Section 8: On the operations which are less needed, <in> 10 chapters

Chapter 1: On <finding> the latitude of a locality from the hours (i.e., the duration) of <its> longest day.

We multiply half the <number of the> hours of the longest day by 15 <degrees>: <The result> is half the day arc. <This result> may be used to find the <maximum> ortive amplitude, which will be <for the sun> in the first of Cancer. Then we divide the Sine of the maximum declination <of the sun> by the Sine of the ortive amplitude, lowered: The result is the Cosine of the latitude of the locality. **Another method**: We find half the day arc and its deficit from 90° or its excess over 90°: This is the equation of daylight. Then we divide the Tangent of the declination of the <ecliptical> degree <of the sun> by the Sine of the equation of daylight, lowered: The result is the Cosine (this should be "Cotangent"; see commentary) of the latitude of the locality. This calculation <method> is generally valid for the <number of the> hours of any day of the year, if we use the declination of the sun on that day.

Chapter 2: On the altitude without a <non-zero> azimuth.

We divide the Sine of the declination of the sun or the distance of the planet from the celestial equator by the Sine of the latitude of the locality, lowered: The result is the Sine of the altitude corresponding to zero azimuth. This altitude can be found if the sun or the planet rises on the northern side of the celestial equator, i.e. <north of> the rising point of the first of Aries or Libra, and passes the meridian circle <at a point> south of the zenith.

Chapter 3: On <finding> the azimuth for any altitude which we assume.

We multiply the Cosine of the declination of the <ecliptical> degree of the sun by the Sine of the ascension of the distance of the <ecliptical> degree of the sun <from the meridian>, and we divide <the product> by the Cosine of the altitude <of the sun>: The result is the Cosine of the azimuth <of the sun>. If the sun is in the northern <zodiacal> signs, and the altitude of the <sun at the given> time is less than the altitude <of the sun> corresponding to its zero azimuth, then the azimuth is eastern or western towards north. If the altitude of the <sun at the given> time greater than the altitude <of the sun> relating to its zero azimuth, and the altitude <of the sun> is eastern or western, then the azimuth is southern. If the sun is in the southern <zodiacal> signs, then the azimuth is southern. <The method presented in> this chapter is less necessary for the planets. <However,> if it is needed

<for them>, <we take> the distance of the planet <from the celestial equator> instead of the declination of the sun, and its (i.e., the planet's) transit degree instead of the <ecliptical> degree of the sun. **Another method**: We multiply the Sine of the altitude <of the sun> by the Sine of the latitude of the locality, and divide <the product> by the Cosine of the latitude of the locality: The result is <called> 'the argument of the azimuth'. If the declination <of the sun> is southern, we add 'the argument of the azimuth' to the Sine of the ortive amplitude. If the declination <of the sun> is northern, we subtract the lesser <of these two values> from the greater <one>. The sum or the remainder is <called> 'the equation of the azimuth'. We divide it by the Cosine of the altitude, lowered. The result is the Sine of the azimuth. Now, if 'the argument of the azimuth' is greater than the Sine of the ortive amplitude, then the azimuth is southern; if it is less than it (i.e., than the Sine of the ortive amplitude), then the azimuth is northern.

Chapter 4: On <finding> the altitude from the azimuth.

We multiply the Cosine of the latitude of the locality by the Cosine of the azimuth, lowered: The result is <called> 'the Sine of the first arc'. We find the corresponding arc. Then we divide the Sine of the latitude of the locality by the Cosine of the first arc, lowered. The result is <called> 'the Sine of the second arc' which is <also> called 'the complement of the argument of the altitude'. Then we multiply the Sine of the declination of the sun by the Sine of the second arc, and divide <the product> by the Sine of the latitude of the locality: The result is <called> 'the Sine of the third arc'. We find the arc corresponding to it. <This arc> is called 'the equation of the altitude'. If the declination <of the sun> is southern, we subtract the third arc from the complement of the second arc. If the declination <of the sun> or the distance <of the planet from the ecliptic> is northern, we add the third arc to the complement of the second arc. The sum or the remainder is the <required> altitude. However, if the azimuth is northern, we always subtract the equation of the altitude from the argument of the altitude.

A <practical> use of these two chapters: If the birth of a baby occurred at a time during the day, and a line is drawn in the direction of the shadow of a vertical gnomon on a horizontal plane <at that time>, and if the return of this shadow to its first azimuth is noted in any <other> day, the altitude of the sun is obtained <by observation> at this <time>, and the azimuth relating to this altitude is computed, then this <azimuth> is the azimuth for the altitude <of the sun at> the time of birth. The altitude <of the sun> relating to this azimuth on the birthday and the position of the sun on that <day> can be computed. This will be the altitude of the sun at the time of birth; and the ascendant and what <else> that may be needed is computed from it.

Chapter 5: On the distance between two stars of which <only> one has a <non-zero> latitude.

We multiply the Cosine of <the longitude difference> in degrees between the two stars by the Cosine of the latitude of the star which has a <non-zero> latitude, lowered: The result is the Cosine of <the distance> between the two stars.

Chapter 6: On the distance between two stars both having <non-zero> latitudes.

We multiply the Cosine of the latitude of the star which has a smaller longitude by the Sine of <the longitude difference> between the two stars in degrees, lowered: The result is the Sine of the first arc. We find the corresponding arc. Then we divide the Sine of this latitude by the Cosine of the first arc, lowered: The result is the Sine of the second arc. We find the corresponding arc, and add <this arc> to the latitude of the star which has a greater longitude, if the two latitudes are in two different directions. If they are in the same direction, we obtain the difference between this latitude and the second arc: <The result> is the third arc. Then we multiply the Cosine of the first arc by the Cosine of the third arc, lowered: The result is the Cosine of <the distance> between the two stars.

Chapter 7: On the extraction of the meridian line.

We level a site on the ground so that its surface becomes parallel to the horizon. We draw a circle on it, we prick a straight needle and we measure its perpendicularity to the surface from three positions on the circumference, distant from each other. Then, near noon, we observe the tip of the shadow of the needle. <The shadow> will be diminishing as we make marks very close to one another by the tip of another needle on those positions <of the tip of the shadow>, while <the shadow> is turning. We check <the marks> carefully until the shadow begins increasing. Then we connect the mark nearest to the center <of the circle> and the center by a straight line. It will be the meridian line. **Another method** is <as follows>. We level the ground and <we take> the circle and the gnomon as we said <before>, except that the circle should be equal to the altitude circle on the back side of an available astrolabe. <Also> the length of the gnomon should be so that its shadow does not fall short of the circumference at noon. Then we extract the azimuth relating to its (i.e., the sun's) altitude on one of the two sides of the meridian <line>. We make a mark on the circumference where the shadow falls, when this altitude is reached <by the sun>. Using compasses, we obtain <the length> equal to the <chord of the> complement of the azimuth from the altitude circle of the astrolabe. We put one leg of the compasses on the mark, and the other on some point on the circumference in the direction of the altitude <of the sun>, which may be eastern or western <while the compass opening is the same>. From where it falls, we draw a line to the center of the circle. It will be the meridian line. If the altitude <of the sun> is <equal to> the altitude for a zero azimuth, the shadow will be on the east-west line. The line drawn from the middle of its two endpoints to the center of the circle is the meridian line. There are many ways to draw this line. However, all of them are less accurate and in practice <less> close to the correct <direction> than these two ways, but theoretically all of them are correct and can be proved.

Chapter 8: On the deviation of the <directions of> localities with known longitudes and latitudes from the meridian of our locality.

This deviation is called 'the azimuth of localities'. Let the locality whose azimuth is desired be Mecca. <To find this azimuth> we multiply the Cosine of the latitude of Mecca by the Sine of the <difference> between the two longitudes, lowered: The result is <called> 'the Sine of the equation of longitude'. We find the corresponding arc. Then we divide the Sine of the latitude of Mecca by the Cosine of the equation of longitude, lowered: The result is the Sine of the equation of latitude. We find the corresponding arc. If this arc is less than the latitude of our locality, we subtract it from the latitude of the locality. The remainder will be the adjusted latitude of the locality, <which latitude is> southern. If it (i.e., the adjusted altitude) is exactly equal to it (i.e., to the latitude of our locality), then the azimuth of Mecca is the east-west line. If it is greater <than the latitude of our locality>, we subtract from it the latitude of <our> locality: The result is the adjusted latitude of the locality, <which latitude is> northern. Then we multiply the Cosine of the equation of longitude by the Cosine of the adjusted latitude of the locality, lowered: The result is the Cosine of the distance between the two localities. Then we divide the Sine of the equation of longitude by the Sine of the distance between the two localities, lowered: The result is the Sine of the deviation of <the direction of> Mecca <from the local meridian>. <For finding> **the direction of the deviation**, we check <the arc> between the two localities and the adjusted latitude of the locality: If <the arc> between the two longitudes is situated in the east-south quadrant and the adjusted latitude of the locality is southern, then the deviation is towards the south-east; if the adjusted latitude of the locality is northern, then the deviation is towards north-east. If the <arc> between the two longitudes is situated in the south-west quadrant, and the adjusted latitude of the locality is southern, then the deviation is towards the south-west. If the adjusted latitude of the locality is northern, then the deviation is towards the north-west. When we carried out this operation for the locality of Rayy, taking its longitude from <Canary Islands in the> West (i.e., the Atlantic Ocean> to be 85°, and its latitude 36°, the longitude of Mecca 77°, and its latitude 21°, the deviation is <found to be> 27; 36° towards west.

Chapter 9: On the names of the fixed stars and their features in order to recognize them by seeing.

We have compiled in a table what we need <to know> about these stars in most <cases>. We have recorded their positions for the beginning of the year 301 of the Yazdigird <era>. <Their> adjustment <for other years> is <merely applying> the equation of the apogees. We have put in front of them <in the table> their latitudes, magnitudes, and their temperaments in relation to the planets. Since we need <to be able> to recognize a star and <sometimes> two stars by seeing <them> in each quadrant of the <sky> and all the quadrants of the ecliptic in order to obtain their altitudes <by observation> for knowing the ascendant and the time, we mention their features, so that the observer may recognize them. They are <as follows:> ***al-Kaff al-khażīb*** (lit. "the dyed palm [of the hand]"; Caph, β Cassiopeiae): a star in Aries, of the third magnitude, in the north, on the hump of the constellation known as *al-Nā*qa ("the she-camel") by the common people; there are two stars of the same magnitude under it, which, together with this star, form a triangle; ***ʿAyn al-thaur*** (lit. "the eye of the bull"; Oculus Tauri, α Tauri): also called *al-Dabarān* (Aldeberan): a red star in Taurus, of the first magnitude, in the south, behind the Pleiades, between some stars which look like <the Arabic letter> *dāl*; ***al-ʿAyyūq*** (Capella, α Aurigae): a big star in Gemini, of the first magnitude, in the north, on the edge of the Milky Way, behind three stars which are in a row; it rises <simultaneously> with the Pleiades; ***Mankib al-jawzāʾ*** (lit. "the shoulder of the Twins"; Betelgeuse, α Orionis): a red star in Gemini, of the first magnitude, in the south; it is in the place of the shoulder of a standing person; ***al-Shiʿrā al-yamānīya*** (Sirius, α Canis Majoris): a white big star in the beginning of the Cancer, of the first magnitude, in the south, behind the stars of Gemini; ***al-Shiʿrā al-shāmīya*** (Procyon, α Canis Minoris): a star in Cancer, of the first magnitude, in the south; it is smaller than Sirius, to the north of it, and in front of it; ***Qalb al-asad*** (lit. "heart of the lion"; Regulus, α Leonis): a star in Leo, of the first magnitude; it is approximately on the ecliptic, on the southern side of four stars standing from south to north in a crooked row; ***al-Ṣarfa*** (Cygnus, β Leonis) also called *Dhanab al-asad* (Denebola): a star in Virgo, on the tail of Leo, of the first magnitude, in the north; there are two bright stars called *al-Zubra* (Zubra, δ and θ Leonis) between it and Regulus; ***al-Simāk al-rāmiḥ*** (lit., "chest of the spearman"; Arcturus, α

Bootis): a star in Libra, of the first magnitude, in the north; there is a star smaller than it, called *al-Rāmiḥ* ("the spearman"), in front of it towards west; ***al-Simāk al-aʿzal*** (lit. "the chest of the unarmed <man>"; Spica, α Virginis): a star in Libra, of the first magnitude, in the south, in front of *al-Rāmiḥ*; ***al-Munīr min al-fakka*** (lit. "the luminous <star> of Coronae Borealis"; Alphacca, α Coronae Borealis): a star in Libra, of the second magnitude, in the north, between <a> circular <array of> stars behind *al-Simāk al-rāmiḥ*, the common people call it *Qaṣʿat al-masākīn* (lit. "the bowl of the poor"); ***Qalb al-ʿaqrab*** (lit. "heart of Scorpion"; Antares, α Scorpii): a red star in Scorpion, of the second magnitude, in the south, between two luminous stars on a curved line; ***al-Nasr al-wāqiʿ*** (lit. 'the falling eagle"; Vega, α Lyrae): a star at the end of Sagittarius, of the first magnitude, in the north; its path is close to the zenith; there are two small stars under it, which together with this star form a triangle called *athāfī* (lit. "andiron/trivet"; α, ε, ζ Lyrae or α, β, γ Lyrae) by the common people; ***al-Nasr al-ṭāʾir*** (lit. "the flying eagle"; Altair, α Aquilae): a star in Capricorn, of the second magnitude, in the north, between two luminous stars on a straight line; ***Dhanab al-dajāja*** (lit. "tail of the hen"; Deneb, α Cygni) <also> called *al-ridf*: a star in Aquarius, of the second magnitude, in the north, behind the luminous stars that cut through the Milky Way; ***Mankib al-faras*** (lit. "shoulder of the horse"; Scheat, Menkib, β Pegasi): a star in Pisces, of the second magnitude, in the north, northern in relation to another star of the same magnitude; they are called <together> as *al-fargh al-muqaddam* (α and β Pegasi), <which is> one of the lunar mansions

Chapter 10: On the names of the lunar mansions, and their rising days.

The 28 <lunar> mansions and their names <are as follows>:

1. *al-sharaṭain* 20th of Nīsān	2. *al-buṭain* 3rd of Ayār	3. *al-thurayyā* 16th of Ayār	4. *al-dabarān* 29th of Ayār	5. *al-haqʿa* 11th of Ḥazīrān
6. *al-hanʿa* 25th of Ḥazīrān	7. *al-dhirāʿ* 8th of Tammūz	8. *al-nathra* 20th of Tammūz	9. *al-ṭarfa* 2nd of Āb	10. *al-jabha* 15th of Āb
11. *al-zubra* 28th of Āb	12. *al-ṣarfa* 10th of Īlūl	13. *al-ʿawwāʾ* 23rd of Īlūl	14. *al-simāk* 6th of Tishrīn I	15. *al-ghafr* 20th of Tishrīn I
16. *al-zubānī* 2nd of Tishrīn II	17. *al-iklīl* 15th of Tishrīn II	18. *al-qalb* 28th of Tishrīn II	19. *al-shawla* 11th of Kānūn I	20. *al-naʿāʾim* 24th of Kānūn I
21. *al-balda* 6th of Kānūn II	22. *saʿd al-dhābiḥ* 19th of Kānūn II	23. *saʿd bulaʿ* 1st of Shabāṭ	24. *saʿd al-suʿūd* 14th of Shabāṭ	25. *saʿd al-akhbiya* 27th of Shabāṭ
26. *al-fargh al-muqaddam* 12th of Ādhār		27. *al-fargh al-muʾakhkhar* 25th of Ādhār		28. *baṭn al-ḥūt* 7th of Nīsān

The parts of the ecliptic corresponding to these mansions are equal. They are taken <subsequently, starting> from the point corresponding to the beginning of Aries. Their constellations are <formed> of fixed stars with different magnitudes and positions in the Zodiac. Their <heliacal> rising days, i.e. <the times of> their apparition <after they exit> from <being> under the rays <of the sun are as follows>: *al-sharaṭain* rises on the 20th of Nīsān around the year 1320 of the Two-Horned (i.e., Alexander) <era>. Then each next mansion rises 13 days later, until *al-Simāk* rises. We take the rising of *al-Ghafr* next to it, after 14 days for compensating the fractions which <are> with the 13 days. Then up to the end of the mansions <we take 13 days>, as before. After 66 years *al-Sharaṭain* rises on the 21st of Nīsān, and similarly all <other> mansions rise one day later. When a mansion <of the moon> rises, its opposite <mansion>, which is the fifteenth <mansion counting> from it, sets. Thus, when *al-Sharaṭain* rises, *al-Ghafr* sets. It is not impossible that there may be a difference of one or two days between the actual apparition <of these mansions> and what we have defined for them. Precise observations do not exist in this <connection> that may lead to major inconsistency, and there is no need to determine that <inconsistency>. <Now,> after we have completed the chapters <on elementary calculations> that we indicated in the preface to the <first> book, and we tried to make them close <to understanding>, and we did our best in making them precise, we finish the first book by this chapter. We ask God for help, and on Him is <our> reliance. This is followed by the second book on the tables.

Commentary

I.8.1 The ortive amplitude can be found from half the day arc by the formula given in I.5.6: $Cos\theta = (Cos\delta_1 Sin\frac{D}{2})/R$, where θ is the ortive amplitude of the sun, δ_1 the declination of the sun, and D the day arc. Then, as mentioned here, we can find the latitude of the locality by the formula $Sin\theta_{max}\,Cos\varphi = RSin\varepsilon$, where θ_{max} is the maximum ortive amplitude of the sun, φ the geographical latitude of the locality, and ε the maximum declination of the sun. The second method can also be deduced from a formula given in I.5.7 for calculating the equation of daylight: $R\,Sin\Delta D = Tg\delta_1 Tg\varphi$, where ΔD is the equation of daylight. Kūshyār erroneously puts Cosφ instead of Cotgφ in the second method. Both methods are demonstrated in IV.8.1. The proof of the formula given in I.5.6 is repeated there. Ptolemy [1984, 77-78] provides a different method for finding the latitude of the locality from the length of the longest day. Al-Battānī [1899-1907, III, 30] first calculates the ortive amplitude in a similar way; then he obtains the latitude of the locality from the ortive amplitude, half the day arc and the excess of half the day arc, by a method equivalent to $Sin\varphi = RSin\Delta DCos\theta / Sin\theta Sin\frac{D}{2}$. This formula can be obtained from the above formulas.

I.8.2 Here Kūshyār calculates the altitude h_0 of the sun if it is due east or west (or has zero azimuth, as Kūshyār puts it) and if its northern declination (δ) is given, for a locality with geographical latitude φ. In modern notation, his method is equivalent to: $Sinh_0 = RSin\delta / Sin\varphi$. Kūshyār provides a proof of this formula in IV.8.2.

I.8.3 The two methods provided by Kūshyār in this chapter for finding the azimuth of the sun when its altitude is known, are equivalent to the following formulas:

(1) $\cos az = \cos\delta_1 \sin A_d(\lambda) / \cos h$,

(2) $\sin az = (\sin\theta \pm \sin h \tan\varphi) / \cos h$,

where az is the required azimuth, δ_1 is the declination of the sun, $A_d(\lambda)$ is the right ascension of the arc between the sun and the meridian, h is the known altitude, φ is the latitude of the locality, and θ is the ortive amplitude. Proofs of these two methods are given in IV.8.3. The second method is provided by al-Battānī [1899-1907, III, 33-34, 53-54] for the sun and the planets or stars.

I.8.4 The method for finding the altitude of the sun when its azimuth is known, is equivalent to the following formulas:

$\cos\varphi \cos az = \sin\alpha_1$

$\sin\varphi / \cos\alpha_1 = \sin\alpha_2$

$\sin\delta_1 \sin\alpha_2 / \sin\varphi = \sin\alpha_3$

$h = 90° - \alpha_2 \pm \alpha_3$

$\alpha_1, \alpha_2,$ and α_3 being auxiliary arcs, the last two of which are called 'the complement of the argument of the altitude' and 'the equation of altitude', respectively. Of course the third formula may also be presented in the simpler form, $\sin\delta_1 / \cos\alpha_1 = \sin\alpha_3$. However, the present form may better reflect the geometrical process of finding the unknown altitude. A proof of the validity of this method is provided in IV.8.4, where, for the case when the azimuth is northern, the above-mentioned simpler formula is actually demonstrated. As Kūshyār mentions in the text, this chapter together with the former chapter may be used to find the altitude of the sun at a certain daytime on the day when a person was born. For this we should have registered the azimuth of the sun at the time of his birth by drawing a line along the shadow of a gnomon at that time. It is interesting that Kūshyār also thinks of the astrological application in this chapter and the former one.

I.8.5 In modern notation, the distance between two stars whose longitude difference is $\Delta\lambda$ and whose latitudes are $0°$ and β, respectively, is found by the formula: $\cos\Delta\lambda \cos\beta = \cos d$, where d is the distance between the two stars. A proof of the validity of this method is provided in IV.8.5. Al-Battānī gives a more lengthy method for this [1899, III, 59].

I.8.6 In modern notation, the distance between two stars whose longitude difference is $\Delta\lambda$, and whose latitudes are β_1 and β_2, respectively, can be found by the following formulas:

$\cos\beta_1 \sin\Delta\lambda = \sin\alpha_1$

$\sin\beta_1 / \cos\alpha_1 = \sin\alpha_2$

$\alpha_2 \pm \beta_2 = \alpha_3$

$\cos\alpha_1 \cos\alpha_3 = \cos d$

α_1, α_2, and α_3 being auxiliary arcs and d being the distance between the two stars. Kūshyār proves this method in IV.8.6. Al-Battānī gives a different method in a more detailed way for this calculation [1899, III, 60].

I.8.7 In this chapter, Kūshyār describes two methods for finding the meridian line, which he mentions as the most accurate among several theoretically correct methods. His first method is based on the fact that the shadow of any gnomon is shortest at true local noon when the sun is on the local meridian. In practice, this method is not very accurate. In the second method, the chord of the angle between the azimuth of the sun and the meridian line is computed by the methods provided in I.8.3 or by the astrolabe, from the altitude of the sun at the time of measuring. The graduation for the altitudes on the rim of the astrolabe is then utilized for obtaining this angle. See also IV.8.7. The phrase in the second method for the case when the shadow is on the east-west line: "from the middle of its two endpoints" is unclear to me. Maybe there is a lacuna here. Ptolemy did not provide any method for drawing the meridian line in the *Almagest*, but he assumes that it can be drawn [1984, 62]. Diodorus of Alexandria (first century B.C.) devised a method for determining the meridian line from any three gnomon shadows [ibid., fn. 72]. His original work is lost, but an Arabic version of his method has recently been published with an English translation [Hogendijk 2001, 68-72]. Al-Battānī describes three methods in his *zīj* for determining the meridian line [1899, III, 35-38], the second of which is similar to Kūshyār's second method, except that al-Battānī finds the chord of the angle between the azimuth of the sun and the east-west line.

I.8.8 In this chapter Kūshyār provides a method for finding the angle between the southern direction and the great circle arc between his locality and another locality (he calls it *inḥirāf*, which means "deviation"), and discusses it for the example of Mecca, because its direction is needed for Islamic prayers. In modern notation, his method is equivalent to the following formulas:

$$\cos\varphi_m \sin\Delta\Lambda = \sin\alpha_1$$

$$\sin\varphi_m / \cos\alpha_1 = \sin\alpha_2$$

$$|\varphi_c - \alpha_2| = \alpha_3$$

$$\cos\alpha_1 \cos\alpha_3 = \cos l$$

$$\sin\alpha_1 / \cos l = \sin d$$

where $\varphi_m, \Delta\Lambda, \varphi_c, l$ and d are, respectively, the latitude of Mecca, the longitude difference between the locality and Mecca, the latitude of the given locality, the arc between the locality and Mecca, and the deviation of the great circle arc from the locality to Mecca from the meridian line, i.e., the angle between the direction of Mecca and the southern direction. The auxiliary parameters $\alpha_1, \alpha_2, \alpha_3$ are respectively, called 'the equation of longitude', 'the equation of latitude', and 'the adjusted latitude'. Kūshyār proves the validity of this method in IV.8.8. Kūshyār's method for finding the direction to Mecca is the same as al-Bīrūnī's fourth method in his *Taḥdīd* [al-Bīrūnī 1967, 253-55], which he calls "the method of the *zīj*es" [Berggren 1985, 1]. Kūshyār's terminology in this chapter from the mss. F, C, Y, B and P is like that of al-Bīrūnī. However, ms. L uses a different terminology in which 'the equation of longitude' is called 'the first arc', etc. and its method is slightly different from a geometrical point of view [ibid, 8]. Al-Battānī describes a simple method for finding the direction of Mecca [1899, III, 206]. Al-Bīrūnī [ibid., 199] says that al-Battānī's method for finding the direction of Mecca is erroneous, because he "treated the meridian circles as parallel straight lines, and the parallels of latitudes as parallel straight lines." At the end of this chapter, Kūshyār calculates the deviation for the locality of Rayy as 27; 36°. I have recalculated this value according to Kūshyār's method and coordinates, and found it to be 27; 8, 20, 20°. The difference may be due to rounding errors. However, the real value of the deviation may be calculated as 38; 40, 58°. In this calculation modern values are used: Mecca (long. 39.83°, lat. 21.41°), Rayy (long. 51.44°, lat. 35.60°. As we see, Kūshyār's obvious error is due to inaccurate coordinates of the localities, especially to incorrect longitudes (based on his data $\Delta\Lambda$ is 8° whereas the correct value is 11.60°). Since in this calculation only $\Delta\Lambda$ is involved, the fact that I take the longitudes with respect to Greenwich and Kūshyār takes them with respect to the Canary Islands is irrelevant.

I.8.9 Kūshyār describes 16 bright stars which occur in the 12 zodiacal signs. In table II.55, he provides the coordinates and characteristics of 48 stars including 16 stars occurring in the zodiacal signs, not exactly the same as those described in I.8.9. This table is for the year 301 of the Yazdigird era (932-33 A.D.). For other years, the positions of the stars are found by applying "the equation of the apogees", by which Kūshyār means the precession of the equinoxes that, according to him, is 54 seconds per year (see I.4.4 and its commentary). For each year, an arc of 54 seconds is subtracted from the ecliptical longitude of any star.

I.8.10 In this chapter Kūshyār mentions the names of the 28 lunar mansions and the days on which they rise just before the sun, for the year 1320 of the Seleucid (Two-Horned, Alexander) era, corresponding to the year 399/400 A.H (1008-09 A.D.). This may be the reason why Prof. Kennedy believes that Kūshyār wrote this *zīj* around 1010 A.D. The same lunar mansions are mentioned by al-Battānī [1899, III, 188-89] where the last mansion is missing. The idea of dividing the zodiac into 28 (or 27) parts comes from ancient Indian astronomy, and the names already existed in Arabic although they were used for unequal lunar mansions. Final remarks by the scribes indicate the date of copying of the final part of the manuscript of Book I. The ms. F bears Maḥmūd b. Aḥmad al-Ḥussain's date, 545 A.H., and C bears the date 1169 A.H.

In the name of God, the Merciful, the Compassionate

Book IV of the *Jāmiʿ Zīj*

Kiyā Abū'l-Ḥasan Kūshyār b. Labbān b. Bāshahrī al-Jīlī –may God illuminate his tomb! – says: When I got through with the third Book on astronomy, I started this fourth Book on proofs, following the order of the chapters of the first Book.

Geometrical demonstration is a reasoning that does not allow any excess or deficiency in <its> exactness, and all those who understand it, equally know what has been proved and learn it by this proof. This is the last Book of this treatise, and when finishing it, I begged God for infallibility, ability, success and guidance, <for> he is really a bestower of these.

List of the chapters <of Book IV> containing 8 sections and 70 chapters

Section 1: On Chords and Sines, <in> 11 chapters
1. On the description of the Chord and Sine.
2. On finding the quantity of the Chord of the complement of an arc when the Chord of the arc is known.
3. On finding the quantity of the Chord of a quarter <of a circle>.
4. On finding the quantity of the Chord of a third <of a circle>.
5. On finding the quantity of the Chord of one-tenth and one-fifth <of a circle>.
6. On a premise for what follows.
7. On finding the quantity of the Chord of the difference between two arcs whose Chords are known.
8. On finding the quantity of the Chord of half an arc whose Chord is known.
9. On finding the quantity of the Chord of the sum of two arcs whose Chords are known.
10. On a premise for what follows.
11. On measuring the Chord of 1° very accurately and the composition of the <table of the> Chords.

Section 2: On Tangents and Cotangents, <in> 3 chapters
1. On the description of Tangents and Cotangents.
2. On finding <the quantity> of the (i.e., any) Tangent.
3. On finding <the quantity> of the (i.e., any) Cotangent.

Section 3: On premises on which the proofs are based, <in> 7 chapters
1. On a general premise for most proofs.
2. On another premise which derives from the first one.
3. Notice on the properties of the proportional magnitudes.
4. On another premise which also derives from the first one.
5. On a premise concerning Tangent<s>, which is a substitute for the first premise in most proofs.
6. Notice on the properties of the Tangent<s>.
7. Another notice also on the properties of the Tangent<s>.

Section 4: On <finding> the true longitudes of the planets and their positions, <in> 10 chapters
1. On the equation of time.
2. On the equation of the sun.
3. On the first equation for the moon.
4. On the second equation for the moon and the planets.
5. On the difference between the <apparent> radius of the epicycle between its maximum and minimum distance <from the earth>.
6. On the first equation for Mercury.
7. On the first equation for the other planets.
8. On the latitude of the moon.
9. On the latitudes of the planets
10. On the retrogradation of the planets.

Section 5: On the operations relating to the ascendants of the day and night, <in> 16 chapters
1. On the first declination.
2. On rising times of the <zodiacal> signs on the equator.
3. On the second declination.
4. On the distance of the stars from the celestial equator.
5. On the latitude of the <given> locality.
6. On the ortive amplitude of the sun and the stars.
7. On the equation of the daylight of the sun and the star<s>.
8. On the rising times <of the signs> in any locality.
9. On the maximum altitude of the sun and the star<s>.
10. On half the day arc of the sun and the star<s>.
11. On the <ecliptical> degree of the transit of a star through the meridian.
12. On the <ecliptical> degree of the rising and setting of a star.
13. On <finding> the arc of revolution <of the celestial equator> since the rising of the sun and the star<s> from the altitude of the <sun or the planet at a given> time.

14. On <finding> the ascendant from the arc of revolution <of e.g., the sun> and <finding> the arc of revolution from the ascendant.
15. On the proof of <using> a base generally applicable to the arc of revolution and what relates to it.
16. On the equalization of the houses.

Section 6: On eclipses and what pertains to them, <in> 14 chapters
1. On the absolute and adjusted magnitudes of a lunar eclipse in digits.
2. On the absolute times of a lunar eclipse.
3. On the correction of times.
4. On drawing the figure of a lunar eclipse.
5. On the distance of the moon from the earth.
6. On the altitude of the pole of the ecliptic.
7. On the altitude of any desired degree of the ecliptic.
8. On the parallax of the two luminaries in the altitude circle.
9. On the six angles needed in <the calculation of> solar eclipses.
10. On <finding> the longitudinal and latitudinal parallax of the moon from these angles.
11. On drawing the figure of a solar eclipse.
12. On <finding> the altitude of the moon according to its latitude.
13. On <finding> the longitudinal and latitudinal parallax of the moon by a proven method.
14. On the visibility arc<s>.

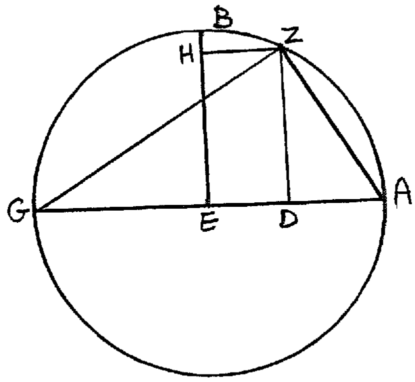
Section 7: On what pertains to astrology, in one chapter
1. On <finding> the projection of the ray taking the latitude of the planet into account.

Section 8: On the operations which are less needed, <in> 8 chapters
1. On <finding> the latitude of a locality from the hours (i.e., the duration) of <its> longest and shortest days.
2. On <finding> the altitude without (i.e., with zero) azimuth.
3. On <finding> the azimuth of <a point of given declination and> any assumed altitude.
4. On <finding> the altitude from the azimuth <and the declination>.
5. On the distance between two stars, one of which has a <non-zero> latitude.
6. On the distance between two stars <both> having <non-zero> latitudes.
7. On the extraction of the meridian line.
8. On the deviation of <the directions of> the <other> localities from the meridian of our locality.

These chapters are sufficient to prove the <contents of> first Book, because what may be beyond this, can be proved for someone who has advanced in astronomy and geometry with a little effort and easy thinking. God grants success and help.

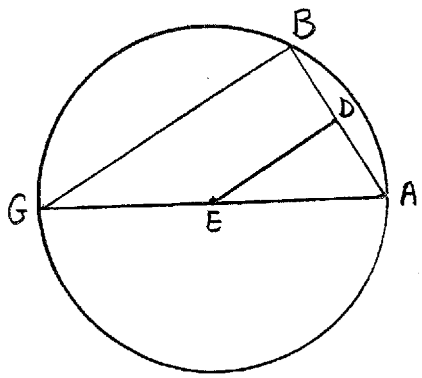
Section 1: On Chords and Sines, <in> eleven chapters
Chapter 1: On the description of the Chord and Sine.

ABG is a circle with *E* as its center, and *AG* its diameter. We draw *EB* at right angles <to the diameter>. We take the arc *AZ* and we draw the line <segment> *AZ*. We draw *ZD* perpendicular to *AG* and *ZH* perpendicular to *EB*. We draw the line <segment> *ZG*. Then the line <segment> *AZ* is the Chord of the arc *AZ* and <the line segment> *ZG* is the Chord of its complement. *ZD* is the Sine of the arc *AZ*, and *ZH* is its Cosine and equal to the line <segment> *DE*. *AD* is the Sagitta of the arc *AZ* and *BH* is the Sagitta of the arc *ZB*. The arc *ZB* is the complement of the arc *AZ* to a quarter of the circle and the arc *ZBG* is the supplement of the arc *AZ* to a semicircle. This is what we wanted to describe.



Chapter 2: On finding the quantity of the Chord of the supplement of an arc when the Chord of the arc is known.

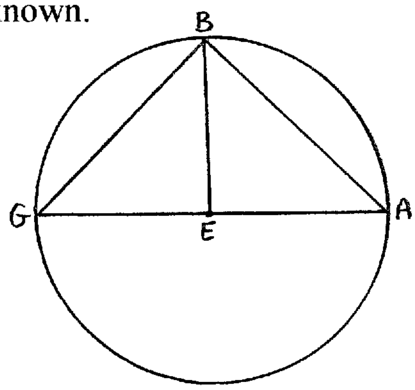
Let *ABG* be a circle and *AG* its diameter. We cut off the arc *AB* from it (i.e., from the circle) and we draw the line <segment>s *AB* <and> *BG*. We assume the Chord *AB* <to be> known; then I say that the Chord *BG* is <also> known.



Proof: The angle *ABG* is a right angle because it is <subtended> in a semicircle. Then the square of *AG* is equal to <the sum of> the squares of *AB* <and> *BG*. If we subtract the square of *AB* from the square of *AG*, the remainder <which is> the square of *BG* <will be> known. Its square root which is the Chord *BG* is <also> known. That is what we wanted to demonstrate. Now it has become clear that the ratio of any Chord to the diameter of the circle is equal to the ratio of the Sine of half the arc of the Chord to the radius of the circle. <For showing> this <we notice that> if we bisect the Chord *AB* at *D* and we draw *DE*, <where> *E* is the center of the circle, <then> *DE* will be parallel to *BG* and *AD* will be the Sine of half the arc *AB*. Then the ratio of *BA* to *AG* is equal to that of *DA* to *AE*. So all calculations that are made on <the basis of> the Chord and the diameter can be carried over to the Sine of half the arc of the Chord and the radius. This is what we wanted to demonstrate.

Chapter 3: On finding the quantity of the Chord of a quarter <of a circle>.

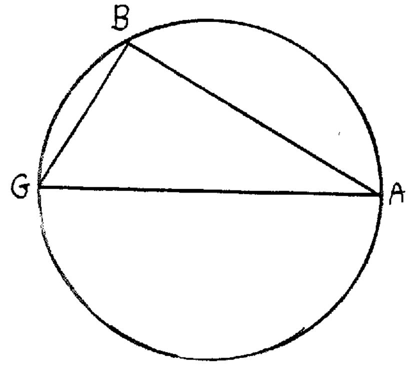
Let *ABG* be a circle centered at *E*, and *AG* its diameter. We draw *EB* at right angles <to the diameter> and we draw <the line segments> *AB* <and> *BG*. Then each of the arcs *AB* <and> *BG* is a quarter of the circle and each of the line <segment>s *AB* <and> *BG* is the Chord of a quarter <of the circle>. I say that they are known.



Proof: Angle *AEB* is right, so the square of *AB* is equal to the <sum of the> squares of *AE* <and> *EB*. Each of <the line segments> *AE* and *EB* is <equal to> the radius. Then the sum of their squares is known and <thence> its square root is known. Therefore, the Chord *AB* is known. That is what we wanted to demonstrate. Now it has become clear that the square of the Chord of a quarter <of a circle> is equal to twice the square of the radius, and that the square of the diameter is equal to four times the square of the radius. <This is> because the square of *AG* is equal to the <sum of the> squares of *AB* <and> *BG*, and each of the squares of *AB* <and> *BG* is equal to twice the square of *AE*. So the square of *AG* <is equal to> four times the square of *AE*. This is what we wanted to demonstrate.

Chapter 4: On finding the quantity of the Chord of a third <of a circle>.

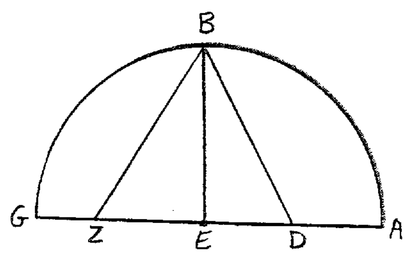
Let *ABG* be a circle and *AG* its diameter. We draw *BG* equal to the radius of the circle; it is <the Chord of> a sixth <of the circle>. We draw *AB*. I say that the Chord of a third <of a circle> is known.



Proof: Angle *ABG* is right because it is <subtended> in a semicircle. Then the square of *AG* is equal to <the sum of> the squares of *AB* <and> *BG*. The square of *AG* is known, and the square of *BG*, which is the Chord of a sixth <of a circle>, is known. So the square of *AB* remaining from the square of *AG* is known. So its square root is <also> known. It is the Chord *AB*. So the Chord *AB* is known. That is what we wanted to demonstrate. Now it has become clear that the square of the Chord of a third <of a circle> is <equal to> three times the square of the radius <of the circle>. The Chord *BG* is equal to the radius <of the circle>. If we subtract the square of *BG* from the square of *AG*, three times the square of the radius <of the circle> is the remainder of *AG*. It is the square of the Chord *AB*. This is what we wanted to describe.

Chapter 5: On finding the quantity of the Chord of one-tenth and one-fifth <of a circle>.

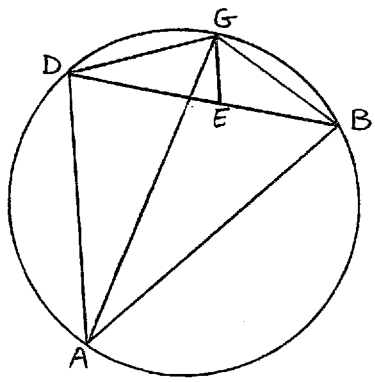
Let *ABG* be a semicircle centered at *E*, and *AG* its diameter. *EB* is perpendicular to *AG*. We bisect *AE* at *D* and we draw *BD*. We make *DZ* equal to *BD*. I say that *EZ* is equal to the Chord of one-tenth of the circle and *BZ* is equal to the Chord of one-fifth of it.



Proof: *AE* is bisected at *D* and *EZ* is added to it. So the product of *AZ* by *ZE* plus the square of *DE* is equal to the square of *DZ* (by *Elements* II.5). But *DZ* is equal to *DB*, and the square of *DB* is equal to <the sum of> the squares of *DE* <and> *EB*. So the product of *AZ* by *ZE* plus the square of *DE* is equal to the <sum of the> squares of *DE* and *EB*. We subtract the common square of *DE*. Then the remaining product of *AZ* by *ZE* is equal to the square of *EB*. But *EB* is equal to *EA*. So *AZ* is divided in mean and extreme ratio <at *E*>, and the greater portion is *AE*. But *AE* is the Chord of one-sixth <of the circle>. So (by *Elements* XIII.9) *EZ* is the Chord of one-tenth <of the circle>. Since the <sum of the> squares of *BE* <and> *EZ* is equal to the square of *BZ*, *BE* is the Chord of one-sixth <of the circle>, and *EZ* is the Chord of one-tenth <of the circle>, therefore (by *Elements* XIII.10) *BZ* is the Chord of one-fifth <of the circle>. This is what we wanted to demonstrate.

Chapter 6: On a premise for what follows.

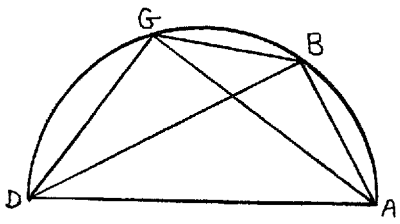
<In> any quadrilateral inscribed in a circle, if we multiply each side by its opposite side, the sum of the products will be equal to the product of the two diagonals. Let *ABG* be a circle and the quadrilateral *AGBD* is <inscribed> in it. I say that the product of *AB* by *GD* and <the product of> *AD* by *GB* when added <together> are equal to the product of *AG* by *BD*.



Proof: We make the angle *DGE* equal to the angle *BGA*. Since the angle *DGE* is equal to the angle *BGA* and the angle *AGE* is common, the angle *DGA* will be equal to the angle *BGE*. But the angle *GAD* is equal to the angle *GBD*, because they are <subtended> in the arc *GD*. So the remaining angle *ADG* is equal to the angle *BEG*. Therefore the ratio of *GB* to *BE* is equal to the ratio of *GA* to *AD*. So the product of *GB* by *AD* is equal to the product of *GA* by *BE*. Again, the angle *DGE* is equal to the angle *BGA*, and the angle *GDB* is equal to the angle *GAB*, because they are <subtended> in the arc *BG*. So the remaining angle *GED* is equal to the angle *ABG*. Therefore the ratio of *GD* to *DE* is equal to the ratio of *GA* to *AB*. Then the product of *GD* by *AB* is equal to the product of *GA* by *DE*. But it has been shown that the product of *GB* by *AD* is equal to the product of *GA* by *BE*. So the product of *AG* by *BD* is equal to the product of *GB* by *AD* and <the product of> *GD* by *AB* <added together>. This is what we wanted to demonstrate.

Chapter 7: On finding the quantity of the Chord of the difference between two arcs whose Chords are known.

Let *ABG* be a semicircle with <a known line segment> *AD* as its diameter. The Chords *AB* <and> *AG* in it (i.e., in the circle) are known. We draw *BG*. Then I say that *BG* is known.

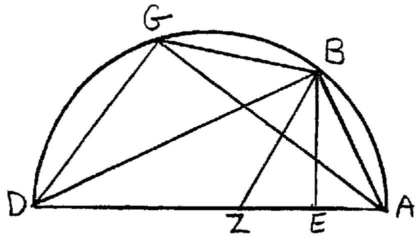


Proof: We draw *BD* <and> *GD*. Then they are both known, because they are the Chords of the supplements of *AB* <and> *AG* <to a semicircle>. So according to what was demonstrated in the premise, the product of *AG* by *BD* is equal to the sum of the product of *AB* by *GD* and <that of> *AD* by *GB*. But the product of *AG* by *BD* is known, and the diameter *AD* is known. So the Chord *BG* is known. This is what we wanted to demonstrate.

Chapter 8: On finding the quantity of the Chord of half an arc of whose Chord is known.

Let *ABGD* be a circle and *AD* is its diameter. We assume the Chord *AG* <to be> known. We bisect the arc *AG* at *B*. We draw *AB* <and> *BG*. Then I say that *AB* is known.

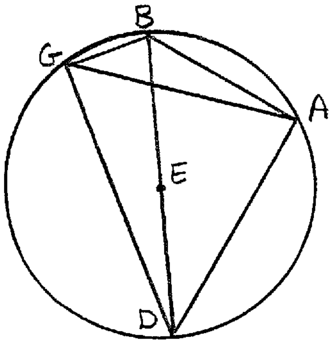
Proof: We draw *GD* and we make *DZ* equal to *GD*. We draw *BD* <and> *BZ*, and we draw *BE* perpendicular to *AZ*. Then *GD* is equal to *DZ* and *DB* is common. Thus, *GD* <and> *DB* are equal to *ZD* <and> *DB* <respectively>, and the angle *ZDB* is equal to the angle *BDG* because they are <subtended> in two equal arcs. So the base *BG* is equal to the base *BZ*. But *AB* is equal to *BG*. So *AB* is equal to *BZ*. So the triangle *ABZ*



is isosceles. The perpendicular *BE* is drawn from <the vertex of> the angle *ABZ*, so *AE* is equal to *EZ*. Since the triangle *ABD* is right-angled and the perpendicular *BE* is drawn from its right angle, the triangles *ABD* <and> *ABE* are similar. So the ratio of *DA* to *AB* is equal to the ratio of *BA* to *AE*. So the product of *DA* by *AE* is equal to the square of *AB*. Each of *DA* <and> *AE* are known, so the square of *AB* is known. So, its square root, i.e. the Chord *AB*, is known. This is what we wanted to demonstrate.

Chapter 9: On finding the quantity of the Chord of the sum of two arcs whose Chords are known.

Let *ABG* be a circle with *E* as its center. We assume the two known Chords *AB* <and> *BG* in it. We draw *AG*. Then I say that *AG* is known.

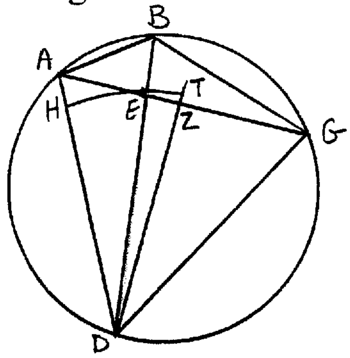


Proof: We draw the diameter *BD* from *B*, and draw *AD* <and> *DG*. Then *AD* is the Chord of the supplement of *AB* <to a semicircle> and *GD* is the Chord of the supplement of *BG* <to a semicircle>, and <hence> they are known. So the product of *AB* by *GD* and <the product of> *BG* by *AD* <added together> is equal to the product of *BD* by *AG*. But each of *AB*, *GD*, *BG*, <and> *AD* is known and the diameter *BD* is known. Then the Chord *AG* is known. This is what we wanted to demonstrate.

Chapter 10: On a premise for what follows.

If there are two unequal chords in a circle, then the ratio of the greater chord to the smaller chord is less than the ratio of the arc of the greater chord to the arc of the smaller chord. Let *ABGD* be the circle circumscribing them (i.e., the chords). <Inscribed> in it are the chords *AB* <and> *BG*, and *BG* is the greater <chord> of the two. I say that the ratio of the chord *BG* to the chord *BA* is less than the ratio of the arc *BG* to the arc *BA*.

Proof: We bisect the angle *ABG* by the line <segment> *BD*. We draw the <line segments> *AG*, *AD*, and *GD*. Since the angle *ABG* is bisected by the line <segment> *BD*, the line <segment> *GD* is equal to the line <segment> *AD*. But the line <segment>

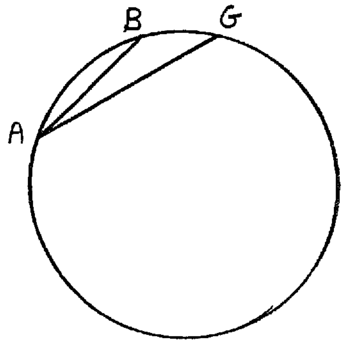


GE is longer than the line <segment> *EA*. We draw from *D* to the line <segment> *AG* the perpendicular *DZ*. Since *AD* is longer than *ED*, and *ED* is longer than *DZ*, the circle drawn with the center *D* and radius *DE* intersects *AD* and goes beyond *DZ*. We draw <an arc such> as *HET* and we extend *DZ* to *T*. Since the sector *DET* is greater than the triangle *DEZ* and the triangle *DEA* is greater than the sector *DEH*, the ratio of the sector *DET* to the sector *DEH* is greater than the ratio of the triangle *DEZ* to the triangle *DEA*. The ratio of the triangle *DEZ* to the triangle *DEA* is equal to the ratio of the line <segment> *EZ* to *EA*. The ratio of the sector *DET* to the sector *DEH* is equal to the ratio of the angle *ZDE* to the angle *EDA*. Then the ratio of the line <segment> *ZE* to the line <segment> *EA* is less than the ratio of the angle *ZDE* to the angle *ADE*. Componendo, the ratio of the line <segment> *ZA* to the line <segment> *EA* is less than the ratio of the angle *ZDA* to the angle *ADE*. The ratio of halves is equal to the ratio of <their>doubles. So, the ratio of the double of *AZ*, which is *GA*, to *AE*, is less than the ratio of the double of the angle *ZDA*, which is the angle *GDA*, to the angle *ADE*. Separando, the ratio of the line <segment> *GE* to *EA* is less than the ratio of the angle *GDE* to the angle *EDA*. The ratio of *GE* to *EA* is equal to the ratio of the chord *GB* to the chord *BA*, and the ratio of the angle *GDB* to the angle *BDA* is equal to the ratio of the arc *GB* to the arc *BA*. So, the ratio of the chord *GB* to the chord *BA* is less than the ratio of the arc *GB* to the arc *BA*. This is what we wanted to demonstrate.

Chapter 11: On measuring the Chord of 1° very accurately and the composition of the <table of the> Chords.

It was shown in Chapter 7 how to find the Chord of the difference between a sixth and a fifth of a circle, which is <equal to> the Chord of

12°. From Chapter 8, <we can find> the Chord of its half and half of its half, up to the Chord of one and half degrees, and the Chord of a half plus a quarter of a degree. After this <introduction>, we draw a circle <with the points> *A*, *B*, <and> *G* on it. First, we take the line <segment> *AB* as the Chord from the circle of the arc of a half plus a quarter of a degree, and *AG* as the Chord of 1°. Then the ratio of the Chord of *AG* to the Chord of *AB* is less than the ratio of the arc *AG* to the arc *AB*. The arc *AG* is equal to the arc *AB* plus one-third of it. So the Chord *AG* is less than the Chord *AB* plus one-third of it. The Chord *AB* plus one-third of it <is> 0; 1, 2, 49, 52. Again we take in this circle the line <segment> *AB* as the Chord of 1° and the line <segment> *AG* as the Chord of one and a half degree. Then the arc *AG* is equal to the arc *AB* plus half of it. So the Chord *AG* is less than the Chord *AB* plus half of it. So the Chord *AB* is



greater than two thirds of the Chord *AG*. Two thirds of the Chord *AG* <is> 0; 1, 2, 49, 48. Since the Chord of 1° is once <found> less and another time more than the same thing exactly, without a <noticeable> magnitude difference, if half the difference is added to the smaller value, the Chord of 1° is found with the closest approximation <equal to> 0; 1, 2, 49, 50. After knowing this, <I add that> in Chapter 7 <finding> the Chord of the sum of two arcs has been explained. <Since> the Chord of 1° is known, the Chord of 2° is <also> known. Again, the Chord of 1° and the Chord of 2° are known, so the Chord of 3° is <also> known. Again, the Chord of 1° and the Chord of 3° are known, so the Chord of 4° is known as well. On this basis we compose the Chords of <any number of> degrees up to 90° and put them in the table. This is what we wanted to demonstrate.

Commentary

Book IV actually has 70 chapters as found in F, V, L, and M. However, F mentions the number of the chapters to be 60, whereas L, M and V mention it to be 66 in the contents list provided in the opening of the Book. The number of the chapters in Y and A, which lack chapters IV.3.4, IV.3.6, IV.3.7, and IV.4.1, is equal to 66. Most parts of this section are direct quotations from the *Almagest*. The contents of IV.1 are also found in al-Battānī's *zīj* [1899, III, 13-14] without any proof.

IV.1.1 This introductory chapter defines the terms Chord, Sine, Sagitta, and complement or supplement of any arc.

IV.1.2 This method is found in Ptolemy's *Almagest* I.10 [1984, 50].

IV.1.3 In Ptolemy's *Almagest* I.10 [1984, 49-50], the approximate value of the Chord of 90° is found by this method.

IV.1.4 Again, the approximate value of the Chord of 120° is found by this method in Ptolemy's *Almagest* I.10 [1984, 49-50].

IV.1.5 This proof is found in Ptolemy's *Almagest* I.10 [1984, 48-49].

IV.1.6 This is usually called Ptolemy's theorem. Kūshyār presents a proof similar to that in Ptolemy's *Almagest* I.10 [1984, 50-51].

IV.1.7 Kūshyār's proof is found in Ptolemy's *Almagest* I.10 [1984, 51].

IV.1.8 This proof is found in Ptolemy's *Almagest* I.10 [1984, 52-53].

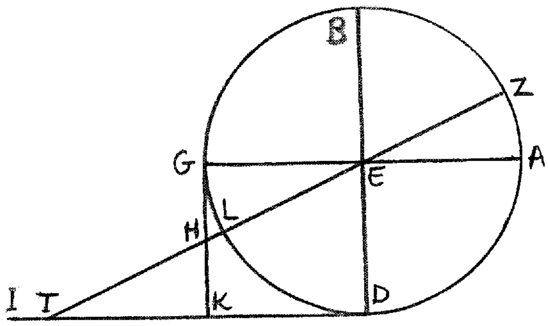
IV.1.9 This theorem is also proved by using Ptolemy's theorem in Ptolemy's *Almagest* I.10 [1984, 53]. However, Kūshyār's proof is simpler.

IV.1.10 When speaking about [the ratios of] sectors or triangles, Kūshyār means [the ratios of] their areas. This proof is found in Ptolemy's *Almagest* I.10 [1984, 54-55].

IV.1.11 Ptolemy presented the same method in *Almagest* I.10 [1984, 55-56] with a less accurate result (1; 2, 50) compared to Kūshyār's result 1; 2, 48, 50). In I.2.1 Kūshyār takes Sine 1° equal to 1; 2, 49, 38, 31. Both Kūshyār's values are correct to 2 sexagesimal digits. Sin 1° is 1; 2, 49, 43, 11 correct to 4 sexagesimals.

Section 2: On Tangents and Cotangents, <in> three chapters
Chapter 1: On the description of Tangents and Cotangents.

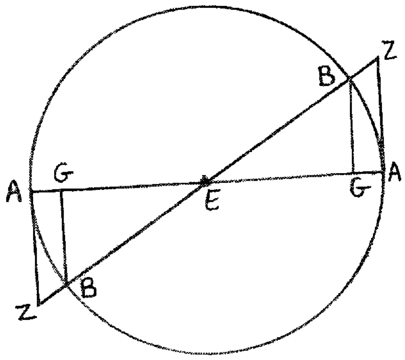
Let *ABGD* be the altitude circle with *E* as its center, and <let> *DI* be the intersection of the plane of the altitude circle and the horizon circle, and <let> *DE* be the vertical gnomon perpendicular <to the horizon> at the point *D*, and <let> *GK* be the intersection of the plane of the altitude circle and the plane perpendicular to the horizon <plane>, and <let> *GE* be the gnomon parallel to the horizon plane, perpendicular to the above mentioned (i.e., the vertical) plane at the point *G*.



We assume *AZ* <as> the altitude arc. We draw *ZET* which is the ray joining the tip of the gnomon and the endpoint of the shadow. *DT* is the shadow of the gnomon *DE*, and it is <called> the Horizontal Shadow and <also> the Cotangent (lit., "Second Shadow") of the altitude *AZ*. *GH* is the shadow of the gnomon *GE*, and it is <called> the Reversed Shadow and <also> the Tangent (lit., "First Shadow") of the altitude *AZ*. If we assume *BZ* as the altitude <arc>, then *GE* will be the gnomon for the Horizontal Shadow (Cotangent) and *DE* will be the gnomon for the Reversed Shadow (Tangent). So, *DT* will be the Tangent of the altitude *BZ* and *GH* will be its Cotangent. But *BZ* is the complement of *AZ*. So the Tangent of any altitude is the Cotangent of the complement of this altitude. The Reversed Shadow (Tangent) is called the First <Shadow> because it begins to appear and to increase <simultaneously> with the appearance and increase of the altitude of the sun. The Second Shadow (Cotangent) decreases with increasing altitude. This is what we wanted to demonstrate.

Chapter 2: On finding the quantity of the (i.e., any) Tangent.

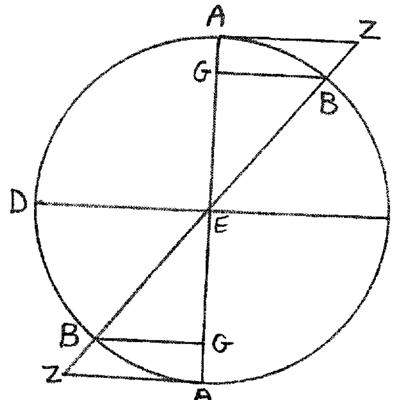
Let *ABGD* be the altitude circle centered at *E* and <let> *AEA* be its diameter, and <let> *AB* be the altitude arc. We draw *EBZ*, and we draw *AZ* perpendicular to *AE*. We also draw *BG* perpendicular to *AE*. Then *AZ* is the Tangent of the altitude *AB*. I say that it is known.



Proof: *ZA* and *BG* are both perpendicular to *AE*, so they are parallel. So the ratio of *ZA* to *AE* is as the ratio of *BG* to *GE*. But *AE* is the radius <of the circle> and it is equal to the gnomon <which may be> assumed <as divided> into any <number of> parts, and *BG* is the Sine of the arc *AB* and *GE* is equal to its Cosine. Therefore *AZ* is known. This is what we awnted to demonstrate.

Chapter 3: On finding the quantity of the (i.e., any) Cotangent.

Let *ABGD* be the altitude circle centered at *E*, and *AEA* and *DED* its two <perpendicular> diameters. We assume arc *DB* as the altitude. We draw *EBZ* and we draw *AZ* perpendicular to *AE*. We also draw *BG* perpendicular to *AE*. Then *AZ* is the Cotangent of the altitude *DB*. I say that it is known.



Proof: *ZA* and *BG* are both perpendicular to *AE*, so, they are parallel. So the ratio of *ZA* to *AE* is as the ratio of *BG* to *GE*. But *AE* is the radius <of the circle> and it is equal to the gnomon <which may be> assumed <as divided> into any <number of> parts, and *BG* is the Cosine of the altitude and *GE* is equal to the Sine of the altitude. Therefore *AZ* is known. This is what we wanted to demonstrate.

Commentary

IV.2.1 In the Islamic-period trigonometry the function 60tgθ was called الظل الاول (*al-ẓill al-awwal*, lit. "the first shadow"), الظل المعكوس (*al-ẓill al-ma'kūs*, lit. "the reversed shadow"), الظل المنكوس (*al-ẓill al-mankūs*, lit. "the inverted shadow"), or الظل المنتصب (*al-ẓill al-muntaṣab*, lit. "<vertically> erected shadow"). Similarly, the function 60cotgθ was called الظل الثاني (*al-ẓill al-thānī*, lit. "the second shadow"), الظل المستوي (*al-ẓill al-mustawī*, lit. "the horizontal shadow"), الظل المبسوط (*al-ẓill al-mabsūṭ*, lit. "the <horizontally> extended shadow"), or الظل البسيط (*al-ẓill al-basīṭ*, lit. "the plain shadow") [Kennedy 1956, 140; al-Bīrūnī 1954, I, 332-54; al-Bīrūnī 1985, 127-29]. Whenever the term "shadow" was used without any adjective, it meant the First Shadow (Tangent) [al-Bīrūnī 1985, 129]. In this work, I always translate "the first shadow, meaning 60tgθ, as the Tangent, and "the second shadow", meaning 60cotgθ, as the Cotangent.

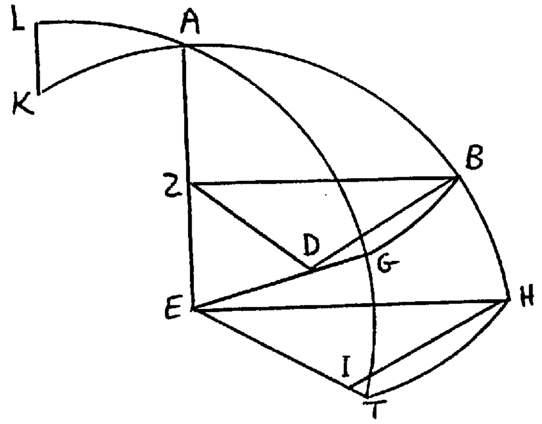
I do not know why Kūshyār uses the expression "horizon circle" and not merely "horizon", while al-Bīrūnī [1985, 127] uses the correct expression "horizon plane". Ptolemy [1984, 80-82] discusses the noon shadows at equinoxes and solstices, but he does not mention "Shadows" as trigonometric functions. Al-Battānī [1899, III, 31-33] discusses the horizontal and vertical shadows in their proper meaning and describes their calculation in terms of the Chords and vice versa. In the commentary to this and the following section, I always refer to al-Bīrūnī's *Maqālīd 'ilm al-hay'a* ("The keys to astronomy") [al-Bīrūnī 1985], because this was a standard work on spherical trigonometry, and the first independent treatise on the subject, whose author was a contemporary of Kūshyār and was aware of Kūshyār's work [al-Bīrūnī 1985, 101, 103, 143, 145].

IV.2.2 A similar method using a different figure is provided by al-Bīrūnī [1985, 129].

IV.2.3 A similar method using a different figure is provided by al-Bīrūnī [1985, 127].

Section 3: On premises on which the proofs are based, <in> 7 chapters
Chapter 1: On a general premise for most proofs.

<In> any triangle consisting of arcs of great circles on the sphere, in which one angle is right and another angle is assumed, the ratio of the Sine of the side subtending the right angle to the Sine of the side subtending the given angle, is equal to the ratio of the greatest Sine to the Sine of the assumed angle. Let the triangle be *ABG*, and its right angle be *G*, and the assumed <angle> be *BAG*. I say that the ratio of the arc *AB* to the Sine of the arc *BG* is equal to the ratio of the greatest Sine to the Sine of the angle *BAG*.



Proof: The center of the sphere is *E*. We draw *AE*. We complete each of the arcs *AB* and *AG* to quadrants, *AH* and *AT*. We take *A* as the pole and we draw the arc *HT* with radius equal to the side of an <inscribed> square. Then angle *HTG* is right. We draw *GE* and *TE* both equal to the radius of the circle *AGT*. Then they are in the plane of that circle. We draw *BD* perpendicular to *GE*, and *HI* perpendicular to *TE*. Then they are perpendicular to the plane of the circle *AGT*. We draw *BZ* perpendicular to *AE* and similarly, *HE* perpendicular to it. Then they are in the plane of the circle *ABH*. We draw *DZ*. Then *BZ* is the Sine of the arc *AB*, *BD* is the Sine of the arc *BG*, *HE* is the greatest Sine, and *HI* is the Sine of the arc *HT*, and it is <also> the Sine of the angle *BAG*. Since *BD* and *HI* are perpendicular to the plane of the circle *AGT*, all lines drawn <in the plane of circle *AGT*> from <any of> the two points *D* and *I* make a right angle with the perpendicular. So, the angles *D* and *I* are right. Thus, *BZ* and *HE* are parallel, *BD* and *HI* are parallel; then, *ZB* and *BD* are parallel to *EH* and *HI*<, respectively>. So, the angle *ZBD* is equal to the angle *EHI*. The angles *D* and *I* are right, then the angles *Z* and *E* of the two triangles are equal. So the two triangles *ZBD* and *EHI* are similar, so the ratio of *ZB* to *BD* is equal to the ratio of *EH* to *HI*. It has previously been said that *ZB* is the Sine of the arc *AB*, *BD* is the Sine of the arc *BG*, *EH* is the greatest

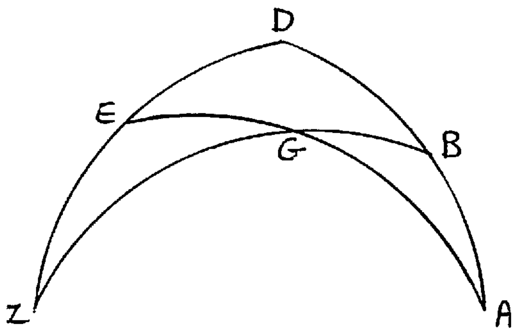
Sine, and *HI* is the Sine of the angle *HAT*. Therefore, the ratio of the Sine of the arc *AB* to the Sine of the arc *BG* is equal to the ratio of the greatest Sine to the Sine of the angle *HAT*. This is what we wanted to demonstrate.

Now it has become clear that in any two triangles in the sphere with two <correspondingly> equal angles, and two right angles, the ratio of the Sine of the side subtending the right angle in one triangle to the Sine of the side subtending the <other> equal angle is equal to the ratio of the Sine of the side subtending the right angle in the other triangle to the Sine of the side subtending the angle corresponding to the first one.

<Proof:> On the basis of this rule (i.e., the general premise), if the angle *L* in the triangle *AKL* is right, we may mount the arc *AL* on the arc *AG* as the arc *AK* may be mounted on the arc *AB*, because the angles *A* <in the two triangles> are equal, and the ratio of the Sine of the arc *AK* to the Sine of the arc *LK* will be equal to the ratio of the Sine of the arc *AH* to the Sine of the arc *HT*. Similarly, if the angle *K* is right, and we mount the arc *AK* on the arc *AG*, as the arc *AL* may be mounted on the arc *AB*, then the ratios are those ratios <which we just described>.

Chapter 2: On another premise which derives from the first one.

<In> any triangle consisting of arcs of great circles on the sphere, in which one angle is right, the ratio of the Cosine of one of the two sides encompassing the right angle to the Cosine of the hypotenuse of the right angle is equal to the ratio of the greatest Sine to the Cosine of the third side. Let the angle *B* in the triangle *ABG* be right; then I say that the ratio of the Cosine of *BG* to the Cosine of *GA* is equal to the ratio of the greatest Sine to the Cosine of *AB*.



Proof: We take *A* as the pole and we draw the circle *DHZ* with distance equal to side of an <inscribed> square. We complete the quadrants *DEZ*, *AGE*, *ABD*, and *BGZ*; then the angle *E* in the triangle *ZGE* is right. So, according to what was demonstrated in the first premise, the ratio of the Sine of *ZG* to the Sine of *GE* is equal to the ratio of the greatest Sine to the Sine of the angle *Z*. But *ZG* is the complement of *BG*, *GE* is the complement of *AG*, and *BD* is the arc of the angle *Z*, and it is the complement of *AB*. Thus the ratio of the Cosine of *BG* to the Cosine of *AG* is equal to the ratio of the greatest Sine to the Cosine of *AB*. This is what we wanted to demonstrate.

Chapter 3: Notice on the properties of the proportional magnitudes.

If there are four proportional magnitudes, and <also> four others making another ratio, so that the ratios are not in continued proportion, and the two means of the first <proportion> are equal to the two means of the other <respectively> (First example), then by composition <of ratios>, the ratio of the <first> antecedent to the <second> antecedent is equal to the ratio of the <first> consequent to the <second> consequent, in inverse order. Also, the ratio of the <first> antecedent to the <second> consequent is equal to the ratio of the <second> antecedent to the <first> consequent, in inverse order. If the two antecedents of the first <proportion> are equal to the two antecedents of the second (Second example), <respectively,> then the ratio of the <first> consequent to the <second> consequent of the first <proportion> is equal to the ratio of the <first> consequent to the <second> consequent of the other <proportion>. If the two consequents of the first <proportion> are equal to the two consequents of the other <proportion> (Third example), then the ratio the <first> antecedent to the <second> antecedent of the first <proportion> is equal to the ratio of the <first> antecedent to the <second> antecedent of the other <proportion>. This is what we wanted to mention.

First example

A	**B**	**G**	**D**
2	4	3	6
E	**W**	**Z**	**H**
1	4	3	12

Second example

A	**B**	**G**	**D**
2	4	3	6
E	**W**	**Z**	**H**
2	8	3	12

Third example

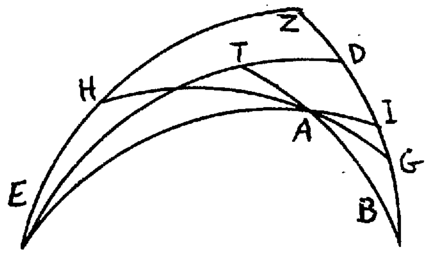
A	**B**	**G**	**D**
2	4	3	6
E	**W**	**Z**	**H**
1	4	1; 30	6

Proof: <In the first example,> the ratio of **A** to **B** is equal to the ratio of **G** to **D**, the ratio of **E** to **W** is equal to the ratio of **Z** to **H**, and **B** is equal to **W** and **G** is equal to **Z**. <So> the product of **B** by **G** is equal to the product of **A** by **D**, and the product of **W** by **Z** is equal to the product of **E** by **H**. We cast out the equal <product of the means>. It follows that the product of **A** by **D** is equal to the product of **E** by **H**. Thus the ratio of **A** to **H** is equal to the ratio of **E** to **D**.
<In> the second example, the product of **B** by **G** is equal to the product of **A** by **D**, and the product of **W** by **Z** is equal to the product of **E** by **H**. But **A** is equal to **E** and **G** is equal to **Z**. Thus the product of **B** by **Z** is equal to the product of **E** by **D** and the product of **G** by **W** is equal to the product of **A** by **H**. We cast out the equal <product of the means>. It follows that the product of **B** by **H** is equal to the product of **W** by **D**. Thus the ratio of **B** to **W** is equal to the ratio of **D** to **H**.
<In the> third example, the product of **B** by **G** is equal to the product of **A** by **D**, and the product of **W** by **Z** is equal to the product of **E** by **H**. <Now,> **B** is equal to **W**, and **D** is equal to **H**. Then the product of **B** by **Z** is equal to the product of **E** by **D**, and the product of **G** by **W** is equal to the product of **A** by **H**. We cast out the equals. There remains that the product of **A** by **Z** is equal to the product of **E** by **G**. Then the ratio of **A** to **E** is equal to the ratio of **G** to **Z**. That is what we wanted to demonstrate.
What <we proved> in this notice on proportions <may be> summarized <as follows:> If the second <term>s are equal and the third <term>s are equal, the ratio of the first <term of the first proportion> to the first <term of the second proportion> will be equal to the ratio of the fourth <term> to the fourth <term>, in inverse order. If the first <term>s are equal and the third <term>s are equal, the ratio of the second <term of the first proportion> to the second <term of the second proportion> will be equal to the ratio of the fourth <term> to the fourth <term>, in the same order. If the second <term>s are equal and the fourth <term>s are equal, the ratio of the first <term> to the first <term> is equal to the ratio of the third <term> to the third <term>, in the same order, and the ratio of the first <term> to the third <term of the first proportion> is equal to the ratio of the first <term> to the third <term of the second proportion, in the same order>.

Chapter 4: On another premise which also derives from the first one.

<In> any triangle consisting of arcs of great circles, the ratio of the Sine of an angle of it to the Sine of another angle <of it> is equal to the Sine of the side subtending the first angle to the Sine of the side subtending the

other angle. <Let> the triangle *ABG* have different sides and angles; then I say that the ratio of the Sine of the angle *B* to the Sine of the angle *G* is equal to the Sine of the arc *AG* to the Sine of the arc *AB*.

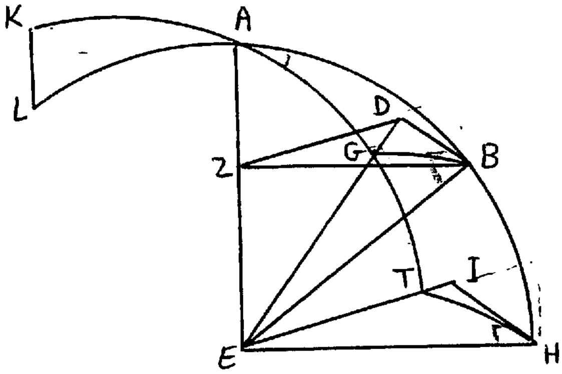


Proof: We take *B* as the pole and we draw <the circular arc> *DTE* with a distance equal to side of an <inscribed> square. We take *G* as the pole and we draw *ZHE* <similarly>. We complete each of <the arcs> *BGZ*, *BAT*, and *GAH*, and we draw *EAI*. Since *B* is the pole of *ETD*, *BT* and *BD* are quadrants. Since *E* is the pole of *BDZ*, *EZ*, *ED*, and *EI* are quadrants. Since *G* is the pole of *ZHE*, each of <the arcs> *GE*, *GZ* are quadrants. Then, in the triangle *BAI* the angle *I* is right. So, the ratio of the Sine of *BA* to the Sine of *AI* is equal to the ratio of the greatest Sine, which is the Sine of *BT*, to the Sine of *TD*. Also the angle *I* in the triangle *GAI* is right. So the ratio of the Sine of *GA* to the Sine of *AI* is equal to the ratio of the greatest Sine, which is the Sine of *GH*, to the Sine of *HZ*. Since the means of the first <proportional> magnitudes, i.e., <the Sines of> *AI* and *BT*, are equal to the means of the other <proportional> magnitudes, i.e., <the Sines of> *AI* and *GH*, therefore the ratio of the Sine of *BA*, which is the side subtending the angle *G*, to the Sine of *AG*, which is the side subtending the angle *B*, is equal to the ratio of the Sine of *HZ*, which is equal to the Sine of the angle *G*, to <the Sine of> *TD*, which is the Sine of the angle *B*. Thus the ratio of the Sine of an angle to the Sine of <another> angle <in any triangle> is equal to the Sine of the side subtending the <first> angle to the Sine of the side subtending the <other> angle. This is what we wanted to demonstrate.

Chapter 5: On a premise concerning the Tangent<s>, which is a substitute for the first premise in most proofs.

<In> any triangle consisting of arcs of great circles, in which an angle is right and another angle is assumed, the ratio of the Sine of the side between the right angle and the assumed angle to the Tangent of the side subtending the assumed angle is equal to the ratio of the greatest Sine to the Tangent of the assumed angle. Let *ABG* be the triangle where angle *B* is right and *BAG* is the assumed angle. Then I say that the ratio of the

Sine of the arc *AB* to the Tangent of the arc *BG* is equal to the ratio of the greatest Sine to the Tangent of the angle *BAG*.

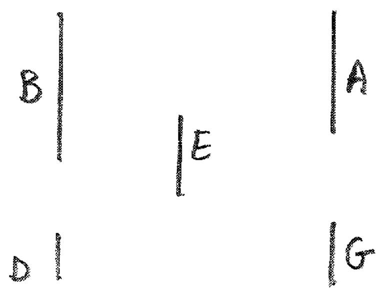


Proof: With *E* as the center of the sphere, we draw *AE* and we complete each one of <the arcs> *AB* and *AG* into the quadrants *AH* and *AT*. We draw *HE* and *BZ* perpendicular to *AE*. We take *A* as the pole and we draw the arc *HT* with a distance equal to the side of an <inscribed> square. We draw *EG* and *ET*, the radii of the circle *AGT*, and we extend them to *D* and *I*. We draw *EB*, a radius of the circle *ABH*. We draw *BD* and *HI* perpendicular to the diameters <containing> *EB* and *EH* at *B* and *H*. We draw *DZ*. Then *BZ* is in the plane of *ABH*, so it is the Sine of the arc *AB*. *HE* is also in that plane, and it is the greatest Sine. *ZB* and *HE* make right angles with the perpendiculars *BD* and *HI*, <respectively>. So the two planes *BZD* and *HET* are parallel (see commentary). *BD* is perpendicular to the diameter <containing> *EB*, and therefore, perpendicular to the plane *ABH*. Then all lines drawn in the plane *ABH* make right angles with the perpendicular *BD*. So the angle *DBZ* is right and the two angles *HEI* and *BZD* are equal. Then the two triangles *HEI* and *ZBD* are similar. So the ratio of *ZB* to *BD* is equal to the ratio of *EH* to *HI*. But *ZB* is the Sine of the arc *AB*, *BD* is the Tangent of the arc *BG*, *EH* is the greatest Sine, and *HI* is the Tangent of the angle *HAT*. So the ratio of the Sine of the arc *AB* to the Tangent of the arc *BG* is equal to the ratio of the greatest Sine to the Tangent of the angle *BAG*. This is what we wanted to demonstrate.
Now it has become clear that in any two triangles in a sphere having two <respectively> equal angles and two right angles, the ratio of the Sine of the side between the right angle and the <other> equal angle to the Tangent of the other one of the two sides containing <the right angle> is equal to the ratio of the Sine of the corresponding <side> to the Tangent of the <other> corresponding side in the other triangle. This rule is valid,

because in the triangle *ALK*, the proof is equal to the ratio of the first premise, whether the angle *K* or the angle *L* is right.
The spherical right triangles will be known by these premises in <the following> three cases.
First <case>: An angle with one of the sides, either the side subtending the right angle or the side subtending the known angle, <are given. Solution:> The ratio of the Sine of the side subtending the right angle to the Sine of the side subtending the known angle is equal to the ratio of the greatest Sine to the Sine of the known angle.
Second <case>: Any two of its sides <are given. Solution:> The ratio of the Cosine of one of the sides containing the right angle to the Cosine of the side subtending the right angle is equal to the ratio of the greatest Sine to the Cosine of the third side; the ratio of the Sine of the side subtending the right angle to the Sine of the other <known> side is equal to the ratio of the greatest Sine to the Sine of the angle <opposite to> the other known side.
Third <case>: An angle with a side adjacent to it, namely one of the two sides containing the right angle, <are given. Solution:> The ratio of the Sine of this side to the Shadow of the other side containing the right angle is equal to the greatest Sine to the Shadow of the known angle.

Chapter 6: Notice on the properties of the Tangent<s>.

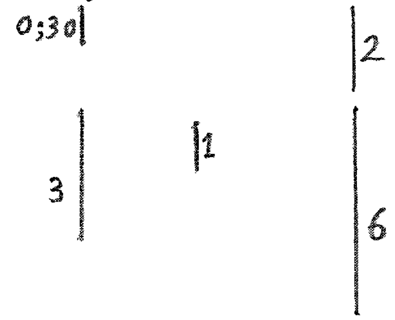
<Given> any two different arcs, their Tangent is the inverse of their Cotangent. Let **A** and **B** be the Tangents of two different arcs, **G** and **D** be their Cotangents, and **E** be the gnomon. For the arc whose Tangent is **A**, let its Cotangent be **G**, and for the arc whose Tangent is **B**, let its Cotangent be **D**; then I say that the ratio of **A** to **B** is equal to the ratio of **D** to **G**.



Proof: The ratio of **A** to **E** is equal to the ratio of **E** to **G**, and the ratio of **B** to **E** is equal to the ratio of **E** to **D**. Then the product of **A** by **G** is equal to the product of **B** by **D**. So, the ratio of **A** to **B** is equal to the ratio of **D** to **G**.

Chapter 7: Another notice also on the properties of the Tangent<s>.

Something divided by the Tangent or Cotangent of any arc is equal to the <same> thing multiplied by its (i.e., the arc's) Cotangent or Tangent <, respectively>. <Example:> Let the Cotangent of a given arc be 2, its Tangent be 0; 30, and <the length of> the gnomon be 1, that is, unity. <If> the magnitude 6 is divided by 2, <the result> is 3, and then I say that 3 is equal to the product of 6 by 0; 30.



Proof: 6 divided by 2, is 3. Then the product of 2 by 3 is 6, and the product of 2 by 0; 30, i.e. a half is 1, because the ratio of 2 to 1 is equal to that of 1 to 0; 30. Then the ratio of 0; 30 to 3 is equal to that of 1 to 6. The product of 0; 30 by 6 is equal to that of 1 by 3. The product of 1 by 3 is 3, because 1 is <the length of> the gnomon which is taken <equal to> unity. Then the product of 0; 30 by 6 is 3. Since the Cotangent of any arc is <equal to> the Tangent of its complement, something divided by the Tangent of any arc is equal to the <same> thing multiplied by the Tangent of its complement. This is what we wanted to demonstrate.

Commentary

This chapter has already been translated into English and published with an introduction, summary and commentary based on ms. L in [Berggren, 1987]. I have mentioned the differences of that translation with mine, whenever they are significant. There are also some minor differences mostly due to the differences between the two mss.

IV.3.1 This is a special case of the Sine Theorem for right spherical triangles. Al-Bīrūnī [1985, 101,103] notes that Kūshyār took this theorem from Abū Maḥmūd Khujandī, named it *al-Mughnī* (lit., "making [one] able to dispense" [with Menelaus' Theorem]), and abridged Khujandī's proof of it. Al-Bīrūnī later quotes Kūshyār's abridged version of the proof [*op. cit.*, 143, 145] and adds that Kūshyār did not find the generalized form of this theorem. However, we find a generalization of it in IV.3.4. The second proof at the end of this chapter seems to be a later addition which exists in F but is not found in A and Y, and has later been added to the mss. V and L.

IV.3.2 This is equivalent to the Cosine Theorem for right spherical triangles. According to al-Bīrūnī [*op. cit.*, 151], Abū'l-ʿAbbās al-Nayrīzī and Abū Jaʿfar al-Khāzin in their non-extant commentaries on Ptolemy's *Almagest* presented this theorem. Al-Bīrūnī then quotes the proof provided by them, which is more complicated than Kūshyār's.

IV.3.3 Here Kūshyār states the following theorems. If A:B=G:D and E:W=Z:H, then:

(a) If B=W, G=Z, we have A:E=H:Z and A:H=E:Z
(b) If A=E, G=Z, we have B:W=D:H.
(c) If B=W, D=H, then A:E=G:Z.

Kūshyār supposes that "the ratios are not in continued proportion", i.e., A:B≠E:W. If A:B=E:W, we would have A=E, B=W, G=Z, and D=H in all three cases (a), (b), and (c).
In the ms. L, the part on the proofs of the three propositions which comes before the final part, the summary of the propositions of this notice, is transferred to Section 5 of this chapter, just before the proof of the Tangent Theorem. Prof. Berggren who has based his English translation of this chapter on ms. L, has restored the misplaced fragment to the end of Chapter 3 [1987, 24, 28]. He shows that this is a piece skipped by the scribe who, after he had noted his error, simply put it at the first available

place. However, the proof for the third example is missing in Prof. Berggren's translation [*op.cit.*, 24].

IV.3.4 This is the general case of the Sine Theorem for a general spherical triangle. Kūshyār's proof is similar to the proof which al-Bīrūnī presents for this theorem [1954, 355-56]. This chapter is missing in the mss. A and Y.

IV.3.5 This is the Tangent Theorem for right spherical triangles which, according to al-Bīrūnī, was discovered by Abū al-Wafā' al-Būzjānī. Kūshyār's proof is similar to al-Būzjānī's as presented by al-Bīrūnī [1985, 131]. The second part of this chapter, which discusses the solution of right spherical triangles, was presented by al-Bīrūnī [*op. cit.*, 169-91] in more details: In proving that the planes *BZD* and *HET* are parallel, it must also be noticed that the spherical angles *B* and *H* are right angles, so the planes *HEI* and *BEG* are perpendicular to the plane of *ABH*. Therefore *BD* and *HI* are both perpendicular to the plane *ABH*. Prof. Berggren has translated the title of this chapter differently: "On a premise called 'the Shadow' established on the basis of the first premise in many of the proofs". The Arabic title in L is like what we read in this edition but written in a less clear form, and this title implies that the Tangent Theorem may be used instead of the first premise (the Sine theorem) in many proofs. At the end of the proof of this chapter, before mentioning the three cases of spherical right triangles, the sentence "This rule is valid, because ..." which involves a reference to a later addition in Chapter 1, also seems to be a later addition to F, and is not found in other mss. So it is not found in Berggren's translation either.

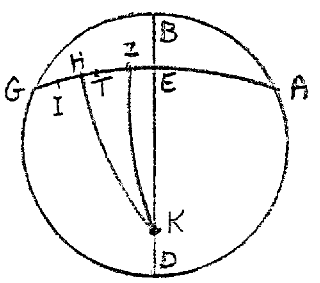
IV.3.6 and IV.3.7 Since the Tangent and Cotangent functions are inversely proportional by definition, these propositions are trivial. It is quite interesting that Kūshyār takes the length of the gnomon equal to unity, as we do now.

IV.3.7 In [Berggren 1987, 27] the Arabic characters involved in this chapter are supposed to denote magnitudes rather than their numerical values in the *Abjad* numeral system, possibly due to the misleading diagram in L, which was the only ms. accessible to Berggren. However, the present interpretation is more consistent and leaves no ambiguity about the content of the chapter.

Section 4: On <finding> the true longitudes of the planets and their positions, <in> 10 chapters

Chapter 1: On the equation of time.

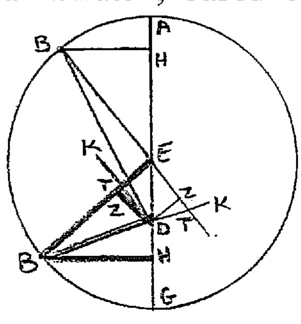
As has been said in Book III, this equation is the difference between the mean day and the true day. The mean day is <the duration of one> rotation of the celestial equator from the (i.e., any) meridian to the <same> meridian, plus the arc of it (i.e., the equator) equal to the mean daily motion of the sun. The true day is <the duration of one> rotation of the celestial equator from the (i.e., any) meridian to the <same> meridian, plus the part of it (the equator) which rises together with the varying <daily> motion of the sun (i.e., the difference between the right ascensions of the true solar longitudes at the end and the beginning of the day). <The maximum magnitude of> (i.e., an upper bound of) this equation is the sum of twice the difference between the <solar> ecliptical degrees and the <corresponding> right ascensions, plus twice the difference between the sun's mean longitude and its <corresponding> true longitude. That part relating to the difference of the <right> ascension <and the true longitude> is <at most> about 5 degrees, and that part relating to the difference of the <positions of the mean and the true> sun <on the ecliptic> is <at most> 4 degrees, approximately. Then the sum of the two <maximum> differences is approximately 9 <time->degrees, which is three-fifths of an equinoctial hour, minus a small amount. <However,> this equation never reaches the total <amount>, because while one of the two differences is maximum, the other is somewhat less than its maximum, except when the apogee is in the middle or in the last decan of Leo, because <the variation of> this equation in one or two days is not noticeable. We may define any position on the ecliptic as the base (i.e., the point of reference or zero point). But if we define the middle decan of Aquarius (as the zero point of the equation of time), the mean days will always exceed the true days, until it (i.e. the sun) reaches the aforesaid apogee. If another <position> is defined as the base (i.e., the zero point), the mean days sometimes exceed the true days and sometimes are less than them. Now I say that the <amount in> hours of the excess of the mean days over the true <days> is known.



Proof: *ABGD* is the horizon circle, *DB* the meridian, and *AEG* the celestial equator with *K* as its pole. Let point *E* be the mean longitude of the base, which is one of the degrees in the middle decan of Aquarius, and *Z* the <right> ascension of its true longitude. We draw *ZK*. Let point *T* be another mean longitude. The <right> ascension relating to its true longitude may be less or more than it. First we let it be more, as *H*. We draw *HK*. In the situation which we described, the difference between the two mean longitudes <here represented as arcs on the celestial equator> is greater than the difference between the <right> ascensions of the two true longitudes. Thus, the arc *ET* is greater than the arc *ZH*. *ZT* is common <to them> so *EZ* is greater than *TH*. Therefore, the time in which the arc *ET* passes the meridian is greater than the time in which the arc *ZH* passes the meridian, in the amount of the excess of *EZ* over *TH*. Each one of the arcs *ET*, *ZH*, *EZ*, and *TH* is known, so the excess of *ET* over *ZH* is known. Each 15 degrees of the equator <corresponds to> one hour. So the magnitude of this excess with respect to 15 degrees is known. So the excess of the mean days over the true days is known. It is the deficit of the true days from the mean days, if we want <to find> the mean days. Also let the point *I* be a mean longitude and the point *H* the right ascension of its true longitude, being less than it (i.e., the mean longitude). Thus *EI* is greater than *ZH* in <the amount of the sum of> the two arcs *EZ* and *HI*. The two arcs *EI* and *ZH* are known, so the sum of the two arcs *EZ* and *HI* is known. The time in which the arc *EI* passes the meridian is greater than the time in which the arc *ZH* passes <the meridian> in the amount of the sum of the arcs *EZ* and *HI*. But their amount with respect to 15 degrees is known. So the excess of the mean days over the true days is known. It is the deficit of the true days from the mean days, if we <to find> the mean days from the true days. This is what we wanted to demonstrate.

Chapter 2: On the equation of the sun.

<Let> *ABG* <be> the circle of the eccentric orb with *E* as its center and *AG* as its diameter, and <let> *D* <be> the center of the orbit representing the ecliptic (i.e., parecliptic; *D* is the center of the earth). Then *DE* is the eccentricity. It has been found to be <equal to> 2 parts and 4 minutes plus half and a quarter <of a minute>, based on <taking> *EA* <equal to>

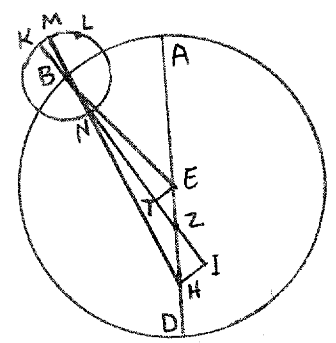


The bold lines are for the case in which the anomaly exceeds 90°

sixty parts. *A* is the position of the apogee, *B* is the body of the sun, and *AB* the solar mean anomaly. We drop *BH* perpendicular to *AE*. It is the Sine of the arc *AB*. <We drop> *DZ* perpendicular to *BZ*. The angle *ZED* is equal to the angle *HEB*, and the two angles *Z* and *H* are right. So the ratio of *EB* to *BH* is equal to the ratio of *ED* to *DZ*. *EB* is <equal to> sixty parts. *BH* and *ED* are known. So *DZ* is known, and *ZE* is known because *HE* is the Cosine of the mean anomaly (and *EB:HE=ED:ZE*). So *BZ* is known. The <sum of the> squares of *BZ* and *ZD* is equal to the square of *BD*. So *BD* is known. The ratio of *BD* to *DZ*, which is known (i.e., which has been computed) based on <taking> *BE* as the radius, is equal to the ratio of sixty to *DZ* in the magnitude in which it (i.e., *DZ*) is desired (i.e., we want *DZ* for *BD* = 60). So *DZ* based on <taking> *BD* as the radius is known. It is the Sine of the angle *ZBD*. So the angle *ZBD* is known, and it is the angle of the equation. That is what we wanted to demonstrate. Since *AEB* is an exterior angle of the triangle *BDE*, the angle *AEB*, which is the value of the mean anomaly, is greater than the angle *EDB*, which is the angle of the true longitude, in the amount of the angle *EBD* which is the angle of the equation. If the equation is to be subtracted from the mean anomaly or the mean longitude, <the angle of> the true longitude (i.e., *EDB*) and the mean anomaly are less than 180 <degrees>. If the mean anomaly is greater than 180 <degrees>, <we do> the opposite (i.e., we add the equation to the angle of the true longitude and the mean anomaly).

Chapter 3: On the first equation for the moon.

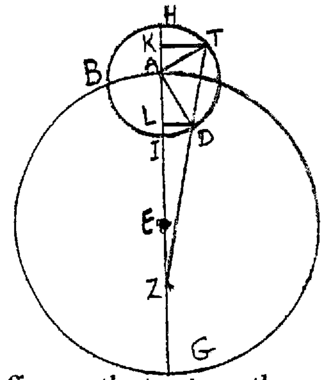
<Let> *ABG* <be> the circle of the eccentric orb (=deferent) with *E* as its center, *AD* as its diameter, *Z* as the center of the inclined orb (i.e., the lunar orbit; *Z* is the earth), *H* as the (prosneusis) point towards which the line joining the apogee and perigee of the epicycle, i.e. *M* and *N* is pointing, *LKG* as the epicycle centered at *B*, and *L* as the body of the moon. The angle *ZBH* is the angle of equation. Then *AZB* is the angle of double elongation. *EZ* and *ZH* are equal, each of them being <equal to> 12 parts and half based on <taking> *AE* <equal to> 60 parts. *ET* and *HI* are perpendicular to *BI*. The angle *EZT* is known, and the angle *T* is right. Then the remaining angle *E* as well as the sides of the triangle *EZT* are known.



EB is <equal to> 60 parts and its square is equal to the <sum of the> squares of *BT* and *TE*. So, *BT* and, therefore, the whole *BZ* are known. The angles of the triangle *EZT* are equal to those of the triangle *ZIH*. Then the ratio of *EZ* to *ZH* is equal to the ratio of *ZT* to *ZI* and to the ratio of *ET* to *HI*. *EZ* and *ZH* are equal. Then *IZ* and *ZT*, and *ET* and *HI* are equal. Then the whole *BI* is known and its square plus the square of *IH* is equal to the square of *BH*. Then *BH* is known. If we take the point *B* as the center and draw a circle with its radius equal to *BH*, then *HI* is the Sine of the angle *IBH* based on <taking> *BH* as the radius whose value is known (i.e., 60). Then *HI* is known, based on <taking> *BH* <equal to> 60 parts. Then the angle *IBH* is known (so *ZBH*, the angle of the first equation, is known). The angle *IBH* is equal to the angle *MBK*; so, the arc *MK* is known. *KL* is the adjusted anomaly based on <taking> *AB*, the double elongation, less than 90 <degrees>. If the double elongation is greater than 90 <but not greater than 180 degrees>, we find the angle of equation in the same way. If it is greater than 180 <degrees>, the equation is subtracted from the mean anomaly. That is what we wanted to demonstrate.

Chapter 4: On the second equation of the moon and the planets.

<Let> *ABG* <be> the circle of the eccentric orb centered at *E*, *Z* the center of the inclined orb, and *HTD* the epicycle centered at *A* (supposed to be the apogee). Let *T* <be> the position of the moon, because the motion of the moon is in this direction (i.e. clockwise on the epicycle). We join *TA* and *TZ*, and <we draw> *TK* perpendicular to *AH*. The angle *TZH* is the angle of equation. *TK* is the Sine of the adjusted mean anomaly, i.e. the arc *TH*, and *KA* is its Cosine. Each one of them is known based on <taking> *TA* <equal to> 5 parts and a quarter. Since the ratio of *AT* to *TK* is equal to that of the greatest Sine to the Sine of the <adjusted> anomaly and *ZA* is <assumed equal to> 60 parts, the whole *ZK* is known. Then its square plus that of *KT* is equal to *TZ* squared. Then *TZ* is known. If we take *Z* as the center and draw a circle with its radius <equal to> *ZT*, then *TK* is the Sine of the arc of equation angle in terms of the known value. But *TK* is known based on <taking> *TZ* <equal to> 60 parts. It is the Sine of the arc of the equation angle. In the same way, if we take the position of the moon at *D*, then *DI* is known, *DL* is its Sine and *AL* is its Cosine. Then *DL* is found by the previous method based on <taking> *ZD* <equal to> 60 parts. It is the Sine of *DZL*, the arc of the equation angle. That is what we wanted to demonstrate.

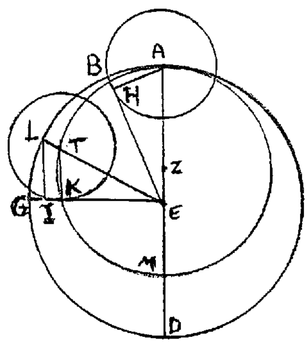


It is apparent from this figure that when the adjusted anomaly is less than 180 <degrees>, the arc of equation is subtracted from the mean longitude of the moon. When the <adjusted> anomaly is greater <than 180 degrees>, it (i.e., the arc of equation) is added to it (i.e., the mean longitude of the moon). For the other planets, if the adjusted anomaly is greater than 180 <degrees>, this equation is subtracted from the adjusted center. If the adjusted anomaly is less <than 180 degrees>, it (i.e., the arc of equation) is added to it (i.e., the adjusted center). This is because the motions of their bodies in the epicycles are in the opposite direction of that of the moon. That is what we wanted to describe.

Chapter 5: On the difference between the <apparent> radius of the epicycle between its maximum and minimum distances <from the earth>.

<If> the center of the epicycle of the moon is supposed <to be> at the maximum distance, and its distance from the center of the inclined orb is <taken as> 60 parts, <then> the radius of the epicycle in terms of this value is <equal to> 5 parts and a quarter. The maximum value of the second equation depends on the radius of the epicycle. The apparent value <of the equation> varies between the maximum and minimum distances, because the angle at the center of the inclined orb and subtended by the radius of the epicycle becomes greater when the center of the epicycle gets nearer to the center of the inclined orb. The same holds for the radius of the epicycles of the planets, however, <in this case> the centers of their orbs (i.e., epicycles) are supposed to be at mean distance when <the distance> between them and the center of the inclined orb is <taken to be> 60 parts. Between the mean distance and the maximum distance, the radius of their epicycles is less than the supposed value <of the radius>. Between the mean distance and the minimum distance, it (i.e., the radius of the epicycle) is greater than the supposed value <of the radius>. This is because the maximum distance for each planet is 60 <parts> plus half the eccentricity, and the minimum distance is 60 <parts> minus half the eccentricity. The ratio of each of these two (i.e., the minimum and the maximum) distances to the Sine of the <maximal> second equation at the mean distance, is equal to the ratio of 60 <parts> to the Sine of the <maximal> second equation at that

<minimum or maximum> distance. The same holds for other distances (i.e., the Sine of the maximal value of the second equation for any distance is inversely proportional to the distance).

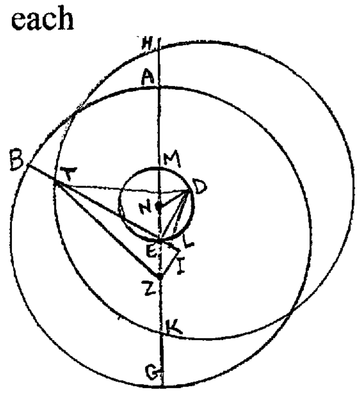


Let *ABGD* centered at *E* be the circle of the inclined orb; *ATM* centered at *Z* be the circle of the eccentric orb; *A* be the center of the epicycle at the maximum distance; and *T* be its center at another distance. We draw *EB* tangent to the circle at *H* and, we join *AH*. We draw *EG* tangent to the circle at *K* and we join *TK* and *LI* both perpendicular to *EG*. The angle *LEG* is greater than the angle *AEB*, because *ET* is smaller than *EA*. If we place it on *EA*, *TK* falls outside the line *EH*. *AH* is the radius of the epicycle at the maximum distance and its arc is *AB*. It subtends the angle *AEB*. Then *AB* is the maximum equation at maximum distance. *TK* is the radius of the epicycle at this <arbitrary> distance. The angle of equation is *LEG* and its arc is *LG*. Then *LG* is the maximum equation at this distance. The angle *LEG* is greater than the angle *AEB*. Then the arc *LG* is greater than the arc *AB*. The ratio of *ET* to *TK* is equal to that of *EL* to *LI*, because the two triangles *TEK* and *LEI* are similar. *ET* is known from the figure relating to the first equation (i.e., IV.4.3). It is <the distance> between the center of epicycle and the center of the inclined orb. The magnitude of *TK* is like *AH*, and *EL* is equal to *EA*. Then *LI* is known. It is the Sine of the arc *LG*. Then *LG* and its excess over *AB* are known. It is the total difference at this distance depending on <the length of> the line <segment> *ET*. The total difference at other distances is known in this <same> way. That is what we wanted to demonstrate.
We have written down the difference for the moon and the planets based on this calculation. We incorporated the approximate <magnitudes> of this equation (i.e., the difference, both for the moon and the planets) in a single table in a manuscript, but we do not intend <to compute a table like this here>. For the moon it (i.e., the difference) is uniformly additive from the maximum distance to the minimum distance. But for the planets <the difference is> diminutive from the maximum distance to the mean distance, and additive from the mean distance to the minimum distance. It is clear that this difference for the moon depends on the double elongation, which corresponds to *AL* in the figure. For the other planets, it

depends on the adjusted centrum. Then we look for the sixtieths, their ratio to 60 minutes being equal to the partial second equation to the total <second> equation. If we multiply these sixtieths by the <total> difference for the moon, the amount of the <partial second> equation in that position will result. Since there is no equation at the apogee of the epicycle, no difference is necessary for it, and at maximum equation, the total difference is necessary. That is why the difference is taken in terms of double elongation for the moon, <but> for other planets <it is taken> in terms of the adjusted center. The sixtieths are found from the <corresponding values of> the mean anomaly. It has become clear for us from this proof that the epicycle radius difference of Mars at maximum distance is one part and a fifth less than what is written in Ptolemy's tables. At minimum distance, it is 2 parts and a fifth <less than Ptolemy's value>. This is something that we used in our calculation of the tables. But the value in the <present> treatise is correct and this difference in the calculations is inevitable.

Chapter 6: On the first equation for Mercury.

<Let> *ABG* <be> the circle of the equant orb; *E* its center; *AG* its diameter; *Z* the center of the inclined orb; *N* the center of the small circle carrying the center of the deferent for the center of the epicycle; and *M* the center of the deferent. We imagine that *M* moves and describes the arc *MD* <in the direction> opposite to the succession <of the zodiacal signs> equal to the <amount of the> motion of the sun <from *A*>, and the center of the epicycle moves <simultaneously> with *M* in the direction <of the zodiacal signs> until it moves from *H* to *T* and describes the arc *AB* of the circle *ABG* similar to the arc *DM*. We take *D* as the center and we draw the deferent <equal> to the equant in magnitude. It is *HTK*. We join *ETB*, *ZT*, *DT*, *DN*, *DE*, <and we draw> *DL* and *ZI* perpendicular to *BI*. *ITZ* is the angle of equation. The two angles *MND* and *AEB* are equal, because their arcs are similar and each

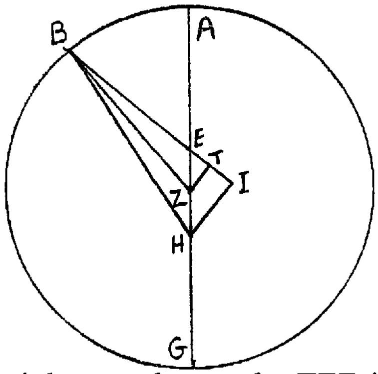


of them is <equal to> the angle of the centrum. So they are known. Each of the two arcs *MD* and *DE* are known. So, the chord *DE* is known in terms of the greatest diameter. Its ratio to it (i.e., to the greatest diameter) is like the chord *DE* to the diameter *EM*. *EM* is equal to 6 parts and a third. Then the chord *DE* is known. The angle *DEN* is half the angle *DNM*. Then the angle *DEN* is half the angle *AEB*. Then the whole angle *DEB* is known. The angle *DLE* is right. Then the angle *LDE* is known. *DE* is known, so, the sides of the triangle *LDE* are known. *DT* is equal to 60 parts and its square is equal to the <sum of the> squares of *DL* and *LT*. Then *LT* and *LE* are known. Then *TE* is known. The angle *ZEI* is also known, because it is equal to the angle *AEB*. The angle *I* is right. Then the angle *IZE* is known. *ZE* is known to be 3 parts and a sixth. Then the sides of the triangle *ZEI* are known. *TE* is known, so, *TI* is known and its square plus the square of *IZ* are equal to the square of *ZT*. Then *ZT* is known. If we take *T* as the center and draw a circle with its radius <equal to> *TZ*, *ZI* is the Sine of the arc of the angle *ITZ* in terms of the radius *ZT*. Then *ZI* is known based on <taking> *ZT* <equal to> 60 parts. It is the Sine of the angle of equation. This is what we wanted to demonstrate.

In this way, we obtain the equation for all sides of the circle. It is found by calculation that the line <segment> *TZ* is equal to 60; 30° for the centrum being <equal to> zero; it is <equal to> 60° for the centrum being <equal to> 66°; it is <equal to> 56; 50° for the centrum being <equal to> 90°; it is <equal to> 55; 20° for the centrum being <equal to> 120° and in this case the line *DT* coincides with the line *ET*; it is again <equal to> 56; 50° for the centrum being <equal to> 180°. Its maximum <value occurs> at the maximum distance. Its mean <value occurs> at the distance <equal to> 66°. Its minimum <value occurs> at the distance 120°. <Its values> are the same at the distances 90°and 180°. Since the angle *AZT* in this figure, being the angle of the centrum, is less than the angle *AET* and the difference between them is <equal to> the angle *ITZ*, it is necessary that we subtract the equation from the centrum and add <the equation> to the mean anomaly, if the centrum is less than 180°. If the center is greater than 180°, we should add <the equation> to the centrum and subtract <the equation> from the mean anomaly.

Chapter 7: On the first equation for the other planets.

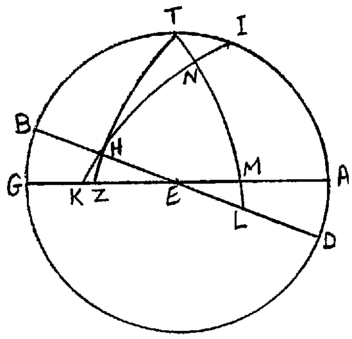
<Let> *ABG* centered at *Z* <be> the circle of the deferent; *AG* its diameter; *E* the center of the equant; and *H* the center of the inclined orb. *EZ* and *ZH* are equal. Each of them is <equal to> 3 parts plus a quarter and a sixth for Saturn, 2 parts and a half and a quarter for Jupiter, 6 parts for Mars, and one part plus 2 and half minutes for Venus <if *AZ* is 60 parts>. *B* is the center of the epicycle. We join the line segments *EB*, *ZB*, and *HB*. *ZT* and *HI* are perpendicular to *BI*. The angle *EBH* is the angle of equation. *AEB* is the <given> angle of the centrum; so the angle *TEZ* is



known. The angle *T* is right; so the angle *TZE* is known. *ZE* is <also> known. So, both *ET* and *TZ* are known. *BZ* is <taken equal to> 60 parts. Its square is equal to the <sum of the> squares of *ZT* and *TB*. Then *TB* is known. Since the triangles *IHE* and *TZE* are similar and *ZE* is half *EH*, then *ZT* is half *HI* and *ET* is half *EI*. Then *TI* and *TB* are known. So, the whole *IB* is known. Its square plus the square of *IH* are equal to the square of *HB*. Then *HB* is known. If we take *B* as the center and draw a circle with *HB* as its radius, *HI* is the Sine of the arc of the angle *IBH* based on <taking> the magnitude of *BH* as the radius. Then *HI* is known based on <taking> *BE* <equal to> 60 parts. It is the Sine of the arc of the equation angle. That is what we wanted to demonstrate.
In this way, we obtain the equation for each side of the circle. Since the angle *EBH* is the difference between the two angles *AEB* and *AHB*, the equation is subtracted and added as explained for Mercury. If the center is less than 180°, <the equation> is subtracted from the center and added to the <mean> anomaly. If the center is greater than 180°, <the equation> is added to the center and subtracted from the <mean> anomaly.

Chapter 8: On the latitude of the moon.

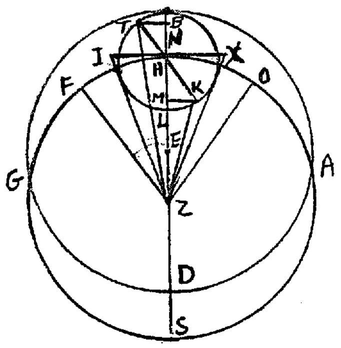
<Let> *ABGD* centered at *E* <be> the circle passing through the poles of the inclined <lunar> orb and the ecliptic; *AEG* the circle of the inclined <lunar> orb with *T* as its pole; *DEB* the circle of the ecliptic with *I* as its pole; the point *E* a lunar node; *H* the position (i.e., the orthogonal projection) of the moon on the ecliptic; and *K* the body of the moon on the inclined orb which is not different from its position on the epicycle, because the plane of the epicycle is in (i.e., it coincides with) the plane of the inclined orb. Then *EH* is the argument of latitude. We pass the arcs *THZ* and *IHK* through *H*. *HK* is the latitude of the moon. Those engaged in the art <of astronomy> take the arc *HZ* <for it> according to their calculations. <However,> *HZ* is not the latitude of the moon. It is actually an arc close <in magnitude> to the latitude of the moon. <Now> I say that *HK* is known.



Proof: As it was demonstrated in the fourth premise, <since> the angle *H* in the triangle *EHK* is right, *HEK* is the total latitude angle, i.e., the arc *BG*; thus the ratio of the Sine of *EH* to the Shadow of *HK* is equal to the ratio of the greatest Sine to the Shadow of the angle *HEK*. *EH* is the <given> argument of latitude, the angle *E* is the total latitude, and the greatest Sine is known. Then the Shadow of *HK* is known, and so is *HK*. But <we may also provide the proof> merely based on Sines: The angle *Z* in the triangle *EHZ* is right, and the angle *E* is the total latitude. As it was demonstrated in the first premise, the ratio of the Sine of *EH* to the Sine of *HZ* is equal to the ratio of the greatest Sine to the Sine of the angle *E*. *EH* is the argument of latitude, the angle *E* is the total latitude, and the greatest Sine is known. Then *HZ* is known. We take the point K as the center and draw the quadrant *NML* with its radius equal to the side of an <inscribed> square. *L* is the center of the circle *IHK*. Then each of <the arcs> *NK* and *HL* are quadrants. Then *EL* is the complement of *EH*. So, in the triangle *ELM*, the angle *M* is right, and the angle *E* is the total latitude angle. Then the ratio of the Sine of *EL* to the Sine of *LM* is equal to the ratio of the greatest Sine to the Sine of the angle *E*. *EL* is the complement of the argument of latitude and the angle *E* is known. Then *LM* is known. So, its complement *MN* is known. It is the magnitude of the angle *HKZ*. <In> the triangle *HKZ*, *Z* is a right angle and the angle *K* is known. Then the ratio of the Sine of *KH* to the Sine of *HZ* is equal to the ratio of the greatest Sine to the Sine of the angle *K*. *HZ* and the angle *K* are known. So, *HK* is known and it is the latitude of the moon. That is what we wanted to demonstrate.

Chapter 9: On the latitudes of the planets.

It was said in Book III that for each of the superior planets there are two anomalies in the latitude. One of them is <due to> the inclination of the inclined orb from the ecliptic and the other is the inclination of the apogee and perigee of the epicycle from the inclined orb. The inclination of the apogee of the epicycle is towards the ecliptic, and the inclination of the perigee is <in the direction> opposite to it. For Venus and Mercury, there are three anomalies. The first and second are those mentioned for the superior planets. The third is <due to> the inclination of the diameter passing through the two mean distances of the epicycle. The magnitudes of these inclinations as found by observation are provided in their (i.e., the planets') descriptions.



Let the circle *ABG* centered at *E* be the ecliptic circle; *AHGS* the circle of the inclined orb centered at *Z*; *A* the ascending node; and *G* the descending node. *AHG* is northward except for Mercury. *BIL* is the epicycle centered at *H*. We suppose *HB* <to be> towards the ecliptic and *HL* in its opposite <direction>. We take *HI*, the radius of the "epicycle", equal to the Sine of the maximum inclination of the apogee or the perigee of the epicycle. Let the circle intersect at right angles the plane of the inclined orb, so that the half on which <falls> *BTL*, is towards the ecliptic and the other half is towards the inclined orb. The angle *HZI* is the maximum inclination of the apogee of the epicycle from the inclined orb towards the ecliptic. The angle *IZF* is the excess of the inclination of the inclined orb <from the ecliptic> over the <maximum> inclination of the apogee of the epicycle. The angle *XZH* is the maximum inclination of the perigee from the inclined orb in the <direction> opposite to the inclination of the apogee <of the epicycle>. The angle *OZX* is the excess of the inclination of the inclined orb <from the ecliptic over the maximum inclination of the epicycle>. These angles are known by observation. *IT* is the "adjusted anomaly" and *TB* is its complement. *TN* is the Sine of *TB*, and *HN* is equal to the Sine of *IT*. Both *TN* and *HN* are known in terms of *HT*, and *ZH* is <taken equal to> 60 parts. Then *ZN* is known. Its square plus the square of *NT* are equal to the square of *ZT*. So, *ZT* is known. Then *TN* is known based on <taking> *TZ* <equal to> 60 parts. It is the Sine of the angle *TZN*. So, the angle *TZN* is known. Then the angle *TZI* is known. Therefore, the whole angle *TZF* (i.e., the required latitude) is known. For Venus and Mercury, the resulting angle *TZN* is subtracted from the angle *HZI* being the maximum inclination angle of the apogee of the epicycle at one of the ascending or descending nodes. Also, the arc *XK* is the excess over the adjusted anomaly towards 90°. <Similarly,> *KL* is <equal to> the "adjusted anomaly". *KM* is the Sine of *KL* and *MH* is equal to the Sine of its complement *XK*. So, both *KM* and *MH* are known in terms of *HK*. *ZH* is <equal to> 60 parts. Then *ZM* is known. Its square plus the square of *MK* are equal to the square of *KZ*. Then *KZ* is known. So, *MK* is known based on <taking> *KZ* <equal to> 60 parts. It is the Sine of the angle *MZK*. So, the angle *MZK* and, therefore, the angle *KZX* are known. Then the whole angle *KZO* (i.e., the required latitude) is known. For Venus and Mercury the resulting angle *MZK* is subtracted from the angle *HZX* as we said before. That is what we wanted to demonstrate.

Description of its calculation: Minutes of the argument of latitude are <a certain number of> minutes whose ratio to 60 minutes is equal to the ratio of partial inclination of (i.e., the inclination of a point somewhere on) the inclined orb to its total (i.e., maximal) <inclination>, and equal to the ratio of partial latitude of the moon to its total magnitude (i.e., the "minutes of latitude" are proportional to the lunar latitude). We divide the partial latitude of the moon by its total magnitude, lowered. The result is the partial <latitude> in terms of the minutes of the argument of latitude. The inclination of Venus and Mercury at mean distance is called 'slant'. The ratio of its partial magnitude to its total (i.e., maximal) magnitude is equal to the ratio of the partial magnitude of the second equation to its total magnitude (i.e., the slant is proportional to the second equation). We multiply the partial magnitude of the second equation by the total slant, which is equal to 2 degrees and half, and divide <the product> by the total magnitude of the second equation. The result is the partial value of the slant. The inclination of the apogee and perigee of the epicycle is also computed from the adjusted anomaly, as indicated by the figure and proof that have been mentioned above.

Description of its tables: On the first <rows> of the tables <of planetary latitudes of the superior planets> is written "north" and "south", where the excess of the inclination of the inclined orb over the inclination of the apogee of the epicycle is <tabulated>. The "north" <column> is for the case when the center of the epicycle is in the northern half of the inclined orb. The "south" <column> is for the case when the center of the epicycle is in the southern half of the inclined orb. The inclinations <tabulated> for Venus and Mercury are their maximum inclination at one of the two nodes: for Venus at the ascending node and for Mercury at the descending node. In both cases the inclination of the apogee of the epicycle is southward.

Description of the operation by <using> tables: We take the minutes of the argument of the latitude from the adjusted centrum, to which (adjusted centrum) we add 50 degrees for Saturn, subtract 20 degrees for Jupiter, and <we take it> as it is for Mars. <We do so,> because the apogee of Saturn is shifted 50 degrees from the point *H* towards *G* which is the descending node, and the apogee of Jupiter <is shifted> 20 degrees from *H* towards *A*, and the apogee of Mars is at *H*. It (i.e., *H*) is the <point of> maximum inclination of the inclined orb. As we have already said, the minutes of the argument of the latitude take account of the <variable> inclination of the inclined orb for <varying> distance of the center of epicycle from the node. Then we take the latitude <corresponding to> the adjusted anomaly <from the "north" or "south" column>. If the adjusted center is in the semicircle *AHG*, the latitude is "north", because the inclination of the epicycle is towards the north in this semicircle. But if the adjusted center is in the semicircle *ASG*, the latitude is "south", because the inclination of epicycle in this semicircle is towards the south. Then we multiply the latitude by the minutes of the argument of latitude to obtain <the latitude> for <arbitrary> distance of the center of the epicycle from <one of> the two nodes.

<The cases with> Venus and Mercury <are as follows>. The apogee of Venus is at *H*, which is the northern extreme, and the apogee of Mercury is at *S*, which is the southern extreme. We take the inclination and slant for the <known> adjusted anomaly. The slant of Mercury at <its> apogee is 2; 15°, and at <the position> opposite to the apogee, 2; 45°. It was difficult to compose two tables for that. So, one table is composed for 2; 30°. Then a tenth of it is subtracted in <the region of> the apogee and a tenth of it is added in <the region> opposite to the apogee. This is sufficient for us. Then we add to the adjusted centrum, 3 <zodiacal> signs for Venus and 9 <zodiacal> signs for Mercury. The result is the distance from the ascending or descending node. If the result is less than 90° or greater than 270°, the distance is <regarded> from the ascending node. If the result is greater than 90° and less than 270°, the distance is <regarded> from the descending node. We use it (i.e., the distance) to find the minutes of the argument of latitude. We multiply it (i.e., the minutes of the argument of latitude) by the inclination for finding it (the latitude component) corrected for the distance from the node (i.e., for a position of the epicycle not coinciding with the node), because the extreme <magnitude> of this inclination is <achieved> at the two nodes. If the augmented center and the true anomaly are in the same half of the inclined orb, this latitude is southward. If their positions are <in> different <halves>, the latitude is northward. <They are so,> because the inclination of the apogee of epicycle is southward, and the inclination of <its> perigee is northward, between *SA* and *AH*. Conversely, if the result is between <the endpoints of the arc>, the center is between *SA* and *AH*. Then, if the true anomaly is also in the upper half, the inclination is southward. If the result is between <the endpoints of the arc> *GSA*, which is the lower half, the center is between <the endpoints of the arc> *HGS*.

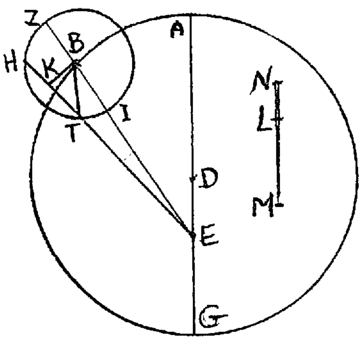
<Now> if the true anomaly is also in the lower half, the inclination is southward. From what we said, it is apparent that if the position of the result and the position of true anomaly are different, the latitude is northward. Then, we take the adjusted centrum of Venus as it is, and that of Mercury by adding 6 <zodiacal> signs. We obtain thereby the minutes of the argument of latitude, and we multiply these by the slant <at the highest point of the deferent> to obtain it (the slant) corrected for the distance of the center <of the epicycle> from the apogee for Venus, and for the distance to the point opposite to the apogee for Mercury. <We do so, > because the extreme of the slant is reached <when the epicycle center is> at the <point of> extreme inclination of the inclined orb. If the center is between <the endpoints of the arc> *AHG*, the upper half, and the true anomaly is between <the endpoints of the arc> *BIL* of the epicycle, then the latitude is northward. If the true anomaly is in the other half, the latitude is southward. <It is so,> because the endpoint *I* of the diameter *HI* is between <the endpoints of the arc> *AHG* <which lies> towards north and its other endpoint <lies> towards south. If the center is between <the endpoints of the arc> *GSA*, and the true anomaly is less than 180°, the latitude is southward. If the true anomaly is more <than 180°>, the latitude is northward. <It is so,> because the endpoint *I* of the diameter *HI* is between <the endpoints of the arc> *GSA* <which lies> towards south and the other endpoint <lies> towards north. Al-Battānī has neglected these directions in the text in his *zīj*, if it is not a scribal mistake. Then we multiply the <number of the> minutes of the argument of latitude which we have found lastly, by 1/6 degrees for Venus, and by a half plus a quarter of a degree for Mercury for finding the inclination of the inclined orb <at the center of the epicycle> from the distance of the center <of the epicycle> from the node. This inclination is always northward for Venus and southward for Mercury. Adding 6 <zodiacal> signs to the center of Mercury in the first and second <steps> is for shifting from the apogee region to its opposite <region>. <In this way,> the assertions about its latitudes and directions <become> like those for Venus. Then the single words (i.e., the rule) for it (i.e., for Venus) will be <valid> in general (i.e., also for Mercury). That is what we wanted to demonstrate.

Chapter 10: On the retrogradation of the planets.

<Let> *ABG* <be> the circle of the deferent with *D* as its center and *AG* as its diameter; *E* the center of the inclined orb; *ZHTI* the circle of the epicycle centered at *B*; and the line <segment> *BE* the distance of the center of the epicycle from the center *E*, <the method for> knowing which was already provided in the chapter on the first equation. We draw *ETH* passing through the first station. We join *BK* perpendicular to *HT*.

Based on what Ptolemy and his predecessors have demonstrated, the ratio of *KT* to *TE* is equal to the ratio of the motion (i.e., the motion during a particular day) of the center of the epicycle to the motion (i.e., the motion during the same day) of the planet on the epicycle.

BZ is the radius of the epicycle adjusted according to the distance of its center from the <position corresponding to the> mean distance (i.e., Kūshyār chooses *EB*=60), and it is known. We join *BT*. Then the arc *IT* is half the arc of retrogradation on the epicycle. The angle *BEK* is half the angle of retrogradation. *BE* and *BI* are known, so the remainder *IE* and the sum *ZE* are known. The product of *ZE* by *EI* is known. According to what has been demonstrated in the *Elements* (III.36), it is equal to the product of *HE* by *ET*. Then the product of *HE* by *ET* is known. The ratio of *KT* to *TE* is known, and *HT* is twice *KT*. Then the ratio of *HT* to *TE* is known. Let it be equal to the ratio of <two given segments> *NL* and *LM*, then the rectangle on *HE* and *ET* is similar to the rectangle on *NM* and *LM*, because their angles are equal and their sides are proportional. According to what has been demonstrated in the *Elements* (VI.23), the ratio of <the area of> the rectangle on *HE* and *ET* to



the <area of the> rectangle on *NM* and *ML* is equal to the ratio of the square of *HE* to the square of *NM*. The <areas of the> rectangles on *HE* and *ET*, and on *NM* and *ML* are known, and so is the square of *NM*. Then the square of *HE* is known; so *HE* is known. The ratio of *ZE* to *ET* is equal to the ratio of *HE* to *EI*, because the product of *ZE* by *EI* is equal to the product of *HE* by *ET*. But *ZE*, *HE*, and *EI* are known. Then *ET* is known. So, both *ET* and *TH* are known. Then *TK* and *KE* are known. So *KE* is known based on <taking> *BE* <equal to> 60 parts. Then its arc <Sine> is known, and it is <corresponding to> the angle *EBK*. Then the angle *EBK* is known. Also, *KT* is known based on <taking> *BT* <equal to> 60 parts. Then its arc <Sine> is known, and it is <corresponding to> the angle *TBK*. Then the angle *TBK* is known. If we subtract it from the angle *EBK*, the remainder is the angle *IBT*. It is the angle <corresponding to> the arc *TI*. Then the arc *TI* is known. It is half the arc of retrogradation on the epicycle. If we subtract the angle *EBK* from the right angle *BKE*, the remainder is the angle *BEK*. It is the angle of half the arc of retrogradation on the ecliptic. If the center of the epicycle had no movement towards east (i.e., if the distance of *B* from *E* was constant), the angle *BEK* and the arc *IT* would be adjusted (i.e., correct). But since it has a movement, we rely on finding a number (for adjustment of the retrogradation arc) whose ratio to the arc *IT* is equal to the ratio of the motion (i.e., the angular velocity) of the center of the epicycle to the motion (i.e., the angular velocity) of the planet on the epicycle. We subtract the obtained number from the angle *BEK* and the arc *IT*. The remainders are adjusted <magnitudes of> the angle *BEK* and the arc *IT* .If we divide the <magnitude of> the adjusted angle *BEK* in degrees by the daily mean motion of the planet, the result is half the <number of the> days of its retrogradation and its double is the whole <number> of the days of retrogradation. That is what we wanted to demonstrate.

Commentary

IV.4.1 In this chapter Kūshyār provides the definition of the equation of time and mentions its approximate bounds. Without going into details, he then shows that it is possible to calculate its magnitude. He presents the method for calculating the equation of time in I.4 and describes this subject in more detail in III.13. What Kūshyār says about the solar apogee reaching the sign of Leo may happen after around 20,000 years from his time. Kūshyār takes the mean solar longitude as the independent variable in his method, whereas al-Kāshī, for example, takes the true solar longitude as the independent variable. Benno van Dalen [1994a, 104] has shown that "as far as the computation of a table for the equation of time or its application is concerned, it makes little difference whether the independent variable is the true or mean solar longitude"; see also [Kennedy, 1988].

Kūshyār's text is confused. The equation of time is actually the sum of the differences between the successive mean and true solar days. The equation of time $E(t)$ at day t can be found by adding the solar equation (true minus mean longitude) at t to the mean solar longitude minus its right ascension at t. The result can be positive or negative. In order to avoid negative values, Kūshyār modifies his definition of equation of time. First he chooses u such that $E(u)$ is minimal. This turns out to be when the sun is in the middle of the sign Aquarius. Then Kūshyār defines his equation of time as $E(t) - E(u)$. This explains why he speaks about "twice" the differences in what is actually the determination of an upper bound of $E(t) - E(u)$. See also the commentary to I.4.5 where I have discussed Kūshyār's displacement method to avoid negative values of the equation of time. Ptolemy deals with the equation of time and its calculation in [1984, 169-72], and al-Battānī discusses this subject in [1907, 73-75]. See also [Pedersen 1974, 157-58; Neugebauer 1975, 61-68; van Dalen 1996, 211-18].

IV.4.2 Kūshyār's model for the solar motion is similar to that of Ptolemy. In this model, the sun B moves uniformly on a circle (the deferent) whose center E does not coincide with the earth D. Line DE points to point A on the ecliptic, which is taken as Gemini 18;31 for the beginning of the Yazdigird era (632 A.D.) and whose motion is very slow (1 revolution in 24,000 years). The radius EA is taken as 60 "parts". Kūshyār supposes that the uniform motion of B around E is known. The eccentricity is found to be 2; 29, 30° by Ptolemy [1984, 155], if the radius of the deferent is taken as 60. Al-Battānī, following Ptolemy's method but using his own observational data, found the eccentricity equal to 2 parts and 4 minutes plus half and a quarter of a minute (i.e., 2+4/60+1/120+1/240= 2;4,45°) [1988-1907, III, 66]. Kūshyār accepted the amount found by al-Battānī. The angle *EDB* in the figure is called "the angle of the true longitude". In the mss. A and M there is the justification that "if we add the true longitude of the apogee to it, the result will be the true longitude of the sun". Kūshyār provides the values of the equation of the sun in the table II.16, and describes its application in I.4.6. For avoiding negative values of the equation, Kūshyār adds 2 degrees to all of the tabular entries in II.16. Then in order to cancel this shift, he subtracts 2 degrees from all the tabular values of the table II.13 for the mean longitude of the sun. See also [Kashino 1998, 7-8; Ptolemy 1984, 157-166; Pedersen, 149-151].

IV.4.3 In this chapter, Kūshyār computes the "equation of center" of the moon, which is the difference between the mean epicyclic apogee *M* and the true epicyclic apogee *K* (which is on *ZB* extended). See [Pedersen 1974, 194]. Kūshyār's lunar model is similar to that of Ptolemy [1984, 226-33]. In this model, the earth is at *Z*. The moon *L* moves on an epicycle with center *B*. This center moves on a deferent circle whose center *E* does not coincide with the earth *Z*. The motion of *L* on its epicycle is contrary to the direction of the motion of *B* on the deferent. To further define the motion of *A* and *B* it is useful to bisect angle *AZB* by line *ZS* (not shown in the figure). Then *ZS* always points toward the mean sun. Points *A* and *B* move uniformly in opposite directions so that *ZS* always bisects the angle *AZB*, and angle *BZS* is the mean elongation, that is the difference between the mean lunar and the mean solar positions. We note that *B* moves on the circle with center *E*, but the motion is uniform with respect to *Z* (so not with respect to *E*). Finally define point *H* on *EZ* extended such that *ZH*=*EZ*, and extend *HB* to meet the epicycle at point *M*. Then the motion of the moon *L* on the epicycle is uniform with respect to *M*. All uniform motions are supposed to be known. See also pp. xxxii-xxxiv above.
However, Kūshyār deviates from Ptolemy and al-Battānī by taking the radius of the deferent *AE* (and not the inclined orb *AZ*) equal to 60 parts. So, his value for the distance between the center of the deferent and the center of the inclined orb is different from that of Ptolemy [1984, 226] and al-Battānī [1907, 82]:

$$10;19\times60/(60-10;19)= (619/60)\times(60-619/60)\approx 12.4589$$

Kūshyār uses the approximate value 12.5. He increases the values of the first equation of the moon tabulated in II.20 by 14 (the smallest integer greater than the maximum negative magnitude of the first equation) so that the first equation is always additive. The application of II.20 is described in I.4.7. In order to compensate the added 14 degrees, Kūshyār subtracts 14 degrees from the entries of the table II.18 for the lunar anomaly [cf. Kashino 1998, 12].

IV.4.4 For the moon, compare [Ptolemy 1984, 233-38; Pedersen 1974, 192-98]. Point *Z* is the earth, and *E* the center of the deferent. Kūshyār supposes that the epicycle is at maximum distance, so its center is at the apogee of the deferent. He computes the second (anomalistic) equation of the moon for any position of the moon on this epicycle. In the notation of Pedersen, Kūshyār computes the function in p. 196 (6.53). In this case, the mean and true anomaly are the same because the points *K* and *M* in the figure for IV.4.3 coincide.

For the planets, compare [Ptolemy 1984, 456-67; Pedersen 1974, 287-88]. Kūshyār's model for the planetary motion is similar to that of Ptolemy. In what follows, we exclude Mercury, for which see Section IV.4.6. The planet *T* moves on an epicycle whose center moves on a deferent, of which the center *E* does not coincide with the earth *Z*. The motion of the planet on its epicycle is in the same direction as the direction of the motion of its center on the deferent. In chapter IV.4.4, Kūshyār assumes that the center of the epicycle coincides with the apogee *A* of the deferent. The motion of the center of the epicycle on the deferent is uniform with respect to the equant point, that is a point *P* on *ZE* extended such that *EP*=*ZE*.

Again the values of the second equation of the moon are tabulated in II.20 and their application is described in I.4.7. Here Kūshyār's description is different from that of al-Battānī who follows Ptolemy exactly. Kūshyār adds 8 degrees to all values of the second equation of the moon to avoid negative values, and subtracts 8 degrees from all entries in the table II.17 for the mean longitudes of the moon. However, for compensating this shift and the 2 degrees shift in the entries for the mean longitudes of the sun, he subtracts 12 degrees from the entries of the table II.19 for the double elongation. See [Kashino 1998, 13].

IV.4.5 This chapter is a preparation for the computation of the "second equation", which is the angle between the center of the epicycle and the moon or planet, as seen from the earth. In the figure for IV.4.5, *E* is the earth, and *Z* the center of the deferent. Kūshyār discusses the case where the center of the epicycle is at *T*, not at the apogee of the deferent, and he only computes the maximum anomalistic equation. First he describes the computation and then he gives a proof. He assumes that the distance *ET* to the center of the epicycle is known. This distance can be computed from the angle *AZT* (mean motion from apogee) by the method of Chapter IV.4.2. Unlike Ptolemy, Kūshyār does not provide an exact computation of the second equation for an arbitrary point on the epicycle

whose center is an arbitrary position on the deferent. However, he presents a computation based on an interpolation method which is somewhat different from the method used by Ptolemy, for which see [Ptolemy 1984, 237-39; Pedersen 1974, 197-8]. Kūshyār briefly describes his interpolation method at the end of Chapter IV.4.5. Chapters IV.4.4 and IV.4.5 are used for the construction of interpolation tables in Book II. G. Van Brummelen [1998] studied Kūshyār's planetary tables and he reconstructed Kūshyār's interpolation method from the tables. Van Brummelen's mathematical reconstruction of Kūshyār's method is confirmed by the text at the end of Chapter IV.4.5 which Van Brummelen apparently did not consult. (Compare [Van Brummelen 1998, 273] "Kūshyār's second function gives the difference" with the text at the end of Chapter IV.4.5.)
Some of Kūshyār's parameter values are different from those of Ptolemy. Kūshyār mentions that he found the difference for the radius of the epicycle of Mars at its maximum distance one and a fifth degree less than the value found by Ptolemy, and that at its minimum distance two and a fifth parts less than that found by Ptolemy. Compare [Van Brummelen, 1998, 268]. For the moon, Kūshyār prescribes an interpolation procedure which is the same as his interpolation procedure for the planets, and different from the procedure in Ptolemy, for which see [Pedersen 1974, 196, (6.58)]. The "equation of center" of the planets is discussed in IV.4.6 for Mercury and in IV.4.7 for the other planets. Similar discussions are found in [al-Battānī 1899-1907,III, 80-81].

IV.4.6 See [Ptolemy 1984, 443-67] and especially [Pedersen 1974, 315-20] for a clear description of the very complicated Ptolemaic model for the motion of Mercury. Kūshyār uses the same model as Ptolemy. *Z* is the earth, and line *ZA* points towards a point on the ecliptic which moves so slowly that its motion is only noticeable after centuries. Points *E*, *N* and *M* are on *ZA* such that $ZE=EN=NM$. A small circle is drawn with center *N* and radius *ND*. On this circle, point *D* moves in the opposite direction of the sun, and with a velocity equal to the solar velocity; the exact position will be defined below. *D* is the center of the deferent with radius *DT*. Point *T* moves on the deferent such that the line *ET* is parallel to the direction of the mean sun (the direction of the line from the center of the solar deferent to the sun in the figure for IV.4.2). This also defines the position of *D*.
The values of the first equation of Mercury are tabulated in II.36 of Kūshyār's *zīj*. The magnitudes provided by Kūshyār for line segment *ZT* corresponding to the center being equal to 0°, 66°, 90°, 120° and 180° are correct and very close to the results of my recalculation. However, at the end of this chapter Kūshyār erroneously mentions the angle *AZT* – and

not *AET* – as the angle corresponding to the center. See also the commentary to I.4.8 and [Kashino 1998, 14-19; Pedersen, 315-28].

IV.4.7 Compare [Ptolemy 1984, 545-54] and [Pedersen 1974, 279-80, 283-85], and see the commentary to section IV.4.5 for a description of the model for the motion of the planets except Mercury. *H* is the earth, *Z* the center of the deferent, *E* the equant point, and *B* the center of the epicycle (which is not drawn). The values of the first equation of Saturn, Jupiter, Mars and Venus are provided in the relevant tables of Book II.

IV.4.8 In table II.37 are provided the values of the latitude of the moon for different values of the argument of latitude. Kūshyār presents two methods for calculating the latitude: one using both Sine and Shadow functions, and another using merely the Sine function. In the figure, *HZ* is the required latitude, but Kūshyār erroneously calculates *HK* as the latitude of the moon. See also the commentary on I.4.9.

IV.4.9 Kūshyār's theory for the latitudes of the planets is essentially Ptolemy's theory of latitudes in the *Almagest* for which see [Ptolemy 1984, 597-647] and [Pedersen 1974, 355-86]. Ptolemy's theory for the latitude of the superior planets (Saturn, Jupiter and Mars) is basically as follows. He supposes that the planes of the deferent and the ecliptic intersect in the nodal line, that is a straight line through the center of the earth, and that the plane of the deferent (Kūshyār's "inclined orb") makes a (small) angle with the plane of the ecliptic. The line through the earth *Z* perpendicular to the nodal line will intersect the deferent in a point *H*, the highest point, where the center of the epicycle has maximal northern distance from the ecliptic. For Mars, the highest point coincides with the apogee of the deferent, as in Kūshyār's figure. For Jupiter and Saturn, if *E* is the center of the deferent, angle *HZE* is equal to 20 or 50 degrees.
The epicycle makes a variable angle with the deferent which is maximal when the center of the epicycle is at the highest point *H* and the lowest point *S* of the deferent, and zero when the center of the epicycle is at the nodes, i.e., the two points of intersection of the deferent and the plane of the ecliptic. The apogee of the epicycle is between the deferent and the ecliptic, but the perigee on the other side of the deferent, so that the latitude of the planet is maximal when it is at the perigee of the epicycle.
As usual, Kūshyār only considers a special situation, where the center of the epicycle is in the highest point *H* of the deferent. He wants to show how the latitude of the planet can be computed from its true anomaly in this situation. He assumes that the angle between deferent and the ecliptic is known, and also the latitudes of the apogee and the perigee of the

epicycle are known. He approximate computation is similar to, but easier than Ptolemy's (which is also an approximation).
Kūshyār's figure is confusing for the modern reader because it is a superposition of two different figures, which are in two different planes. Some of the points in the paper play different and inconsistent roles in the two figures. The procedure to superpose different figures in two or sometimes even three planes in plane of the paper was sometimes used in ancient Greek and medieval Islamic geometry in a type of constructions which are called analemma-construction by modern historians of science.
The first figure (which we call the "horizontal figure") consists of the earth *E*, the ecliptic *ABGD*, the deferent *ASGH* with center *Z*, the nodes *A* and *G*, and the epicycle *BILX* with apogee *B* and perigee *L*. The center *H* of the epicycle is supposed to be at the highest point of the deferent.
The second figure (which we call the "vertical figure") describes the situation in the plane through *HZ* perpendicular to the ecliptic. In this second figure, *ZF* is the intersection with the ecliptic, and one considers a circle with center *H* and radius the "sine of the maximum inclination" of the apogee and perigee of the epicycle from the plane of the deferent. This has to be understood in the sense that the radius is equal to the (equal) distances of the apogee and perigee of the epicycle to the deferent. This small circle is essentially an interpolation device. Kūshyār also indicates this new circle by the same letters *BILX*. However, in the "vertical" figure, *I* is the position of apogee (which has approximately the minimum latitude in this model), and *X* the position of the perigee (the point on the epicycle with approximately the maximal latitude). Therefore, *I* in the vertical figure corresponds to *B* in the horizontal figure and vice versa. In the vertical figure, the angle *FZH* is the inclination of the deferent.
If *T* in the vertical figure is the position of the planet, *IT* is its "adjusted anomaly", and *TB* is its complement (we have emended the manuscript texts which say that *BT* is the adjusted anomaly and *TI* is its complement-this is true in the horizontal figure). Kūshyār can now easily compute the angle *TZF*.
It is instructive to express his result in the notation in [Pedersen 1974, 365-67], so that it can be compared to Ptolemy's much more complicated computation. In the notation of Pedersen, the quantity to be computed is $\beta(90^\circ, a_v)$= angle *FZT*. The radius of the epicycle is $r \sin j_m$, where j_m is the angle between the epicycle and the deferent. In Kūshyār's vertical figure we have $TI = a_v$ (the "adjusted anomaly"), $HN = r \sin j_m \sin a_v$,
$TN = r \sin j_m \cos a_v$, $ZH = \rho$, $ZT^2 = (\rho + r \sin j_m \sin a_v)^2 + (r \sin j_m \cos a_v)^2$,
and finally $\beta(90^\circ, a_v)$= (angle) *FZH* – (angle) *TZH*
$= i - \text{arc sin}\,[r \sin j_m \cos a_v / ((\rho + r \sin j_m \sin a_v)^2 + (r \sin j_m \cos a_v)^2)^{1/2}]$,

where i is the angle between the deferent and the ecliptic.
Then Kūshyār wants to compute the latitude when the adjusted anomaly of the planet is between 90 and 180 degrees. He now constructs a new vertical figure, in which angle *XZO* is the latitude of the center of the epicycle (and also of the point *X*). Kūshyār now considers the position of the planet at *K* between *X* and *O*, in the horizontal and at the same time in the vertical figure. He computes the angle *KZO* and considers this to be the latitude. However, the resulted latitude function is strange, increasing slowly when *K* moves away from *X*, and sharply when *K* approaches the perigee *L*. This is in contrast with Ptolemy's more sensible method of interpolation. It is likely that Kūshyār copied the latitude tables of Ptolemy and did not use his own method of computation.
For reasons of space, I will not describe the complicated geometric theory of the latitude of the inferior planets here. However, I will provide some formulas. To compute the latitude of any (superior or inferior) planet, one needs the adjusted anomaly (of the planet on its epicycle, reckoned from the true apogee), and the difference x between the ecliptical longitude of the center of the epicycle minus the longitude of the ascending node (for all planets except Mercury this can be defined as the node where the epicycle center passes from southern to northern latitude). For the superior planets, Ptolemy's tabular computation of latitudes boils down to

$\sin(x)\, f(a)$ for x between 0 and 180 degrees, and

$\sin(x)\, g(a)$ for x between 180 and 360 degrees.

Kūshyār calls x-90 the "argument of latitude", and the function $|\sin(x-90)| = \cos(x)$ "minutes for the argument of latitude"; the function is expressed in minutes (1/60) of unity. Kūshyār tabulates $f(a)$, $g(a)$ and $|\sin(x-90)|$ for a and x between 0 and 360 degrees. For x between 0 and 180 degrees, the latitude is northern, and $f(a)$ is tabulated in the column "north". For x between 180 and 360 degrees, the latitude is southern, and $g(a)$ is tabulated in the column "south".
For Venus and Mercury, the latitude is the sum of three components: $c\sin^2 x + \sin x\, f(a) + k \cos x\, g(a)$ for a and x between 0 and 360 degrees. Here c =6 for Venus and c =0.75 for Mercury. The "inclination" $f(a)$ and "slant" $g(a)$ are tabulated in the third and fourth columns. They are the differences between the latitude of the planet and the latitude of the center of the epicycle, at x =90 (inclination) or x = 0 (slant of Venus). For Mercury $g(a)$ is a hypothetical slant with maximum value 2.5; the real slant is computed by multiplication by the constant k, which is 0.9 for x between 0 and 180 degrees, and 1.1 for x between 180 and 360 degrees. The constant k is 1 for Venus. The fifth column displays again values of $\sin x$ in "minutes" for the "argument of latitude" $x - 90$.
The last part called "description of the operation by tables" closely resembles *Almagest* [Ptolemy 1984, 635-36]. Kūshyār reproaches

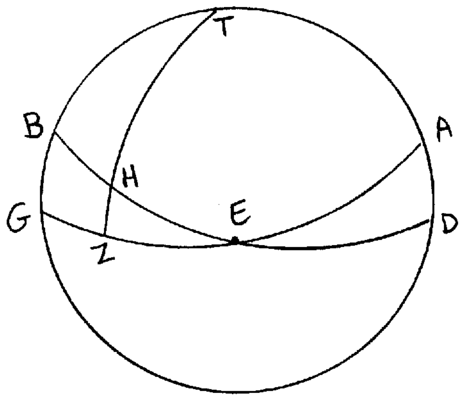
al-Battānī of not giving the correct rules for the determination of the direction of the latitude components (northern or western). However, these rules are found in the text of al-Battānī [1899, 174]. To find the third latitude component, Kūshyār multiplies the minutes of latitude by 1/6 degrees for Venus and by 0.75 degrees for Mercury (so he obtains $c\sin x$). Kūshyār then forgets to multiply the result again by the "minutes of latitude". The same mistake is found in al-Battānī's *zīj* [1899, 175, lines 14-16]. Ptolemy explains the computation correctly in *Almagest* [Ptolemy 1984, 636]: "Then we take these same sixties which were found by the second entry with the longitude, calculate the amount which is the same fraction of them as they are of 60, and for Venus, take 1/6th of this and set it out ...."
The tables for the latitudes of the five planets are presented in II.38 to II.42.

IV.4.10 Kūshyār starts this chapter with the Apollonius Theorem provided also in the *Almagest* [Ptolemy 1984, 555-62; Pedersen 1974, 329-51], where he says that this preliminary lemma was demonstrated by a number of mathematicians, notably Apollonius of Perga [ibid, 555]. Kūshyār's wording corresponds to the generalized theorem of Apollonius [Pedersen 1974, 341-43]. Ptolemy [1984, 562-81] also provides the numerical calculation of the retrogradation arc for each planet at mean, greatest and least distances. Then he uses his usual interpolation method for other distances. See also [Pedersen 1974, 329-54], where the author also quotes a beautiful and interesting proof of Apollonius Theorem devised by B. L. van der Waerden [ibid, pp. 331-32]. The correction to the retrograde arc at the end of this chapter (to account for the eastward motion of the epicycle) corresponds to formula (11.40) in [Pedersen 1974, 347].

Section 5: On the operations relating to the ascendants of the day and night, <in> 16 chapters

Chapter 1: On the first declination.

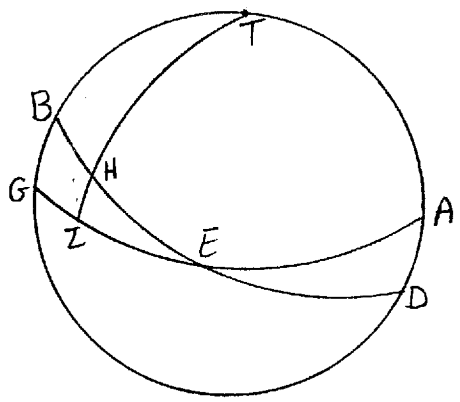
Let *ABGD* be the circle passing through the two poles of the <celestial> equator and the ecliptic, *AEG* the celestial equator with its pole at *T*, *BD* the ecliptic, and *E* one of the two equinoxes. We take *EH* as <the arc> from the ecliptic, whose first declination we want. We draw the arc *THZ*. Then *HZ* is the first declination of the arc *EH*. I say that it (i.e., *HZ*) is known.



Proof: The angle *Z* of the triangle *EHZ* is right and the angle *E* is <equal to> the greatest declination. So, the ratio of the Sine of *EH* to the Sine of *HZ* is equal to the ratio of the greatest Sine to the Sine of the angle *E*. But *EH* is known, and the greatest Sine is known through observation, so *HZ* is known. This is what we wanted to demonstrate.

Chapter 2: On the right ascensions of the <zodiacal> signs.

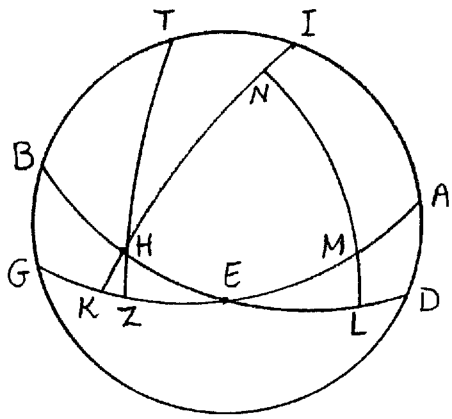
Let *ABGD* be the circle passing through the poles <of the celestial equator and the ecliptic>, *AEG* the celestial equator with its pole at *T*, *BED* the ecliptic, and *E* one of the two equinoxes. We take *EH* as <the arc> from the ecliptic, whose right ascension we want. We draw the arc *THZ*. Then *EZ* is the <right> ascension of the arc *EH*. I say that it is known.



Proof: The angle *Z* of the triangle *EZH* is right. *E* is the angle of the greatest declination, and *HZ* the declination of *HE*. According to what was demonstrated in the fourth premise, the ratio of the Sine of *EZ* to the Tangent of *ZH* is equal to the ratio of the greatest Sine to the Tangent of the angle *E*. But *ZH* and the greatest Sine are known, so *EZ* is known. This is what we wanted to demonstrate.
Another method: Again, the angle *Z* of the triangle *EZH* is right. Then, according to what was demonstrated in the second premise, the ratio of the Sine of the complement (i.e., Cosine) of *EZ* to the Sine of the complement (i.e., the Cosine) of *EH* is equal to the ratio of the greatest Sine to the Sine of the complement (i.e., the Cosine) of *ZH*. The complements of *EH* and *ZH* are known. Then the complement of *EZ* is known. So, *EZ* <itself> is known. This is what we wanted.

Chapter 3: On the second declination.

ABGD centered at *E*, is the circle passing through the poles <of the celestial equator and the ecliptic>. *AEG* is the celestial equator with pole at *T*, *BED* the ecliptic with pole at *I*, and *E* one of the two equinoxes. We take <the arc> *EH* from the ecliptic whose second declination we want. We draw the arc *IHK*. Then *KH* is the second declination of the arc *EH*. I say that it is known.



Proof: The angle *H* of the triangle *EHK* is right and the angle *E* is <equal to> the angle of the greatest declination. Then the ratio of the Sine of *EH* to the Tangent of *HK* is equal to the ratio of the greatest Sine to the Tangent of the angle *E*. But *EH* is known. Therefore the Tangent of *HK* is known. So, <*HK*> itself is known. This is what we wanted.
Another method: Again, we draw the arc *THZ*. We <also> draw the arc *NML*, taking its pole at *K* and its radius equal to the side of the <inscribed> square. Then *L* is the pole of the circle *IHK*, and both *KN* and *NL* are quadrants. *HZ* is the first declination of the arc *EH*, and *EL* is the complement of *EH*. *ML* is the first declination of the arc *EL*, and *MN* is

the complement of *ML*, <equal to the> magnitude of the angle *NKM*. So the angle *Z* of the triangle *HKZ* is right and the angle *K* is known. So the ratio of the Sine of *KH* to the Sine of *HZ* is equal to the ratio of the greatest Sine to the Sine of the angle *K*. But *HZ* is known and the angle *K* is known, therefore, *HK* is known. This is what we wanted.

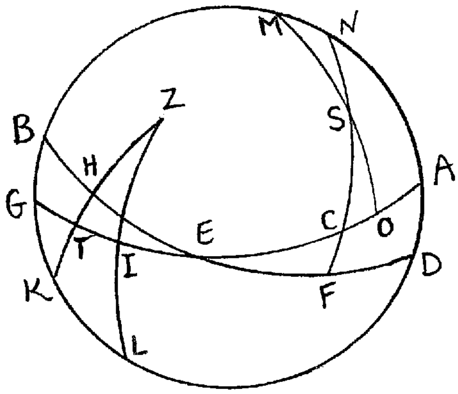
Another method: If we take the known <arc> *EH* <as a part> of the celestial equator, and <the arc> *EK* <as a part of> the ecliptic, <then> *KH* will be the first declination of the arc *EK*. If we find the arc corresponding to *EH* in <the table for> right ascensions, *EK* will become known, and it is called 'the inverse ascension'. If we take its first declination, it will be <equal to> *HK*, being the second declination of the arc *EH*. Then *HK* is known. This is what we wanted to demonstrate.

<Finding right> ascensions from the two declinations: *EH* is <an arc> of the ecliptic and *EZ* is <an arc> of the celestial equator. *HZ* is the first declination of the arc *EH* and the second declination of the arc *EZ*. If we find the arc corresponding to *HZ* in the table of the second declinations, *EZ* will become known. It is the right ascension of *EH*.

So the right ascension is known from the two declinations. This is what we wanted to demonstrate.

Chapter 4: On the distance of the stars from the celestial equator.

ABGD is the circle passing through the poles <of the celestial equator and the ecliptic>. *AEG* is the celestial equator with poles at *L* and *M*. *BED* is the ecliptic with poles at *K* and *N*. First, we suppose the star <to be at> the point *Z*, so that the latitude and the second declination are in the <same> direction. We draw the arcs *KTZ* and *LIZ*. Then *HZ* is the latitude of the star, *HT* is its second declination, and *ZI* is its distance from the celestial equator. I say that it (i.e., *ZI*) is known.

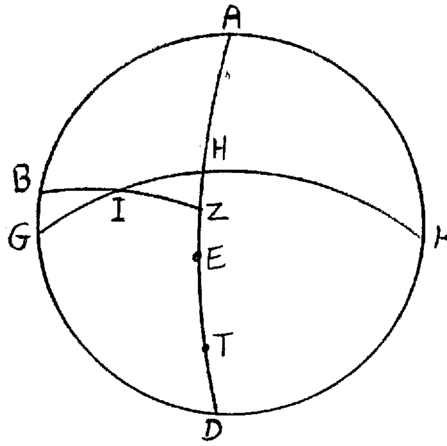


Proof: The two triangles *ZTI* and *KTG* are similar, because their angles *T* are equal and the angles *I* and *G* are right. Then the ratio of the Sine of *TZ* to the Sine of *ZI* is equal to the ratio of the Sine of *TK* to the Sine of *KG*. *TZ* is known, being <equal to> the latitude plus the second declination

<of *Z*>. *TK* is the complement of the second declination. *KG* is the complement of the greatest declination, because *KL* is <equal to> the greatest declination. So, *ZI* is known.
Again, we suppose the star <to be at> the point *S*, so that the latitude and the second declination are in two <opposite> directions. We draw the arcs *MSO* and *NSF*. Then *FS* is its latitude, *FC* its second declination, and *SO* its distance from the celestial equator. I say that it (i.e., *SO*) is known.
Proof: The triangles *CSO* and *CNA* are similar, because the angle *C* is common <to them> and the angles *A* and *O* are right. Then the ratio of the Sine of *CS* to the Sine of *SO* is equal to the ratio of the Sine of *CN* to the Sine of *NA*. *CS* is known and *CN* is the complement of the second declination. *NA* is the complement of the greatest declination because *NM* is <equal to> the greatest declination. Therefore *SO* is known. This is what we wanted to demonstrate.

Chapter 5: On the latitude of a <given> locality.

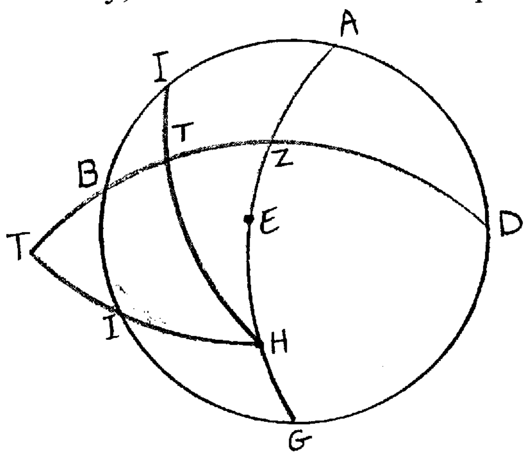
ABGD is the horizon circle, *E* the zenith, *AED* the meridian, *GIK* the celestial equator with *T* as pole, and *BIZ* the ecliptic. Then *EH* is <equal to> the latitude of the locality. I say that it is known.



Proof: *AZ* is the maximum altitude of the sun, found by some altitude <measuring> instrument. *ZH* is the declination of the sun. Therefore *AH* is known, but it is the complement of *EH*. Therefore, *EH* is known and it is <equal to> the latitude of the locality. If the point *H* is on the ecliptic, <the point> *Z* on the celestial equator, and *EZ* the latitude of the locality, then *AH* is the maximum altitude of the sun and *ZH* the declination of the sun. So, the sum *AZ* is known, but it is the complement of *EZ*. Thus *EZ*, the latitude of the locality, is known. This is what we wanted to demonstrate.

Chapter 6: On the ortive amplitude of the sun and the stars.

ABGD is the horizon circle, *E* the zenith, *AEG* the meridian, and *BZD* the celestial equator with its pole at *H*. Let *I* be the rising point of the sun or the star on that day, then *BI* is the ortive amplitude. I say that it is known.



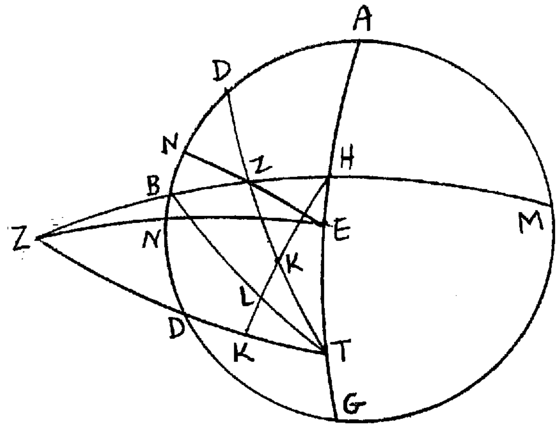
Proof: We draw the arc *HTI*. Arcs *BZ* and *BA* are both quadrants. *AZ* is the complement of the latitude of the locality. It is equal to the magnitude of the angle *ZBA*. In the triangle *BIT*, *T* is a right angle, the angle *B* is known, and *IT* is the declination of the sun or the star from the celestial equator. Then the ratio of the Sine of *BI* to the Sine of *IT* is equal to the ratio of the greatest Sine to the Sine of the angle *B*. So *BI* is known. This is what we wanted.

Another method: Based on what we will demonstrate in Chapter 10, *TZ* is half the day arc. *Z* and *A* in the two triangles *HTZ* and *HIA* are right angles. *TZ* is <equal to> the magnitude of the angle *H*, because *TH* is a quadrant. The ratio of the Sine of *HI* to the Sine of *IA* is equal to the ratio of the greatest Sine, being the Sine of *HT*, to the Sine of the angle *H*. The Sine of *HI* is equal to the Cosine of *IT*, and *IT* is the declination of the sun or the distance of the star from the celestial equator. The angle *H* is known, since it is <equal to the magnitude of> the arc *TZ*. Then *IA* is known, but it is the complement of *IB*, and *IB* is the ortive amplitude. Therefore the ortive amplitude is known. If we substitute the point *G* for the point *A*, the proof for finding the complement of the ortive amplitude in the northern and southern directions will be the same. This is what we wanted to demonstrate.

Chapter 7: On the equation of daylight of the sun and the star<s>.

ABGD is the horizon circle, *E* the zenith, *AEG* the meridian, and *BHM* the celestial equator with its pole at *T*. Let the point *D* be the degree whose equation of daylight is desired. We draw the arc *TZD* through it. Then *DZ* is the declination of the point *D*, *BZ* its equation of daylight, and *BD* its ortive amplitude. I say that *BZ* is known.

Proof: *BZD* is a right triangle, and <its right angle> is *Z*. Then the ratio of the Cosine of *DZ* to the Cosine of *DB* is equal to the ratio of the greatest Sine to the Cosine of *BZ*. But *DZ* and *DB* are known. Then *BZ* is known. This is what we wanted to demonstrate.

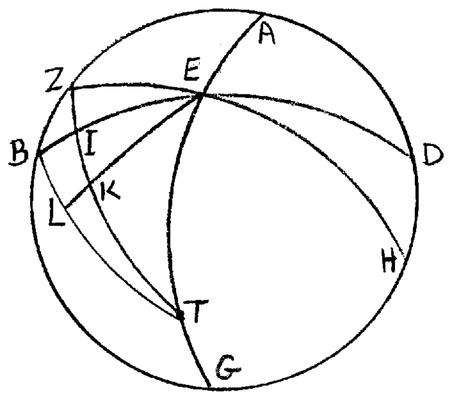


Another method: We draw the arc *EZN*. The angles *D* in the two triangles *DZN* and *DTG* are equal, and *N* and *G* are right angles. Therefore, according to what was demonstrated in the first premise, the ratio of the Sine of *DZ* to the Sine of *ZN* is equal to the ratio of the Sine of *DT* to the Sine of *TG*. *DZ* is the declination <of the sun> or the distance <of the star from the celestial equator>. <For northern *D*,> *DT* is the complement of the declination or of the distance. *TG* is the latitude of the locality. Therefore *ZN* is known. Again, the ratio of Sine of *BZ* to the Sine of *ZN* is equal to the ratio of the Sine of *BH* to the Sine of *HA*. But *ZN* is known, *BH* is a quadrant, and *HA* is the complement of the latitude of the locality. Therefore *BZ* is known. This is what we wanted.

Another method: In the triangle *BZD*, *Z* is a right angle. The angle *B* is equal to the complement of the latitude of the locality, being <equal to the magnitude of the arc> *HA*, because the arcs *BH* and *BA* are both quadrants. In view of what was demonstrated in the fourth premise, the ratio of the Sine of *BZ* to the Tangent of *ZD* is equal to the ratio of the greatest Sine to the Tangent of the angle *B*. So, we have to divide the Tangent of the declination, i.e. *ZD*, by the Tangent of the complement of the latitude of the locality, lowered. According to what was demonstrated in the third notice, being Chapter 7 of Section 3 <of Book IV in this *zīj*> on the premises, that <quotient> is equal to the product of the Tangent of the declination by the Tangent of the latitude of the locality, lowered. The result is the Sine of *BZ*. So *BZ* is known. This is what we wanted to demonstrate.

Another method <applicable> if the equation of daylight for the solstices is known: We draw the arc *BT*, being a quadrant. The circle *DZT* takes the place of the celestial equator, because if we fix the point *Z* and we rotate the arc *TZD*, it coincides with the arc *HBZ* of the celestial equator. We take on it the <right> ascension of the arc whose equation of daylight is desired. Let it be *TK*. We pass the arc *HKL* through it. It (i.e., *HKL*) is a quadrant, because the arc *BT* is drawn with *H* as its pole. Then *L* is a right angle. In the triangle *TKL*, *L* is a right angle and the angle *T* is equal to the total (i.e., maximal) equation <of the daylight>. On the basis of what was demonstrated in the first premise, the ratio of the Sine of *TK* to the Sine of *KL* is equal to the ratio of the greatest Sine to the Sine of the angle *T*. But *TK* is the ascension that was assumed, and the angle *T* is known. Then *KL* is known, and it is the partial equation of daylight. This is what we wanted.

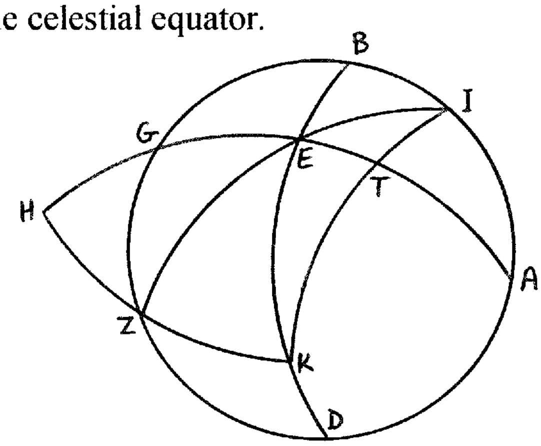
Another method for <finding> the equation of daylight <applicable> when the total equation <of daylight> is known: *ABGD* is the horizon circle, *AEG* the meridian, *BED* the celestial equator with its pole at *T*, and *ZEH* the ecliptic. Let the point *Z* be the first of Capricorn, and let us draw the arc *TIZ*. Then *BI* is the total (maximal) equation of daylight. We draw the arc *TB*. Both of the arcs *TI* and *TB* may take the place of the celestial equator, because if the intersection point *I* is fixed and the arc is rotated, it will coincide with the celestial equator. We take from the arc *TI* a magnitude <equal to the arc> whose equation of daylight is desired. Let



it be *TK*. We draw E*KL* <which> intersects the arc *TB* at right angles, because *TB* is drawn with *E* as the pole. So, *EL* is a quadrant. Then the ratio of the Sine of *TK* to the Sine of *KL* is equal to the Sine of *TI*, being the greatest Sine, to the ratio of the Sine of *IB*. But *TK* is taken from the celestial equator, *TI* a quadrant, and *IB* the total equation. So *KL* is known. This is what we wanted to demonstrate.

Chapter 8: On ascensions for a locality (i.e., oblique ascensions).

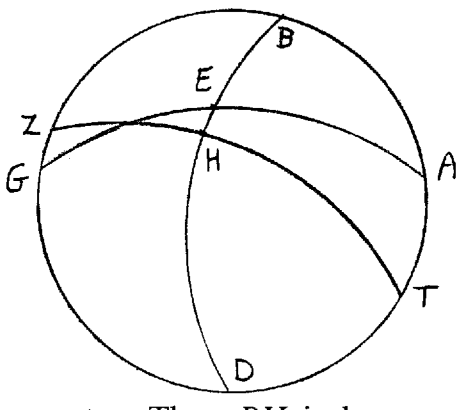
Let *ABGD* be the horizon circle, *BED* the meridian (this is not necessary; see commentary), *AEG* the celestial equator, *ZEI* the ecliptic, and *K* the pole of the celestial equator.



We draw the two arcs *KTI* and *KZH*. Then the arcs *TA* and *GH* are the equation of daylight for the points *I* and *Z*. The arcs *ET* and *EH* are the right ascensions of the two arcs *EI* and *EZ*. Let *EZ* be northern and *EI* southern. If we add *TA* to *ET*, the result is *EA*, the oblique ascension of *EI*. If we subtract *GH* from *EH*, the result is *EG*, the oblique ascension of *EZ*. This is what we wanted to demonstrate.

Chapter 9: On the maximum altitude of the sun and the star<s>.

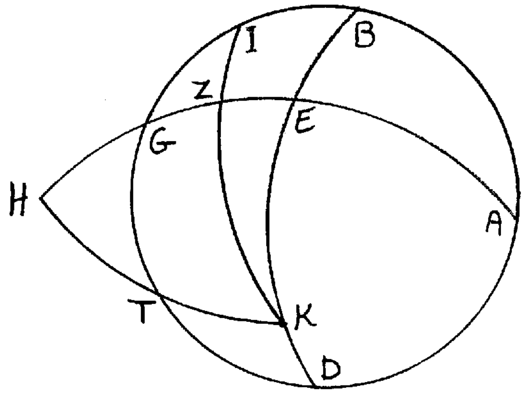
ABGD is the horizon circle, *BED* the meridian, *AEG* the celestial equator, and *ZHT* the ecliptic. We suppose the sun or the star <to be at> the point *H*. Then the arc *BH* is its maximum altitude, *BE* the complement of the latitude of the locality, and *EH* the declination of the sun or the star



from the celestial equator. Then *BH* is known. Again, let *ZHT* be the celestial equator, *AEG* the ecliptic, and *E* the position of the sun or the star. Then *BH* is the complement of the latitude of the locality, and *EH* the declination or the distance. So, *BE* is known. This is what we wanted to demonstrate.

Chapter 10: On half the day arc of the sun and the star<s>.

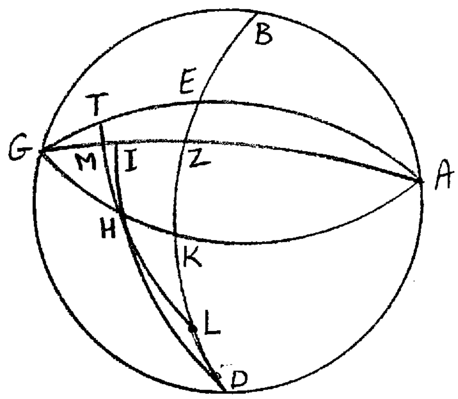
ABGD is the horizon circle, *BED* the meridian, and *AEG* the celestial equator. We assume the two points *I* and *T* as <two> rising positions of



the ecliptic, and *K* as the pole of the celestial equator. We draw the two arcs *KZI* and *KTH*. Then *GZ* is the equation of daylight for the point *I*, which is southern, and *ZE* is half its day arc. But *EG* is a quadrant, so *ZE* is known. *GH* is the equation of daylight for the point *T*, which is northern. *EH* is half its day arc and *EG* is a quadrant. So *EH* is known. This is what we wanted to demonstrate.

Chapter 11: On the <ecliptical> degree of the transit of a star through the meridian.

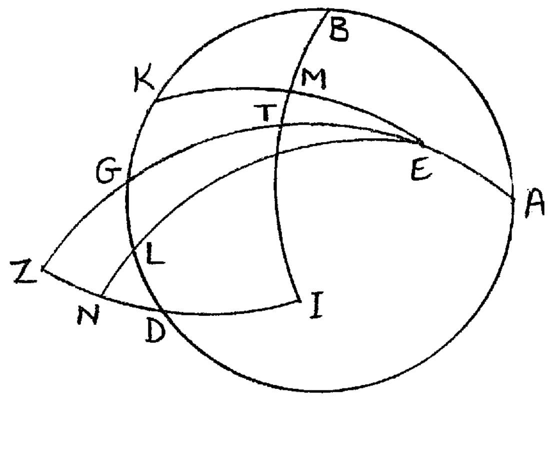
ABGD is the horizon circle (not necessary; see commentary), *BED* <the circle> passing through the poles <of the ecliptic and the celestial equator>, *AEG* the celestial equator with its pole at *L*, *AZG* the ecliptic with its pole at *D*, and *H* the body of the star. We draw <the arcs> *LHT*, *DHI*, and *AHG*. Then *I* is the <ecliptical> degree of the star, *IH* its latitude, *HT* its distance from the celestial equator, *M* the <ecliptical> degree of its transit, and *Z* the point of one of the two solstices. In the triangle *KDH*, *K* is a right angle, and the angle *D* is known, being <equal to the magnitude of> the arc *ZI*, because *DHI* is a quadrant.



ZI is the <longitude> distance of the <ecliptical> degree of the star from the solstice, and *DH* is the complement of the latitude <of the star>. The ratio of the Sine of *DH* to the Sine of *HK* is equal to the ratio of the greatest Sine to the Sine of the angle *D*. So, *HK* is known. Again, in the triangle *LHK*, *K* is a right angle, *LH* is the complement of the distance of the star from the celestial equator, and *HK* is known. The ratio of the Sine of *LH* to the Sine of *HK* is equal to the ratio of the greatest Sine to the Sine of the angle *L*. It (i.e., the Sine of *L*) is <equal to the magnitude of> the Sine of the arc *TE*. The arc *TE* is the right ascension of *ZM* <starting> from the first <degree> of the solstitial <sign> (i.e., Cancer or Capricorn). In view of what was demonstrated in the first notice (i.e., VI.3. 3), <since> the two middle terms in the <ratio of the> first <four> magnitudes are equal to the two middle terms in the <ratio of the> other <four> magnitudes, by ex aequali <of ratios>, the ratio of the Sine of *DH*, the complement of the latitude, to the Sine of *TE*, the right ascension of the <ecliptical> degree of the transit <taken> from the first <degree> of the solstitial <sign>, is equal to the ratio of the Sine of *LH*, the complement of the distance from the celestial equator, to the Sine of *IZ*, the distance of the <ecliptical> degree of the star from the solstice. *TE* is known, so *ZM* is known. Therefore the point *M* is known. This is what we wanted to demonstrate.

Chapter 12: On the <ecliptical> degree of the rising and setting of a star.

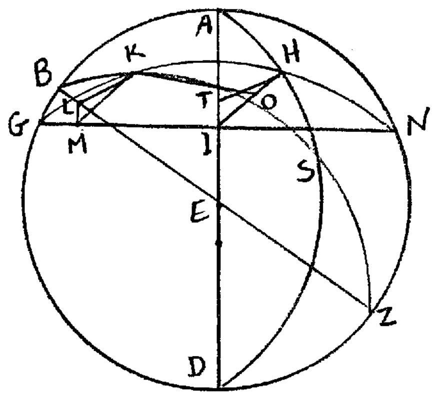
ABGD is the horizon circle, *AEG* is the celestial equator with pole at *I*, *E* is one of the two equinoxes, *EK* a southern <part> of the ecliptic, and *EL* a northern <part> of it. We assume *B* as the body of the star in the south, and *D* as its body in the north. *M* and *N* are the <ecliptical> degree of the transit of the star <in each of these two cases>. We draw *IBT* and *IDZ*. Then *GT* is the equation of daylight for *B*. If it (i.e., *GT*) is added to *ET*, the right ascension of the <ecliptical> degree of the transit of the star *B*, the result is *EG*, the oblique ascension of *EK*, and *K* is the <ecliptical> degree that rises <simultaneously> with the star.



Again, *EZ* is the right ascension of the <ecliptical> degree of the transit of the star *D*, and *GZ* is its equation of daylight. If it (i.e., *GZ*) is subtracted from *EZ*, *EG* remains, the oblique ascension of *EL*. But *L* is the <ecliptical> degree that rises <simultaneously> with the star. So *EG* is the <oblique> ascension of the <ecliptical> degree that rises <simultaneously> with the star *B*, and <with> the star *D*. If we imagine that the point *G* moves according to the general motion <of the celestial sphere> and arrives at the western horizon, i.e., at the point *A*, <then> it would have moved by the amount of the day arc of the star. <Then a certain> point of the celestial equator will have arrived at the eastern horizon, which <point> is (i.e., defines) the <oblique> ascension of the opposite to the <ecliptical> degree which sets <simultaneously> with the star. This is what we wanted to demonstrate.

Chapter 13: On <finding> the arc of revolution <of the celestial equator> since the rising of the sun and the star<s> from the altitude of the <sun or the star at a given> time, and <finding> the altitude from the arc of revolution.

ABGD is the horizon circle, *ASD* the meridian and *AED* its diameter, *BSZ* the altitude circle, and *BEZ* its diameter. Then *S* is the zenith. Arc *GHN* is <a part> of the parallel circle <to the equator> above the earth, and *GN* its chord. Then *H* is the intersection <point> of the parallel circle and the meridian, and *K* the intersection <point> of the parallel circle and the altitude <circle>. We draw *HT* perpendicular to *AE*. Then it (i.e., *HT*) is the Sine of the arc *AH*, the meridian altitude for the point *H* of the parallel circle. We join *HI*; it is the Sagitta of the arc *GKH*, and *GKH* is half the day arc. We draw *KL* perpendicular to *BE*; it is the Sine of the arc *BK*, and *BK* is the altitude of the <sun or the star at a given> time. We draw *KM* perpendicular to the chord *GN*. It (i.e., *KM*) is the 'arrangement Sine' of the arc of revolution. If we imagine *ASD* and *BSZ* to be vertical <semicircles> in the <celestial> sphere, with diameters *AD* and *BZ*, it is clear that, as we said, *HT* is the Sine of the arc *AH*, *HI* the Sagitta of the arc *GKH*, *KL* the Sine of the altitude, and that *KM* is parallel to *HI*.

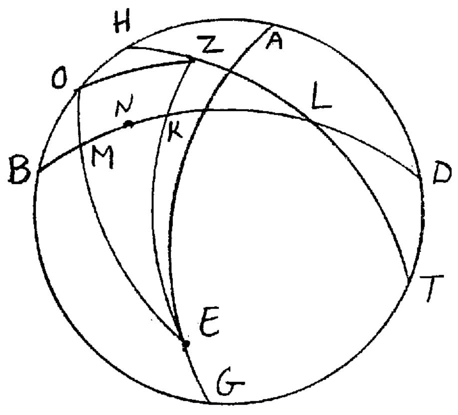


Then the two triangles *IHT* and *MKL* are similar, because *T* and *L* are right angles, and the two angles *H* and *K* are equal because the two line <segment>s *IH* and *HT* are parallel to the line <segment>s *MK* and *KL*, <respectively>. Then the ratio of *HT*, the Sine of the meridian altitude, to *KL*, the Sine of the altitude of the <sun or the star at a given> time, is equal to the ratio of *HI*, the Sagitta of half the day arc, to *KM*, the 'arrangement Sine' of the arc of revolution. So *KM* is known. We join *KO* perpendicular to *HI*. *KM* is equal to *OI* in the <celestial> sphere. The rest <from *HI*> is *HO*, the Sagitta of the arc *HK* <so *HK* is known>. *HK* is the excess of the arc of revolution, <and *HS*, half the day arc is known>, so *KG*, the arc of revolution, is known. This is what we wanted to demonstrate.

<Finding> the altitude from the arc of revolution: Again, if *KG*, the arc of revolution, is known, <then> the altitude *BK* is known. That is <based on the following proof:> The arc *GKH*, being half the day arc is known. The arc of revolution *GK* is also known. So *KH*, the excess of the arc of revolution, is known. Thus its Sagitta is known, and it is the excess of *IH* over *MK*. <*IH* is known> so *MK* is known. But the ratio of *MK*, the 'arrangement Sine' of the arc of revolution, to *KL*, the Sine of the altitude, is equal to the ratio of *IH*, the Sagitta of half the day arc, to *HT*, the Sine of the meridian altitude. So *KL* is known. So the altitude is known. This is what we wanted.

Chapter 14: On <finding> the ascendant from the arc of revolution <of e.g., the sun> and <finding> the arc of revolution from the ascendant.

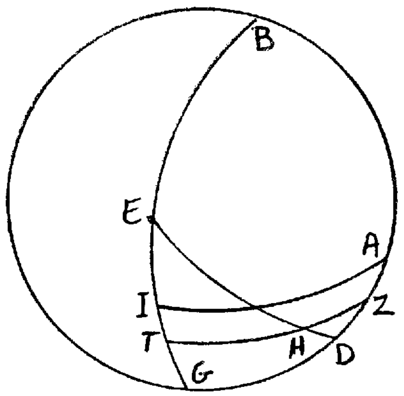
ABGD is the horizon circle, *AEG* the meridian, *BLD* the celestial equator with *E* as its pole. *HLT* is the ecliptic, its point *H* is <situated> on the horizon, and it (i.e., *H*) is required. I say that it is known. So we assume *Z* as the <known> position of the sun or the star on the ecliptic, and the arc *ZO* is its <known celestial> parallel. We draw two arcs passing through the pole of the celestial equator and the points *Z* and *O*. They intersect the celestial equator at *K* and *M*.



Then the arc *MK* is similar to the arc *ZO*, and *ZO* is the <known> arc of revolution, so *MK* is <also> the arc of revolution. *BM* is the equation of daylight for the point *Z*, and *LK* is the right ascension of *LZ*. We cut off *KN* equal to *BM*. Then *LN* is the oblique ascension of *LZ*, *BM* is equal to *KN*, and *MN* is a common <part>. So *MK* is equal to *BN*. But *MK* is the arc of revolution, so *BN* is equal to the arc of revolution. If *BN* is added to *LN*, the result is *LB*, which is known, because it is the oblique ascension of *LH*. Then the point *H*, which is the <ecliptical> degree of the ascendant, is known. This is what we wanted to demonstrate.
<Finding> the arc of revolution from the ascendant. Again, if the point *H* is known, and *Z* is the <ecliptical> degree of the sun or the star, <then> both <oblique> ascensions of *LH* and *LZ* are known, and they are *LB* and *LN* <respectively>. Then *BN*, equal to the arc of revolution, is known. This is what we wanted.

Chapter 15: On the proof <using> a base generally applicable to the arc of revolution and to what is related to it.

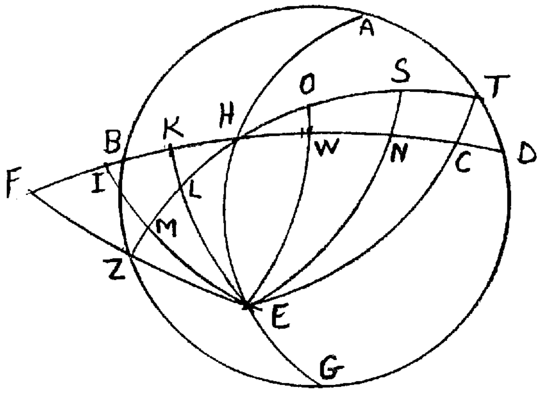
It is known from the forty-first figure (i.e., IV.5.13, see commentary), on the proof of the <method for finding the> arc of revolution from the altitude, that the ratio of the Sine of any altitude to the 'arrangement Sine' of its <corresponding> arc of revolution is equal to the ratio of the Sine of any other altitude to the 'arrangement Sine' of its <corresponding> arc of revolution. It is known that through any point from the ecliptic which is assumed on the horizon, a circle can be drawn passing through the two poles of the celestial equator. <The arc> between the assumed point and the celestial equator, of the circle passing through the two poles of the celestial equator, is the declination of the assumed point. The line drawn from the assumed point perpendicular to the diameter of the celestial equator is the Sine of the declination of the point. The diameter is <the line> drawn from the intersection of the celestial equator and the circle passing through its poles. <The line segment> between the foot of the perpendicular on this diameter and the complement of the radius is the Cosine of the declination of the <assumed> point, and it is equal to the radius of the parallel circle passing through the assumed point. The diameter <of the parallel circle> is drawn from that point, so the radius of any parallel circle is equal to the Cosine of its declination.
After this premise, let *ABGD* be the horizon circle, *BEG* the meridian, *EHD* <a part> of the altitude circle, *AI* the celestial equator, and *ZHT* <a part> of the parallel <circle>. Then the ratio of the Sine of *DH*, the altitude, to the 'arrangement Sine' of *HZ*, is equal to the Sine of *IG*, i.e. the altitude of the point *I*, to the 'arrangement Sine' of *AI*. But *AI* is a quadrant, and *IG* is the complement of the latitude of the locality.



So the product of the Sine of the altitude and the greatest Sine is equal to the product of the Cosine of the latitude of the locality and the 'arrangement Sine' of *HZ*. So the 'arrangement Sine' is known in terms of the magnitude (i.e., unit) for which the radius of the circle *AI* is 60 parts. We want to know this in terms of the radius of the circle *ZT*. The radius of the circle *ZT* is equal to the Cosine of the declination. So the ratio of the 'arrangement Sine' of *HZ* to the Cosine of the declination is equal to the ratio of the desired base (i.e., the 'arrangement Sine' *HZ* in terms of the radius of *ZT*) to the greatest Sine. So, the product of the known 'arrangement Sine' of *HZ* and the greatest Sine is equal to the product of the base (i.e., the value of the 'arrangement Sine' of *HZ*) in terms of the desired magnitude (i.e., the radius of *ZT*) and the Cosine of the declination of the degree (i.e., the point *H* on the parallel circle). Then the 'arrangement Sine' of *HZ* in terms of the radius of the circle *ZHT* is known. So, <to carry out> the multiplication, we may <actually> multiply the Sine of the altitude by the greatest Sine and <then> divide it by the Cosine of the latitude of the locality. Then we multiply the result by the greatest Sine and we divide it by the Cosine of the declination of the <ecliptical> degree. This is as if we multiply the Sine of the altitude by the greatest Sine twice and divide <the result> by the Cosine of the latitude of the locality, and then <we divide the final result> by the Cosine of the declination. That is <also> equal to multiplying it (i.e. the Sine of the altitude) by the greatest Sine twice and dividing it by the product of the Cosine of the latitude of the locality and the Cosine of the declination of the <ecliptical> degree. If we multiply the Cosine of the latitude of the locality by the Cosine of the declination of the <ecliptical> degree, lowered twice, because it should be multiplied by the greatest Sine twice, the result is the base from which the arc of revolution and what relates to it may be derived (i.e., computed). This is what we wanted to demonstrate.

Chapter 16: On the equalization of the houses.

ABGD is the horizon circle, *AEG* the meridian, *BHD* the celestial equator and *E* its pole, *ZHT* the ecliptic, the point *Z* the ascendant, *H* the midheaven, and *T* the descendant. We draw <the arcs> *EZF* and *ECT*.



Then *HF* is half the day arc of the ascending degree, and *HC* is half its night arc. If we divide *HF* into the three divisions *FI*, *IK* and *KH*, each division thereof will be equal to twice <the number of> the parts (i.e., degrees) <corresponding to> the hours of the ascendant. If we divide *HC* into three <equal> parts *HW*, *WN* and *NC*, each part will be equal to twice the <number of the> degrees <corresponding to> the hours of the descendant. <This is> because the time <-degrees corresponding to> each of <the arcs> *HF* and *HC* are 6 seasonal hours. If we draw circles from the pole of the celestial equator passing through these division <point>s, they cut the ecliptic at the division <point>s being the equal <i.e., ecliptical> degrees for the first division <point>s of the celestial equator. They are the divisions *HL*, *LM* and *MZ*, and the divisions *HO*, *OS* and *ST*. If we subtract 90° from the <right> ascension of the ascendant, i.e. *HB*, the right ascension of the tenth <house> remains. If we write down the right ascension of the tenth <house> in two positions, and add to it twice the <number of the> parts <corresponding to> the hours of the ascendant repeatedly, and if we subtract <in the other position> from it twice the <number of the> parts <corresponding to> the hours of the descendant repeatedly, the result of additions will be the right ascensions of the eleventh <house>, the twelfth <house>, and the ascendant. The result of the subtractions will be the right ascensions of the ninth <house>, the eighth <house>, and the descendant. If we write down the right ascension of the ascendant in two positions, and subtract from it twice the <number of the> parts <corresponding to> the hours of the ascendant repeatedly, and if we add <in the other position> to it twice the <number of the> parts <corresponding to> the hours of the descendant repeatedly, the result of these subtractions will be the right ascensions of the twelfth <house>, the eleventh <house>, and the tenth <house>. The result of the additions will be the right ascensions of the second <house>, the third <house>, and the fourth <house>. This is what we wanted to demonstrate.

Commentary

IV.5.1 The subject of first declination (the distance of any ecliptical degree from the celestial equator) is also discussed by Ptolemy [1984, 69-70]. Kūshyār's method based on the Sine theorem is general and simple, but Ptolemy's proof is longer and he performs the calculation for the specific arcs 30° and 60°. See also the commentary on I.5.1. The distance of a star to the equator is nowadays also called "declination".

IV.5.2 Kūshyār applies what he has proved in the premises IV.3.5 and IV.3.2 in the first and the second proofs, respectively. They are the Tangent Theorem and the Cosine Theorem for the right spherical triangles. See also the commentary on I.5.2.

IV.5.3 Here Kūshyār first proves his second method in I.5.3. See the commentary on I.5.3.

IV.5.4 In this chapter, Kūshyār supposes that the (ecliptical) latitude *HZ* and second declination *HT* are known. The application of the concept "similarity" for spherical triangles by Kūshyār is rather strange. He applies the so called "Rule of Four" for spherical right triangles that have an acute angle in common or equal acute angles. The same concept of "similarity" of spherical triangles is used in the solution of the problem in al-Kāshī's *Khāqānī Zīj*, see [Kennedy 1985, 9]. See also the commentary on I.5.4.

IV.5.5 Kūshyār uses the same figure for the two cases in which the declination of the sun is towards north and south, thereby interchanging the circles representing the ecliptic and the celestial equator. See also the commentary on I.5.5.

IV.5.6 See the commentary on I.5.6.

IV.5.7 In the first method for finding the equation of daylight for the sun and the stars, it is assumed that the declination and the ortive amplitude of the sun or the star are known. In the second and the third methods, it is assumed that the latitude of the city and the declination of the sun or the star are known. In the fourth method, it is assumed that the right ascension of the arc whose equation of daylight is requested and the equation of daylight for the solstices, *BZ*, are known. It is interesting that Kūshyār applies a rotation in this method. The last method is not mentioned in I.5.7 where the first four methods are provided. However, this additional method is similar to the fourth one, except that *TK* is taken

equal to the arc whose equation of daylight is desired, and not its right ascension as taken in the fourth method. In both fourth and fifth methods, the values found for the equinoxes and the solstices are accurate, and the intermediate values vary uniformly. The results can be good approximations for the desired equation of daylight. Kūshyār's proofs for these two methods show that *KL* can be found from the existing data. But he does not prove that *KL* is equal to the desired equation of daylight. See also the commentary on I.5.7. The fourth and the fifth methods are mathematically correct.
Proof: since $\mathrm{Sin}\Delta = \mathrm{Tg}\delta \times \mathrm{Tg}\varphi / R$ and $\mathrm{Sin}M = \mathrm{Tg}\varepsilon \times \mathrm{Tg}\varphi / R$, we have $\mathrm{Sin}\Delta \,/\, \mathrm{Sin}M = \mathrm{Tg}\delta \,/\, \mathrm{Tg}\varepsilon = \mathrm{Sin}\alpha / R$, where M is the known equation of daylight for solstices. Kūshyār's proof seems to imply some sort of geometrical transformation, but the idea escapes us.

IV.5.8 In this chapter, if we take *BED* as the meridian, then the problem will lose its generality. So this is a superfluous phrase possibly inserted into the text because it exists in the next chapters. *BED* can be any great circle through the celestial pole *K*. See also the commentary on I.5.8.

IV.5.9 Again, for using the same figure for the cases in which the declination of the sun or a star from the celestial equator is towards the north or south, Kūshyār interchanges the circles representing the ecliptic and the celestial equator for the second case. See also the commentary on I.5.9.

IV.5.10 See also the commentary on I.5.10.

IV.5.11 In this chapter, the starting phrase "*ABGD* is the horizon circle" is superfluous. It has possibly been inserted to the text by mistake, because it appears at the beginning of the former and the next chapters. *ABGD* is a great circle through the intersections *A* and *B* of the ecliptic and the equator. Kūshyār assumes that the ecliptical latitude *HI* and the ecliptical distance along the ecliptic to the solstice *IZ* are known (this distance can be derived from the ecliptical longitude).
Kūshyār concludes the desired proportion $\mathrm{Sin}DH : \mathrm{Sin}TE = \mathrm{Sin}LH : \mathrm{Sin}IZ$ by IV.3.3 from $\mathrm{Sin}DH : \mathrm{Sin}HK = R/\mathrm{Sin}D$, $\mathrm{Sin}LH{:}\mathrm{Sin}HK = R/\mathrm{Sin}L$ and from the fact that the magnitudes of the angles *D* and *L* are measured by arcs *IZ* and *TE* respectively. The technical term "ex aequali" is defined in definition 17 of Book V of Euclid's *Elements*. On the problem of finding *TE* (the right ascension of the star at *H*), see also al-Kāshī's *Khāqānī Zīj* [Kennedy 1985, 10-14]. See also the commentary on I.5.12.

IV.5.12 In the last part: "If we imagine...", the description is confusing. Apparently, Kūshyār means that "If we imagine that the point G moves according to the general motion <of the celestial sphere> and B arrives at the western horizon, then the point G would have moved by the amount of the day arc of the star. Then a certain point of the celestial equator will have arrived at the eastern horizon, which is the oblique ascension of the <ecliptical degree> opposite to the <ecliptical> degree which sets <simultaneously> with the star." See also the commentary on I.5.13.

IV.5.13 For a modern mathematical formula for finding the arc of revolution from the altitude of the time, and a definition of the 'arrangement Sine', see the commentary on I.5.14.

IV.5.14 Here Kūshyār provides the proof for the method described in I.5.16 and for its inverse described in I.5.17.

IV.5.15 The chapter on the proof of the validity of the method for finding the arc of revolution from the altitude of the sun or the star, IV.5.13, corresponds to the 41st figure (*shakl*) in Book IV. However, in the ms. A, it is referred to as the '44th figure'. IV.5.13 is Chapter 44 in the ms. A (which presents the chapters of Book IV as numbered consecutively) and apparently the word *shakl* in the reference is used in its other meaning "theorem".
In this chapter Kūshyār demonstrates that the 'arrangement Sine' can be found by dividing the Sine of the altitude by $\mathrm{Cos}\varphi\,\mathrm{Cos}\delta\,/R.R$. He calls this expression the "base". In I.5.21 this base is used in short efficient methods for finding different astronomical quantities. For a modern mathematical formulation of these methods see the commentary on I.5.21.
In finding the value of the arrangement Sine of HZ in terms of the radius of the parallel ZT, all manuscripts erroneously put '60' instead of 'the Cosine of the declination of the degree (i.e., the point H)', and the correct version is only mentioned in a marginal note of the ms. M.
The concept of "base" was widespread in medieval Islamic astronomy, see for example [King 2004, 114,909].

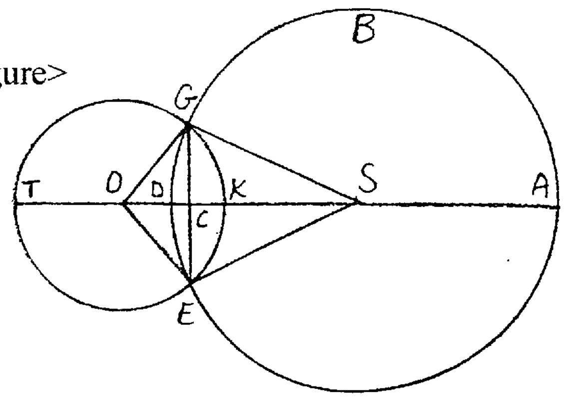
IV.5.16 Here Kūshyār actually explains the geometrical basis of the method for the equalization of the houses which was the most popular one among the medieval Islamic authors of astrological works. See also the commentary to I.5.22.

Section 6: On eclipses and what pertains to them, <in> 14 chapters

Chapter 1: On the absolute and adjusted magnitudes of a lunar eclipse in digits.

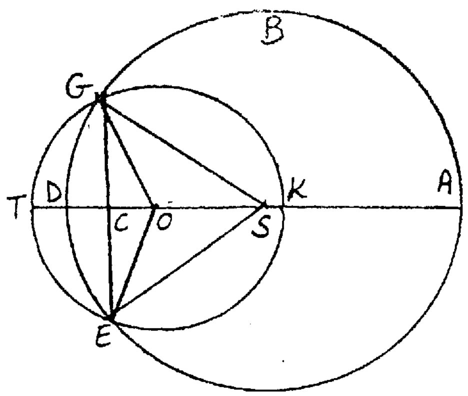
ABGD is the disk of the <earth's> shadow in the transition position <of maximum obscuration> of the moon, and *GTEK* is the disk of the moon's surface. They are perceived as being in the same plane. *AD* is the diameter of the disk of the shadow, and *KT* is the diameter of the disk of the moon. *SD* <plus> *KO* is the sum of the two radii. *SO* is the latitude of the moon. *KD* is the excess of *SD* <plus> *KO* over *SO*. So, *KD* the magnitude of the lunar eclipse in minutes, is known. *KT* is <also> known. So *KD* is known by taking *KT* <equal to> 12 digits. It (i.e., the magnitude of 12*KD*/*KT*) is <called> 'the absolute magnitude of the lunar eclipse in digits'. The area of <the segment> *DGEK*, the surface of the moon's disk is <called> the 'adjusted magnitude of the lunar eclipse in minutes'. It is <called> the 'adjusted magnitude of the lunar eclipse in digits', by assuming the area of the moon's disk <equal to> 12 digits. It (i.e., the adjusted magnitude in digits) is desired <if the absolute magnitude is known>. <To this end>, we join *EG* and we draw the line <segments> *SG*, *SE*, *OG* and *OE*. Since *AD* and *GE* intersect in a circle, the product of *AC* by *CD* is equal to the product of *GC* by *CE* (*Elements*, III.36), and the product of *TC* by *CK* is equal to the product of *GC* by *CE*. Thus the product of *AC* by *CD* is also equal to that of *TC* by *CK*. So, the ratio of *AC* to *CT* is equal to that of *KC* to *CD*. If we subtract *KD* from both diameters *AD* and *KT*, the ratio of the remainder, *AK*, to *DT* is equal to that of *KC* to *CD* (*Elements*, V.17). By composition of ratios, the ratio of the sum of *AK* and *DT* to *TD* is equal to that of *KD* to *DC*. The sum of *AK* and *DT*, *DT*, and *KD* are known. So, *DC* is known, and it is the Sagitta of the <arc *GE* of the> shadow's disk. *KC* is <also> known, and it is the Sagitta of the <arc *GE* of the> the moon's disk. So both *TC* and *CK*

The first <figure>



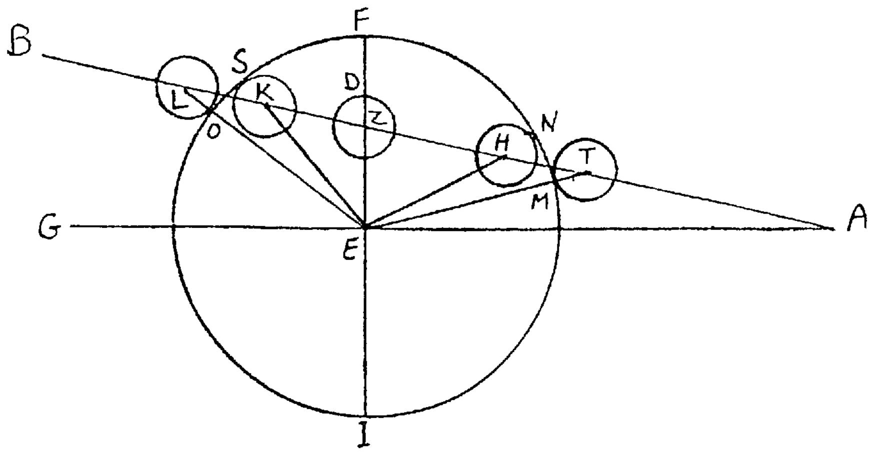
are known. The product of *TC* by *CK* is equal to the square of *GC*, because *GC* is equal to *CE*. So, *GC* is known, and it is the Sine of the arc *GK* by assuming *OG* as the radius of the moon. Thus it (i.e., *GC*) is known by assuming *OG* <equal to> 60 parts. Hence the arc *GK* of the great circle is known, and it is the complement of the arc *GT* towards 180 <degrees> in the second figure. Its ratio to 360 <degrees> is equal to the ratio of the arc *GK* on the circumference of the moon's disk to the whole circumference of the moon's disk. Thus <the ratio of the length of> the arc *GK* to <the circumference of> the moon's disk is known. *OK* is <also> known. So the area of the sector *OGKE* is known. *OC* and *GC* are known, so, the area of the triangle *OGE* is known. So the area of the segment *GKEC* of the moon's disk is known. Again, *GC* is the Sine of the arc *GD* <based> on <taking> *SG* the radius of the disk of the shadow. So, it (i.e., *GC*) is known <based> on <taking> *SG* <equal to> 60 parts. So the arc *GD* of the great circle is known. Its ratio to 360 <degrees> is equal to the ratio of the arc *GD* of the disk of the shadow to the whole circumference of the disk <of the shadow>. Therefore the arc *GD* of the disk of the shadow is known and *SD* is known. Thus the area of the sector *SGDE* is known. <Also> *SC* and *GC* are known. So, the area of the triangle *SGE* is known. Then the area of the segment *GDEC* of the disk of the shadow is known. <Now> the sum of *GKEC* and *GDEC* is known, and it is the 'adjusted magnitude of the lunar eclipse in minutes'. Its ratio to the area of the surface of the moon's disk is equal the 'adjusted magnitude of the lunar eclipse in digits' to 12. This is what we wanted.

The second <figure>



Chapter 2: On the absolute times of a lunar eclipse.

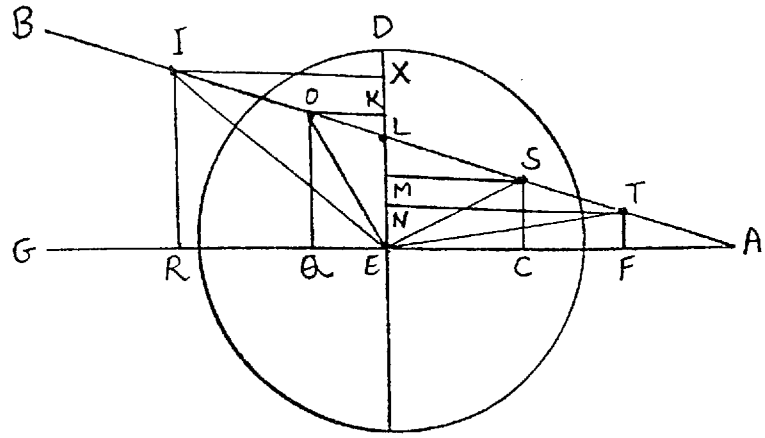
Let *AB* be a segment of the inclined <lunar> orb, *AG* a segment of the ecliptic, and the point *E* the center of the <earth's> shadow disk. *ED* is a segment of the circle *IEF* passing through the two poles of the ecliptic, <the part> *EZ* on it being the latitude of the moon in the middle of the eclipse. *T* is the center of the moon's disk at the beginning of the eclipse when it is tangent to the shadow's disk at the point *M*. <Then the moon> proceeds to enter the eclipse. *H* is the center of the moon's disk when it is totally eclipsed and the beginning of its total immersion when it is tangent to the shadow's disk at *N*. <Then the moon> proceeds to enter the <total> eclipse, and *Z* is the center of the moon's disk in the middle of the eclipse, being nearest to the center of the shadow <disk>. *K* is the center of the moon's disk at the end of the total immersion and at the beginning of emersion, and it is tangent to the shadow's disk at *S*. <Then the moon> proceeds to go out of it (i.e., out of the shadow) and *L* is the center of the moon's disk at the end of emersion, and it is tangent to the shadow's disk at *O*. <Then the moon> proceeds to separate from the shadow. Both the line segments *ET* and *EL* are the sum of the two radii, and both *EH* and *EK* are <equal to> the radius of the shadow <disk> minus the radius of the moon <disk>. *ZT* <corresponds to the duration of> immersion in minutes, from the beginning of the eclipse up to its middle. *ZH* <corresponds to> the duration <of totality> in minutes, from the beginning of the total immersion up to the middle of the eclipse. *ZK* <corresponds to> the duration <of totality> in minutes, from the middle of the eclipse up to the beginning of emersion. *ZL* <corresponds to the duration of> immersion in minutes, from the middle of the eclipse up to the end of emersion. <The lengths of> these line <segments> are desired, because each of them being divided by the lunar gain, provides the hours (i.e., time interval) corresponding to these minutes <of arc>. We assume all <the arcs> *AB*, *AG* and *ED* to be straight line <segments>, because they are small, and there is no <noticeable> difference between taking them as arcs or as straight line <segments> in the eclipses. *ET* is the sum of the two radii, *EZ* the latitude of the moon at the middle of the eclipse, and *Z* is approximately a right angle. If the square of *EZ* is subtracted from the square of *ET*, the result is the square of *TZ*. So *TZ* is known, and it is <corresponding to the duration of> the immersion. If we subtract its <time in> hours from the <time in> hours of the middle of the lunar eclipse, the <time in> hours of the beginning of the lunar eclipse will result.



If it is added <to the time of the middle of the eclipse>, the <time in> hours of the end of emersion will be obtained, because *ET* is equal to *EL*. Again, *EH* is <equal to> the radius of the shadow <disk> minus the radius of the moon <disk>. If we subtract the square of *EZ* from its square (i.e., from *EH* squared), the result is the square of *HZ*. Thus *HZ* is known. If its <time in> hours are subtracted from those of the middle of the lunar eclipse, the result is the <time in> hours of the beginning of its total immersion. If it is added <to the time of the middle of the eclipse>, the result is the <time in> hours of the beginning of emersion, because *EH* is equal to *EK*. These are the five <desired> times. If there is no total immersion in the eclipse, the <time in> hours of the beginning of the total immersion and of the beginning of emersion will be deleted. This is what we wanted.

Chapter 3: On the correction of the times.

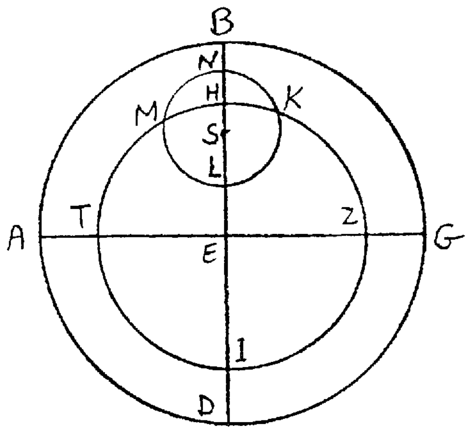
Let *AB* be a segment of the inclined <lunar> orb, *EG* a segment of the ecliptic, and *E* the center of the shadow's disk. *ED* passes through the two poles of the ecliptic, and <the part> *EL* of it is the latitude of the moon in the middle of the eclipse. The points *T*, *S*, *L*, *O* and *I* are the centers of the moon at the beginning of the lunar eclipse, the beginning of total immersion, the middle of the lunar eclipse, the beginning of the emersion, and the end of emersion, <respectively>. From these points, we draw the line <segments> *TF*, *SC*, *OQ* and *IR* parallel to the line <segment> *LE*. Each of them is the latitude of the moon corresponding to these centers. We draw the line <segments> *TN*, *SM*, *OK* and *IX* parallel to the line <segment> *AG*. *LEG* is a right angle. We join the line <segments> *TE*, *SE*, *OE* and *IE*. Since *TN* is parallel to *FE*, *EN* is equal to *TF*, both being the latitude of the moon at the beginning of the lunar eclipse (whose time is known from Chapter 2). *TE* is the sum of the two radii, and *ENT* is a right angle. Then *TN* is known, and *NL* is <the difference> between the latitude <of the moon> at the beginning <of the eclipse> and its latitude in the middle <of the eclipse>. Then *LT* is known, and it is <corresponding to> the adjusted <duration of> immersion in minutes.



Again, *ES* is the remainder of subtracting the radius of the moon from that of the shadow. *SC* is the latitude <of the moon> at the beginning of total immersion. *ECS* is a right angle. Then *EC* is known, being equal to *SM*. So, *SM* is known. *LM* is <the difference> between the latitude <of the moon> at the beginning of total immersion and <the latitude in> the middle of the eclipse. *LMS* is a right angle. Then *LS* is known, and it is <corresponding to the adjusted> duration <of totality> in minutes. Also, *EO* is the remainder of subtracting the radius of the moon <disk> from that of the shadow <disk>. *OQ* is the latitude of the moon at the beginning of emersion. Then *QE* is known, being equal to *OK*. *KL* is <the difference> between the latitude <of the moon> in the middle <of the eclipse> and <the latitude at> the beginning of emersion. *OKL* is a right angle. Then *LO* is known, being <corresponding to the adjusted> duration <of totality> up to the beginning of emersion in minutes. Again, *EI* is the sum of the two radii, and *IR* the latitude <of the moon> at the end of emersion. Then *RE* is known, being equal to *IX*. *LX* is <the difference> between the latitude <of the moon> at the middle <of the eclipse> and <its latitude at> the end of emersion. Then *LI* is known, and it is <corresponding to the adjusted duration of> immersion from the middle <of the eclipse> to the end of emersion. So, the five adjusted times are known. This is what we wanted.

Chapter 4: On drawing the figure of a lunar eclipse.

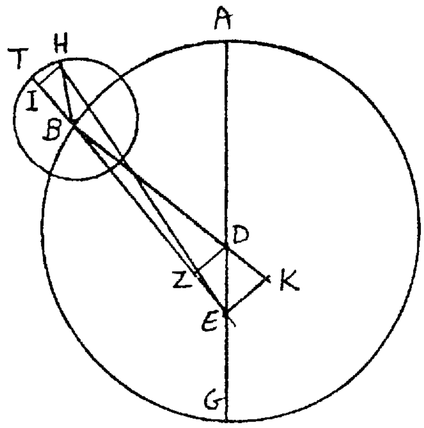
Let *ABGD* be a circle whose radius is equal to the sum of the two radii in minutes, centered at *E*. <Also let> *ZHTI* <be> a circle centered at this <point *E*>, its radius being equal to that of the shadow. The two line <segments> *AG* and *BD* intersect at *E* at right angles. Let *EB* be the south line, *ED* the north line, *EA* the east line, and *EG* the west line, and let *ES* be the latitude of the moon in the middle of the lunar eclipse.



KLMN is the moon circle centered at *S*. Its arc *KLM* is situated in the shadow circle and it (i.e., the lunule *KLM*) is the magnitude of the eclipsed part of the surface of the moon, by taking its whole surface <equal to> 12 digits. *LH* is the non-adjusted (i.e., absolute) <magnitude of the> lunar eclipse in digits (based on taking *NL* equal to 12 digits). *EB* is <equal to> the radius of the shadow plus the radius of the moon. Its <part> *EH* is the radius of the shadow. The remaining <part> *HB* is the radius of the moon, and *LH* is the <absolute magnitude of the> lunar eclipse in digits. This is what we wanted to demonstrate.

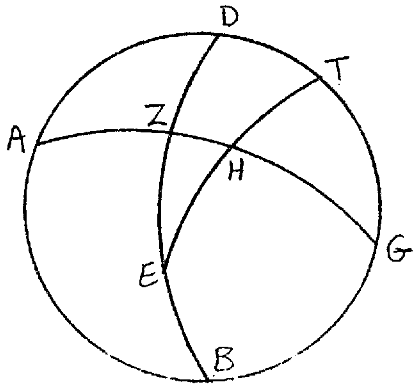
Chapter 5: On the distance of the moon from the earth.

Let the circle *ABG* be the eccentric orb centered at *D*, *AG* its diameter, *E* the center of the ecliptic, *B* the center of the epicycle of the moon, *T* the apogee of the epicycle, and *H* the body of the moon. We join the lines. *EH* is, as desired, the distance of the moon from the earth. The two lines *DZ* and *HI* are perpendicular to *ET*. The angle *AEB* is known, being the double elongation. *Z* is a right angle. So the angle *EDZ* is known. *DE* is <equal to> 10 parts and a third, if we take *AE* <equal to> 60 parts. So both *DZ* and *ZE* are known. *DB* is <equal to> 49 parts and two thirds, and its square is equal to <the sum of> the squares of *BZ* and *ZD*. Then *BZ* and *ZE* are known. So is *EB*, being the distance of the center of the epicycle of the moon from the earth. Also, the angle *TBH* is the adjusted mean anomaly (i.e., true anomaly) of the moon. *I* is a right angle. So the angle *BHI* is known. *BH* is the radius of the epicycle in terms of the distance of its center from the point *A*. Both *HI* and *IB* as well as *EB* are known. So, *EI* is known, its square plus the square of *HI* being equal to the square of *EH*. So, *EH* is known, being the distance of the moon from the earth. This is what we wanted to demonstrate.



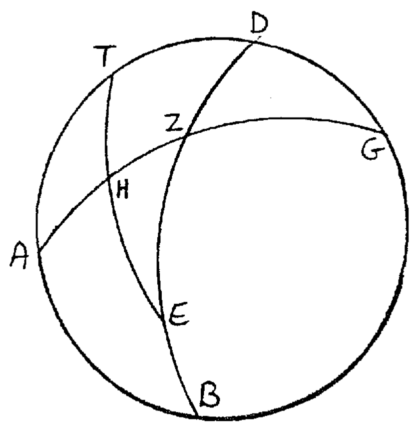
Chapter 6: On the altitude of the pole of the ecliptic.

<Let> *ABGD* <be> the horizon circle, *BED* the meridian, *AHG* the ecliptic, and *ET* <a part> of the altitude circle drawn with its pole at *A* and its radius equal to the side of an <inscribed> square. Then *HT* is <equal to> the magnitude of the angle *HAT*, for *AT* and *AH* are both quadrants. <Now> *EH* is desired, for it is equal to the altitude of the pole of the ecliptic. In the triangle *ADZ*, *D* is a right angle, the side *AZ* <the distance> between the ascendant and the midheaven along the ecliptic, and *ZD* the altitude of the <ecliptical> degree of the midheaven. The ratio of the Sine of *AZ* to that of *ZD* is equal to the ratio of the greatest Sine, being the Sine of *AH*, to the Sine of *HT*. So *HT* is known. Therefore its complement *EH* is known. This is what we wanted to demonstrate.



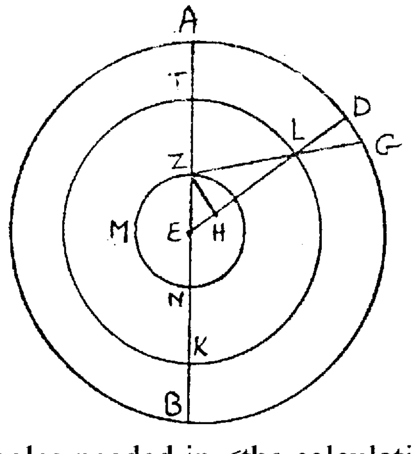
Chapter 7: On the altitude of any desired degree of the ecliptic.

<Let> *ABGD* <be> the horizon circle, *BED* the meridian, *AZG* the ecliptic, the two points *A* and *Z* the ascendant and the tenth <house, *E* the zenith>, *ET* <a part> of the altitude circle, and *H* the <ecliptical> degree whose altitude is desired. <So,> the arc *HT* is desired. In the two triangles *AHT* and *AZD*, *D* and *T* are right angles. *AH* is <the distance> between the ascending <ecliptical> degree and the <ecliptical> degree whose altitude is desired. *ZD* is the magnitude of the altitude of the midheaven degree. The ratio of the Sine of *AH* to that of *HT* is equal to the ratio of the Sine of *AZ* to that of *ZD*. So, *HT* is known. This is what we wanted to demonstrate.



Chapter 8: On the parallax of the two luminaries in the altitude circle.

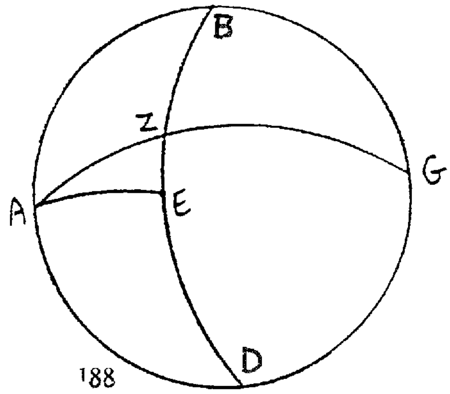
<Let> *ABGD* <be> the altitude circle on the surface of the Sphere of the Whole (i.e., the Universe), *TKL* the altitude circle on the surface of the sphere of the moon (i.e., a sphere through the moon with center the center of the earth) with the point *L* as the moon on it, and the circle *ZMN* the surface of the earth. The three circles are in the same plane and centered at *E*. The line <segments> *AE*, *TE*, and *ZE* are the radii of these circles passing through the zenith. From the two points *E* and *Z*, we draw two lines intersecting at the point *L* and ending at the two points *G* and *D*. The angle *ELZ* is the parallax, because it is the excess of the angle *LZT*, i.e. the arc *AG* visible on the surface of the earth, over the angle *LET*, i.e. the arc *AD* visible from the center of the earth. From the point *Z*, we draw *ZH* perpendicular to *EL*. The arc *AD* is the complement of the altitude visible from the center of the earth. So, the angle *ZEH* is known <from the computation of the lunar position>. *ZHE* is a right angle, and *ZE* is the radius of the earth, being <equal to> one degree (i.e. the unit of length). So, both *EH* and *HZ* are known. *EL* is the distance of the moon from the earth. Then *HL* is known, and so is *LZ*. Then *ZH* is known by taking *LZ* <equal to> 60 parts, and its arc is known. Then the angle *ELZ* is known and its magnitude is <equal to> the arc *GD*. This is what we wanted to demonstrate.



Chapter 9: On the six angles needed in <the calculation of> solar eclipses.

First <case>, when the position of the moon is at the first <degree> of Aries or Libra and it is at the ascending <ecliptical> degree of the time:

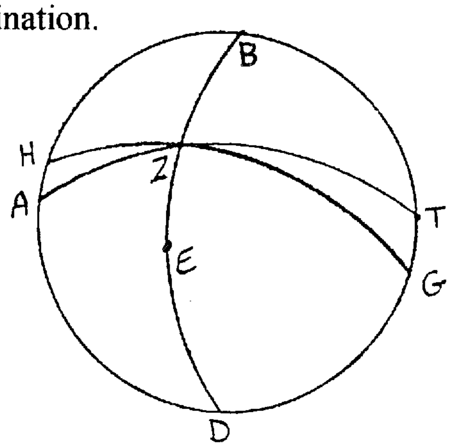
<The first figure>



Let *ABGD* in the first figure be the horizon circle, *BED* the meridian, *E* the zenith, *AZG* the ecliptic, *EA* <a part> of the altitude circle, and the point *A* the rising position of the equinox. The angle *EAZ* is desired, its magnitude being <equal to> the arc *EZ*. Both *AZ* and *EA* are quadrants, and *EZ* is known, because the point *Z* is the first of Cancer or the first of Capricorn. Therefore the angle *EAZ* is known (see commentary).

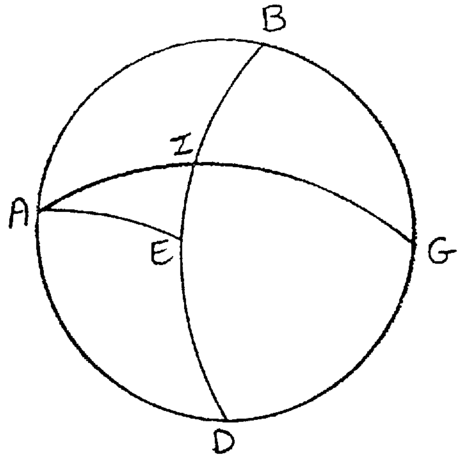
Second <case>, when the position of the moon is at the first <degree> of Aries or Libra and it is at the <ecliptical> degree of the tenth <house> of the time: It (i.e., the desired angle) is the angle *GZD* in the second figure, taking *AZT* as the celestial equator, *HZG* as the ecliptic, the point *Z* as the equinox, and *BED* as the meridian. *TZD* is a right angle, and the angle *TZG* is the total declination. So the remaining angle *GZD* is the complement of the <total> declination.

<The second figure>



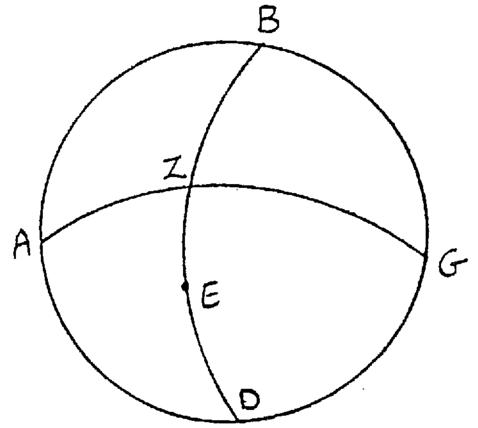
Third <case>, when the position of the moon is other than the first <degree> of Aries or Libra, and it is at the ascending <ecliptical> degree of the time: It is the angle *ZAE* in the third figure, assuming *A* to be other than the rising position of the equinox, and the circle *BED* drawn with its pole at *A* and its radius <equal to> the chord of the quadrant of <a great> circle. Then the arc *EZ* is equal to the altitude of the pole of the ecliptic (computed in Chapter 6), and its magnitude equal to the angle *EAZ*.

<The third figure>



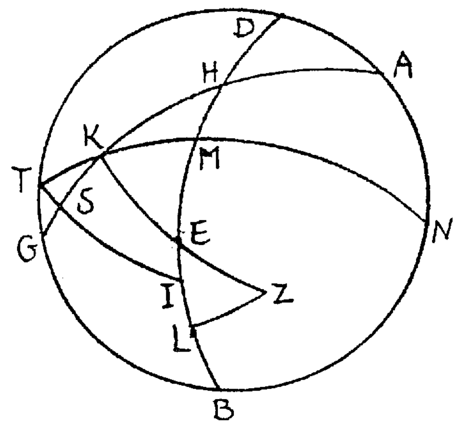
Fourth <case>, when the position of the moon is the first <degree> of Cancer or the first <degree> of Capricorn, and it is the <ecliptical> degree of the tenth <house> of the time: It is the angle *AZD* in the fourth figure, *AZG* being the ecliptic and *BED* the meridian. It (i.e., the desired angle) is a right angle, for *AZ* is a quadrant (*A* is an equinox).

<The fourth figure>



Fifth <case>, when the position of the moon (*H* in the fifth figure) is other than an equinoctial or solstitial point, and it is the <ecliptical> degree of the tenth <house> of the time: Let *ABGD* in the fifth figure be the horizon circle, *BED* the meridian, *L* the pole of the celestial equator, *GKA* the ecliptic, and *Z* its pole. *KHE* is the desired angle. In the triangle *KHE*, *K* is a right angle. The side *EH* is <the distance> between the zenith and the ecliptic along the meridian. *EK* is equal to the altitude of the pole of the ecliptic. The ratio of the Sine of *HE* to that of *EK* is equal to the ratio of the greatest Sine to that of the angle *H*. So the angle *H* is known. (Some manuscripts contain an alternative method which uses the points *I* and *S*, see the commentary to this chapter.)

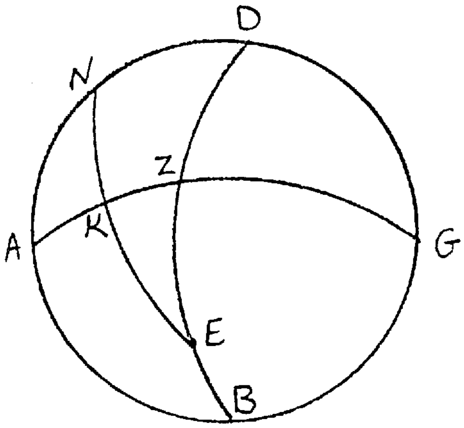
<The fifth figure>



Sixth <case>, when the position of the moon is any <arbitrary ecliptical> degree between the ascendant and the descendant: Let *ABGD* in the sixth figure be the horizon circle, *AZG* the ecliptic, its point *K* the <ecliptical> degree of the moon, *BED* passing through its (i.e., ecliptic's) poles <and

the zenith *E*>. *EKN* is <a part> of the altitude circle. *EKZ* is the desired angle. In the triangle *EKZ*, *Z* is a right angle. The side *KE* is the complement of the altitude of the <ecliptical> degree of the moon (computed in Chapter 7). The side *EZ* is equal to the altitude of the pole of the ecliptic (computed in Chapter 6).

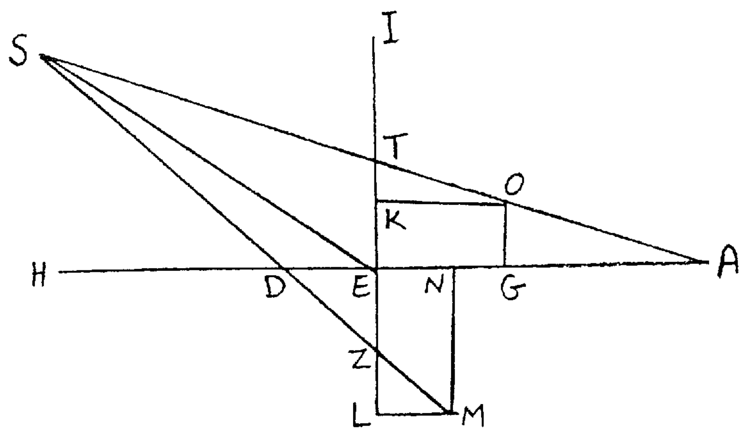
<The sixth figure>



The ratio of the Sine of *KE* to that of *EZ* is equal to the ratio of the greatest Sine to that of the angle *K*. Then the angle *K* is known. This is what we wanted.

Chapter 10: On <finding> the longitudinal and latitudinal parallax of the moon from these angles.

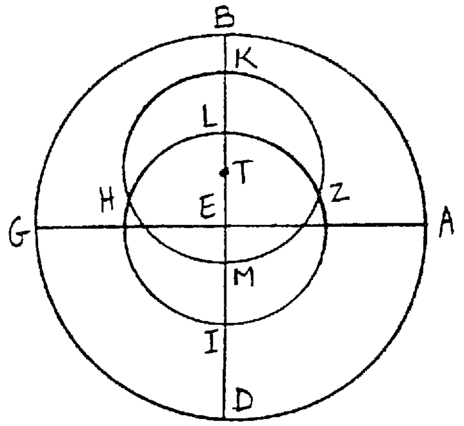
Let *AH* be an arc of the ecliptic, *IL* an arc of the latitude circle, *ET* the northern latitude of the moon, *E* the <ecliptical> degree of the moon, *T* the body of the moon, and *S* the zenith. We draw two arcs of the <two> altitude circle<s> passing through the two points *T* and *E*. They are the arcs *SA* and *SE*. Let *TO* be the parallax in the altitude circle. We draw *OK* parallel to *AH*, and *OG* parallel to *IE*. The lines in this figure are arcs. However, there is no <noticeable> difference between taking them as arcs or as straight lines, because they are small at the time of eclipses (because the lunar latitude is almost zero).



The two line <segments> *EG* and *OG* are desired. *EG* is the difference in longitude, and *OG* the apparent latitude. The angle *SEH* is the latitude angle (computed in Chapter 9). There is no noticeable <difference> between it and the angle *SAH*. The angle *SAH* is equal to the angle *TOK*, because *OK* is parallel to *AE*. Both the angles *SAH* and *TOK* are equal to the angle *SEH*; so they are known. *OKT* is a right angle; so the angle *OTK* is known. Since *OT*, the hypotenuse of the right angle is known, both *OK* and *KT* are known. *KT*, the difference in latitude, <is known>, so *KE* is known, and it is equal to *OG*. Then *OG* is known, and it is the apparent latitude. *OK* is equal to *GE*. So *GE* is known, and it is the difference in longitude. So because of its latitude *ET*, the moon is seen at the point *G* of the ecliptic. Again, let *EZ* be the latitude in the south <direction>, and *ZM* the parallax in the altitude circle. We join *ML* parallel to *AH* and *MN* parallel to *EL*. The two line <segments> *MN* and *NE* are desired. The angle *SDH* is approximately equal to the angle *SEH*. The angle *ZML* is equal to the angle *SDH*, because *ML* is parallel to *EH*. So the angle *ZML* is equal to the angle *SEH*. *L* is a right angle, and *ZM* is known. Then the angle *MZL* is known; so the sides of the triangle *MLZ* are known. *EZ* is known; so *EL* is known, and it is equal to *MN*. So *MN* is known, and it is the apparent latitude. *ML* is known, so *EN* is known, and it is the difference in longitude. So because of its latitude *EZ*, the moon is seen at the point *N* of the ecliptic. This is what we wanted to demonstrate.

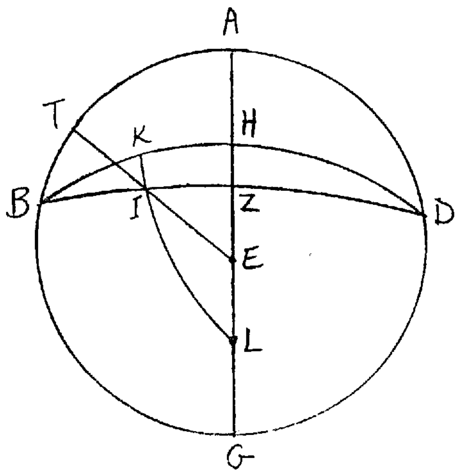
Chapter 11: On drawing the figure of a solar eclipse.

<Let> *ABGD* <be> a circle <with its radius equal to> the sum of the two radii, centered at *E*, *EB* equal to the radius of the moon plus the radius of the sun, *EL* the radius of the sun and *ZLHI* the circle of its surface. *ET* is the <apparent> latitude of the moon, *TK* its radius, and *ZKHM* the circle of its surface. Then <the segment> *ML* of the diameter of the sun is <the magnitude of> the solar eclipse in digits. The line <segment> *AG* is the east-west line, and the line <segment> *BD* is the north-south line. This is what we wanted.



Chapter 12: On <finding> the altitude of the moon, taking account of its latitude.

<Let> *ABGD* <be> the horizon circle, *BHD* the ecliptic with its pole at *L*, *AEG* passing through its poles <and the zenith>, and *I* the body of the moon. We draw through it (i.e., the point *I*) <the arcs> *LIK*, *BID* <,and> *EIT*. The arc *IT* is desired. <The arc> *IK* is the latitude of the moon. In the two triangles *LIZ* and *LKH*, the angle *L* is common, and *Z* and *H* are right angles.



So the ratio of the Sine of *LI* to the Sine of *IZ* is equal to that of the Sine of *LK* to the Sine of *KH*. But *LI* is the complement of the latitude of the moon, *LK* is a quadrant, and *HK* is the complement of the distance of the <ecliptical> degree of the moon from the ascendant. So *IZ* is known, therefore, its complement *IB* is known. Again, in the two triangles *BIK* and *BZH* the angle *B* is common, and *K* and *H* are right angles. So the ratio of the Sine of *BI* to that of *IK* is equal to the ratio of the Sine of *BZ* to the Sine of *ZH*. But *BI* is known, *IK* is the latitude of the moon, and *BZ* is a quadrant. So *ZH* is known. But *AH* is the complement of the altitude of the pole of the ecliptic <, so it is> known. Thus the whole *AZ* is known. Also, in the two triangles *BIT* and *BZA*, the angle *IBT* is common, and *T* and *A* are right angles. So the ratio of the Sine of *BI* to that of *IT* is equal to the ratio of the Sine of *BZ* to the Sine of *ZA*. But *BI* is known, *BZ* is a quadrant, and *ZA* is known. So *IT* is known. This is what we wanted to demonstrate.

Chapter 13: On <finding> the longitudinal and latitudinal parallax of the moon by a method <the validity of> which can be proved.

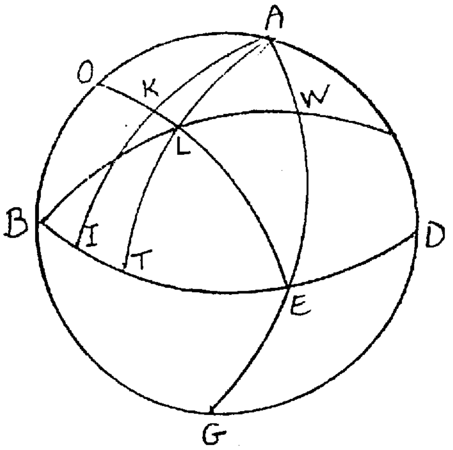
It was said in Chapter 11 of Section 6 in the First Book that the altitude obtained from the calculation is true altitude. <If> the parallax is subtracted from it, <the remainder> is the apparent altitude. Having said that, <I add that the subject of> this chapter occurs in five cases.

First <case>, when the altitude of tenth house at the time is 90 degrees and the moon has no <non-zero> latitude: The parallax in the altitude circle is longitudinal parallax alone.

Second <case>, when the distance of the <ecliptical> degree of the moon from the ascendant of the time is 90 degrees, whether the moon has or does not have a <non-zero> latitude: The parallax in the altitude circle is the apparent latitude alone (there is no longitudinal parallax).

Third <case>, when the altitude of the tenth <house> at the time is 90 degrees and the moon has a <non-zero> latitude. Let *ABGD* in the first figure be the horizon circle, *BED* the ecliptic and the two points *A* and *G* its poles (*E* is the zenith). *AEG* passes through these two poles. *EO* is <a part> of the altitude circle, *L* the body of the moon, and *LK* the parallax in the altitude circle.

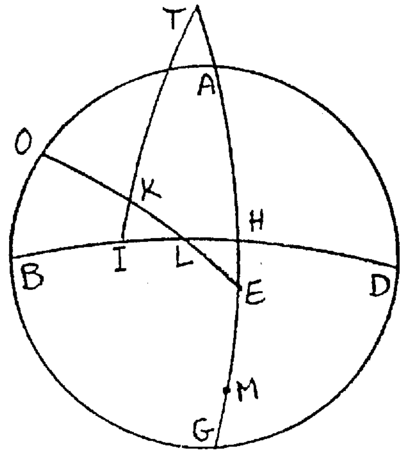
The first <figure>



We draw the arcs *ALT*, *AKI* and *BLW* through the points *L* and *K*. Then *LT* is the southern latitude of the moon (the method is also valid for northern latitudes), *KI* the apparent latitude, and *TI* the difference in longitude. In the two triangles *ELT* and *EKI* the angle *E* is common, and *T* and *I* are right angles. So the ratio of the Sine of *EL* to the Sine of *LT* is equal to the ratio of the Sine of *EK* to the Sine of *KI*. *EL* is the complement of the true altitude (computed in Chapter 12), *LT* the latitude of the moon, and *EK* the complement of the apparent altitude. So *KI* is known, and it is the apparent latitude. Also the angle *A* is common to the two triangles *AKO* and *AIB*, and *O* and *B* are right angles. So the ratio of the Sine of *AK*, the complement of the apparent latitude, to the Sine of *KO*, the apparent altitude, is equal to the ratio of the Sine of *AI*, the greatest Sine, to the Sine of *IB*. So *IB* is known, and *TB* is the distance of the <ecliptical> degree of the moon from the ascendant. So *TI* is known, being the difference in longitude.

Fourth <case>, when the altitude of the tenth <house> of the time is less than 90 degrees and the moon has no <i.e., zero> latitude. Let *ABGD* in the second figure be the horizon circle, *BHD* the ecliptic and the two points *T* and *M* its poles through which passes the circle *AEG* (*E* is the zenith). *EO* is <a part> of the altitude circle, *L* the body of the moon, and *LK* the parallax in the altitude

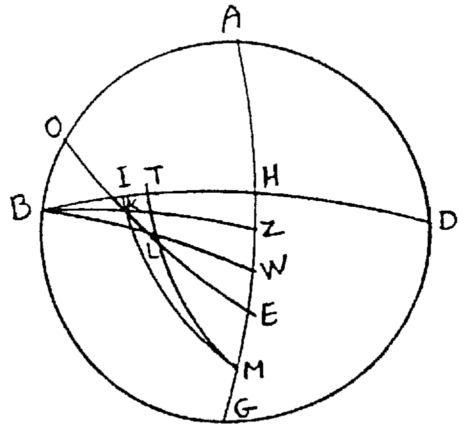
The second <figure>



circle. We draw the arc *TKI* and through the two points *L* and *K*. So *KI* is the apparent latitude, and *LI* is the longitudinal parallax. In the two triangles *KLI* and *HLE* their two angles *L* are equal, and *I* and *H* are right angles. So the ratio of the Sine of *LE* to the Sine of *EH* is equal to the ratio of the Sine of *LK* to the Sine of *KI*. *LK* is the parallax in the altitude circle, *LE* is the complement of the true altitude, and *EH* is the altitude of the pole of the ecliptic. So *KI* is known, and it is the apparent latitude.
Difference in longitude: In the triangle *LKI*, *I* is a right angle. So the ratio of the Cosine of *IK*, the apparent latitude, to the Cosine of *KL*, the parallax in the altitude circle, is equal to the ratio of the greatest Sine to the Cosine of *LI*, the longitudinal parallax. So *LI* is known, <and it is the difference in longitude>.

Fifth <case>, when the altitude of the tenth <house> of the time is less than 90 degrees and the moon has a <non-zero> latitude. Let *ABGD* in the third figure be the horizon circle, *BHD* the ecliptic and *M* its pole. *AEG* is the circle passing through it (i.e., *M*) <and the zenith *E*>. *EO* is <a part> of the altitude circle, *L* the body of the moon, and *LK* the parallax in the altitude circle. We draw the arcs *BKZ*, *BLW*, *MLT*, and *MKI* through the points *L* and *K*. Then *TL* is the northern latitude of the moon, *KI* the apparent latitude, and *TI* the longitudinal parallax.

The third <figure>

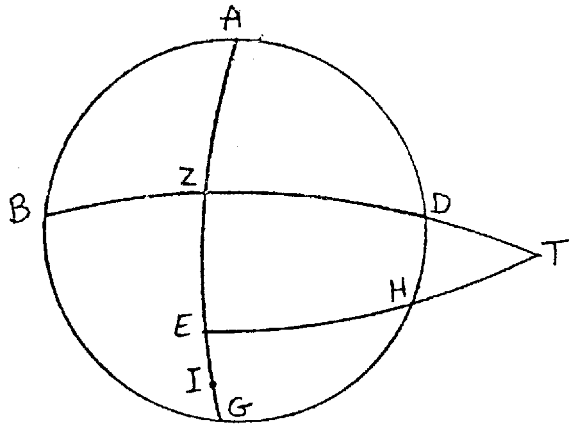


In the two triangles *MLW* and *MTH* the angle *M* is common, and *W* and *H* are right angles. So the ratio of the Sine of *ML* to the Sine of *LW* is equal to the ratio of the Sine of *MT* to the Sine of *TH*. *ML* is the complement of the latitude of the moon, *MT* is a quadrant, and *TH* is the complement of the distance of the <ecliptical> degree of the moon from the ascendant. So *LW* is known. Also, in the triangles *ELW* and *EKZ* the angle *E* is common, and *W* and *Z* are right angles. So the ratio of the Sine of *EL* to the Sine of *LW* is equal to the ratio of the Sine of *EK* to the Sine of *KZ*. *EL* is the complement of the true altitude, *LW* is known, and *EK* is the complement of the apparent altitude. So *KZ* is known, and its complement *KB* is known. Also, in the triangles *BKO* and *BZA* the angle *B* is common, and *O* and *A* are right angles. So the ratio of the Sine of *BK* to the Sine of *KO* is equal to the ratio of the Sine of *BZ* to the Sine of *ZA*. *BK* is known, *KO* is the apparent altitude, and *BZ* is a quadrant. So *ZA* is known. *AH* is the complement of the altitude of the pole of the ecliptic. So *HZ* is known. Also, in the triangles *BKI* and *BZH* the angle *B* is common, and *H* and *I* are right angles. So the ratio of the Sine of *BK* to the Sine of *KI* is equal to the ratio of the Sine of *BZ* to the Sine of *ZH*. *BK* is known, *BZ* is a quadrant, and *ZH* is known. So *KI* is known, being the apparent latitude.

Difference in longitude: In the triangles *MKZ* and *MIH* the angle *M* is common, and *Z* and *H* are right angles. So the ratio of the Sine of *MK* to the Sine of *KZ* is equal to the ratio of the Sine of *MI* to the Sine of *IH*. *MK* is the complement of the apparent latitude, *KZ* is known, and *MI* is a quadrant. So *IH* is known. *TH* is known, and so is *TI*, which is the longitudinal parallax. So the apparent latitude and the longitudinal parallax in <all> different situations are known. That is what we wanted to demonstrate.

Chapter 14: On the visibility arc<s>.

<Let> *ABGD* <be> the horizon circle, *BZD* the ecliptic and *I* its pole. *AEG* passes through the two poles of the ecliptic <and the zenith *E*>. *EHT* is the altitude circle <through the sun *T*>, *D* the <ecliptical> degree with which the moon sets <simultaneously>, *ZA* is the complement of the altitude of the pole of the ecliptic, and it is <equal to> the magnitude of the angle *ADZ* which is equal to the angle *TDH*. The arc *TH* is desired. In the triangles *DHT* and *DZA* the two angles *D* are equal, and *H* and *A* are right angles. So the ratio of the Sine of *DT* to the Sine of *TH* is equal to the ratio of the Sine of *DZ* to the Sine of *ZA*. *DT* is the distance between the sun, being the point *T*, and the <ecliptical> degree with which the moon sets <simultaneously>, being the point *D*. *DZ* is a quadrant and *ZA* is the complement of the altitude of the pole of the ecliptic. So *TH* is known, being the desired <arc>. As found up to now, its minimum value <for the visibility of the lunar crescent> is 6½ degrees to 7 degrees. That is what we wanted to demonstrate.



Commentary

IV.6.1 This is a proof of the validity of the method for converting 'absolute digits' (=linear digits) into 'adjusted digits' (=area digits) in the eclipses. Here, and in the subsequent "proofs", Kūshyār assumes that all arcs of great circles on the sphere can be represented as straight lines in a plane. See also I.6.4 and its commentary.

IV.6.2 In the figure, the angle *Z* is usually very close to 90 degrees. So to simplify the calculations, Kūshyār assumes it to be a right angle. This implies that *AB* and *AG* are approximately parallel, although they actually intersect in the figure. See also I.6.5 and its commentary.

IV.6.3 Here, *AB* is no longer assumed parallel to *AG*, and the small change of the moon's latitude during the eclipse has been taken into account. Since point *L* is the midpoint of the chord through *S* and *O*, which is not parallel to *AG*, the angle *LEA* is not exactly equal to 90 degrees. According to my computation, the maximum difference between the results from the methods of Chapter 2 and Chapter 3 is less than 0.08%.

IV.6.4 This is merely an illustration of what Kūshyār describes in I.6.6. The figure illustrates the situation in the middle of the eclipse. The references to the four cardinal directions are merely symbolic, since the moon can be on every side of the shadow cone of the earth (depending on the lunar latitude).

IV.6.5 The proof is correct. The determination of the distance*EB* of the epicycle center corresponds to the procedure which Kūshyār describes in I.6.7. Here Kūshyār deviates from the Ptolemaic lunar model by measuring the adjusted mean anomaly from the epicyclic (or true) apogee, while Ptolemy takes it from another point on the epicycle (mean apogee) which lies on the extension of the line connecting *B* and a new point *E'*, so that *E* is the midpoint of the line segment *DE'* [Ptolemy 1984, 249-251; Kashino 1998, 9].
Kūshyār then determines an "adjusted radius" of the epicycle which does not seem to play a role in the proof of IV.6.5. In the light of the proof of IV.6.5, the "adjusted radius" in I.6.7 seems superfluous, and can be replaced by the normal value of the epicycle radius (which Kūshyār also used in the computation of the solar distance).

IV.6.6 In the figure, *E* represents the zenith and *EH* is drawn perpendicular to the ecliptic. Since *E* is the pole of the horizon, the pole *A* of circle *ET* is on the horizon. Since circle *EHT* is perpendicular to the ecliptic, the pole of ecliptic is on circle *EHT*, 90 degrees away from point *H*. Thus the altitude of the pole of the ecliptic is equal to the complement of *EH* of the zenith distance.

IV.6.7 This is a proof of the validity of the method provided in I.6.9

IV.6.8 This is a proof of the validity of the method provided in I.6.11. In a personal communication, Prof. E. S. Kennedy remarks: "'The Sphere of the Whole' (in Arabic *kurrat al-kull*) means 'the celestial sphere together with its contents': the immobile terrestrial sphere at the center (containing us), next the moon, next the five planets, comets etc. in between, and finally the fixed stars on the inside of the celestial sphere."
The figure is very similar to that used by Ptolemy [1984, 248] for finding the distance of the moon from a known lunar parallax. Since the computation of parallax is mostly used for the prediction of lunar eclipses, which happen when the center of the lunar epicycle is at maximal distance from the earth, it is reasonable to take the lunar distance equal to 60 earth radii. Kūshyār's assumption that the magnitude of the angle *L* is equal to arc *GD* is approximately true, since the radius *EA* of the universe is very large compared to the radius of the lunar sphere *ET*.

IV.6.9 The proofs of the validity of the methods provided in I. 6.12 for the six cases of the angle between the ecliptic and the altitude circle passing through the ecliptical degree of the moon are presented in this chapter. These angles are used in Chapter IV.6.10 for finding the longitudinal and the latitudinal parallax of the moon.
In the first case, the required angle is equal to $90^{\circ} - \varphi \pm \varepsilon$, where φ is the geographical latitude of the locality and ε is the obliquity of the ecliptic. In the second case, this angle is equal to $90^{\circ} - \varepsilon$ (note that we always take the smaller angle). The tenth house is the point of intersection of the ecliptic with the meridian above the horizon.
In the fifth case, the ms. A only contains an alternative proof which is found as marginal note in F, V and L. The points *I* and *S* in the figure are related to this alternative proof. The ms. Y only provides a proof almost similar to that of A, but using the triangle *KHM*. The ms. M only contains the proof of the main method in the text of F, V and L. As mentioned in the footnote of the Arabic text, the mss. F, V , L and M use the first figure for the cases one to four. But in A and Y one figure is drawn for each case, and we have followed them in this regard.

IV.6.10 This chapter presents the proofs of the validity of the methods for the two cases provided in I.6.13 in which the latitude of the moon is towards north or south, respectively. In the figure, *O* is the apparent lunar position, as seen by the observer, and *T* is the "real" lunar position, as seen from the center of the earth. See IV.6.12 for a more refined treatment (in which *SHE* and *SAH* are no longer considered as equal).

IV.6.11 This is an example of the drawing described in I.6.16. Also see the commentary to IV.6.4.

IV.6.12 This is a proof of the method provided in I.6.17. In that method, 'the first arc' is *IB*, 'the second arc' is *HZ*, and 'the result from the complement of the altitude of the pole' is *AZ*. Here we have provided the version found in A and Y. Other mss. contain proofs of the case in which *BHD* is the celestial equator. They correspond to 'another method' in I.6.17 which I have deleted, because that method and its proofs do not seem authentic. See I.6.17 and the footnote to the Arabic text of IV.6.12.

IV.6.13 In this chapter more accurate methods of computation of the longitudinal and latitudinal parallax are provided, in contrast to the approximate methods of Chapter 10. We have not seen this refined determination of the longitudinal and latitudinal parallax in other Arabic sources. Of course, since the arcs involved in this subject are very small, the results of the approximate methods are sufficient for practical purposes and the accurate methods merely have theoretical significance.
The first and the second cases are trivial. The proof of the third and the fourth cases are provided only for the first method of these cases in I.6.18. No proof is given for the alternative methods relating to the third and the fourth cases in I.6.18. In the proof of the validity of the method for finding the apparent latitude in the fifth case, the arcs *LK*, *KZ*, *ZA* and *ZH* are corresponding to the 'first', 'second', 'third' and 'fourth' arcs mentioned in the description of the method in I.6.18. For finding the difference in longitude in this case, *IH* represents 'the first arc'.

IV.6.14 This is a proof of the procedure of I.6.20 for finding the visibility arc. The minimum value of the visibility arc for the moon is mentioned to be 6 degrees in I.6.20 (less than 6½ to 7 degrees in this chapter). The limit values of the visibility arc for the planets are mentioned in I.6.20.
In the single chapter of Book III on the definition of the astronomical terms, Kūshyār defines the visibility arc (*qaus al-ru'ya*) as follows: "The arc <which is part> of the altitude circle, between the horizon on which lies the planet, and the sun <situated> under the earth (i.e., the horizon); it may also be regarded as the arc of the altitude circle between the planet

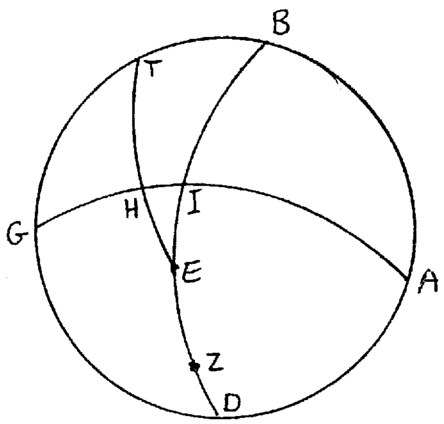
<situated> above the earth (i.e., the horizon) and the horizon on which lies the sun." Ptolemy gives the limit magnitudes of *arcus visionis* (visibility arc) for the stars and the planets as the sun's depression at the moment the celestial body is on the horizon [Ptolemy 1984, 413-415, 639-640]. In I.6.20, Kūshyār uses both cases of the visibility arc, below and above the horizon. Bīrjandī states that the arc of visibility was considered above the horizon because the depression is a calculated value, whereas an altitude could be observed [al-Ṭūsī 1993, II, 464]. For a description of the visibility arc (arc of vision, *arcus visionis*) see [Pedersen 1974, 388; Neugebauer 1975, I, 234-236].

Section 7: On what pertains to astrology, in one chapter

On the projection of the ray taking the latitude of the planet into account.

Those earlier astrologers who had some knowledge of astronomy said that when the planet has a <non-zero> latitude, its rays are not taken from the ecliptic, but from the <great> circle passing through the planet and cutting the ecliptic at <a distance equal to> the supposed distance. Al-Battānī (in Chapter 54 of his *al-Zīj al-Ṣābī* [1899, 196-197]), wishing to calculate this, has gone to great lengths to calculate and describe this. If this (i.e. the projection of the ray) is influential in astrology, and if it is needed in astrology, the way to <find> it is really short and its calculation is as I have demonstrated in Book I.

Proof: Let *ABGD* be the ecliptic circle, *E* its pole, and the point *H* the body of the planet. *EHT* passes through the two poles of the ecliptic. So *T* is the <ecliptical> degree of the planet, *TH* is its latitude, and *EH* is the complement of the latitude. Let *AIG* pass through the planet, and *Z* <be> its pole. *BED* passes through the two poles (i.e., *E* and *Z*). *GH* is assumed to be 60 degrees, being the sextile arc on this circle, and its complement *HI* is 30 degrees. In the two triangles *EHI* and *ETB*, the angle *E* is common, and *I* and *B* are right angles. So the ratio of the Sine of *EH* to that of *HI* is equal to ratio of the Sine of *ET*, being the greatest Sine, to the Sine of *BT*. Then *BT* is known, being the desired magnitude on the ecliptic. If it is subtracted from *BG* (i.e., 90°), the remainder *TG* is the sextile arc. If it (i.e., *BT*) is added to *BG*, the result is the trine arc. If the arc *GT* is assumed <to be> 60 degrees, and the arc *GH* is desired, the ratio is the same as before. The arc *HI* will be known, its complement *HG* will be the sextile arc, and its addition to *GI* <will result> in the trine arc. *GI* and *GB* are the quartile arcs, any of which may be taken. That is what we wanted to demonstrate.



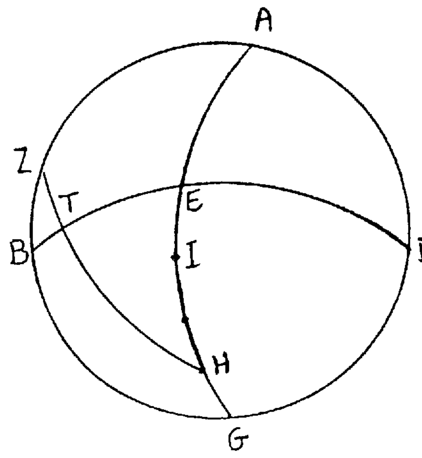
Commentary

IV. 7 This is a proof of the method presented in I.7.2 for the calculation of the projection of the rays when the planet has non-zero latitude. See I.7.2 and its commentary.

Section 8: On the operations which are less needed, <in> 8 chapters

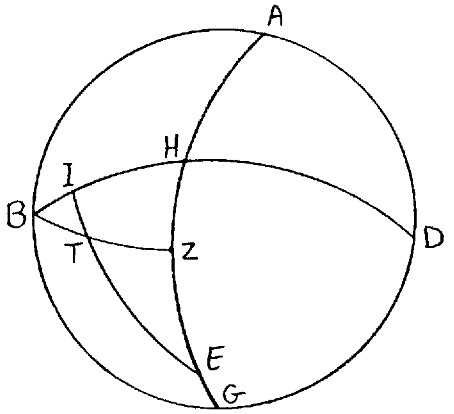
Chapter 1: On <finding> the latitude of a locality from the hours (i.e., the duration) of <its> longest and shortest days.

<Let> *ABG* <be> the horizon circle, *AEG* the meridian, *BED* the celestial equator, *H* its pole, and *Z* the rising position of <the beginning of> Capricorn or Cancer. We draw *HTZ*. *BZ* is the ortive amplitude and *ZT* is the total declination (i.e., the obliquity of the ecliptic). *TE* is half the day arc. It is known from the multiplication of half the <number of equinoctial> hours of the day by 15. *EA* is the complement of the latitude of the locality. *EI* is the desired latitude of the locality (*I* is the zenith). In the two triangles *THE* and *HZA* the angle *H* is common, and *E* and *A* are right angles. So the ratio of the Sine of *HZ* to that of *ZA* is equal to the ratio of the Sine of *TH* to that of *TE*. The Sine of *HZ* is equal to the Sine of the complement of the <total> declination. *HT* is a quadrant and *TE* is known. So *ZA* is known, and it is the complement of the ortive amplitude. Then *ZB*, the ortive amplitude, is known. In the two triangles *BTZ* and *BEA* the angle *B* is common, and *T* and *E* are right angles. So the ratio of the Sine of *BT*, the equation of the daylight, to the Tangent of *TZ*, the total declination, is equal to the ratio of the Sine of *BE*, the greatest Sine, to the Tangent of *EA*, the complement of the latitude of the locality (see the premise proved in IV.3.5).
<Proof of another method:> In the two triangles *BZT* and *BAE* the angle *B* is common, and *T* and *E* are right angles. So the ratio of the Sine of *BZ* to that of *ZT* is equal to the ratio of the Sine of *BA* to that of *AE*. *BZ* is the ortive amplitude, *ZT* is the <total> declination, and *BA* is a quadrant. Then *AE* is known, and it is the complement of the latitude of the locality. Then *IE* is known, and it is the latitude of the locality. This is what we wanted to demonstrate. Now it has become clear that this proof is generally valid for the hours of every day of the year, if the declination of the sun corresponding to its ecliptical degree is taken (instead of the total declination).



Chapter 2: On <finding> the altitude without (i.e., with zero) azimuth.

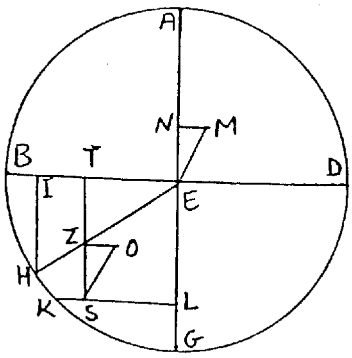
<Let> *ABGD* <be> the horizon circle, *Z* the zenith, *AZG* the meridian, and *BHD* the celestial equator. *ZTB* is <a part> of the altitude circle passing through the rising position of the equinox (i.e., the East point with zero azimuth), and *T* is the body of the sun or the planet. So *IT* is the declination of the sun or the distance of the planet from the celestial equator. In the two triangles *BTI* and *BZH*, the angle *B* is common, and *I* and *H* are right angles. So the ratio of the Sine of *BT* to that of *TI* is equal to the ratio of the Sine of *BZ* to that of *ZH*. *TI* is the declination or the distance, *BZ* is a quadrant, and *ZH* is the latitude of the locality. So *BT* is known, and it is the altitude without (i.e., with zero) azimuth. This is what we wanted to demonstrate.



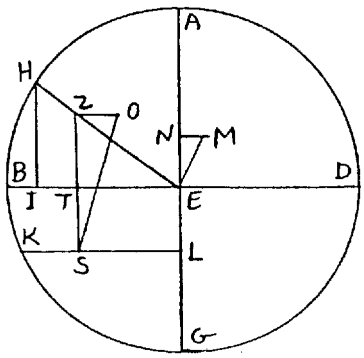
Chapter 3: On <finding> the azimuth of <a point of given declination and> any assumed altitude.

<Let> *ABGD* <be> the horizon circle, *AEG* the intersection of the meridian <plane> and the horizon, *BED* the intersection of the celestial equator <plane> and the horizon, *LK* the intersection of the parallel circle <plane> and the horizon, and *EH* the radius of the altitude circle. *OZ* is the perpendicular drawn from the intersection <point> of the altitude circle and the parallel circle to the horizon plane. So it is the Sine of the altitude. We draw *ZT* perpendicular to *EB*. It is also perpendicular to *LK*, because *BE* and *KL* are parallel. We draw *HI* perpendicular to *BE*. It is the Sine of the azimuth. *MN* is the perpendicular drawn from the intersection <point> of the meridian and the celestial equator to the horizon plane. It is the Cosine of the latitude of the locality. We join *ME* and *OS*. The <corresponding> sides of the two triangles *MNE* and *OZS* are parallel. So the ratio of *MN*, the Cosine of the latitude of the locality,

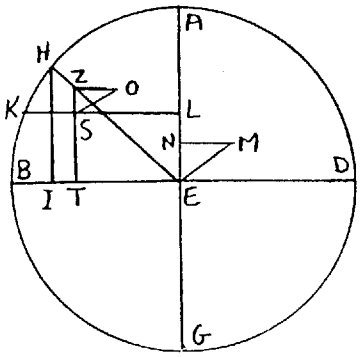
to *NE*, the Sine of the latitude of the locality, is equal to the ratio of *OZ*,



southern declination
and azimuth



northern declination
and southern azimuth

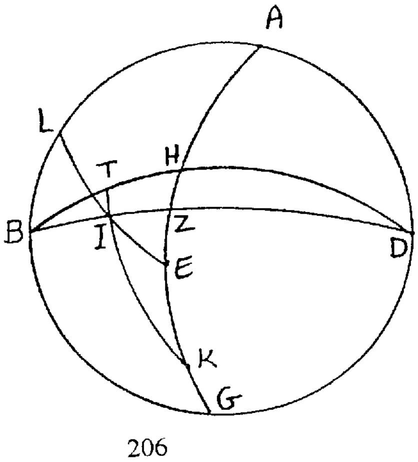


northern declination
and azimuth

the Sine of the altitude, to *ZS*, <which is called> 'the argument of the azimuth'. So *ZS* is known, and *ST* is the Sine of an arc equal to *BK*, the ortive amplitude. Then *ZT* is known, and it is <called> 'the equation of the azimuth'. Again, in the two triangles *EZT* and *EHI*, the two bases *ZT* and *HI* are parallel. So the ratio of *EZ*, the Cosine of the altitude, to *ZT*, the equation of the azimuth, is equal to the ratio of *EH*, the greatest Sine, to *HI*, the Sine of the azimuth. So *HI* is known, and it is the required azimuth. This is what we wanted to demonstrate.

In the second figure, *ZS*, the argument of the azimuth, is greater than *TS*, which is equal to the Sine of the ortive amplitude. If *ST* is subtracted from *SZ*, the remainder is the equation of the azimuth, and the azimuth *BH* is southern. In the third figure, *ZS*, the argument of the azimuth, is less than *TS*, which is equal to the Sine of the ortive amplitude. If *ZS* is subtracted from *ST*, the remainder is *ZT*, which is the equation of the azimuth, and the azimuth is northern. This is what we wanted to demonstrate.

Another method <for the case> when the ascendant and the tenth <house> are known: <Let> *ABGD* <be> the horizon circle, *AEG* the meridian, *K* the pole of the celestial equator, and *BHD* the celestial equator.

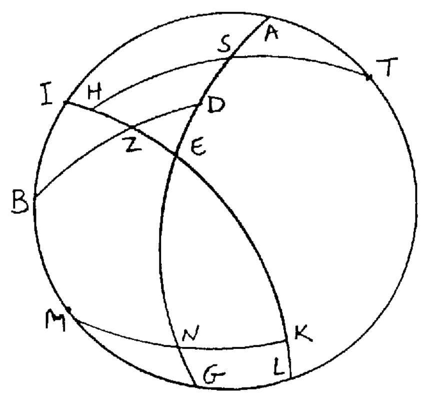


I is the <ecliptical> degree of the sun, and *BID* passes through it. *EIL* is <a part> of the altitude circle. *KIT* passes through the pole of the celestial equator and the <ecliptical> degree of the sun. The arc *BL* is required, and I say that it is known.

Proof: In the two triangles *KIZ* and *KTH*, the angle *HKT* is common, and *Z* and *H* are right angles. So the ratio of the Sine of *KI* to that of *IZ* is equal to the ratio of the Sine of *KT* to that of *TH*. So *IZ* is known, because *KI* is the complement of the declination and *TH* is the right ascension (in the ancient sense) of the distance of *I* from the meridian (*TH* is known because the ascendant and the tenth house are known). Again, in the two triangles *EIZ* and *ELA*, the angle *ZEI* is common, and *Z* and *A* are right angles. So the ratio of the Sine of *EI* to that of *IZ* is equal to the ratio of the Sine of *EL* to that of *LA*. Then *LA*, the complement of *BL*, is known. So *BL* is known. This is the case, because *EI* is the complement of the altitude, and *IZ* is known. Since the two middle terms in the first proportion are equal to the two middle terms in the other proportion, the ratio of the Sine of *KI* to that of *EI* is equal to the ratio of the Sine of *AL* to that of *HT*. So *AL*, the complement of the azimuth, is known. Therefore, *BL* is known. This is what we wanted.

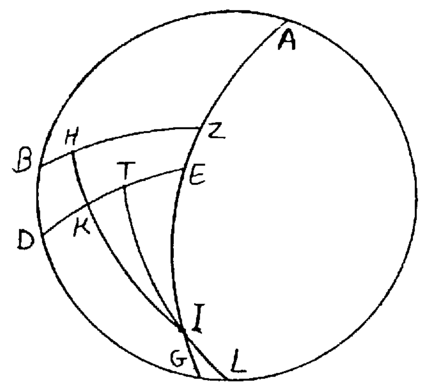
Chapter 4: On <finding> the altitude from the azimuth <and the declination>.

Premise: If the circle of the celestial equator and an altitude circle intersect, and we take an arc of the meridian <starting> from the horizon and equal to the latitude of the locality, then <the distance > between the zenith and the celestial equator on the altitude circle is equal to <the distance> between the horizon and the circle passing through the pole of the altitude circle and <the endpoint of an arc> equal to the latitude of the locality.

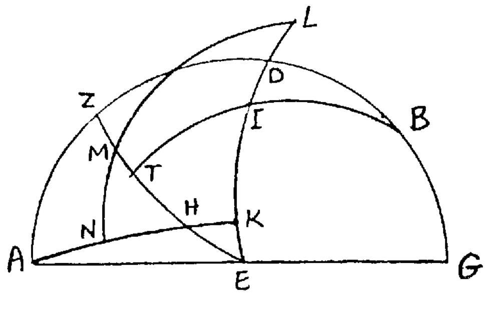


<Proof:> Let *ABG* be the horizon circle, *AEG* the meridian, *BZD* the celestial equator and *N* its pole. *IEL* is the altitude circle, and each one <of the arcs> *GN*, *ED* and *SA* is <equal to> the latitude of the locality. *T* and *M* are the two poles of the altitude circle. We draw <the arcs> *MNK* and *TSH*. I say that *EZ* is equal to *HI*. <Proof:> The point *Z* is the pole of the circle *MNK*; <so> *KZ* is a quadrant. *LE* is <also> a quadrant. We subtract the common <arc> *KE*; the remaining <arcs> *LK* and *EZ* are equal. The ratio of the Sine of *MN* to that of *NG* is equal to the ratio of <the Sine of> *MK* to that of *KL*. *MN* is equal to *TS*, *NG* is equal to *SA*, and *MK* is equal to *TH*. So the remaining arcs *HI* and *KL* are equal. Hence it is demonstrated that *KL* is equal to *EZ*. Then *HI* is equal to *EZ*. This is what we wanted to demonstrate.

When the azimuth is northern: <Let> *ABGD* <be> the horizon circle, *AEG* the meridian, *I* the pole of the celestial equator, *BHZ* <a part> of the celestial equator, *ED* <a part> of the altitude circle, <*E* is the zenith,> and *K* the <ecliptical> degree of the sun. Then *BD* is the given azimuth, and *DG* is its complement. *IG* is the latitude of the locality. We take *GL* equal to *BD*, and we draw *LIT* and *HKI*. The arc *KD*, the altitude <of the sun> at the <given> time, is desired. *KH*, the declination of the sun, is always northern, and *KI* is its complement. So the ratio of the Sine of *EI* to that of *IT* is equal to the Sine of *EG* to that of *GD*. *EI* is the complement of the latitude of the locality, and *DG* is the complement of the azimuth. So *IT* is known, and its complement, *LI*, is known. This is the case (i.e., the above proportion holds), because in the two triangles *EIT* and *EGD* the angle *E* is common, and *T* and *D* are right angles. Also, in the two triangles *LIG* and *LTD* the angle *L* is common, and *G* and *D* are right angles. So the ratio of the Sine of *LI* to that of *IG* is equal to the ratio of the Sine of *LT* to that of *TD*. *LI* is known, and *IG* is the latitude of the locality. So *TD* is known. Also, in the triangle *IKT*, *T* is a right angle. So the ratio of the Cosine of *TI* to the Cosine of *IK* is equal to the ratio of the greatest Sine to the Cosine of *KT*. *TI* is known, and *IK* is the complement of the declination of the sun. So the complement of *TK* is known. Then, *TK* is known. *TD* was already known, so the remainder *KD*, which is desired, is known.

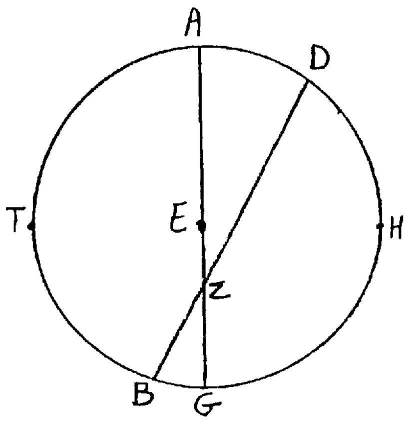


When the azimuth is southern: <Let> *ADG* <be> the southern horizon semicircle, *ED* the meridian, *AK* <a part> of the celestial equator and *L* its pole, and *EZ* <a part> of the altitude circle. Then *AZ* is the given azimuth. We take *DB* equal to *AZ*, and *DI* equal to *EK*, which is the latitude of the locality. We draw *BIT*. *M* is the body of the sun. We draw *MLN*. *HZ* is the argument of the altitude, *MH* the equation of the altitude, *AH* the argument of the arc of revolution, and *NH* the equation of the arc of revolution. Based on what was demonstrated in the premise, *EH* is equal to *TZ*. If *EH* and *ZD* are known, *HK* is <also> known, and so is its complement *HA*. So the circle of the celestial equator and the altitude circle intersect at a point <that can be> known either from the two triangles *EHK* and *EZD*, or from the two triangles *AHZ* and *AKD*. In the triangle *MNH*, *N* is a right angle. So the ratio of the Cosine of *MN* to the Cosine of *MH* is equal to the ratio of the greatest Sine to the Sine of *HN*. So *HN* is known. In this figure, the arc *MZ* is desired. In the two triangles *ETI* and *EZD*, the angle *E* is common, and *T* and *Z* are right angles. So the ratio of the Sine of *EI*, the complement of the latitude of the locality, to the Sine of *IT* is equal to the ratio of *ED*, the greatest Sine, to the Sine of *DZ*, the complement of the azimuth. Then *TI* is known, and its complement *BI* is known. Also, in the two triangles *BID* and *BTZ* the angle *B* is common, and *D* and *Z* are right angles. So the ratio of the Sine of *BI*, which is known, to that of *ID*, the latitude of the locality, is equal to the ratio of the Sine of *BT*, which is the greatest Sine, to the Sine of *TZ*, which is unknown. So *ZT* is known. It was already demonstrated to be equal to *EH*. So *EH* is known. Then *ZH*, the argument of the altitude, is known. Also, *L* is the pole of the celestial equator and *M* is the <ecliptical> degree of the sun. Then, *LMN* is the declination circle, and *MN*, the declination, is southern. In the two triangles *HMN* and *HEK*, their angles *H* are equal, and *K* and *N* are right angles. The ratio of the Sine of *HM*, which is the equation of the altitude, to the Sine of *MN*, the declination of the sun, is equal to the ratio of the Sine of *HE*, the complement of the argument of the altitude, to the Sine of *EK*, the latitude of the locality. So *HM* and *ZH* are known. Then *MZ*, which is the desired altitude, is known. This is what we wanted to demonstrate.



Chapter 5: On the distance between two stars, one of which has a <non-zero> latitude.

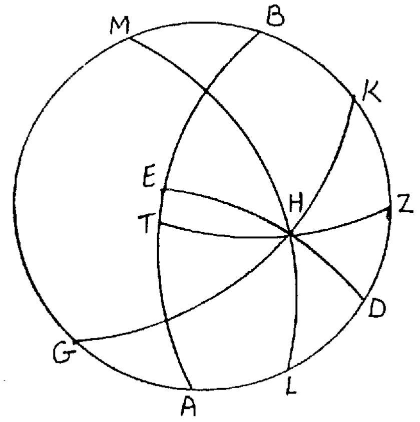
<Let> *ABGD* <be> the latitude circle centered at *E*, *AEG* the ecliptic with *H* and *T* as its poles and centered at *E*. We assume the point *B* as the star having a <non-zero> latitude. *DZB* is the circle passing the star and cutting the ecliptic at *Z*, where *Z* is a degree on the ecliptic, or the position of the star without a (i.e., with zero) latitude. The arc *BZ* is desired. *GB* is the latitude of the star, and *GZ* is <the distance> between the <ecliptical> degree of the star and the <ecliptical> degree from which the distance of the star is desired. In the triangle *ZGB*, *G* is a right angle. So the ratio of the Cosine of *GZ* to that of *ZB* is equal to the ratio of the greatest Sine to the Cosine of *GB*. So *BZ* is known. This is what we wanted to demonstrate.



Chapter 6: On the distance between two stars <both> having <non-zero> latitudes.

<Let> *ABG* <be> the latitude circle, *AEB* the ecliptic and *Z* its pole. First, we assume the two stars <to be> the points *G* and *H* <with latitudes> in opposite directions. We draw <the arcs> *GHK*, *ZHT*, and *EHD* (*E* is the pole of the circle *ABG*). The required arc is <the arc> *GH* which passes through the two stars. In the triangle *HET*, *T* is a right angle. So the ratio of the Cosine of *TE*, the complement of <the distance> between the two stars <measured> in terms of the ecliptical degrees, to the Cosine of *EH*, is equal to the ratio of the greatest Sine to the Cosine of *HT*, the latitude of one of the two stars. Thus *EH* is known. In the two triangles *EHT* and *EDA*, the angle *E* is common, and *T* and *A* are right angles. So the ratio of the Sine of *EH*, which is known, to that of *HT*, the latitude of the first star, is equal to the ratio of the Sine of *ED*, the greatest Sine, to that of *DA*. So *DA* is known. *AG* is the latitude of the other star. So the sum *GD* is known. In the triangle *GHD*, *D* is a right angle. So the ratio of the

Cosine of *GD*, which is known, to the Cosine of *HG*, which is desired, is equal to the ratio of the greatest Sine to the Cosine of *HD*, which is already known. So *HG* is known.

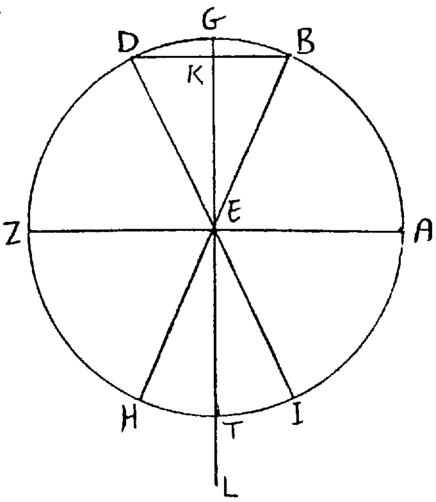


Again, we assume the two stars <to be> the points *H* and *L* <with latitudes> in the same direction. We draw *LHM*. Based on what has already been explained above, *EH*, *HD*, and *DA* are known. *AL* is the latitude of the other star. So *LD* is known. In the triangle *LHD*, *D* is a right angle. So the ratio of the Cosine of *LD*, which is known, to the Cosine of *LH*, which is desired, is equal to the ratio the greatest Sine to the Cosine of *DH*, which is known. So *LH* is known. It is <also> known that if *DA* is less than *AL*, and this is <possible> if the point *D* is situated between the points *A* and *L*, then <*DA*> must be subtracted from *AL*. Then the remainder *DL* is known. In the triangle *DHL*, *D* is a right angle, so *LH* is known. This is what we wanted.

Chapter 7: On the extraction of the meridian line.

Let *AGZ* be the horizon semicircle, and *ATZ* the semicircle of the celestial equator coinciding with the <plane of the> horizon circle or perpendicular to it; and let their <common> center <be> *E*. We suppose the sun rotating on it (i.e., the equator) or parallel to it in its universal rotation. *T* is the midheaven. The arcs *TH* and *TI* are equal. A gnomon should be mounted evenly and vertical at *E*. When the sun is at the point *H*, the shadow of the gnomon is *EB*. When it is at the point *I*, its shadow is *ED*. We join *BD*, and we bisect it at *K*. We draw *KE* and extend it to *L*. Then the line *KL* is the meridian line. This is because the shadow of the gnomon is always opposite to the body of the sun. So *ED* is in the direction of *EI*, and *EB* in the direction of *EH*. So the arc *BD* is equal to the arc *HI*. When we bisect it at *G* and join *GEL*, <it> will be the intersection of the plane of the meridian circle and the plane of the horizon circle. This is what we wanted to demonstrate.

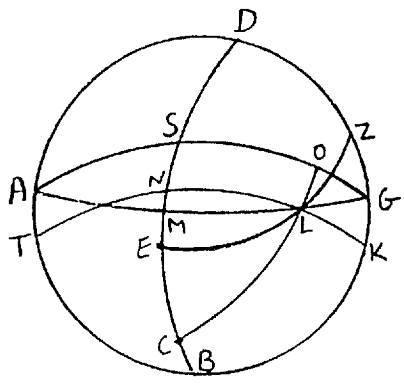
Another method <for the case> when the azimuth of the sun is known: Let *AGZT* be the horizon circle, *A* the equinoctial rising point <of the sun>, and *Z* its <equinoctial> setting point. If the azimuth of the sun is along the point *I*, the shadow of the gnomon will be *ED*. *AI* is the known azimuth, and it is equal to *ZD*. Then *ZD* is known, and so is its complement *DG*. If we draw the line *GET*, <it> will be the intersection of the plane of the meridian circle and the plane of the horizon circle. This is what we wanted to do.



Chapter 8: On the deviation of <the directions of> the <other> localities from the meridian of our locality.

<Let> *ABGD* <be> the horizon circle, *E* the zenith, *A* and *G* the equinoctial rising position and setting position, *BED* the meridian, *C* the pole of the celestial equator, *ASG* the celestial equator, and *ES* the latitude of our locality (then *E* is the zenith). We assume *SO* <to be the difference> between the longitude of our locality and that of Mecca. We draw *CO* as a quadrant of the meridian of Mecca. We take *C* as the pole, and we draw a circle with its distance (i.e., radius) <equal to> the chord of the colatitude of Mecca, and parallel to the celestial equator. It is <marked as> *TNK*. It intersects the arc *CO* at *L*. Then *L* is the zenith of the inhabitants of Mecca. The two arcs *SN* and *OL* are <equal to> the latitude of Mecca. So *EN* is <the difference> between the two latitudes (i.e., that of our locality and of Mecca). We draw *ELZ* <as a part> of the circle passing through <our> zenith. It is the circle of the distance between the two localities, because *EL* is the distance <on the sphere> between our locality and Mecca. Then the arc *ZD* is the deviation of <the direction of> Mecca <from the meridian of our locality>. We draw the semicircle passing through the equinoctial rising position and the point *L*. It is <marked by> *AMLG*. In the two triangles *CLM* and *COS*, the angle *C* is common, and *M* and *S* are right angles. <So> the ratio of the Sine of *CL*, the colatitude of Mecca, to the Sine of *ML*, <which is called> 'the

equation of longitude', is equal to the ratio of the Sine of *CO*, the greatest Sine, to the Sine of *OS*, <the difference> between the two longitudes. So *LM* and its complement *LG* are known. In the two triangles *GLO* and *GMS*, the angle *G* is common, and *O* and *S* are right angles. So the ratio of the Sine of the known *GL*, to that of *LO*, the latitude of Mecca, is equal to the ratio of the Sine of *GM*, the greatest Sine, to the Sine of *MS*, <which is called> 'the equation of the latitude'. Then *MS* is known, and *SE* is the latitude of our locality. Then, *ME* is known, and it is <called> 'the adjusted latitude of the locality'. Then *MD* is known. In the two triangles *GLZ* and *GMD*, the angle *G* is common, and *Z* and *D* are right angles.



So the ratio of the Sine of the known *GL* to that of *LZ*, the complement of the distance between the two localities, is equal to the Sine of *GM*, the greatest Sine, to the Sine of the known *MD*. Then *LZ* is known, and it is the complement of the distance between the two localities. So its complement *LE* is known, and it is the distance <between the two localities>. In the two triangles *ELM* and *EZD*, the angle *E* is common, and *M* and *D* are right angles. So the ratio of the Sine of *EL*, the distance between the two localities, to the Sine of the known *LM*, is equal to the ratio of the Sine of *EZ*, which is a quadrant, to the Sine of *ZD*, the deviation of Mecca. Then, the deviation of <the direction of> Mecca from the meridian of our locality is known. This is what we wanted to demonstrate. Based on this figure, it is possible that while the two latitudes are equal, <the direction of> Mecca be not along the east-west line. It is because <the direction of> the zenith of the inhabitants of Mecca will be situated on the side inclined towards the <north> pole (i.e., *C*) from the circle passing through the equinoctial rising position and our zenith. That will be demonstrated if we draw a parallel circle with the same pole as the celestial equator and with its distance (i.e., radius) equal to the chord of the colatitude of our locality.

Having kept our promise at the beginning of the Book on Chapters and Proofs, we finish the section in this chapter, the book in this section, and the treatise in this book. Praise be to Allāh alone, and that is enough, and His blessing be on the best of His creatures, Muḥammad the Chosen <prophet by Allah>!

[The copying was finished on 18 Ramażān of the year 545 of Hejira, by the hands of Maḥmūd b. Aḥmad b. al-Ḥusayn al-Mu'allimī.]

Commentary

IV.8.1 The first proof corresponds to the second method in I.8.1. Cf. the second proof in IV.5.6. The second proof corresponds to the first method in I.8.1. Cf. the first proof in IV.5.6. Both methods are applicable for the solstices and for any other day for which the length of the day and the declination of the sun is known. See also the commentary of I.8.1.

IV.8.2 This is a proof of the method provided in I.8.2 for the calculation of the altitude for zero azimuth. In the figure, *E* is the pole of the celestial equator.

IV.8.3 The first proof is for the second method provided in I.8.3, and the second proof is for the first method thereof. See I.8.3 and its commentary. In the first proof, Kūshyār applies the archaic "analemma" construction which was known in classical times and the Middle Ages. In this case, the plane of the horizon is the plane of the paper. Points may be projected perpendicularly on the horizon plane. For performing any operation upon an arc or line in a different and nonparallel plane, it is rotated into the plane of projection. Then arc or line will appear in its true shape and size. For more details about the "analemma" construction see [Neugebauer 1969, 214-20; Id 1969, 669-72]. The terms 'argument of azimuth' and 'equation of azimuth' used by Kūshyār are rather strange. Al-Battānī who uses chords instead of sines, applies the term *watar ikhtilāf al-ufuq* (i.e., "chord of the horizon difference") in his description of a similar method [al-Battānī 1989-1907, III, 33-34]. For a discussion of the terms *ḥiṣṣa* ("argument") and *ta'dīl* ("equation") as used by al-Sijzī, Abu'l-Wafā and Ḥabash al-Ḥāsib see [Kennedy-Kunitzsch-Lorch 1999, 101].

IV.8.4 In the premise, point *Z* is the pole of the great circle through *M* and *N* since *M* and *N* are the poles of the circles *TL* and *BD*, which intersect at *Z*. In the proof itself, arc *LT* is 90 degrees since point *L* is the pole of *DE* (because arc *LD* is assumed to be 90 degrees and arc *LE* is 90 degrees since *E* is the zenith) The proof of the validity of the methods provided in I.8.4 for finding the altitude of the sun when its northern or southern azimuth is known are presented here. For a modern formulation of these methods see the commentary of I.8.4. In the figure for the northern azimuth, *IT* is the first arc, *TD* is the second arc or the complement of the argument of the altitude, and $(90^\circ - TK)$ is the third arc or the equation of altitude, mentioned in I.8.4. In the figure for the southern azimuth, *IT* is the first arc, *ZT* is the second arc or the complement of the argument of the altitude, and *HM* is the third arc or the equation of the altitude. In the figure for the northern azimuth, *TD* is greater than *TK*; so, the argument of the altitude is smaller than the equation of the altitude. Therefore, we should subtract the argument of the altitude from the equation of the altitude to obtain a positive value for the altitude. We are unaware of other occurrences of this proof in the medieval Arabic astronomical literature.

In a short fragment following the proof for the northern azimuth, only found in the manuscript A, we read:

و لايظن ان نقطة **ك** يقع بين نقطتى **ه ط** لان الشمس هما (لما؟) زاد ارتفاعها فرعه قل سمت **ب د**
و قرب نقطة **ط** من نقطة **ه** الى ان يجوز الارتفاع بنقطة **ب** فيصير قوس **د ط** تسعين

("It is not be expected that the point *K* lies between the two points *E* and *T*, because when the altitude of the sun increases, it leads to decrease of the azimuth *BD*, and to the approach of the point *T* towards the point *E*, until the altitude <circle> passes through the point *B*, and the arc *DT* becomes <equal to> 90°").

IV.8.5 See I.8.5 and its commentary. In the figure, *E* is the pole of the celestial sphere and the vertical projection of the ecliptic passes through its projection.

IV.8.6 See I.8.6 and its commentary. The arcs *DH* and *DA* are the first arc and second arc mentioned in I.8.6, respectively. The arcs *GD* and *LD* are the "third arc" for the cases in which the latitudes are in different or in the same directions, respectively.

IV.8.7 Here Kūshyār first shows the validity of the method of the Indian circle for finding the meridian line, but not the validity of the shortest shadow method provided in I.8.7. Then he demonstrates the validity of the second method in I.8.7, based on the known azimuth of the sun. This is of course trivial. Here Kūshyār applies the "analemma" method again (see the commentary to IV.8.3). The equinoctial rising/setting points are East/West points.

IV.8.8 A proof for the method of finding the direction of Mecca, essentially al-Bīrūnī's 'method of the *zīj*es', provided in I.8.8, is presented here. The distance between two localities is described as the arc of the terrestrial great circle passing through them. This is equal to the arc joining the zenith points of the two localities on the celestial sphere. It is supposed that the longitude difference and the two latitudes are known. In the figure, *ML*, *MS* and *ME* are the arcs of the "equation of longitude", the "equation of latitude", and the "adjusted latitude of the locality", respectively. The figure is drawn for a locality North-East of Mecca, of course, since Kūshyār lived in Iran. See also I.8.8 and its commentary. It is interesting that Kūshyār shows here that in a locality having the same latitude as Mecca, the direction of Mecca is not necessarily on the east-west line. For an account of different texts that use the terms "equation [or correction] of latitude" (*ta'dīl al-'arż*) and "adjusted [or corrected] latitude" (*al-'arż al-mu'addal*) in the calculation of the direction of Mecca, see [Berggren 1985]. Abū Ja'far al-Khāzin stated in his *Zīj al-Ṣafā'iḥ* that if Mecca is at the same latitude as your locality, the *qibla* is towards the East or the West. This is why Kūshyār may have discussed the error at the end of this chapter [Abū Naṣr ibn 'Irāq 1948, 34].