that the total number of objects of a relation may not be sufficient to express the interrelatedness of its complexes. In particular, certain complex relations cannot adequately be defined in it. This makes the polymorphic theory preferable from a metaphysical perspective.

5.5 Conclusions

Whether relations are really 'out there' or whether they are constructions of our mind may be hard to say, but in all cases it is of metaphysical importance that they are a primary means for structuring the world. In this structuring, substituting objects for occurrences of other objects apparently plays a fundamental role. I find it difficult to imagine how to understand the world without the idea of substitution.

Different theories could be developed around the idea that substitution takes place in *relational complexes*. If complexes are conceived as a structured perspective on what there is 'out there', and in particular on *relational states*, then in principle the following views are possible for the correspondence between complexes and states:

View 1: Every relational state corresponds to exactly one relational complex.

View 2: Every relational state corresponds to one or more relational complexes, but to no more than one within a given relation.

View 3: Every relational state corresponds to one or more relational complexes, possibly even within the same relation.

The first view requires an ultra-refined notion of relational states or a very limited class of relations. Also for the second view the class of relations is seriously limited, among others because conjunction of relations might not always be defined (see Example 5.2.3). Moreover, in both the first and the second view, substitution might lead to a coalescence of occurrences for certain conjunctions of relations, which in some cases appears to be quite unnatural (see Example 5.3.5). I regard the third view as the most promising.

In this paper, I worked out a *polymorphic theory of relations* based on the third view in combination with a coalescence-free account of substitution. It is a very general theory in which no assumptions have been made about the existence of atomic facts or universals, nor any assumptions regarding a specific nature of relational states. In the remainder of this concluding section, I will highlight some of the distinguishing features of the theory.

Substitution works in such a way that it yields a one-to-one correspondence between the occurrences of the objects involved. A clear advantage of such a completely coalescence-free account of substitution is that relational complexes that can be obtained from each other by substitution have a uniform structure. For set-like relations, however, multiple occurrences of the same object in a complex may look a bit artificial. Although I see no objection in principle to make an exception for set-like relations by allowing for them a coalescence of occurrences, I have chosen not to do so since it would make the theory a bit more complicated.

In the polymorphic theory each complex corresponds to one state 'out there'. What complexes are, is completely determined by the network of substitutions between complexes and the corresponding states. This purely extensional view of complexes allows us to formulate a clear identity criterion for relations. Complementary to this criterion, we also have an identity criterion for relational complexes, where two complexes of a single relation are taken to be identical if substitutions in them are fully matching. These criteria enable us to define various operations on relations in a straightforward way in terms of operations on states, in particular conjunction, disjunction, negation and restrictions of relations.

Although the theory does not insist on a particular account of states 'out there', they most likely will be empirical entities. Nevertheless, mathematical relations can easily be represented in the theory. For let us suppose that states 'out there' are conceived as sets of possible worlds. Then a mathematical relation—modulo a certain equivalence—corresponds to the relational complexes of a relation (or of a relation-like structure) with two states, namely the set of all possible worlds and the empty set. (The equivalence is a consequence of the identity principles for relations and relational complexes. We get, for example, the same representation for the mathematical *less than* relation as for the *greater than* relation.) I am not sure whether any specific value should be attached to such representational possibilities of the polymorphic theory.

The principles of the polymorphic theory do not say which notions should be taken as primitive. They leave open the possibility that states 'out there' are merely abstractions from relational complexes. They do not even exclude the possibility that states are nothing but equivalence classes of relational complexes. But by conceiving of complexes as a structured perspective on a relational state (as I did earlier), I certainly suggest that, in my view, states 'out there' are at least as primitive as complexes.

To the polymorphic theory of relations an objection could be raised that is superficially similar to an objection to what Fine [Fin00] calls the *standard view* on relations. According to the standard view, arguments of a relation always come in a certain order. This implies that certain states 'out there' do not belong to a *unique* relation. But, according to Fine [Fin00, pp. 5–6], "we are much more inclined to suppose that there is a single underlying relation connecting the things together". Obviously, also for the polymorphic theory a state can belong to more than one relation. However, the non-uniqueness of the polymorphic theory is not objectionable in the same way. In the first place, in this theory the non-uniqueness is not a representational *artifact*, as is the case in the standard view. Secondly, the polymorphic theory does not exclude the existence of a class of *basic* relations for which the uniqueness of relational states applies.

It is good to contrast the polymorphic theory also to what Fine [Fin00] calls the *positionalist view* on relations. In the positionalist view each relation comes with argument-places. An objection raised by Fine is that such a view cannot cope well with strictly symmetric relations, since switching objects assigned to argument-places would give a different complex. The polymorphic theory does not have this kind of problem. The identity criterion for relational complexes P-10 guarantees single relational complexes for the states of such relations.

There is a close relationship between the polymorphic theory and Fine's *antipositionalist view* on relations [Fin00]. This is of course not very surprising, since—as I said in Section 5.2.2—the substitution mechanism of the theory is directly based on the notion of substitution espoused by that view. The polymorphic theory could be regarded as an elaboration of the antipositionalist view.

An interesting issue is how the states and complexes in the polymorphic theory relate to Fregean thoughts. Perhaps disputes about the question whether thoughts are polymorphic stem from not making a proper distinction between states and complexes. It might not be a bad idea to divide Fregean thoughts into two types: one type corresponding to states 'out there' and one corresponding to relational complexes. Then the state of *Oedipus's loving Jocasta* could be identified with a single thought 'out there', and each of the numerous complexes of this state with a thought having a determinate structure.

The polymorphic theory of relations developed in this paper is quite detailed. Nevertheless, more principles may need to be introduced and some may require adjustments. A drastic departure that might be considered, is to renounce any reference to states 'out there', and to use instead the notion of empirical indistinguishability for complexes. As discussed in Section 5.4.2, with this approach a very similar theory can be developed. The required modifications of the polymorphic theory are for the most part straightforward.

Should we choose a relational theory with states or one without states? For both options there seem to be strong arguments. For example, it seems natural to think of complexes as a structured perspective on states 'out there', and therefore to admit states as real entities. On the other hand, a theory with fewer entities is also appealing, although fewer entities does not necessarily mean that the theory is simpler.

In conclusion, the ontological status of states 'out there' is controversial. But what I consider the main contribution of the analysis in this paper is that it shows that relational complexes and relational states are essentially different notions, and that taking the differences into account is necessary in understanding the logical structure of relations.

Acknowledgments. I am very grateful to Kit Fine for his detailed comments and for asking to what extent states are required to develop a theory of complex relations. Furthermore, I thank Albert Visser and José Luis Bermúdez for their comments on a manuscript of this paper.

A Principles of the polymorphic theory

Structural principles

P-1 Each relation 'has' one or more relational complexes and each relational complex belongs to one and only one relation.

P-2 Each relational complex corresponds to one state 'out there' and each state 'out there' may correspond to one or more relational complexes.

P-3 Each relational complex may contain one or more occurrences of objects and each occurrence belongs to one and only one relational complex.

P-4 Each occurrence is the occurrence of one object and each object may be the content of one or more occurrences.

Substitution principles

P-5 Any substitution of objects for occurrences in a relational complex results in exactly one relational complex of the same relation.

P-6 For any complex, the identity substitution results in the same complex.

P-7 *Composition principle*: If a substitution σ in a relational complex ξ results in ξ', then there is a bijection μ from the occurrences in ξ to the occurrences in ξ', such that

(a) μ maps each occurrence α in ξ to an occurrence of the object substituted by σ for α,

(b) any substitution σ' in ξ' gives the same result as substituting in ξ for each occurrence α the object that σ' substitutes for $\mu(\alpha)$. (Or in symbols, $(\xi \cdot \sigma) \cdot \sigma' = \xi \cdot (\mu \cdot \sigma')$.)

Connectivity principle

P-8 For any relation $\Re$ of fixed finite degree, any relational complex of $\Re$ can be obtained from any other relational complex of $\Re$ via substitution.

Identity criteria

P-9 Relations are identical if and only if they are qualitatively the same.

P-10 Relational complexes that belong to the same relation are identical if and only if they are qualitatively the same.

Operations principles

P-11 Relation $\Re \owedge \Re'$ exists for any relations $\Re, \Re'$.

P-12 Relation $\Re \ovee \Re'$ exists for any relations $\Re, \Re'$.

P-13 Relation $-\Re$ exists for any relation $\Re$.

P-14 Any restriction of a relation is also a relation.

Chapter 6

Epilogue

To what extent did I succeed in revealing the logical structure of relations? In this epilogue I take a look at the main results, discuss the reliability of the results, and say a few words about ideas for future investigations.

6.1 Main results

The results of this thesis strongly support the following claims:

1. Antipositionalism is the superior view on relations.
2. Positional representations of relations are justified.
3. Complexes and states of a relation should not be identified.

I briefly discuss these claims in the next subsections.

6.1.1 Superiority of antipositionalist view

In one way or another, all four papers of this thesis provide support for a view on relations in which the notion of substitution plays a central role.

In the first paper, 'Modeling relations', I made a detailed comparison of mathematical models for the standard view on relations, the positionalist view, and the antipositionalist view. One of the results was that for the class of *simple* relations[1] of finite degree the notion of substitution has the same expressive power as the notion of argument-places. Because the simple relations form a large class and because substitution is a primitive kind of operation (or of a low logical type), this result is a strong argument in favor of the antipositionalist view.

In the second paper, 'The identity of argument-places', I showed that for every simple relation of finite degree, a unique construction of argument-places can be given if we are prepared to go beyond the limits of ordinary set theory. The fact that such a unique construction is possible, shows that there is no need for taking argument-places as a primitive notion. As such, this result provides additional support for the antipositionalist view.

The way the antipositionalist view was modeled in the first two papers had a serious shortcoming, because objects were substituted for objects and not for *occurrences* of objects. In the third paper, 'Modeling occurrences of objects in relations', I examined a more refined account of substitution based on occurrences. The main technical results of the paper support the possibility of a maximally refinedness principle for occurrences, which makes the notion of occurrences more accessible.

The fourth paper, 'Relational complexes', contains what I think to be a powerful elaboration of the antipositionalist view. Here, substitution does not take place in the states of a relation but in complexes, which are conceived as a structured perspective on states 'out there'. With this approach it becomes possible to define operations on relations in a very natural way.

Some of the work done in the first paper can be redone with the original substitution models replaced by models for the theory developed in the fourth paper. This will show that even for a larger class of relations the notion of substitution has the same expressive power as the notion of argument-places. More specifically, it can be shown that *any* positional model has a unique corresponding model with substitution taking place in complexes. Conversely, not every substitution model has a corresponding positional model. In particular, substitution models for relations with a variable adicity have no positional counterparts. This makes the antipositional view more general than the positional view.

Apparently, from a technical perspective, the antipositionalist view has no

[1]Simple relations are defined in terms of substitution working directly on the objects of states. A distinguishing feature of a simple relation is that it has an *initial state*, that is a state from which all other states of the relation can be obtained by substitution.

weaknesses compared to the other two views on relations. There is also a strong metaphysical argument for antipositionalism, which is similar to an argument given by Fine [Fin00, p. 17]. It concerns symmetric relations. For the adjacency relation the state of *a's being adjacent to b* with a and b distinct has *two* complexes under positionalism. We may consider these complexes as indiscernible, but nevertheless the fact that there are two of them strikes me as a defect for a fundamental account of relations. With the antipositionalist view, and in particular with the polymorphic theory developed in the paper 'Relational complexes' we do not have this problem. There we have only one complex for this symmetric state. In the polymorphic theory a state may have multiple complexes in a relation, but then these complexes are genuinely different.

If we combine these considerations with an intuition that the notion of substitution is more primitive than the notion of argument-places, everything seems to be in favor of antipositionalism. However, some caution is required. The superiority is only with respect to the other two views. It is not excluded that in the future better views may emerge.

6.1.2 Justification for positional representations

Adopting an antipositional view on relations does not imply that we should no longer make use of argument-places. On the contrary, this thesis gives an explicit justification for their use.

The main justification is given in the paper 'The identity of argument-places'. There I proposed to identify argument-places with abstract points in a structure exemplified by positional frames. The reason for doing this was that ordinary set theory turned out to be too limited as a modeling medium for a *neutral* construction of argument-places.

There is, however, a point of concern: The proposed identification was restricted to *simple* relations of finite degree. For this class of relations, I proved in 'Modeling relations' that there is a natural one-to-one correspondence between positional and antipositional models. I remarked that for positional models that do not correspond to substitution models it seems highly unlikely that they could adequately model 'real' relations, and that these models probably have no metaphysical significance at all [Leo08a, p. 383]. It was only in writing the paper 'Relational complexes' that I found that this is not correct. For consider the positional model $\mathcal{F}$ with positions

p_1, p_2, p_3 and with

$$\begin{pmatrix} p_1 & p_2 & p_3 \\ x & y & z \end{pmatrix} \quad \overset{\Gamma}{\longmapsto} \quad x \mathrel{-\!\heartsuit\!\!\rightarrow} y \owedge z \mathrel{-\!\heartsuit\!\!\rightarrow} d,$$

where d is a fixed object. I would say that $\mathcal{F}$ definitely models a 'real' conjunctive relation. But this relation cannot be modeled by a substitution model with substitution working directly on states. The reason is that substituting b for d in the state of $a \mathrel{-\!\heartsuit\!\!\rightarrow} d \owedge c \mathrel{-\!\heartsuit\!\!\rightarrow} d$ should, on the one hand, give the state $a \mathrel{-\!\heartsuit\!\!\rightarrow} b \owedge c \mathrel{-\!\heartsuit\!\!\rightarrow} d$ and, on the other hand, the state $a \mathrel{-\!\heartsuit\!\!\rightarrow} d \owedge c \mathrel{-\!\heartsuit\!\!\rightarrow} b$ (cf. Example 5.2.3).

Apparently, the substitution models as defined in 'Modeling relations' are too limited. With a refined substitution mechanism, where substitution works on occurrences of objects in *complexes*, we do not have this problem. As I mentioned in the previous subsection, then *any* positional model has a unique corresponding substitution model. Conversely, any substitution model with an *initial complex*, that is, a complex from which all other complexes of the relation can be obtained by substitution, also has a corresponding positional model. So, from an antipositionalist view as worked out in 'Relational Complexes', we have a complete justification of positional representations for relations.

6.1.3 Complexes and states should not be identified

It seems obvious that a single state 'out there' may belong to more than one relation. For example, the state of Narcissus's loving Narcissus may be regarded as a state of the binary relation of love and also of the unary relation of self-love. In all my papers in this thesis I implicitly assumed this to be the case. In the first three papers of this thesis, however, I also implicitly assumed that for any given relation each state has *at most one* corresponding complex. It is only in the fourth paper that I dropped this constraint.

A compelling reason for dropping the constraint was that certain relations simply need multiple complexes for their states. Examples of such relations are the parthood relation and the conjunction of the binary love relation with the unary relation of loving God (cf. Examples 5.2.2 and 5.2.3).

The following ideas form the cornerstone of the *polymorphic theory of relations* developed in my paper 'Relational complexes':

1. Complexes are a structured perspective on states ‘out there’.
2. For a given relation, more than one complex may correspond to a single state.
3. The complexes of a relation form a network interrelated by substitutions, where two complexes are distinct only if substitutions in them are not fully matching.

Allowing more than one complex for a single state of a relation also has a number of advantages that I missed in my previous attempts to characterize the logical structure of relations. In particular, it made it possible to give the complexes of a relation a very uniform structure by choosing a coalescence-free account of substitution. Furthermore, it made it possible to define all kinds of operations on relations directly in terms of operations on states ‘out there’.

6.2 Reliability of the results

Using a logico-mathematical approach to solve a metaphysical problem may give the impression that the results are necessarily very reliable. There is, however, a serious pitfall, namely in *interpreting* the formal results things can easily go wrong. In particular, the formal notions may not correspond in an appropriate way to the metaphysical notions. In my thesis I tried to be very careful in this respect. I will illustrate this by considering a few cases.

In the paper ‘Modeling relations’ I gave a formal definition of *corresponding frames*, and proved a correspondence between substitution frames and positional frames. In interpreting these results, I drew a conclusion with respect to the expressive power of substitution and the use of positions. But I did not give an explicit justification of my definition of corresponding frames. In fact, I do not know how in this case an explicit justification should look like. However, this should not unduly worry us, since a close examination of the definition will probably convince the reader that it is indeed appropriate. But to play on the safe side, I added the proviso that adequate substitution models and adequate positional models for a relation correspond.

Another example I would like to give comes from ‘The identity of argument-places’. In that paper I used the notion of *neutrality* in an intuitive sense and in a formal sense. I introduced a formal definition of neutrality to *prove* that certain constructions are not possible. The formal definition is quite abstract. It talks about sets in the cumulative hierarchy with urelements being neutral with respect to other sets. But how do we *know* this definition indeed matches our intuitive notion of neutrality? I did not provide evidence for this. In fact, for the conclusions drawn in the paper this was not needed. It is sufficient to see that if a construction of a set Y given a set X is neutral in an intuitive sense, then Y is neutral with respect to X in the formal sense. And this is evident (although it may take some time to see this). I actually *proved* that for some relations no neutral construction of argument-places is possible *if the states and objects of the relations are treated as urelements*, but I only concluded that it is *highly unlikely* that for every relation argument-places can be defined in a canonical way in ordinary set theory.

At some points in this thesis, I made a wrong judgement. As I already mentioned in Section 6.1.2, I erroneously wrote in 'Modeling relations' that positional models that do not correspond to substitution models probably have no metaphysical significance at all. A directly related issue is that I thought that the states and complexes of any relation could be identified. It was only in developing the third paper that I began to doubt the correctness of this view. In the conclusion of that paper I mentioned this issue as something that required further investigation. It was the main challenge of the fourth paper to see if a more adequate theory of relations could be developed if we would no longer identify the states and complexes of a relation. As I argued in the previous section, this resulted in a promising polymorphic theory of relations. But I am sure that further research will provide new insights that may lead again to substantial improvements.

6.3 Future investigations

A continuation of the mathematical analysis of this thesis would be very useful. In my opinion a great benefit of this approach is that it makes the issues as explicit as possible. There are interesting fundamental questions for future investigations, like: What exactly characterizes the unity of relations? Is subtraction a primitive complementary operation of substitution? What are additional constraints for the relationship between complexes and states?

A natural choice for applying the results of this thesis is in the field of linguistics. I think in particular of situation semantics, which provides a relational theory of meaning [BP83]. A problem with the original theory is that it uses a notion of relation that appears to be too simplistic. The theory of relations as developed in this thesis may shed new light on situation semantics, and may give it a push in the right direction.

Besides a purely theoretical research program, a more practically oriented psychological investigation of *the way we learn relations* would be extremely useful. Such an investigation could fit in nicely in the research field of the way we think. Interesting research in this more general field has been done by Philip Johnson-Laird [JL83] and Jonathan Evans, Stephen Newstead and Ruth Byrne [ENB93]. Research on learning relations has also been carried out. For example, Marc Tomlinson and Bradley Love [TL06] developed a model of relational learning, but as far as I know, the role that the notion of substitution may play in learning relations has never been explicitly considered.

There is encouraging research with respect to developing algorithms for learning relations [RM92; Hey+01; KA07]. In some case the goal is to build systems that 'discover' relations, and in other cases to find instances for which a given relation holds. Because I think that substitution may be an essential ingredient for any truly intelligent system, the theoretical results of this thesis may also be very relevant for the field of artificial intelligence. The challenge will be to investigate whether and how a general notion of substitution (and of subtraction) can be implemented in artificial systems. If successful, such systems may subsequently be taught various kinds of relations by feeding them with large sets of samples. My expectation is that in this way sophisticated systems can be developed that may discover all kinds of regularities and similarities in different scenes or in raw data by exploiting the notion of substitution.

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Curriculum Vitae

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Education

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Work experience

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2006 - 2009	PhD Student (AIO), Utrecht University, Dept. of Philosophy
2004 - 2009	Lecturer, HU University of Applied Sciences Utrecht, ILC
2004 - 2004	Coordinator, Dutch Graduate School in Logic (OzsL)
2001 - 2002	Program Manager, Lucent Technologies
1999 - 2000	Business Unit Manager, Solvision
1996 - 1999	Consulting Manager, Siemens
1993 - 1996	Senior Consultant/Trainer, Oracle Netherlands
1992 - 1993	Development Manager, Oracle US
1989 - 1992	Consultant, Oracle Netherlands
1986 - 1989	Ass. Professor, Radboud University, Comp. Science Dept.
1984 - 1986	Researcher, Netherlands Org. for Scient. Research (NWO)

Publications

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Samenvatting

Een van de leuke kanten van mijn onderzoek voor mijn proefschrift is dat ik altijd een geïnteresseerde reactie krijg als ik vertel dat het over relaties gaat. Als ik vervolgens vertel dat het over de logische structuur van relaties gaat, begint men vaak wat glazig te kijken. Als ik dan ook nog wat zeg over de wiskundige aanpak, dan lijk ik in de ogen van velen van een andere planeet te komen. Dit is jammer, want mijn onderzoek gaat over zulke fundamentele zaken dat het voor iedereen verruimend kan zijn om hier iets over te weten.

In deze samenvatting wil ik proberen de kern van mijn onderzoek op een zeer informele manier uit te leggen. Ik zal me hierin pas geslaagd voelen als de lezer niet alleen een globaal idee van mijn onderzoek krijgt, maar ook ziet dat het niet los staat van de alledaagse werkelijkheid. Ik volg hierbij de indeling in hoofdstukken van het proefschrift.

1 Inleiding

Ik wil je voorstellen aan Patrick. Patrick is iemand die in de wereld geen enkele regelmatigheid ziet. Als jij hem een hand geeft en ik dat even later ook doe, ziet hij in deze twee gebeurtenissen totaal geen overeenkomst.

Wij hebben het gevoel dat Patrick aardig wat mist. Er is ook geen gesprek met hem te voeren. Je kunt je zelfs afvragen of er bij hem enige vorm van denken aanwezig is en of hij wel enige bewuste ervaring heeft.

In tegenstelling tot Patrick zien wij wel voortdurend verschillen en overeenkomsten tussen allerlei gebeurtenissen en dingen. Wij zien bijvoorbeeld zelfs een overeenkomst tussen een kopje op tafel en een auto op straat.

Relaties zijn voor ons een middel bij uitstek om de wereld te ordenen. Maar mogelijk zijn relaties meer dan alleen een hulpmiddel. Velen—onder wie ikzelf—zijn geneigd om te denken dat de wereld zélf gestructureerd is en dat relaties in deze structuur een sleutelrol vervullen.

Maar wat zijn precies relaties? Hoe zitten ze in elkaar? Een nogal simpele opvatting is dat relaties niets anders zijn dan verzamelingen van geordende paren of rijtjes objecten. Voor wiskundige relaties is dit correct, maar voor de liefdesrelatie niet. De feitelijke toestand dat Paris van Helena houdt is duidelijk wat anders dan het geordend paar $\langle \text{Paris}, \text{Helena} \rangle$.

Nu probeer ik in mijn onderzoek niet de liefdesrelatie of welke relatie dan ook in detail te analyseren. Het gaat mij alleen om de logische structuur van relaties. Ik laat zoveel details buiten beschouwing dat het meeste van wat ik over de liefdesrelatie zeg evengoed opgaat voor de haatrelatie. Maar het is zeker niet uitgesloten dat de logische structuur van deze relaties zeer verschillend is. Het zou bijvoorbeeld kunnen zijn dat je alleen van iemand kunt houden als je van jezelf houdt, maar dat je om iemand te haten niet jezelf hoeft te haten.

Als je van zoveel zaken abstraheert, dan wordt het er niet in alle opzichten gemakkelijker op. Regelmatig ontglipt mij waarover ik het heb. Dan komen concepten op losse schroeven te staan en valt alles als een kaartenhuis in elkaar. Maar ook op zulke momenten van verwarring zie ik mijn onderzoek nooit als een zinloze bezigheid. Eerder als iets dat te hoog gegrepen is. Evolutionair gezien zijn wij wellicht niet geweldig goed geëquipeerd om metafysische vraagstukken op te lossen. Onze verre voorouders hadden wel iets anders aan hun hoofd.

Ik geloof dat fundamenteel onderzoek naar de logische structuur van relaties ook vanuit een praktisch oogpunt bijzonder zinvol is. Het kan helpen bij het verkrijgen van inzicht in de manier waarop wij en andere intelligente systemen relaties leren kennen. En dit kan weer leiden tot nieuwe manieren voor ons en voor artificieel intelligente systemen om relaties te ontdekken en ermee om te gaan. Misschien is er een vergelijking te maken met natuurkundig onderzoek. Inzicht in de structuur van atomen en moleculen is duidelijk niet alleen theoretisch interessant. Het heeft een enorme impact op ons dagelijks leven, zoals blijkt uit de ontwikkeling van kunststoffen, medicijnen, computers, etc.

2 Modelleren van relaties

Om inzicht te krijgen in de logische structuur van relaties, heb ik wiskundige modellen ontwikkeld voor drie visies op relaties: (1) de standaard visie, (2) de positionele visie, en (3) de antipositionele visie. Een sterk punt van wiskundige modellen is dat hiermee allerlei subtiliteiten heel precies geformuleerd kunnen worden. Ik geef hier een korte beschrijving van deze visies, hun modellen en de metafysische interpretatie van de technische analyse van deze modellen.

2.1 Visies op relaties

De eerste visie, de *standaard visie*, gaat ervan uit dat in een relatie de objecten altijd in een zekere volgorde voorkomen. Echter de toestand uitgedrukt door "Antonius bemint Cleopatra" is dezelfde toestand als uitgedrukt door "Cleopatra wordt bemind door Antonius". In deze toestand komt Antonius niet eerder of later voor dan Cleopatra. De standaard visie biedt alleen geen mogelijkheid om deze toestand op een neutrale manier uit te drukken. Dit maakt de standaard visie zwak.

Een betere visie lijkt de *positionele visie* op relaties. Hierin wordt ervan uitgegaan dat elke relatie argument-plaatsen (posities) heeft, bijvoorbeeld de argument-plaatsen *minnaar* en *beminde* waartussen geen volgorde bestaat. Maar ook deze visie is niet zonder problemen. Waarom zouden we aannemen dat relaties op een *fundamenteel* niveau argument-plaatsen hebben?

Kit Fine heeft een radicaal andere visie op relaties voorgesteld, de zogenaamde *antipositionele visie.* Hierin komen de objecten niet in een bepaalde volgorde voor en nemen ze ook geen argument-plaatsen in. In plaats daarvan wordt een zeker onderling verband tussen de toestanden van een relatie als essentieel aspect van de logische structuur gezien. Bijvoorbeeld het relevante verband tussen de toestand dat Antonius van Cleopatra houdt en de toestand dat Abélard van Héloïse houdt, is dat we de tweede toestand krijgen als we in de eerste Antonius door Abélard vervangen en Cleopatra door Héloïse. Dit *substitueren* van voorkomens (*occurrences*) van objecten door andere objecten wordt als een primitieve operatie opgevat.

Wat de antipositionele visie met name aantrekkelijk maakt is dat de relevante structuur uitsluitend bestaat uit een netwerk van verbindingen tussen de voorkomens van de objecten van verschillende toestanden; er worden geen additionele aannames gemaakt over de intrinsieke structuur van de toestanden zelf.

2.2 Modellen

De modellen die ik voor de drie visies ontwikkeld heb, schets ik aan de hand van de liefdesrelatie:

1. Het model in de standaard visie bestaat uit: de verzameling van alle liefdestoestanden, de verzameling van alle personen, en de functie die aan elk geordend paar van personen (a, b) de toestand $a \overset{\heartsuit}{\rightarrow} b$ toevoegt.

2. Het model in de positionele visie bestaat uit: de verzameling van alle liefdestoestanden, de verzameling van alle personen, de argument-plaatsen *minnaar* en *beminde*, en de functie die aan $\{(minnaar, a), (beminde, b)\}$ de toestand $a \overset{\heartsuit}{\rightarrow} b$ toevoegt.

3. Het model in de antipositionele visie bestaat uit: de verzameling van alle liefdestoestanden, de verzameling van alle personen, en de functie die aan elke toestand $a \overset{\heartsuit}{\rightarrow} b$ en substitutie van a door c en b door d de toestand $c \overset{\heartsuit}{\rightarrow} d$ toevoegt.

Ik neem in de modellen aan dat er voor *elk* tweetal personen a, b een liefdestoestand is waarin a van b houdt, dus ook Kaïn $\overset{\heartsuit}{\rightarrow}$ Abel. Een toestand hoeft dus niet altijd feitelijk het geval te zijn. Als een toestand dat wel is, dan spreken we van een *feit*.

De typen modellen die ik in mijn proefschrift definieer, zijn wat algemener van karakter dan de hierboven geschetste modellen. Zo laat ik voor de standaard visie bijvoorbeeld modellen toe die aan rijtjes van oneindige lengte toestanden toevoegen. De antipositionele modellen zijn zelfs zo ruim gedefinieerd dat hierin toestanden kunnen voorkomen die geen unieke set van objecten hebben waarop substitutie werkt.

2.3 Metafysische interpretatie

Een vraag die ik mijzelf gesteld heb, is hoe de modellen zich ten opzichte van elkaar verhouden. Hieruit is veel te leren over de verhouding van de onderliggende visies op relaties. Ik heb bewezen dat zogenaamde *simpele* positionele modellen op een natuurlijke manier corresponderen met *simpele* antipositionele modellen. Dit resultaat is te beschouwen als een ondersteuning voor de antipositionele visie, maar ook als een rechtvaardiging voor het gebruik van argument-plaatsen in representaties van relaties.

3 De identiteit van argument-plaatsen

Argument-plaatsen spelen een belangrijke rol in hoe wij in het dagelijks leven met relaties omgaan. Het zou mooi zijn als we voor elke willekeurige relatie een *unieke* verzameling argument-plaatsen kunnen reconstrueren. Voor veel relaties is dit geen probleem, maar voor sommige lukt dit niet, althans niet op een *neutrale* manier in de gewone verzamelingstheorie. Als modelleringsmedium blijkt gewone verzamelingstheorie in bepaalde gevallen te beperkt te zijn. Dit is nogal verrassend en het vinden van een overtuigend bewijs hiervoor heeft me aardig wat inspanning gekost.

Dat wij in de gewone verzamelingstheorie niet altijd op een non-arbitraire manier argument-plaatsen lijken te kunnen definiëren, plaatst ons in een situatie die vergelijkbaar is met die van de ezel van de 14e-eeuwse filosoof Buridan. De ezel, die zich precies in het midden van twee even aantrekkelijke balen hooi bevond, stierf de hongerdood omdat hij geen rationele keuze kon maken tussen de balen.

Dit negatieve resultaat wil nog niet zeggen dat er *buiten* de gewone verzamelingstheorie ook geen neutrale reconstructie van argument-plaatsen mogelijk is. Mijn voorstel is om argument-plaatsen op te vatten als abstracties van de argument-plaatsen van positionele modellen waarbij deze abstracties geen interne structuur hebben. Ik beschouw dit als een zeer natuurlijke kijk op argument-plaatsen.

Wat metafysisch interessant is, is dat de aldus verkregen argument-plaatsen niet in alle gevallen onderscheidbaar zijn. Bijvoorbeeld de symmetrische relatie met toestanden van de vorm *x bevindt zich naast y* heeft twee argument-plaatsen die in geen enkel opzicht van elkaar te onderscheiden zijn. Het is de vraag of we zulke posities als 'echte' objecten mogen opvatten. Als we dit wel doen, dan is dat duidelijk in tegenspraak met Leibniz' principe van de *Identiteit van Ononderscheidbaren*. Ik verwacht echter niet dat een verdere analyse van argument-plaatsen op zichzelf een basis biedt om dit principe van Leibniz op te geven.

4 Modelleren van voorkomens van objecten

In de voorgaande hoofdstukken ben ik in mijn modellering van de antipositionele visie op relaties uitgegaan van een ongedifferentieerd substitutiemechanisme. Dit betekent dat voor de toestand dat Narcissus van Narcissus houdt, de modellen geen substitutie toelaten die resulteren in de toestand waarin Echo van Narcissus houdt. Dit is een serieuze beperking.

Een vollediger beeld wordt verkregen door substitutie op individuele voorkomens van objecten te laten werken. De toestand waarin Narcissus van Narcissus houdt, kan dan gezien worden als een toestand met twee voorkomens van Narcissus, die los van elkaar vervangbaar zijn.

Voor een willekeurige relatie is op voorhand nog niet zo duidelijk wat precies voorkomens van objecten zijn. Om hier meer helderheid over te krijgen heb ik modellen voor relaties gedefinieerd met een substitutiemechanisme dat op voorkomens van objecten werkt. Voor een gegeven relatie blijken er echter vaak meerdere van zulke modellen te zijn. De kernvraag is: *wat is het meest adequate model?*

Om hierachter te komen, heb ik voor een gegeven relatie de bijbehorende modellen met een verfijnd substitutiemechanisme onderling vergeleken. Zeer verwonderlijk was dat zelfs de gewone liefdesrelatie oneindig veel echt verschillende van zulke modellen bleek te hebben. Ook voor een willekeurige relatie heb ik bewezen dat dit het geval is, tenminste als er bepaalde zeer grote oneindigheden bestaan (*meetbare kardinaalgetallen*). Echter, als we ons beperken tot *normale* modellen, dan ziet het er gelukkig een stuk overzichtelijker uit. In het bijzonder heb ik bewezen dat veel relaties een *uniek* maximaal verfijnd normaal model hebben. Dit resultaat kan gezien worden als een aanmoediging voor de veronderstelling dat elke toestand van elke 'echte' relatie een maximaal aantal voorkomens van objecten heeft.

5 Relationele complexen

Wat is de onderliggende relatie van de toestand waarin Socrates voetbal speelt? In mijn ogen zijn hiervoor verschillende keuzes mogelijk, waaronder de relatie *spelen* en de relatie *voetbal spelen*. Een gegeven toestand in de wereld kan blijkbaar vanuit verschillende kanten bekeken worden. Elk van deze perspectieven op een toestand van een relatie noem ik een *relationeel complex*.

Wat ik niet eerder in mijn benadering toeliet, maar in dit hoofdstuk wel, is dat een toestand binnen één relatie meerdere corresponderende relationele complexen kan hebben. Ik wijk hiermee sterk af van de gangbare opvatting over toestanden en complexen. Zo zegt Bertrand Russell bijvoorbeeld dat er zeker een één-op-één correspondentie is tussen complexen en feiten.

Het onderscheid dat ik maak tussen relationele complexen en relationele toestanden biedt grote voordelen. Zo maakt dit onderscheid het mogelijk om substitutie zo te definiëren dat de voorkomens van de objecten in de betrokken complexen altijd één-op-één met elkaar corresponderen. Hierdoor krijgen de complexen van een relatie een zeer uniforme structuur.

Deze basisideeën zijn in hoofdstuk 5 uitgewerkt in een theorie die ik de *polymorfe theorie van relaties* noem. In deze theorie zijn logische operaties op relaties op een heldere manier te definiëren. Bijvoorbeeld startend met de liefdesrelatie krijgen we via eenvoudige operaties een relatie met complexen van de vorm *x bemint y en wordt bemind door z*. Elk complex van deze samengestelde relatie heeft drie voorkomens van objecten. Interessant is dat het complex *x bemint y en wordt bemind door y* met dezelfde toestand correspondeert als het complex *y bemint x en wordt bemind door x*.

De theorie betreft ook relaties waarvan de toestanden een variabel aantal voorkomens van objecten hebben, zoals de relatie *vormen een groep*. Om de onderlinge samenhang tussen de complexen van deze relatie te karakteriseren, zou naast substitutie ook *subtractie* van voorkomens van objecten als primitieve operatie opgenomen kunnen worden. Bijvoorbeeld voor de groep bestaande uit Caesar, Pompeius en Crassus kan dan uit een bijbehorend complex het voorkomen van Crassus weggehaald worden. Het betreffende subtractiemechanisme lijkt geschikt voor relaties waarin de toestanden zijn op te vatten als verzamelingen. Echter, het is een open vraag hoe we voor een *willekeurige* relatie de onderlinge samenhang tussen de complexen kunnen karakteriseren.

Ik laat zien dat er ook een theorie ontwikkeld kan worden die veel lijkt op de polymorfe theorie, maar waarin op geen enkele manier naar relationele toestanden verwezen wordt. Dit roept de vraag op of relationele toestanden wel *fundamentele* entiteiten zijn. Ik heb hierop geen overtuigend antwoord. Enerzijds lijkt het arbitrair om relationele complexen wél als fundamenteel te zien, maar relationele toestanden niet. Anderzijds is het een goed uitgangspunt om zo min mogelijk entiteiten als fundamenteel te poneren.

6 Epiloog

Het is gevaarlijk om te beweren dat we de essentie van relaties helemaal gevonden hebben, maar mijn analyse ondersteunt wel in sterke mate een antipositionele visie op relaties. Bovendien lijkt het door mij gemaakte onderscheid tussen relationele toestanden en complexen fundamenteel voor een beter begrip van de logische structuur van relaties. Misschien een minstens zo belangrijk resultaat is dat het onderzoek laat zien dat een wiskundige benadering bijzonder vruchtbaar kan zijn voor het aanpakken van metafysische vraagstukken.

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