WHAT IS THE RIGHT NOTION OF SEQUENTIALITY?

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Abstract. In this paper we give an informally semi-rigorous explanation of the notion of *sequentiality*. We argue that the classical definition, due to Pudlák, is slightly too narrow. We propose a wider notion *m-sequentiality* as the notion that precisely captures the intuitions behind *sequentiality*. The paper provides all relevant separating examples minus one.

1. Introduction

Sequential theories are, as a first approximation, theories with sufficient coding machinery. We have a reasonably robust definition of this class of theories. Yet, we will argue that this definition is slightly too narrow to capture the intuitive motivations behind the notion of *sequentiality*. In essence the point is that the current definition contains the unmotivated constraint that the definition of the sequences in the given theory must be one-dimensional. We propose a wider notion, *m-sequentiality* of which we claim that it captures the intuitions precisely. In fact, *m-sequentiality* is simply *sequentiality* without the unmotivated constraint.

In the rest of this introduction, we explain carefully what the intuitions behind *sequentiality* are. In Section 2, we provide some basic definitions and facts needed to understand the rest of the paper. We discuss the current definition of *sequentiality* in Section 3. We briefly present the state of the art in Subsection 3.2, and, in Subsection 3.3, we give reasons why the state of the art is not quite satisfactory.

In Section 4, the definition of *m-sequentiality* is given, plus several characterizations. For example, it is shown that a one-sorted theory is m-sequential iff it is bi-interpretable with a sequential theory. It follows that m-sequential theories are closed under bi-interpretability. In Section 5, it is proved that, on the one hand, there are sequential theories that essentially involve parameters to witness their sequentiality, and, on the other hand, that parameters can always be eliminated to witness m-sequentiality. So, in a sense, m-sequentiality is a parameter-free notion. Finally, in Section 6, we give a proven example of an m-sequential theory that is not sequential. The fact that it is not all that easy to find such an example, is, perhaps, an indication that the usual notion is only *slightly* wrong. The m-sequential theories we meet in practice are all sequential. On the other hand, the example is not 'unnatural', which could mean that the practice we are looking at is, until now, rather restricted.

Sequentiality is an explication of the idea of *theory with coding*. Can we give an informally rigorous determination of this idea? We think that such a rigorous determination is not to be had, at least not quite. The notion does have a lot of

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intuitive content, but not al choices follow unavoidably from that content. We can only give, say, an informally semi-rigorous determination that will have some gaps in the explanation. Such an explanation cannot quite stand on its own and should be supplemented with other considerations. The considerations are in this case that the successful employment of *sequentiality* in theorems and the robustness of sequentiality with respect to variations on its definition. We will return to these points later. We proceed with an informally semi-rigorous determination of *sequentiality.*

As a first step, towards the determination of the proper notion of *theory with coding*, we must ask: coding for what? We can have different needs for coding and we should expect different answers, depending on those needs, to the question *what is a theory with coding?* A first approximation to the question to which sequentiality is (supposed to be) the answer is to say: *we want coding in order to provide* partial satisfaction predicates *and, as a consequence,* restricted consistency statements. Even this answer is not quite satisfactory: in theories of pairing we also have partial satisfaction predicates. See e.g. [Vau67] and [Vis10]. However, these satisfaction predicates are, in a sense, more finitistic than the ones provided by sequentiality. So, we could add to our specification: *these satisfaction predicates concern classes of formulas with decent verifiable closure properties.*

What do we need to specify partial satisfaction predicates? Clearly, the very formulation of such predicates involves the possibility to code formulas, so we will need a modicum of the theory of syntax. Moreover, this formulation involves the possibility to define finite functions or finite sequences. Hence, the demand that an elementary theory of sequences is present follows directly from our desire to define partial satisfaction. The demand for decent closure properties makes it plausible that such sequences are closed under the operation of pushing an element on top of a sequence considered as stack. To have projections we need numbers as inputs of the projection, hence we need a modicum of number theory.

A further desideratum is this: *we want partial satisfaction for the full language and also restricted consistency for the theories under consideration themselves.* This means that we want to have sequences involving *all* objects of the original theory. It is easy to give examples that fail this condition.

Do we need more? The good news is that these meager means are sufficient also for coding formulas and the like. We can start with a bare minimum and build everything we need in addition from that. For example, we do not need addition and multiplication for our numbers since we can define these operations using the sequences to code the recursive mechanisms.

Well, but couldn't we also have started with a theory of finite functions? Here the prediction is that all such alternative formulations are equivalent to the one involving sequences. Our notion is substantially robust w.r.t. alternative formulations and this robustness is an argument for the correctness of the choices made. We will say a bit more about robustness in Section 3.

Finally, it seems reasonable to allow parameters in the definition of our sequences. So, in the general definition, we allow parameters.

2. Basic Notions

In this section, we formulate the basic notions employed in the paper.

2.1. **Theories.** The primary focus in this paper is on one-sorted theories of first order predicate logic. Specifically, we analyze sequentiality for the case of one-sorted theories. However, many-sorted theories will occur as an auxiliary. They could be eliminated but the approach is less natural if we do so. We only consider theories with a finite number of sorts. We use $\mathfrak{a}, \mathfrak{b}, \ldots$ for sorts. We take identity to be a logical constant. In the many-sorted case, we have identity for each sort.

Our official signatures are relational, however, via the term-unwinding algorithm, we can also accommodate signatures with functions.

Our signatures need not be finite and we need no constraint on the complexity of the axiom set.

2.2. **Interpretations: the One-sorted Case.** We describe the notion of an m-dimensional interpretation for a one-sorted language. In Subsection 2.4 we will indicate how to adapt the definition to the many-sorted case.

An interpretation $K : U \to V$ is given by the theories U and V and a translation τ from the language of U to the language of V. The translation is given by a domain formula $\delta(\vec{x})$, where $\vec{x}$ is a sequence of m variables, and a mapping from the predicates of U to formulas of V, where an n-ary predicate P is mapped to a formula $A(\vec{x}_0, \ldots, \vec{x}_{n-1})$, where the $\vec{x}_j$ are appropriately chosen, pairwise disjoint, sequences of m variables. We lift the translation to the full language in the obvious way making it commute with the propositional connectives and quantifiers, where we relativize the translated quantifiers to the domain δ. We demand that V proves all the translations of sentences of U.

We can compose interpretations in the predictable way. Note that the composition of an n-dimensional interpretation with an m-dimensional interpretation is $m + n$-dimensional.

An m-dimensional interpretation is *m-identity preserving*, if it translates identity to pointwise-identity of the corresponding elements of the sequences assigned to the relata. I.o.w., $x = y$ translates to $\bigwedge_{i<m} x_i = y_i$. An m-dimensional interpretation is *m-unrelativized*, if its domain consists of all the sequences of length m of the interpreting theory. A m-dimensional interpretation is *m-direct* if it is unrelativized and preserves identity. Note that all these properties are preserved by composition. We use *identity preserving*, *unrelativized* and *direct*, for: 1-identity preserving, 1-unrelativized and 1-direct.

Each interpretation $K : U \to V$ gives us an inner model construction that builds a model $\widetilde{K}(\mathcal{M})$ of U out of a model $\mathcal{M}$ of V. Note that $\widetilde{(\cdot)}$ behaves contravariantly.

2.3. **Sameness of Interpretations.** If we want to use interpretations to analyze e.g. sameness of theories, we need to be able to say when two interpretations are 'equal'. Strict identity of interpretations is too fine grained: it is too much dependent of arbitrary choices like which bound variables to use. We specify a first notion of equality between interpretations: two interpretations are *equal* when the *target theory thinks they are*. Modulo this identification, the operations identity and composition give rise to a category INT_0, where the theories are objects and the interpretations arrows.[1]

[1]For many reasons, the choice for the reverse direction of the arrows would be more natural. However, our present choice coheres with the extensive tradition in degrees of interpretability. So, we opted to adhere to the present choice.

For each sufficiently good notion of sameness of interpretations there is an associated category of theories and interpretations: the category of interpretations modulo that notion of sameness. Any such a category gives us a notion of isomorphism of theories which can function as a notion of sameness.

We present a list of notions of sameness. For all notions, it is easily seen that sameness is preserved by composition. (There are many more more notions, but they are irrelevant for this paper.)

2.3.1. *Equality.* The interpretations $K, K' : U \to V$ are equal when V 'thinks' K and K' are identical. By the Completeness Theorem, this is equivalent to saying that, for all V-models $\mathcal{M}$, $\widetilde{K}(\mathcal{M}) = \widetilde{K'}(\mathcal{M})$. This notion gives rise to the category INT_0. Isomorphism in INT_0 is *synonymy* or *definitional equivalence.*

2.3.2. *i-Isomorphism.* An i-isomorphism between interpretations $K, M : U \to V$ is given by a V-formula F. We demand that V verifies that "F is an isomorphism between K and M", or, equivalently, that, for each model $\mathcal{M}$ of V, the function $F^{\mathcal{M}}$ is an isomorphism between $\widetilde{K}(\mathcal{M})$ and $\widetilde{M}(\mathcal{M})$. Two interpretations $K, K' : U \to V$, are *i-isomorphic* if there is an i-isomorphism between K and K'. Wilfrid Hodges calls this notion: *homotopy.* See [Hod93], p222.

The notion of i-isomorphism can be characterized in a third way. We need only demand that, for every V-model $\mathcal{M}$, there is a V-formula F, which defines in $\mathcal{M}$ an isomorphism between $\widetilde{K}(\mathcal{M})$ and $\widetilde{K'}(\mathcal{M})$. By a simple compactness argument one may show that, for finite signatures, the third definition defines the same notion as the first two.

Clearly, if K, K' are equal, they will be i-isomorphic. The notion of i-isomorphism give rise to a category of interpretations modulo i-isomorphism. We call this category INT_1. Isomorphism in INT_1 is *bi-interpretability.*

2.3.3. *Isomorphism.* Our third notion of sameness of the basic list is that K and K' are the same if, for all models $\mathcal{M}$ of V, the internal models $\widetilde{K}(\mathcal{M})$ and $\widetilde{K'}(\mathcal{M})$ are isomorphic. We will simply say that K and K' are isomorphic. Clearly, i-isomorphism implies isomorphism. We call the associated category INT_2. Isomorphism in INT_2 is *iso-congruence.*

2.3.4. *Elementary Equivalence.* The fourth notion is to say that two interpretations are the same if, for each $\mathcal{M}$, the internal models $\widetilde{K}(\mathcal{M})$ and $\widetilde{K'}(\mathcal{M})$ are elementary equivalent. We will say that K and K' are elementary equivalent. By the Completeness Theorem, we easily see that this notion can be alternatively defined by saying that K is the same as K' iff, for all U-sentences A, we have $V \vdash A^K \leftrightarrow A^{K'}$. It is easy to see that isomorphism implies elementary equivalence. We call the associated category INT_3. Isomorphism in INT_3 is *elementary congruence* or *sentential congruence.*

2.3.5. *Identity of Source and Target.* Finally, we have the option of abstracting away from the specific identity of interpretations completely, simply counting any two interpretations $K, K' : U \to V$ the same. The associated category INT_4 is simply the preorder of the degrees of interpretability. Isomorphism in this preorder is *mutual interpretability.*

2.4. **The Many-sorted Case.** Interpretability can be extended to interpretability between many-sorted theories. The *profile* of an interpretation is a mapping from the sorts of the interpreted theory to sequences of the sorts of the interpreting theory. We write $(\mathfrak{a}_0 : \vec{\mathfrak{b}}_0) \ldots (\mathfrak{a}_{\ell-1} : \vec{\mathfrak{b}}_{\ell-1})$ to present a profile. (The order of the sorts is chosen arbitrarily.) We abbreviate a sequence of n $\mathfrak{a}$'s to $\mathfrak{a}^n$. The default sort of one-sorted theories is $\mathfrak{o}$, the sort of objects.

For each sort $\mathfrak{a}$ of the interpreted theory, we specify a domain $\delta^{\mathfrak{a}}(x_0^{\mathfrak{b}_0}, \ldots x_{k-1}^{\mathfrak{b}_{k-1}})$, where $\vec{\mathfrak{b}}$ is the sequence associated to $\mathfrak{a}$ by the profile and where $\vec{\mathfrak{b}}$ has length k. We write $\vec{x}^{\vec{\mathfrak{b}}}$ for $x_0^{\mathfrak{b}_0}, \ldots x_{k-1}^{\mathfrak{b}_{k-1}}$.

With each predicate P of the interpreted theory of type $\mathfrak{a}_0, \ldots \mathfrak{a}_{n-1}$, we associate a formula $A(\vec{x}_0^{\vec{\mathfrak{b}}_0}, \ldots, x_{n-1}^{\vec{\mathfrak{b}}_{n-1}})$, where $(\mathfrak{a}_i : \vec{\mathfrak{b}}_i)$ is in the profile. We demand that the target theory verifies, for $i < n$, the formula $A(\vec{x}_0^{\vec{\mathfrak{b}}_0}, \ldots, \vec{x}_{n-1}^{\vec{\mathfrak{b}}_{n-1}}) \to \delta^{\mathfrak{a}_i}(\vec{x}_i^{\vec{\mathfrak{b}}_i})$.

Our translation commutes with the propositional connectives. The translation of a quantifier $\forall x^{\mathfrak{a}}$ is $\forall \vec{x}^{\vec{\mathfrak{b}}}\, (\delta^{\mathfrak{a}}(\vec{x}^{\vec{\mathfrak{b}}}) \to \ldots)$. Similarly, for the existential quantifier.

An interpretation is $(\mathfrak{a} : \vec{\mathfrak{b}})$-direct if $(\mathfrak{a} : \vec{\mathfrak{b}})$ is in the profile and if identity translates to pointwise identity and if the translated domain consists of all sequences of sort $\vec{\mathfrak{b}}$.

2.5. **Parameters.** We can extend our notion of interpretation to *interpretation with parameters* as follows. Say our interpretation is $K : U \to V$. In the target theory, we have a parameter domain $\alpha(\vec{z})$, which is provably non-empty. The definition of interpretation remains the same but for the fact that the parameters $\vec{z}$ may occur in the domain formula and in the translations of the predicate symbols. We have to take appropriate measures to avoid variable clashes. It is clear, but somewhat laborious to work out, that this can be done in a systematic way. Our condition for K to be an interpretation becomes:

$$U \vdash A \;\;\Rightarrow\;\; V \vdash \forall \vec{z}\, (\alpha(\vec{z}) \to A^{K,\vec{z}}).$$

We note that an interpretation $K : U \to V$ with parameters provides a parametrized set of inner models of U inside a model of V.

3. Sequentiality

In this section we provide the definition of sequentiality for one-sorted theories (Subsection 3.1). We present the state of the art of the subject in Subsection 3.2. Finally, in Subsection 3.3, we give reasons why the present notion is not general enough.

3.1. **Definitions of Sequentiality.** We start with the primary definition in terms of a theory of sequences seq. This definition is in direct accordance with the intuitive picture we provided for sequential theories.

3.1.1. *Sequences.* We define the theory seq as follows. The language of seq has three sorts $\mathfrak{o}$, $\mathfrak{n}$ and $\mathfrak{s}$. We have a unary predicate symbol Z of type $\mathfrak{n}$, a unary predicate symbol E (for: *empty sequence*) of type $\mathfrak{s}$, two binary predicate symbols S and $<$ of type $\mathfrak{nn}$, a binary predicate symbol L of type $\mathfrak{sn}$, a binary predicate symbol Pu (for: *push*) of type $\mathfrak{sos}$, and a ternary predicate symbol Pr (for: *projection*) of type $\mathfrak{sno}$. The variables $x, x_0, x', y, z, u \ldots$ will be of sort $\mathfrak{o}$, the variables $n, n_0, n', m, k, \ldots$ will

will be of sort $\mathfrak{n}$, and the variables $s, s_0, s', t, \ldots$ will be of sort $\mathfrak{s}$. As usual we write $m \leq n$ for: $m = n \vee m < n$. The axioms are:

seq1 $\vdash (n < m \wedge m < k) \to n < k$
seq2 $\vdash \neg\, n < n$
seq3 $\vdash n \leq m \vee m < n$
seq4 $\vdash \exists n\, \mathsf{Z}(n)$
seq5 $\vdash \mathsf{Z}(n) \to n \leq m$
seq6 $\vdash \mathsf{S}(n,m) \leftrightarrow (n < m \wedge \forall k\, (n < k \to m \leq k))$
seq7 $\vdash \exists m\, \mathsf{S}(n,m)$
seq8 $\vdash \mathsf{Z}(n) \vee \exists m\, \mathsf{S}(m,n)$
seq9 $\vdash \exists n\, \mathsf{L}(s,n)$
seq10 $\vdash (\mathsf{L}(s,n) \wedge \mathsf{L}(s,m)) \to n = m$
seq11 $\vdash (\mathsf{L}(s,n) \wedge m < n) \to \exists x\, \mathsf{Pr}(s,m,x)$
seq12 $\vdash (\mathsf{L}(s,n) \wedge \mathsf{Pr}(s,m,x)) \to m < n$
seq13 $\vdash (\mathsf{Pr}(s,n,x) \wedge \mathsf{Pr}(s,n,y)) \to x = y$
seq14 $\vdash \exists s\, \mathsf{E}(s)$
seq15 $\vdash (\mathsf{E}(s) \wedge \mathsf{L}(s,n)) \to \mathsf{Z}(n)$
seq16 $\vdash \exists t\, \mathsf{Pu}(s,x,t)$
seq17 $\vdash (\mathsf{Pu}(s,x,t) \wedge \mathsf{L}(s,n) \wedge \mathsf{S}(n,m)) \to (\mathsf{L}(t,m) \wedge \mathsf{Pr}(t,n,x))$
seq18 $\vdash (\mathsf{Pu}(s,x,t) \wedge \mathsf{Pr}(s,m,y)) \to \mathsf{Pr}(t,m,y)$

We note that we may add extensionality for sequences by reinterpreting identity as extensional equivalence. There are many variants on seq that also would have served our purposes. Specifically, note that one can reduce the signature to just $<$ and Pr in the obvious way. Moreover, we could have avoided the sort $\mathfrak{n}$ by working with a linear preorder on sequences for 'is a shorter sequence than'.

Definition 3.1. We call a one-sorted theory U *sequential* iff there is a $(\mathfrak{o} : \mathfrak{o})$-direct interpretation of seq in U, that is one-dimensional for the three sorts of seq. ❑

Trivially, a one-sorted theory U is sequential iff there is a $(\mathfrak{o} : \mathfrak{o})$-direct interpretation of seq in U, that is one-dimensional for sorts $\mathfrak{o}$ and $\mathfrak{s}$, since we can compress the dimension of interpretation of the numbers using the sequences.

Remark 3.2. We note that the *random access* to the elements of sequences provided by the projection function is essential. Suppose we modify seq to a theory of stacks, say stack, where we omit the sort of numbers and replace the projection function by a function Top that reads the top element of the stack. Let's say that a one-sorted U is a *stack theory* iff there is a $(\mathfrak{o}, \mathfrak{o})$-direct interpretation of stack in U, that is one-dimensional for the two sorts of stack. One can easily show that there are decidable stack theories[2], where, in contrast, sequential theories are essentially undecidable. ❑

3.1.2. *Sets and Classes.* We define the theories *adjunctive class theory* ac and *adjunctive set theory* AS.[3] The theory ac has two sorts $\mathfrak{o}$, the sort of objects, and $\mathfrak{c}$, the sort of classes. It has a binary predicate $\in$ of type $\mathfrak{o}\mathfrak{c}$. We use $x, y, z, \ldots$ as

[2]It is well known that there are decidable theories of non-surjective pairing. Any theory of non-surjective pairing is a stack theory.

[3]The use of capitals in the name of AS registers the fact that AS is a theory with 'iterated' elementhood.

variables for objects and $X, Y, Z, \ldots$ as variables for classes. We have the following axioms.

ac1 $\vdash \exists X \, \forall x \, x \notin X$,
ac2 $\vdash \forall X, x \, \exists Y \, \forall y \, (y \in Y \leftrightarrow (y \in X \vee y = x))$.

We note that we get extensionality for free in ac by replacing identity for classes by extensional identity. The theory ac allows finite models.

The theory AS is a one sorted theory with a binary relation $\in$.

AS1 $\vdash \exists x \, \forall y \, y \notin x$,
AS2 $\vdash \forall x, y \, \exists z \, \forall u \, (u \in z \leftrightarrow (u \in x \vee u = y))$.

The theory AS is much stronger than ac. E.g., it interprets Robinson's Arithmetic Q. Hence, it is essentially undecidable. Note that both ac and AS have 'random access' to the objects contained in the classes/sets.

We have:

Theorem 3.3. *Let U be a one-sorted theory. The following are equivalent.*

i. U is sequential.
ii. There is an $(\mathfrak{o} : \mathfrak{o})$-direct interpretation of ac *in U that is one-dimensional for the sorts $\mathfrak{o}$ and $\mathfrak{c}$.*
iii. There is a direct interpretation of AS *in U.*
iv. There is an extension V of AS *in the same language, such that U is synonymous with V.*

Proof. The equivalence of (ii) and (iii) is easy and so is the implication from (i) to (iii). To prove the implication from (iii) to (i) one shows that seq has an $(\mathfrak{o} : \mathfrak{o})$-direct interpretation in AS that is one-dimensional for the sorts $\mathfrak{s}$ and $\mathfrak{n}$, using an elaborate bootstrap. We refer the reader to the literature. See, e.g., [Pud83], [MPS90], [HP91]. See also Subsection 3.2 for comments on ingredients of the proof. The equivalence of (iii) and (iv) is proved in [Vis08]. ❑

We note that AS is itself sequential. Moreover, sequentiality is preserved by synonymy. Of course, our definition does not work for many-sorted theories, but if we extend it in an appropriate way, it will follow that ac is not sequential, since it has finite models.

Remark 3.4. An alternative to AS was provided by Harvey Friedman. Here is Friedman's theory that we will call *reduced adjunctive subtractive set theory* ASS^-. The axioms are as follows.

ASS^-1 $\vdash \forall x, y \, \exists z \, \forall u \, (u \in z \leftrightarrow (u \in x \vee u = y))$.
ASS^-2 $\vdash \forall x, y \, \exists z \, \forall u \, (u \in z \leftrightarrow (u \in x \wedge u \neq y))$.

We can directly interpret ASS^- in AS using an argument in the style of the proof of Lemma 4.1. We can interpret AS in ASS^- by defining

$$x \in_z^* y :\leftrightarrow (x \in y \wedge x \notin z) \vee (x \notin y \wedge x \in z).$$

An $\in_z^*$-empty set is given by z. We define $\in_z^*$-adjunction of y as $\in$-adjunction of y in case $y \notin z$, and as $\in$-subtraction of y in case $y \in z$. ❑

Open Question 3.5. Is ASS^- essentially parametrically sequential? In other words: can we find a direct interpretation of AS without parameter? My conjecture is: *no.* ❑

3.2. **State of the Art concerning Sequentiality.** The notion of sequential theory was introduced by Pavel Pudlák in his paper [Pud83]. Pudlák uses his notion for the study of the degrees of local multi-dimensional parametric interpretability. He proves that sequential theories are prime in this degree structure. In [Pud85], sequential theories provide the right level of generality for theorems about consistency statements. Let n-provability stand for provability from axioms with code below n and with a proof only involving formulas of complexity at most n, where the complexity measure is, e.g., *depth of quantifier alternations.* We write $\mathsf{con}_n(U)$ for an appropriate arithmetization of the consistency of U w.r.t. n-consistency. For any sequential RE theory U, and any n, we can find a definable number system N such that $U \vdash \mathsf{con}_n^N(U)$. This is a formalization of the soundness theorem. On the other hand, if U is finitely axiomatized, for every definable number system N, we can find an n, such that $U \nvdash \mathsf{con}_n^N(U)$. This is a version of the second incompleteness theorem.

The notion of sequential theory was independently invented by Friedman who called it *adequate theory.* See Smoryński's survey [Smo85]. An important difference is that in the definition, as given by Smoryński, Elementary Arithmetic EA (aka EFA or $I\Delta_0 + \mathsf{exp}$) is stipulated to be interpretable in adequate theories. This demand is too strong. Friedman uses the notion to provide the Friedman characterization of interpretability among finitely axiomatized sequential theories. (See also [Vis90] and [Vis92].) Moreover, he shows that ordinary interpretability and faithful interpretability among finitely axiomatized sequential theories coincide. (See also [Vis93] and [Vis05].)

Examples of sequential theories are Adjunctive Set Theory AS, Buss' theory S^1_2 (see: [Bus86]) and synonymous variants of it like a theory of strings due to Ferreira (see: [Fer88], [Fer90], [FO06]) and a theory of sets and numbers due to Zambella (see: [Zam96]), Wilkie and Paris' theory $I\Delta_0 + \Omega_1$ (see: [WP87]), PRA, Elementary Arithmetic EA (aka Elementary Function Arithmetic EFA, or $I\Delta_0 + \mathsf{exp}$), $I\Sigma^0_1$, Peano Arithmetic PA, ACA_0, ZF, GB.

Here are some memorable results concerning sequential theories.

I. Sequential theories are essentially undecidable (since they interpret Q).
II. Pudlák in his [Pud85] proves a variant of Dedekind's theorem that all models of second order arithmetic are isomorphic. In Pudlák's variant any pair of number systems definable in a sequential theory has a pair of definably isomorphic definable cuts.
III. Sequential theories U have a normal form $\mho(U)$ modulo mutual local interpretability. Here $\mho(U) = \mathsf{S}^1_2 + \{\mathsf{con}_n(U) \mid n \in \omega\}$. We have $U \equiv_{\mathsf{loc}} V$ iff $\mho(U) = \mho(V)$. An alternative normal form is the theory $\Pi(U) := \mathsf{S}^1_2 + \{P \in \Pi^0_1 \mid U \rhd (\mathsf{S}^1_2 + P)\}$. See e.g., [Vis09a]. The normal form theorem is related to both the Orey-Hájek characterization and the Friedman characterization.
IV. For any finitely axiomatized sequential theory U, there is an interpretation of S^1_2 in U that is Σ^0_1-sound. See e.g. [Smo85], [Kra87], [Vis93], [Vis05].
V. There is a sequential theory U with p-time decidable axiom set such that the predicate logic of U is complete Π^0_2. The theory was first given in [Kra87]. The result is proved in [Vis05].

VI. Let's say that U model-interprets V if all models of U contain an internal model of V. It is easily seen that model-interpretability is between interpretability and local interpretability. One can show that there are sequential theories U and V such that U model-interprets V, but U does not interpret V, and that there are sequential theories T and W, such that U locally interprets V, but U does not model-interpret V.

Open Question 3.6. It is unknown whether there is anything like a normal form modulo mutual global interpretability. An even more modest question is the following: is every sequential theory U mutually interpretable with an extension of S^1_2 in the same language? ❑

We have provided several characterizations in Subsection 3.1. An important part of the story of the characterization involving AS is the interpretability of Robinson's Arithmetic Q in AS. Here are some historical notes about that characterization.

(1) In [ST52], Wanda Szmielew and Alfred Tarski announce the interpretability of Q in AS plus extensionality. See also [TMR53], p34.
(2) A proof of the Szmielew-Tarski result is given by George Collins and Joseph Halpern in [CH70].
(3) Franco Montagna and Antonella Mancini, in [MM94], give an improvement of the Szmielew-Tarski result. They prove that Q can be interpreted in an extension of AS in which we stipulate the functionality of empty set and adjoining of singletons.
(4) In appendix III of [MPS90], Jan Mycielski, Pavel Pudlák and Alan Stern provide the ingredients of the interpretation of Q in AS.
(5) A new proof of the interpretability of Q in AS is given in [Vis09b].

A very nice presentation of the converse interpretability of (an extension of) AS plus extensionality in Q, is given in [Nel86]. This is an interpretation with absolute identity.

For further work concerning sequential theories, see, e.g., [Pud85], [Smo85], [MPS90], [HP91], [Vis93], [Vis98], [JV00], [Vis05], [Vis09c], [Vis09d], [Vis09a].

3.3. **Why the Notion of Sequentiality is not Quite Right.** The most direct argument that the notion of sequentiality is not quite right is to notice that there is an unmotivated and unnecessary component in the definition. Why do we demand that the interpretation of seq is one-dimensional for the sorts $\mathfrak{s}$ and $\mathfrak{o}$? The fact that the interpretation is $(\mathfrak{o} : \mathfrak{o})$-direct, and, hence, one-dimensional for $\mathfrak{o}$, follows from our intuitive motivation: we want our sequences to be sequences of *all* objects of the original domain of the interpreting theory. However, there is no reason to put any constraint on our representation of sequences. We are just interested in the accessibility of the elements contained in the sequences, but not in the way the sequences are represented as long as it fulfills its desired function.

A second argument is that sequentiality fails to be preserved under the relation of bi-interpretability. The notion of *bi-interpretability* is a very good notion of sameness of theories. It preserves such diverse properties of theories as finite axiomatizability and κ-categoricity. In the parameter-free case it preserves automorphism groups modulo isomorphism. There are good reasons that sequentiality should be closed under bi-interpretability. If U is bi-interpretable with a sequential theory V, we have the following situation. For any model $\mathcal{M}$ of U, we have an internal model $\mathcal{N}$ of V, and *inside* this model we have again an internal model $\mathcal{M}'$ of U, definably isomorphic, say via F, with the original $\mathcal{M}$. Now we have sequences for $\mathcal{M}'$ provided by $\mathcal{N}$. These sequences are inherited by $\mathcal{M}$, in a definable way via F. So, $\mathcal{M}$ has sequences. Inspection of this argument shows that it fails as a proof precisely because of multi-dimensionality: the definition of the sequences in $\mathcal{M}$ may be more-dimensional, where we demanded it to be one-dimensional. Thus, the arbitrary demand of one-dimensionality blocks a very good closure condition of our theories.

In Section 6, we will show in detail that sequentiality is not closed under bi-interpretability. We will show in Subsection 4.3 that every m-sequential theory is bi-interpretable with a sequential theory. We claim that *m-sequentiality* is the right notion. So, by combining the theorem and the claim, we only need to close off the sequential theories under bi-interpretability to arrive at the right notion.

Remark 3.7. It is easy to see that sequentiality is closed under synonymy.

Sequential theories are known not to be closed under mutual interpretability, however this does not constitute an argument against the definition. E.g., consider the disjoint union $U \boxplus U$ of a sequential theory U with itself. This disjoint union is mutually interpretable with U but it is not sequential.[4] Yet, $U \boxplus U$ proves, on different interpretations of number theory, its own restricted consistency statements. Shouldn't we also call $U \boxplus U$ sequential? Well, in the light of our original motivation the answer should be *no*. The fact that we do have restricted consistency statements is inherited from U, which does have the means to verify these statements. The theory $U \boxplus U$ itself does not have partial satisfaction for all objects in the domain of the theory.

A second example of a non-sequential theory that is mutually interpretable with a sequential one, is as follows. Robinson's arithmetic Q is mutually interpretable with AS but not itself sequential. See [Vis08].

It seems to me that even for *iso-congruence* there is no strong reason that sequential theories should be closed under that equivalence relation. If we trace the argument above concerning the closure of sequentiality under bi-interpretability, we see that in a model $\mathcal{M}$ of U, we have sequences of elements of the isomorphic internal model $\mathcal{M}'$. However, we cannot make it visible in $\mathcal{M}$ that these sequences can also function as sequences of elements of $\mathcal{M}$. Moreover, where, by compactness, we were guaranteed uniformity of the isomorphisms, in the case of bi-interpretability, conceivably,for iso-congruence, the isomorphisms could be wildly different if we vary our models of U. (See also Question 4.11.) ❑

4. What is m-Sequentiality?

Our improved version of sequentiality is *m-sequentiality.* In this section we give the official definition of m-sequentiality and prove some equivalents. We define *m-sequentiality* as follows.

Definition 4.1. A one-sorted theory U is *m-sequential* iff there is an $(\mathfrak{o}, \mathfrak{o})$-direct interpretation of seq in U. ❑

[4] This follows e.g. from the results of [Pud83] or [MPS90].

4.1. **Some useful Results.** As a preparation of our characterizations in the subsequent subsections, we prove some useful results about seq. We distinguish between *internal sequences*, which are objects of type $\mathfrak{s}$, and *external sequences*, which exist only because we can externally form sequences of terms (as in the definition of m-dimensional interpretations).

Theorem 4.2. *Consider any number m. Then, the theory seq interprets ac by an $(\mathfrak{o} : \mathfrak{o}^m)(\mathfrak{c} : \mathfrak{s}^2)$-direct interpretation. In other words, there is a a seq-definable predicate $(z_0, \dots, z_{m-1}) \in (s_0, s_1)$, where the z_i are of type $\mathfrak{o}$ and the s_j are of type $\mathfrak{s}$, such that:*

- $\mathsf{seq} \vdash \exists \vec{s}\, \forall \vec{z}\ \ \vec{z} \notin \vec{s}$
- $\mathsf{seq} \vdash \forall \vec{z}, \vec{s}\, \exists \vec{t}\, \forall \vec{u}\, ((\vec{u} \in \vec{t} \leftrightarrow (\vec{u} \in \vec{s} \vee \bigwedge_{i<m} u_i = z_i))$

Here, of course, all (external) sequences are supposed to be of the right length.

Proof. We work in seq. It is easy to see that there are at least two objects. For arbitrary x and y with $x \neq y$, we define an *m, x, y-sequence* as follows. It is a sequence s such that

i. $\forall n, u\, (\mathsf{Pr}(s, n, u) \rightarrow (u = x \vee u = y))$
ii. $\forall n\, (\mathsf{Pr}(s, n, x) \rightarrow \bigvee_{i<m}((\mathsf{Z}(n-i) \vee \mathsf{Pr}(s, n-i-1, y)) \wedge$
$(\mathsf{L}(s, n-i+m) \vee \mathsf{Pr}(s, n-i+m, y)) \wedge \bigwedge_{j<m} \mathsf{Pr}(s, n-i+j, x))$
iii. $\forall n\, (\mathsf{Pr}(s, n, y) \rightarrow \bigvee_{i<m}((\mathsf{Z}(n-i) \vee \mathsf{Pr}(s, n-i-1, x)) \wedge$
$(\mathsf{L}(s, n-i+m) \vee \mathsf{Pr}(s, n-i+m, x)) \wedge \bigwedge_{j<m} \mathsf{Pr}(s, n-i+j, y))$

Here, e.g., $\mathsf{Z}(n-i)$ abbreviates $\exists k_0, \dots k_{i-1}\, (\mathsf{Z}(k_0) \wedge \mathsf{S}(k_0, k_1) \wedge \dots \wedge \mathsf{S}(k_{i-1}, n))$.

We define $(z_0, \dots, z_{m-1}) \in (s_0, s_1)$ by: for some x, y, we have $x \neq y$ and s_0 is an x, y, m-sequence and s_0 and s_1 have the same length and, for some n, $(\bigwedge_{i<m} \mathsf{Pr}(s_0, n+i, x)$ or $\bigwedge_{i<m} \mathsf{Pr}(s_0, n+i, y))$ and $\bigwedge_{i<m} \mathsf{Pr}(s_1, n+i, z_i)$.

If we take s_0 and s_1 both the empty sequence, we see that (s_0, s_1) satisfies the empty class axiom. We fulfill the adjunction axiom by pushing m times x or m times y on top of s_0, where the choice of x and y depends on the last element of s_0, if any, and where we push subsequently $z_0, z_1, \dots, z_{m-1}$ om s_1. ❑

We provide a strengthening of Theorem 4.2 that will be only used in an alternative proof of Theorem 5.2. We start with a well-known lemma. Let ac^+ be ac plus the following axioms.

$\mathsf{ac3} \vdash \forall X, Y\, \exists Z\, \forall z\, (z \in Z \leftrightarrow (z \in X \wedge z \in Y))$
$\mathsf{ac4} \vdash \forall X, x\, \exists Y\, \forall y\, (y \in Y \leftrightarrow (y \in X \wedge y \neq x)$

Lemma 4.1. *There is an $\mathfrak{o} : \mathfrak{o}$-direct interpretations, of ac^+ in ac, with profile $(\mathfrak{o} : \mathfrak{o})(\mathfrak{c} : \mathfrak{c})$, that is one-dimensional and identity preserving for $\mathfrak{c}$.*

Proof. We work in ac. To simplify the presentation we replace identity on classes by extensional identity. Then we can justify the use of functional notations $\cap$ for intersection and $\setminus$ for subtraction. We define the virtual class $\mathcal{X}_0$ as consisting of all those classes X such that, for all Y, $X \cap Y$ exists. We show that $\mathcal{X}_0$ is closed under empty class, adjunction, and intersection.

Clearly the empty class is in $\mathcal{X}_0$.

We have: $(X \cup \{x\}) \cap Z = (X \cap Z) \cup \{x\}$, if $x \in Z$, and $(X \cup \{x\}) \cap Z = X \cap Z$ if $x \notin Z$. In both cases $(X \cup \{x\}) \cap Z$ exists. Ergo, $X \cup \{x\}$ is in $\mathcal{X}_0$.

Suppose X and Y in $\mathcal{X}_0$. We note that $X \cap Y$ exists, since $X \in \mathcal{X}_0$. Consider any Z. We have $(X \cap Y) \cap Z = X \cap (Y \cap Z)$. Since $Y \in \mathcal{X}_0$, $Y \cap Z$ exists and since $X \in \mathcal{X}_0$, $X \cap (Y \cap Z)$ exists. Hence $X \cap Y$ is in $\mathcal{X}_0$.

By relativizing the classes to $\mathcal{X}_0$ we get an interpretation of ac+ac3. We now work in ac+ac3. As before, we add extensionality. Define $\mathcal{X}_1$ as the class of all X such that $X \setminus \{x\}$ exists, for all x. We show that $\mathcal{X}_1$ is closed under empty class, adjunction, intersection and subtraction.

It is clear that the empty class is in $\mathcal{X}_1$.

Suppose X is in $\mathcal{X}_1$. Consider x. We have $(X \cup \{x\}) \setminus \{y\} = (X \setminus \{y\}) \cup \{x\}$, if $x \neq y$, and $(X \cup \{x\}) \setminus \{y\} = X$, if $x = y$. In both cases $(X \cup \{x\}) \setminus \{y\}$ exists. So $X \cup \{x\} \in \mathcal{X}_1$.

Suppose X is in $\mathcal{X}_1$. Consider any Y in $\mathcal{X}_1$. We have $(X \cap Y) \setminus \{z\} = (X \setminus \{z\}) \cap Y$. So $X \cap Y$ is in $\mathcal{X}_1$.

Suppose X is in $\mathcal{X}_1$. Consider any x. We have

$$(\dagger) \quad (X \setminus \{x\}) \setminus \{y\} = (X \setminus \{x\}) \cap (X \setminus \{y\}).$$

The right hand side of (†) clearly exists. Hence, $X \setminus \{x\} \in \mathcal{X}_1$.

We relativize to $\mathcal{X}_1$ in ac+ac3. This gives us the desired interpretation. ❑

Theorem 4.3. *Consider any number m. Then, the theory* seq *interprets* ac^+ *by an interpretation of profile* $(\mathfrak{o} : \mathfrak{o}^m)(\mathfrak{c} : \mathfrak{s}^2)$*, that is* $(\mathfrak{o} : \mathfrak{o}^m)$*-direct and* $(\mathfrak{c} : \mathfrak{s}^2)$*-identity preserving. In other words, there are a* seq*-definable predicates* $D(s_0, s_1)$ *and* $(z_0, \ldots, z_{m-1}) \in (s_0, s_1)$*, where the* z_i *are of type* $\mathfrak{o}$ *and the* s_j *are of type* $\mathfrak{s}$*, such that:*

- $\mathsf{seq} \vdash \vec{x} \in \vec{s} \to D(\vec{s})$
- $\mathsf{seq} \vdash \exists \vec{s}\, (D(\vec{s}) \wedge \forall \vec{z}\ \ \vec{z} \notin \vec{s})$
- $\mathsf{seq} \vdash \forall \vec{z}, \vec{s}\, (D(\vec{s}) \to \exists \vec{t}\, (D(\vec{t}) \wedge \forall \vec{u}\, (\vec{u} \in \vec{t} \leftrightarrow (\vec{u} \in \vec{s} \vee \bigwedge_{i<m} u_i = z_i))))$
- $\mathsf{seq} \vdash \forall \vec{s}, \vec{s}'\, ((D(\vec{s}) \wedge D(\vec{s}')) \to \exists \vec{t}\, (D(\vec{t}) \wedge \forall \vec{z}\, (\vec{z} \in \vec{t} \leftrightarrow (\vec{z} \in \vec{s} \wedge \vec{z} \in \vec{s}'))))$
- $\mathsf{seq} \vdash \forall \vec{z}, \vec{s}\, (D(\vec{s}) \to \exists \vec{t}\, (D(\vec{t}) \wedge \forall \vec{u}\, ((\vec{u} \in \vec{t} \leftrightarrow (\vec{u} \in \vec{s} \wedge \bigvee_{i<m} u_i \neq z_i))))$

Here all (external) sequences are supposed to be of the right length.

Proof. We compose the interpretations of Theorem 4.2 and Lemma 4.1. ❑

4.2. **m-Direct and Relaxed Direct Interpretations.** Consider one-sorted theories U and V. An interpretation $M : U \to V$ is *relaxed direct* iff there is a V-definable relation R, such that:

- $V \vdash x\, R\, \vec{y} \to \delta_M(\vec{y})$,
- $V \vdash \forall x\, \exists \vec{y}\ \ x\, R\, \vec{y}$,
- $V \vdash (x\, R\, \vec{y} \wedge u\, R\, \vec{v} \wedge \vec{y} =_K \vec{v}) \to x = u$.

If the interpretation contains parameters with domain α, the definition of R will also contain parameters. E.g., the first item becomes:

$$V \vdash (\alpha(\vec{z}) \wedge x\, R_{\vec{z}}\, \vec{y}) \to \delta_{M,\vec{z}}(\vec{y}),$$

etcetera. We have the following theorem.

Theorem 4.4. *Any m-direct interpretation is relaxed direct.*

Proof. Suppose $M : U \to V$ is m-direct. Let $x\, R\, \vec{y} :\leftrightarrow \bigwedge_{i<m} y_i = x$. Clearly, R witnesses that M is relaxed direct. ❑

Here is our first characterization of m-sequentiality.

Theorem 4.5. *The following are equivalent:* (i) *U is m-sequential;* (ii) *for some m, there is an m-direct interpretation of* AS *in U,* (iii) *there is a relaxed direct interpretation of* AS *in U.*

Proof. We give the proof for the case without parameters. The case with parameters is an easy adaptation.

(i) $\Rightarrow$ (ii). Suppose $M : \mathsf{seq} \to U$ is $(\mathfrak{o} : \mathfrak{o})$-direct. Suppose the interpretation of type $\mathfrak{s}$ is k-dimensional. Put $m := 2k$. We define:

$$(z_1, \ldots, z_m) \in (x_0, \ldots, x_{k-1}, y_0, \ldots, y_{k-1}) :\leftrightarrow \delta^{\mathfrak{s}}_M(x_0, \ldots, x_{k-1}) \wedge \delta^{\mathfrak{s}}_M(y_0, \ldots, y_{k-1}) \wedge ((z_1, \ldots, z_m) \in (x_0, \ldots, x_{k-1}, y_0, \ldots, y_{k-1}))^M$$

where Theorem 4.2 provides the relation $\in$ inside M. (By coding the pair of internal sequences as one internal sequence, we can get the more efficient $m := k$.)

(ii) $\Rightarrow$ (iii). This is immediate from Theorem 4.4.

(iii) $\Rightarrow$ (i). Suppose $M : \mathsf{AS} \to U$ is relaxed direct. Let R be the witnessing relation. As we mentioned in the proof of Theorem 3.3, we can $(\mathfrak{o} : \mathfrak{o})$-directly interpret seq in AS, say via K. See, e.g., [Pud83], [MPS90], [HP91]. Suppose $L := M \circ K : \mathsf{seq} \to U$. By the $(\mathfrak{o} : \mathfrak{o})$-directness of K, we find that $\delta^{\mathfrak{o}}_L$ is, U-provably, equal to δ_M, and $=^{\mathfrak{o}}_L$ is, U-provably, the same as $=_M$. We build the desired $(\mathfrak{o} : \mathfrak{o})$-direct interpretation, say, $Q : \mathsf{seq} \to U$, by defining:

- $\delta^{\mathfrak{o}}_Q(x) :\leftrightarrow x = x$,
- $\delta^{\mathfrak{s}}_Q(\vec{y}) :\leftrightarrow \delta^{\mathfrak{s}}_L(\vec{y})$,
- $\delta^{\mathfrak{n}}_Q(\vec{z}) :\leftrightarrow \delta^{\mathfrak{n}}_L(\vec{z})$,
- $=^{\mathfrak{s}}$, $=^{\mathfrak{n}}$, Z, E, S, $<$ L, are interpreted according to Q as they are according to L,
- $x =^{\mathfrak{o}}_Q y :\leftrightarrow x = y$,
- $\mathsf{Pu}_Q(\vec{s}, x, \vec{t}) :\leftrightarrow \exists \vec{y}\,(x \mathrel{R} \vec{y} \wedge \mathsf{Pu}_L(\vec{s}, \vec{y}, \vec{t}))$,
- $\mathsf{Pr}_Q(\vec{s}, \vec{n}, x) :\leftrightarrow \exists \vec{y}\,(x \mathrel{R} \vec{y} \wedge \mathsf{Pr}_L(\vec{s}, \vec{n}, \vec{y}))$.

By the U-provable, totality and $(=, =^{\mathfrak{o}}_L)$-injectivity of R, we can easily verify that Q is indeed an interpretation of seq. ❑

It is clear that we should have the following theorem.

Theorem 4.6. *Suppose $K : \mathsf{AS} \to V$ is relaxed direct and one-dimensional. Then V is sequential.*

Proof. Suppose $K : \mathsf{AS} \to V$ is relaxed direct and one-dimensional. Suppose R witnesses that K is direct relaxed. We define a new direct interpretation M of AS in V by setting $x \in_M y :\leftrightarrow \exists z\,(x \mathrel{R} z \wedge z \in_K y)$. ❑

Remark 4.7. We call a theory U *conceptual* if there is an $(\mathfrak{o}, \mathfrak{o})$-direct interpretation of ac in U. It is easily seen that, if U is conceptual via a one-dimensional interpretation, then U is sequential. However, if the interpretation of the classes is allowed to be more-dimensional, then there are conceptual theories with finite models. So *conceptual $\neq$ m-sequential.* As we hope to illustrate elsewhere the notion of conceptuality is useful in its own right. ❑

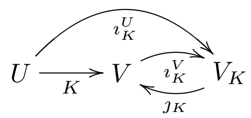
4.3. **bi-Interpretability.** Any interpretation between one-sorted theories can be split up into first an identity-preserving interpretation and then an 1-isomorphism.

Theorem 4.8. *Suppose $K : U \to V$, where U and V are 1-sorted. Suppose V proves that there are at least two objects. Then, there is a theory V_K and interpretations $\imath_K^U : U \to V_K$ and $\jmath_K : V_K \to V$, such that $\imath_K^U$ is identity-preserving, $\jmath_K$ is an 1-isomorphism (i.e., a bi-interpretation), and $K = \jmath_K \circ \imath_K^U$.*

As a general methodology, one should strive —to quote Kreisel— *to say simple things simply*. Regrettably, the proof of the present theorem is a case where I failed miserably to fulfill this dictum. It seems to me that, with the right eyes, one can immediately see that it is true, but when you try to write the proof down with any precision ... My solution to the problem presented by this elaborate proof of a triviality is as follows. I provide a detailed proof of the theorem in Appendix A. Here I give a sketchy proof highlighting the main ideas of the construction.

Proof. We ignore the possibility of parameters. The theory V_K is formed as the disjoint sum of the theories U and V extended with a new predicate F. The obvious embeddings of U and V in V_K are respectively $\imath_K^U$ and $\imath_K^V$. The theory V_K also contains axioms stating that F is an isomorphism between U as interpreted via $\imath_K^V \circ K$ and U as interpreted via $\imath_K^U$ in V_K. In other words, $F : \imath_K^V \circ K \to \imath_K^U$ witnesses the INT_1-identity between $\imath_K^V \circ K$ and $\imath_K^U$.

We want to construct an interpretation $\jmath_K : V \to V_K$, such that $\jmath_K$ and $\imath_K^V$ are inverses in the sense of INT_1. Thus, V and V_K will be bi-interpretable. Moreover, since $\imath_K^V \circ K =_1 \imath_K^U$, we have $K = \jmath_K \circ \imath_K^V \circ K =_1 \jmath_K \circ \imath_K^U$. Here the standard embedding $\imath_K^U$ is clearly one-dimensional and identity-preserving.



We turn to the construction of $\jmath_K$, i.e., of a uniform internal model of V_K in V. We want to reproduce the objects of V and the objects of δ_K as a disjoint sum. To do that more-dimensionality comes into play. The objects of δ_K are given as (external) sequences of length, say m. As a first move we also represent the objects of V as sequences of length m where we take the last component to represent the intended object. The rest is padding. We are given that V produces two distinct objects x and y, so one option is to make our domains disjoint by raising the dimension with 1 and represent the V-objects as $(x, \ldots, v)$ and the U-objects as $(y, u_0, \ldots, u_{m-1})$. The disadvantage of this construction is that we make use of parameters. This is an inessential use, however, and we can replace it by a construction where we do not use parameters. We raise the dimension with two and represent the V-objects by $(x, x, \ldots, v)$, where x is now arbitrary, and we represent the U-objects by $(x, y, u_0, \ldots, u_{m-1})$, where x and y are arbitrary and different.

Thus we produced disjoint copies of the objects of V and the objects of δ_U and the copies are given by sequences of the same length. It is now easy to extend this construction with define predicates to obtain an $m + 2$-dimensional interpretation $\jmath_K$ of V_K in V.

The verification that $\jmath_K$ is inverse to $\imath_K^U$, finally, is just a matter of careful tracing of obvious details. ❑

We use Theorem 4.8 to prove that any m-sequential theory is bi-interpretable with a sequential theory.

Theorem 4.9. *Suppose V is m-sequential. Then V is bi-interpretable with a sequential theory $V^\star$.*

Proof. Suppose $K : \mathsf{AS} \to V$ is a relaxed direct interpretation. Let R be the witness of relaxedness. We take $V^\star := V_K$ as provided by Theorem 4.8.

We start with an arbitrary object x from the domain of V_K, or, in other words, of id_{V_K}. The bi-interpretation between V and V_K is given by $\imath_K^V : V \to V_K$ and $\jmath_K : V_K \to V$. We have an isomorphism $G : \imath_K^V \circ \jmath_K \to \mathsf{id}_{V_K}$. We relate x via G^{-1} to a sequence $\vec{y}$ of $\imath_K^V \circ \jmath_K$. Suppose $\imath_K^V$ is k-dimensional and $\jmath_K$ is ℓ-dimensional. We can view $\vec{y}$ as a sequence $\vec{y}_0, \ldots, \vec{y}_{\ell-1}$ of $\imath_K^V$-objects $\vec{y}_i$ of length k. Each $\imath_K^V$-object $\vec{y}_i$ is related via $R^{\imath_K^V}$ to an $\imath_K^V \circ K$-object $\vec{u}_i$. So we can relate $\vec{y}_0, \ldots, \vec{y}_{\ell-1}$ via $R^{\imath_K^V}$ pointwise to $\vec{u}_0, \ldots, \vec{u}_{\ell-1}$. The sequence $\vec{u}_0, \ldots, \vec{u}_{\ell-1}$ is an external sequence in $\imath_K^V \circ K : \mathsf{AS} \to V_K$. Inside $\imath_K^V \circ K$ we can work in AS. Thus, we may relate the external sequence $\vec{u}_0, \ldots, \vec{u}_{\ell-1}$ to an internal one $\vec{v}$ in $\imath_K^V \circ K$. Let's call the internalizing relation $I^{\imath_K^V \circ K}$ (for the given length ℓ). We have $\vec{v} \in_{\imath_K^V \circ K} \vec{w}$ for some $\vec{w}$ in $\imath_K^V \circ K$. Finally let $H : \imath_K^V \circ K \to \imath_K^U$ be the isomorphism between $\imath_K^V \circ K$ and $\imath_K^U$. We can relate $\vec{w}$ to z via H.

Thus, we can define:

$$\begin{aligned} x \in^\star z \quad :\leftrightarrow \quad & \exists \vec{y}_0, \ldots, \vec{y}_{\ell-1} \, (\, x \, G^{-1} \, (\vec{y}_0, \ldots, \vec{y}_{\ell-1}) \wedge \\ & \exists \vec{u}_0, \ldots, \vec{u}_{\ell-1} \, (\, \bigwedge_{i<\ell} \vec{y}_i \, R^{\imath_K^V} \, \vec{u}_i \wedge \\ & \exists \vec{v} \, (\, (\vec{u}_0, \ldots, \vec{u}_{\ell-1}) \, I^{\imath_K^V \circ K} \, \vec{v} \wedge \\ & \exists \vec{w} \, (\, \vec{v} \in_{\imath_K^V \circ K} \vec{w} \wedge \\ & \vec{w} \, H \, z \,)))) \end{aligned}$$

In this definition, we assume that domain and range are 'built in' in each relation involved. E.g., $\vdash x \, G^{-1} \, (\vec{y}_0, \ldots, \vec{y}_{\ell-1}) \to \delta_{\imath_K^V \circ \jmath_K}(\vec{y}_0, \ldots, \vec{y}_{\ell-1})$.

To see that the definition works, note that each of G^{-1}, $R^{\imath_K^V}$ considered pointwise and $I^{\imath_K^V \circ K}$ is total and injective (w.r.t. the appropriate equivalence relations) and that H is an isomorphism. ❑

Finally we show that, 1-retracts of sequential theories are m-sequential. We note that it follows that U is m-sequential iff U is bi-interpretable with a sequential V iff U is a 1-retract of a sequential V.

Theorem 4.10. *if U is a 1-retract of a sequential theory V, then U is m-sequential.*

Proof. Suppose $K : U \to V$ and $M : V \to U$ and $F : M \circ K \to \mathsf{id}_U$ witness that U is a 1-retract of V. Let $N : \mathsf{AS} \to V$ be direct. Let M be m-dimensional and let K be k-dimensional.

We have $M \circ N : \mathsf{AS} \to U$. Note that $M \circ N$ is m-dimensional. Let I be the relation between external and internal sequences in AS, for the given length k. We take:

$$x \, R \, \vec{z} :\leftrightarrow \exists \vec{y}_0, \ldots, \vec{y}_{k-1} \;\; x \, F^{-1} \, (\vec{y}_0, \ldots, \vec{y}_{k-1}) \, I^{M \circ N} \, \vec{z}.$$

The relation R witnesses that $M \circ N$ is relaxed direct. ❑

Open Question 4.11. It seems to me that there is no strong reason that m-sequential theories should be closed under iso-congruence. However, it would be interesting to know whether they are. I conjecture that the answer is *no*. ❑

4.4. **Summarizing.** We have the following characterization theorem.

Theorem 4.12. *The following are equivalent.*

char(i) *U is m-sequential.*
char(ii) *For some m, there is an m-direct interpretation of* AS *in U.*
char(iii) *There is a relaxed direct interpretation of* AS *in U.*
char(iv) *U is bi-interpretable with a sequential theory V.*
char(v) *U is a 1-retract of a sequential theory V.*
char(vi) *U is bi-interpretable with an extension of* AS *in the same language.*

The last equivalence follows immediately using (iv) of Theorem 3.3.

5. Parameters

In this section, we will show that some sequential theories are not parameter-free sequential. In contrast, we show that all m-sequential theories are parameter-free m-sequential.

Example 5.1. Let us start with the Ackermann interpretation of the hereditarily finite sets in the natural numbers. We call this $\in_{\mathsf{ack}}$. Let α be a bijection between the integers and the natural numbers. We define, on the integers, $x \in^* y :\leftrightarrow \alpha(x) \in_{\mathsf{ack}} \alpha(y)$. Finally, we define, on the integers, $x \in_z y :\leftrightarrow (x+z) \in^* (y+z)$.

We consider the structure, say $\mathcal{M}$, on the set of the integers, given by $x \in_z y$. Clearly, $\mathcal{M}$ is sequential with one parameter.

Suppose a binary predicate $x \in y$ were definable in $\mathcal{M}$, without parameters, satisfying AS. Let m be a singleton. Since successor is an automorphism of $\mathcal{M}$ it follows that all elements are singletons, contradicting the assumption that we have AS for $\in$. ❑

We prove that parameters can always be eliminated from the witnessing interpretations of m-sequentiality.

Theorem 5.2. *Suppose V is m-sequential. Then V is m-sequential without parameters.*

Proof. Our proof is an adaptation of the proof that if V is m-sequential via a k-dimensional interpretation of the sequences, then there is an $2k$-direct interpretation of AS in V.

Suppose U is m-sequential. Let $M : \mathsf{seq} \to U$ be $(\mathfrak{o} : \mathfrak{o})$-direct. Suppose M has dimension k for the interpretation of sort $\mathfrak{s}$. Suppose M has an ℓ-dimensional parameter domain α.

Put $m := 2k + \ell$. We define:

$$(z_1, \ldots, z_m) \in (p_0, \ldots, p_{\ell-1}, x_0, \ldots, x_{k-1}, y_0, \ldots, y_{k-1}) :\leftrightarrow \alpha_M(p_0, \ldots, p_{\ell-1}) \wedge \delta^{\mathfrak{s}}_M(x_0, \ldots, x_{k-1}) \wedge \delta^{\mathfrak{s}}_M(y_0, \ldots, y_{k-1}) \wedge ((z_1, \ldots, z_m) \in (x_0, \ldots, x_{k-1}, y_0, \ldots, y_{k-1}))^{M,\vec{p}}$$

using Theorem 4.2 to provide the $\in$ inside $M, \vec{p}$. (By coding the triple of external sequences as one internal sequence, we can get the more efficient $m := k$.) ❑

The above proof is a bit disappointing, since we have to detour over the characterization theorem. This means that, in a sense, we throw away the interpretations of the successor theory that we were given and have to regain it via an bootstrap. There is a more elegant and more interesting proof that we will give now. The construction shows how we can, in the presence of finite functions, glue together different systems of the ordering of the natural numbers.

Proof. Suppose $M : \mathsf{seq} \to U$ is $(\mathfrak{o} : \mathfrak{o})$-direct. Let M be k-dimensional for sort $\mathfrak{n}$. Let $\vec{p}$ and $\vec{q}$ be in α_M. Using Theorem 4.3, we can define finite functions from $\delta^{\mathfrak{n}}_{M,\vec{p}}$ to $\delta^{\mathfrak{n}}_{M,\vec{q}}$ as finite sets of $2k$-sequences. Note that by Theorem 4.3, we can both adjoin pairs to a function and substract pairs from a function. We consider two such functions *equal*, when, whenever their inputs are $=_{M,\vec{p}}$-equal, then their outputs are $=_{M,\vec{q}}$-equal.

We say such a function f is a $\vec{p},\vec{q}$-isomorphism on $\vec{n}$ (in $\delta^{\mathfrak{n}}_{M,\vec{p}}$) iff the domain of f consists of all $\vec{n}'$ such that $\vec{n}' <_{M,\vec{p}} \vec{n}$, and the range is $<_{M,\vec{q}}$-downwards closed, and, if $\vec{n}'' <_{M,\vec{p}} \vec{n}' <_{M,\vec{p}} \vec{n}$, then $f(\vec{n}'') <_{M,\vec{q}} f(\vec{n}')$. Note that it follows that f commutes with successors. Clearly, we can find a $\delta^{\mathfrak{n}}_{M,\vec{q}}$-number $\vec{n}^*$, such that we can consider f as function from $\vec{n}$ to $\vec{n}^*$.

Define $N_{\vec{p}}$ as the class of those $\vec{n}$ in $\delta^{\mathfrak{n}}_{M,\vec{p}}$, such that, for all $\vec{n}' \leq_{M,\vec{p}} \vec{n}$, and for all $\vec{q}$ in α_M:

i. there is a $\vec{p},\vec{q}$-isomorphism f on $\vec{n}'$, which is unique modulo equality. We will call the unique $\vec{p},\vec{q}$ isomorphism on $\vec{n}$: $f_{\vec{n},\vec{p},\vec{q}}$.
ii. if $n'' <_{M,\vec{p}} n'$, then $f_{\vec{n}'',\vec{p},\vec{q}}$ is the restriction of $f_{\vec{n}',\vec{p},\vec{q}}$ to $\vec{n}'$.
iii. $f_{\vec{n}',\vec{p},\vec{p}}$ is the identity.

It is clear that $N_{\vec{p}}$ is downwards closed under $<_{M,\vec{p}}$ and that it contains the zero of $\delta^{\mathfrak{n}}_{M,\vec{p}}$. We show that $N_{\vec{p}}$ is closed under successor. Consider any $\vec{n}$ in $N_{\vec{p}}$. Suppose $\mathsf{S}_{M,\vec{p}}(\vec{n},\vec{n}^+)$. Either $\vec{n}$ is an $M,\vec{p}$-zero or an $M,\vec{p}$-successor. We leave the case that $\mathsf{Z}_{M,\vec{p}}(\vec{n})$ to the reader. Let the $<_{M,\vec{p}}$-predecessor of $\vec{n}$ be $\vec{n}^-$. Suppose $f_{\vec{n},\vec{p},\vec{q}}(\vec{n}^-) = \vec{n}^*$ and $\mathsf{S}_{M,\vec{q}}(\vec{n}^*,\vec{n}^\star)$. We obtain $f_{\vec{n}^+,\vec{p},\vec{q}}$ by adjoining $(\vec{n},\vec{n}^\star)$ to $f_{\vec{n},\vec{p},\vec{q}}$. It is easy to see that $f_{\vec{n}^+,\vec{p},\vec{q}}$, thus defined, is a $\vec{p},\vec{q}$-isomorphism for $\vec{n}^+$. Also property (ii) and (iii) are evident. Suppose g is a second $\vec{p},\vec{q}$-isomorphism for $\vec{n}^+$. We subtract the pair corresponding to input $\vec{n}$ from g, thus obtaining a $\vec{p},\vec{q}$-isomorphism h for $\vec{n}$. It follows that h is equal to $f_{\vec{n},\vec{p},\vec{q}}$. Hence g must be equal to $f_{\vec{n}^+,\vec{p},\vec{q}}$.

We define: $F_{\vec{p},\vec{q}}(\vec{n}) = \vec{n}^*$ iff $\mathsf{S}_{M,\vec{p}}(\vec{n},\vec{n}^+)$ and $f_{\vec{n}^+,\vec{p},\vec{q}}(\vec{n}) = \vec{n}^*$. Clearly, $F_{\vec{p},\vec{q}}$ is an initial embedding of $N_{\vec{p}}$ in $\delta^{\mathfrak{n}}_{M,\vec{q}}$. Also $F_{\vec{p},\vec{p}}$ is the identity on $N_{\vec{p}}$.

We want to improve the $N_{\vec{p}}$ and the $F_{\vec{p},\vec{q}}$ to a system of classes of 'numbers' with designated isomorphisms between them where the isomorphisms are closed under identity, inverse and composition. We proceed as follows. We define $N^\star_{\vec{p}}$ as the class of those $\vec{n}$ in $N_{\vec{p}}$, such that, for all $\vec{n}' \leq_{M,\vec{p}} \vec{n}$,

a. for all $\vec{q}$ in α_M, we have: $F_{\vec{p},\vec{q}}(\vec{n}') \in N_{\vec{q}}$,
b. for all $\vec{q},\vec{r}$ in α_M, we have: $F_{\vec{q},\vec{r}}(F_{\vec{p},\vec{q}}(\vec{n}')) = F_{\vec{p},\vec{r}}(\vec{n}')$,

It is easy to see that $N^\star_{\vec{p}}$ is downwards closed w.r.t. $<_{M,\vec{p}}$, that it contains the zero of $N_{\vec{p}}$, and that it is closed under $\mathsf{S}_{M,\vec{p}}$.

Suppose $\vec{n}$ is in $N^\star_{\vec{p}}$. We show that $\vec{n}^\star := F_{\vec{p},\vec{q}}(\vec{n})$ satisfies the conditions (a) and (b). We note that (a) for $\vec{n}^\star$, follows from (b) and (a) for $\vec{n}$. We verify (b) for $\vec{n}^\star$.

For $\vec{s}$ in α_M, we have:

$$\begin{array}{rcl} F_{\vec{r},\vec{s}}(F_{\vec{q},\vec{r}}(\vec{n}^\star))) & = & F_{\vec{r},\vec{s}}(F_{\vec{q},\vec{r}}(F_{\vec{p},\vec{q}}(\vec{n})))) \\ & = & F_{\vec{r},\vec{s}}(F_{\vec{p},\vec{r}}(\vec{n}))) \\ & = & F_{\vec{p},\vec{s}}(\vec{n}) \\ & = & F_{\vec{q},\vec{s}}(F_{\vec{p},\vec{q}}(\vec{n}))) \\ & = & F_{\vec{q},\vec{s}}(\vec{n}^\star) \end{array}$$

Consider any $\vec{n}$ in $N^\star_{\vec{p}}$. Suppose $\vec{n}^* \leq_{M,\vec{q}} \vec{F}_{\vec{p},\vec{q}}(\vec{n})$. Since, $F_{\vec{p},\vec{q}}$ is an initial embedding, it follows that, for some $\vec{n}' \leq_{M,\vec{p}} \vec{n}$, we have $\vec{n}^* = F_{\vec{p},\vec{q}}(\vec{n}')$. Clearly, $\vec{n}'$ is in $N^\star_{\vec{p}}$. Ergo by the above $\vec{n}^*$ satisfies conditions (a) and (b). We may conclude that $F_{\vec{p},\vec{q}}(\vec{n})$ is in $N^\star_{\vec{q}}$.

Let $F^\star_{\vec{p},\vec{q}}$ be $F_{\vec{p},\vec{q}}$ restricted to $N^\star_{\vec{p}}$. Clearly, we find $F^\star_{\vec{p},\vec{p}} = \mathsf{id}_{N^\star_{\vec{p}}}$, and $F^\star_{\vec{q},\vec{r}} \circ F^\star_{\vec{p},\vec{q}} = F^\star_{\vec{p},\vec{r}}$, and $F^\star_{\vec{p},\vec{q}} = (F^\star)^{-1}_{\vec{q},\vec{p}}$.

We assume that the $F^\star$ are defined in such a way that we have:

- $U \vdash \vec{n}\ F^\star_{\vec{p},\vec{q}}\ \vec{n}^* \to (N^\star_{\vec{p}}(\vec{n}) \wedge N^\star_{\vec{q}}(\vec{n}^*))$
- $U \vdash \vec{n} =_{M,\vec{p}} \vec{n}'\ F^\star_{\vec{p},\vec{q}}\ \vec{n}^* =_{M,\vec{q}} n^\star \to \vec{n}\ F^\star_{\vec{p},\vec{q}}\ \vec{n}^\star$

We define our parameter-free interpretation $M^\star : \mathsf{seq} \to U$.

- $\delta^{\mathsf{o}}_{M^\star}(x) :\leftrightarrow x = x$,
- $x =^{\mathsf{oo}}_{M^\star} y :\leftrightarrow x = y$,
- $\delta^{\mathsf{n}}_{M^\star}(\vec{p},\vec{n}) :\leftrightarrow \alpha_M(\vec{p}) \wedge N^\star_{\vec{p}}(\vec{n})$,
- $\vec{p},\vec{n} =^{\mathsf{nn}}_{M^\star} \vec{q},\vec{n}^* :\leftrightarrow \vec{n}\ F^\star_{\vec{p},\vec{q}}\ \vec{n}^*$,
- $\mathsf{Z}_{M^\star}(\vec{p},\vec{n}) :\leftrightarrow \mathsf{Z}_{M,\vec{p}}(\vec{n})$,
- $\mathsf{S}_{M^\star}(\vec{p},\vec{n},\vec{q},\vec{n}^*) :\leftrightarrow \exists \vec{n}'\,(S_{M,\vec{p}}(\vec{n},\vec{n}') \wedge \vec{n}'\ F_{\vec{p},\vec{q}}\ \vec{n}^*)$,
- $\vec{p},\vec{n} <_{M^\star} \vec{q},\vec{n}^* :\leftrightarrow \exists \vec{n}'\,(\vec{n} <_{M,\vec{p}} \vec{n}' \wedge \vec{n}'\ F^\star_{\vec{p},\vec{q}}\ \vec{n}^*)$,
- $\delta^{\mathsf{s}}_{M^\star}(\vec{p},\vec{s}) :\leftrightarrow \alpha_M(\vec{p}) \wedge \delta^{\mathsf{s}}_{M,\vec{p}}(\vec{s})$,
- $\vec{p},\vec{s} =_{M^\star} \vec{q},\vec{t} :\leftrightarrow \bigwedge_i p_i = q_i \wedge \vec{s} =_{M,\vec{p}} \vec{t}$,
- $\mathsf{E}_{M^\star}(\vec{p},\vec{s}) :\leftrightarrow \mathsf{E}_{M,\vec{p}}(\vec{s})$,
- $\mathsf{L}_{M^\star}(\vec{p},\vec{s},\vec{q},\vec{n}^*) :\leftrightarrow \exists \vec{n}\,(\vec{n}\ F^\star_{\vec{p},\vec{q}}\ \vec{n}^* \wedge \mathsf{L}_{M,\vec{p}}(\vec{s},\vec{n}))$,
- $\mathsf{Pr}_{M^\star}(\vec{p},\vec{s},\vec{q},\vec{n}^*,x) :\leftrightarrow \exists \vec{n}\,(\vec{n}\ F^\star_{\vec{p},\vec{q}}\ \vec{n}^* \wedge \mathsf{Pr}_{M,\vec{p}}(\vec{s},\vec{n},x))$,
- $\mathsf{Pu}_{M^\star}(\vec{p},\vec{s},x,\vec{q},\vec{t}) :\leftrightarrow \bigwedge_i p_i = q_i \wedge \mathsf{Pu}_{M,\vec{p}}(\vec{s},x,\vec{t})$.

It is not difficult to see that our interpretation delivers the goods. ❑

6. Sequentiality is not preserved under Bi-interpretablity

We consider the following model $\mathcal{M}$ in a signature with unary predicate symbols A, B, S, C, a binary predicate symbol $\in$ and a ternary symbol app. The domain is partitioned in three infinite sets B, S and C. The set A is an infinite subset of S. Here A is the interpretation of A, etc. The union of the sets A and B forms the urelements. The set S consists of the hereditarily finite sets over A plus A. The relation $\in$ (that corresponds, par abus de langage, with the symbol $\in$) is the element relation on S. Here urelements are treated as empty sets. Note that we can define 'the true empty set' $\emptyset$: it is the unique empty set not in A. The set C consists of partial bijections from A to B. We treat the empty bijection in C, say $\square$, as distinct from the true empty set $\emptyset$ in S. Finally app stands for the application relation that we will again call app.

We use $a, a', a_0, \ldots$ to range over elements of A, and $b, b', b_0, \ldots$ for elements of B, and $s, s', s_0, t, \ldots$ for elements of S, and $\chi, \chi', \chi_0, \nu, \ldots$ for elements of C. Finally, $x, x', x_0, y, z, \ldots$ are general variables. There will be some ad hoc uses of other letters for variables.

We write $\chi(a) = b$ for $\mathsf{app}(\chi, a, b)$. We will use $\langle s, t\rangle$ for internally defined pairing on S and (x, y) for 'external pairing' as used in more-dimensional interpretations.

We first show that (the theory of) our model is m-sequential. To do that we need some preparation. We start to work in S. Clearly, in S we have the familiar machinery available to implement certain inductive definitions. Let's say that the class of *substitution pairs* is the minimal class such that a pair $\langle c, s\rangle$ is a substitution pair, if c is a partial bijection from A to A, coded in the usual way as a set of pairs, and s is a set of substitution pairs such that for every element $\langle d, t\rangle$ of s we have $\mathsf{range}(d) \subseteq \mathsf{dom}(c)$.

We will prove that there is a relaxed direct interpretation of AS in (the theory of) $\mathcal{M}$ to show that (the theory of) $\mathcal{M}$ is m-sequential. We first specify our interpretation of AS. The elements of our domain, say the class of set*s, are of the form (χ, s), where s is a set of substitution pairs for any $\langle d, t\rangle$ in s, we have $\mathsf{range}(d) \subseteq \mathsf{dom}(\chi)$.

We define: $(\nu, t) \in^* (\chi, s)$ iff, for some d, we have $\langle d, t\rangle \in s$ and $\nu = \chi \circ d$. In fact d is uniquely determined by this condition: we could also say $(\nu, t) \in^* (\chi, s)$ iff $\langle \chi^{-1} \circ \nu, t\rangle \in s$.

We note that for any $\langle d, t\rangle \in s$, we have that $(\chi \circ d, t)$ is a set*. Consider any $\langle e, u\rangle \in t$. Since $\langle e, u\rangle$ is a substitution pair, we have $\mathsf{range}(e) \subseteq \mathsf{dom}(d) = \mathsf{dom}(\chi \circ d)$.

We easily see that any pair $(\chi, \emptyset)$ will be an empty set*.

We define $\{(\chi, s)\}^* := (\chi, \{\langle j, s\rangle\})$, where j is the identity on $\mathsf{dom}(\chi)$, or, in other words, $j = \chi^{-1} \circ \chi$. We easily see that $\{(\chi, s)\}^*$ is indeed an $\in^*$-singleton containing (χ, s).

Next we want to show that —not necessarily unique— unions exist. We need the following lemmas.

Lemma 6.1. *Suppose $\chi \subseteq \rho$ and that (χ, s) is a set*. Then, (ρ, s) is also a set*. Moreover, (ρ, s) is $\in^*$-extensionally equal to (χ, s).*

Lemma 6.2. *Suppose that (χ, s) and (χ, t) are sets*, then $(\chi, s \cup t)$ is a set*, which is an $\in^*$-union of (χ, s) and $\chi, t)$.*

Lemma 6.3. *Suppose χ and ρ have the same range. Consider (χ, s). We can find an u such that (ρ, u) is a set* that is $\in^*$-extensionally equal to (χ, s).*

Lemmas 6.1 and 6.2 are easy. We prove Lemma 6.3.

Proof. Suppose χ and ρ have the same range and consider (χ, s). Define:

$$u = \{\langle \rho^{-1} \circ \chi \circ d, t\rangle \mid \langle d, t\rangle \in s\}.$$

We note that $\langle \rho^{-1} \circ \chi \circ d, t\rangle$ is a substitution pair, since $\mathsf{dom}(\rho^{-1} \circ \chi \circ d) = \mathsf{dom}(d)$. The pair (ρ, u) is a set*, since $\mathsf{range}(\rho^{-1} \circ \chi \circ d) \subseteq \mathsf{dom}(\rho)$. Finally, we have:

$$\begin{array}{rcl}(\nu,t)\in^\star(\chi,s) & \Leftrightarrow & \langle\chi^{-1}\circ\nu,t\rangle\in s\\ & \Leftrightarrow & \langle\rho^{-1}\circ\chi\circ\chi^{-1}\circ\nu,t\rangle\in u\\ & \Leftrightarrow & \langle\rho^{-1}\circ\nu,t\rangle\in u\\ & \Leftrightarrow & (\nu,t)\in^*(\rho,u)\end{array}$$

This ends our proof. ❑

Consider any set*s (χ_0, s_0) and (χ_1, s_1). Consider any ρ with as range the union of the ranges of the χ_i. Let ρ_i be the restriction of ρ to the range of χ_i. By Lemma 6.3, we can find a set u_i such that (ρ_i, u_i) is a set* and such that (ρ_i, u_i) is $\in^*$-extensionally equal to (χ_i, s_i). By Lemma 6.1, (ρ, u_i) is extensionally equal to (χ_i, s_i). Finally, by Lemma 6.2, $(\rho, u_0 \cup u_1)$ is a union of the (ρ, s_i) and, ipso facto, of the (χ_i, s_i).

We may conclude that we have indeed produced an interpretation of AS. We still need a witnessing relation R. We define by recursion the following mapping F from S to substitution pairs. $F(a) := \langle\emptyset, a\rangle$, $F(s) := \langle\emptyset, \{F(t) \mid t \in s\}\rangle$. For s in S, we set: $s\ R\ (\Box, F(\{s\}))$. For b in B, we set $b\ R\ (\tau_{ab}, F(a))$, where τ_{ab} is the bijection with domain $\{a\}$ that maps a to b. For $\chi \in C$, we set $\chi\ R\ (\chi, F(\emptyset))$. It is easy to see that F is total, has set*s as values and is injective.

We conclude that our model is m-sequential.

We show that our model is not sequential, i.e., that there is no one-dimensional direct interpretation, possibly with parameters, of AS in $\mathcal{M}$. Suppose there was. Let the elementhood predicate be $\in^\star$. Since all elements of the model can be defined in terms of urelements, we may assume that all parameters are urelements. Suppose the number of parameters in A is n and the number of parameters in B is m.

Consider any object c representing a set* (modulo $\in^\star$-extensionality) of the form

$$\{a_0, \ldots, a_{n+m}, b_0, \ldots, b_n\}^\star$$

with a_i in A en b_j in B, where the a_i and b_j are not parameters.

Can c be in S? Consider any b' in B disjoint from the b_j and the parameters. Consider the automorphism σ generated by interchanging b_0 and b'. Since $b_0 \in^\star c$, we find $b' = \sigma(b_0) \in^\star \sigma(c) = c$. Quod non. By a similar argument c is not in B.

So c is a bijection χ. Suppose a_i is not in the domain of χ. Let σ be the automorphism generated by interchanging a_i and a', where a' is disjoint of the parameters and of the a_i. It follows that: $a' = \sigma(a_i) \in^\star \sigma(\chi) = \chi$. Quod non. So each of the a_i is in the domain of χ. By a similar argument, each of the b_j is in the range of χ.

By our choice of the number of a_i and b_j, we have, for some k, $\chi(a_k) = b'$, where b' is neither a parameter nor one of the b_j. Similarly, we have, for some ℓ, $\chi(a') = b_\ell$, where a' is not a parameter. Let τ be the automorphism generated by interchanging a_k and a', and b' and b_ℓ. We find: $b' = \tau(b_\ell) \in^\star \tau(\chi) = \chi$. A contradiction.

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Appendix A. Proof of Theorem 4.8

We repeat the theorem:

Theorem 4.8. *Suppose $K : U \to V$, where U and V are 1-sorted. Suppose V proves that there are at least two objects. Then, there is a theory V_K and interpretations $\imath^U_K : U \to V_K$ and $\jmath_K : V_K \to V$, such that $\imath^U_K$ is identity-preserving and $\jmath_K$ is an 1-isomorphism (i.e., a bi-interpretation) and $K = \jmath_K \circ \imath^U_K$.*

Proof. We give the proof for the case without parameters and then sketch how is should be adapted if there are parameters.

Suppose $K : U \to V$, where U and V are 1-sorted and K is m-dimensional. We define V_K as follows. The signature of V_K is the disjoint union of the signatures of U and V, plus a unary predicate D and a $m+1$-ary predicate F. As axioms we take the axioms of U relativized to D, the axioms of V relativized to the complement of D, plus axioms that express that F defines, in each model $\mathcal{M}$ of V_K, an isomorphism between the internal model $\widetilde{K}(\mathcal{P})$ of U in $\mathcal{M}$, where $\mathcal{P}$ is the internal model of V with domain $D^c_{\mathcal{M}}$, and the internal model $\mathcal{N}$ of U in $\mathcal{M}$ with domain $D_{\mathcal{M}}$.

We can view this as follows. There are the obvious identity-preserving interpretations $\imath^U_K : U \to V_K$ and $\imath^V_K : V \to V_K$. The axioms for F precisely make F a witness for the identity in INT_1 of $\imath^V_K \circ K$ and $\imath^U_K$.

We define an $m + 2$-dimensional interpretation $\jmath_K : V_K \to V$ as follows.

- $\delta_{\jmath_K}(x, y, \vec{z}) :\leftrightarrow x = y \vee \delta_K(\vec{z})$
- $(x_0, x_1, \vec{x}_2) =_{\jmath_K} (y_0, y_1, \vec{y}_2) :\leftrightarrow$
  $(x_0 = x_1 \wedge y_0 = y_1 \wedge x_{2(m-1)} = y_{2(m-1)}) \vee (x_0 \neq x_1 \wedge y_0 \neq y_1 \wedge \vec{x}_2 =_K \vec{y}_2)$
- Suppose P is in (the disjoint version of) the signature of V:
  $P_{\jmath_K}(x_{00}, x_{01}, \vec{x}_{02}, \ldots, x_{(n-1)0}, x_{(n-1)1}, \vec{x}_{(n-1)2}) :\leftrightarrow$
  $\bigwedge_{i<n} x_{i0} = x_{i1} \wedge P(x_{02(m-1)}, \ldots, x_{(n-1)2(m-1)})$
- Suppose P is in (the disjoint version of) the signature of U:
  $P_{\jmath_K}(x_{00}, x_{01}, \vec{x}_{0,2}, \ldots, x_{(n-1)0}, x_{(n-1)1}, \vec{x}_{(n-1)2}) :\leftrightarrow$
  $\bigwedge_{i<n} x_{i0} \neq x_{i1} \wedge P_K(\vec{x}_{02}, \ldots, \vec{x}_{(n-1)2})$
- $D_{\jmath_K}(x, y, \vec{z}) :\leftrightarrow x \neq y \wedge \delta_K(\vec{z})$

- $F_{\jmath K}(x_{00}, x_{01}, \vec{x}_{0,2}, \ldots, x_{(m-1)0}, x_{(m-1)1}, \vec{x}_{(m-1)2}, y_0, y_1, \vec{y}_2,) :\leftrightarrow$
$\bigwedge_{i<m}(x_{i0} = x_{i1} \wedge x_{i2(m-1)} = y_{2i}) \wedge y_0 \neq y_1 \wedge \delta_K(\vec{y}_2)$

We want to show that $\jmath_K$ is an 1-isomorphism with inverse $\imath^V_K$. We first compute $M := \jmath_K \circ \imath^V_K$.

- $\delta_M(x, y, \vec{z}) \leftrightarrow x = y$.
- $(x_0, x_1, \vec{x}_2) =_M (y_0, y_1, \vec{y}_2) \leftrightarrow (x_0 = x_1 \wedge y_0 = y_1 \wedge x_{2(m-1)} = y_{2(m-1)})$
- $P_M(x_{00}, x_{01}, \vec{x}_{0,2}, \ldots, x_{(n-1)0}, x_{(n-1)1}, \vec{x}_{(n-1)2}) :\leftrightarrow$
$\bigwedge_{i<n} x_{i0} = x_{i1} \wedge P(x_{02(m-1)}, \ldots, x_{(n-1)2(m-1)})$

It is clear that G with $G(x_0, x_1, \vec{x}_2, y) :\leftrightarrow x_0 = x_1 \wedge x_{2(m-1)} = y$ gives us the desired isomorphism $G : M \to \mathsf{id}_V$.

Next we compute $N := \imath^V_K \circ \jmath_K$. We write $\vec{x} : D^c$, where $\vec{x}$ has length k, for $\bigwedge_{i<k} \neg D(x_i)$. We write A^{D^c} for A with its quantifiers relativized to D^c.

- $\delta_N(x, y, \vec{z}) :\leftrightarrow x, y, \vec{z} : D^c \wedge (x = y \vee \delta^{D^c}_K(\vec{z}))$
- $(x_0, x_1, \vec{x}_2) =_N (y_0, y_1, \vec{y}_2) :\leftrightarrow x_0, x_1, \vec{x}_2, y_0, y_1, \vec{y}_2 : D^c \wedge$
$(x_0 = x_1 \wedge y_0 = y_1 \wedge x_{2(m-1)} = y_{2(m-1)}) \vee (x_0 \neq x_1 \wedge y_0 \neq y_1 \wedge \vec{x}_2 =_K \vec{y}_2)$
- Suppose P is in (the disjoint version of) the signature of V:
$P_N(x_{00}, x_{01}, \vec{x}_{0,2}, \ldots, x_{(n-1)0}, x_{(n-1)1}, \vec{x}_{(n-1)2}) :\leftrightarrow$
$x_{00}, x_{01}, \vec{x}_{0,2}, \ldots, x_{(n-1)0}, x_{(n-1)1}, \vec{x}_{(n-1)2} : D^c \wedge$
$\bigwedge_{i<n} x_{i0} = x_{i1} \wedge P(x_{02(m-1)}, \ldots, x_{(n-1)2(m-1)})$
- Suppose P is in (the disjoint version of) the signature of U:
$P_N(x_{00}, x_{01}, \vec{x}_{0,2}, \ldots, x_{(n-1)0}, x_{(n-1)1}, \vec{x}_{(n-1)2}) :\leftrightarrow$
$x_{00}, x_{01}, \vec{x}_{0,2}, \ldots, x_{(n-1)0}, x_{(n-1)1}, \vec{x}_{(n-1)2} : D^c \wedge$
$\bigwedge_{i<n} x_{i0} \neq x_{i1} \wedge P^{D^c}_K(\vec{x}_{02}, \ldots, \vec{x}_{(n-1)2})$
- $D_N(x, y, \vec{z}) :\leftrightarrow x, y, \vec{z} : D^c \wedge x \neq y \wedge \delta^{D^c}_K(\vec{z})$
- $F_N(x_{00}, x_{01}, \vec{x}_{0,2}, \ldots, x_{(m-1)0}, x_{(m-1)1}, \vec{x}_{(m-1)2}, y_0, y_1, \vec{y}_2) :\leftrightarrow$
$x_{00}, x_{01}, \vec{x}_{0,2}, \ldots, x_{(m-1)0}, x_{(m-1)1}, \vec{x}_{(m-1)2}, y_0, y_1, \vec{y}_2 : D^c \wedge$
$y_0 \neq y_1 \wedge \delta^{D^c}_K(\vec{y}_2) \wedge \bigwedge_{i<m}(x_{i0} = x_{i1} \wedge x_{i2(m-1)} = y_{2i})$

We now may take to define our isomorphism $H : N \to \mathsf{id}_{V_K}$, the formula:

- $H(x_0, x_1, \vec{x}_2, y) :\leftrightarrow x_0, x_2, \vec{x}_2 : D^c \wedge$
$((x_0 = x_1 \wedge x_{2(m-1)} = y)) \vee (x_0 \neq x_1 \wedge \delta^{D^c}_K(\vec{x}_2) \wedge F(\vec{x}_2, y))$

To add parameters, suppose the parameter domain of K is α and the number of parameters is k. We define the theory V_K as follows. For every predicate of V we have a copy in the predicates of V_K of the same arity. For every predicate of U of arity n we have a corresponding $k + n$-ary predicate. Again we keep the predicates corresponding to those of V and U disjoint. We have a unary predicate D_V and a $k + 1$-ary predicate D_U. We have a $k + 2$-ary predicate F. We have the following axioms.

- $\vdash D_U(\vec{p}, x) \to (\vec{p} : D_V \wedge \alpha^{D_V}(\vec{p}))$
- $\vdash \exists \vec{p}\, D_U(\vec{p}, x) \leftrightarrow \neg D_V(x)$
- $\vdash (D_U(\vec{p}, x) \wedge D_U(\vec{q}, x)) \to \vec{p} = \vec{q}$
- if A is an axiom of V: $\vdash A^{D_V}$
- $\vdash F(\vec{p}, x, y) \to (\alpha^{D_V}(\vec{p}) \wedge D_V(x) \wedge \delta^{D_V}_{K,\vec{p}}(x) \wedge D_U(\vec{p}, y))$
- $\vdash (\alpha^{D_V}(\vec{p}) \wedge D_V(x) \wedge \delta^{D_V}_{K,\vec{p}}(x)) \to \exists y\, F(\vec{p}, x, y)$
- $\vdash (\alpha^{D_V}(\vec{p}) \wedge D_U(\vec{p}, y)) \to \exists x\, F(\vec{p}, x, y)$
- $(F(\vec{p}, x, y) \wedge D_V(u) \wedge \delta^{D_V}_{K,\vec{p}}(u) \wedge x =^{D_V}_K u) \to F(\vec{p}, u, y)$

- $(F(\vec{p}, x, y) \wedge F(\vec{p}, u, v)) \rightarrow (x =_K^{D_V} u \leftrightarrow y = v)$
- If P is an n-ary predicate of U:
  $\vdash \bigwedge_{i<n} F(\vec{p}, x_i, y_i) \rightarrow (P_{K,\vec{p}}^{D_V}(x_0, \ldots, x_{n-1}) \leftrightarrow P(y_0, \ldots, y_{n-1}))$

Note that the axioms of U will take care of themselves because of the isomorphism. The rest of the development is precisely as expected. ❑

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