Algorithms and Obstructions for Linear-Width and Related Search Parameters*

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Abstract

The *linear-width* of a graph G is defined to be the smallest integer k such that the edges of G can be arranged in a linear ordering $(e_1, \ldots, e_r)$ in such a way that for every $i = 1, \ldots, r-1$, there are at most k vertices incident to edges that belong both to $\{e_1, \ldots, e_i\}$ and to $\{e_{i+1}, \ldots, e_r\}$. In this paper, we give a set of 57 graphs and prove that it is the set of the minimal forbidden minors for the class of graphs with linear-width at most two. Our proof also gives a linear time algorithm that either reports that a given graph has linear-width more than two or outputs an edge ordering of minimum linear-width. We further prove a structural connection between linear-width and the mixed search number which enables us to determine, for any $k \geq 1$, the set acyclic forbidden minors for the class of graphs with linear-width $\leq k$. Moreover, due to this connection, our algorithm can be transfered to two linear time algorithms that check whether a graph has mixed search or edge search number at most two and, if so, construct the corresponding sequences of search moves.

1 Introduction

A *graph parameter* is a function which maps each graph to a positive integer. Given a graph parameter f and a positive integer k, we denote as $\mathcal{G}[f, k]$ the class of graphs for which the value of f does not exceed k.

Let $\mathcal{G}$ be a class of graphs. We say that $\mathcal{G}$ is *closed under taking of minors* if all the minors of graphs in $\mathcal{G}$ belong also in $\mathcal{G}$ (we say that a graph H is a minor of a graph G if it can be obtained by G after a number of vertex/edge removal or/and edge contractions – for the formal definitions, see subsection 2.1). We also say that a graph parameter f is *closed under taking of minors* if, for any k, $\mathcal{G}[f, k]$ is closed under taking of minors.

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The *obstruction set* of a graph class $\mathcal{G}$ – namely $\mathrm{ob}(\mathcal{G})$ – is defined to be the set of the minor minimal graphs that do not belong in $\mathcal{G}$. According to the result of Robertson and Seymour in their Graphs Minors series of papers (see [28] for a survey), the minor minimal elements of any graph class are finite. It follows that if a graph class $\mathcal{G}$ is closed under taking of minors then, for any graph G, $G \in \mathcal{G}$ iff none of the graphs in $\mathrm{ob}(\mathcal{G})$ is a minor of G. In the same series of papers, Robertson and Seymour prove that there exist a $O(n^3)$ time algorithm checking if a given n-vertex graph G contains a fixed graph H as a minor [29, 31, 30]. A quite important consequence of that is that for any graph class that is closed under taking of minors there exist an polynomial time membership checking algorithm. Moreover, according to the result of Bodlaender in [4], this membership check can be done in linear time if some of the excluded minors is planar (see also [14, 6]).

Many interesting graph classes/parameters have been proved to be closed under taking of minors. Unfortunately, the membership algorithm we mentioned above preassume the knowledge of the obstruction set. As there exist no general method to find the obstruction set of a graph class (see [16, 17]), the research on this topic has been oriented to the specification of the obstruction set of individual graph classes (see [2, 13, 15, 23, 26]). Clearly, given a graph parameter f that is closed under taking of minors, each value of k corresponds to a different obstruction set, i.e. $\mathrm{ob}(\mathcal{G}[f,k])$. To our knowledge, obstruction sets have been found for the following graph parameters: treewidth, for $k \leq 3$ (see [1, 18, 32]), branchwidth, for $k \leq 3$ (see [8]), node search number, for $k \leq 3$ (see [10, 20]), and mixed search number, for $k \leq 2$ (see [34]).

The *linear-width* of a graph G is defined to be the least integer k such that the edges of G can be arranged in a linear ordering $(e_1, \ldots, e_r)$ in such a way that for every $i = 1, \ldots, r \Leftrightarrow 1$, there are at most k vertices incident to edges that belong both to $\{e_1, \ldots, e_i\}$ and to $\{e_{i+1}, \ldots, e_r\}$. Linear-width was first mentioned by Thomas in [36] and is strongly connected with the notion of *crusades* introduced by Bienstock and Seymour in [3]. In this paper we prove that several variants of problems appearing on graph searching can be reduced to the problem of computing linear-width.

In a graph searching game a graph represents a system of tunnels where an agile, fast, and invisible fugitive is resorting. We desire to capture this fugitive by applying a search strategy while using the fewest possible searchers. In short terms, the search number of a graph is the minimum number of searchers a searching strategy requires in order to capture the fugitive. Several variations on the way the fugitive can be captured during a search, define the the parameters of the *edge, node,* and *mixed search number* of a graph (namely, $\mathrm{es}(G)$, $\mathrm{ns}(G)$, and $\mathrm{ms}(G)$). The first graph searching game was introduced by Breisch [9] and Parsons [27] and is the one of *edge searching. Node searching* appeared as a variant of edge searching and was introduced by Kirousis and Papadimitriou in [22]. Finally, *mixed searching* was introduced in [35] and [3] and is a natural generalisation of the two previous variants (for the formal definitions see Subsection 5.1).

The problem of computing $\mathrm{es}(G), \mathrm{ns}(G), \mathrm{ms}(G)$, or linear-width$(G)$ is NP-complete (see [24, 22, 35] and Theorem 5.i of this paper). On the other hand, since all of these parameters is closed under taking of minors, we know that there exist a linear algorithm checking membership in $\mathcal{G}[f, k]$ where f is ms, es, ns, or linear-width. Such a linear time algorithm has been constructed for the node search number [5] (actually, the result in [5] concerns the parameter of pathwidth which is known to be equal to the node search number minus one – see [21, 19, 25]). Recently, a linear algorithm, checking if a graph belongs to $\mathcal{G}[\text{linear-width}, k]$, was found (see [7]). Moreover, the algorithm in [7] is constructive: for any fixed k, one can construct, if exists, an optimal edge arrangement. On the other hand, the algorithm in [7] appears to be difficult to be implemented and rather impractical, even for small values of k, as the contribution of the fixed k on the "hidden" part of their linear time complexity is heavily exponential.

In order to overcome the above problems one needs practical "tailor-made" algorithms for specific (usually small) values of k. Mainly, such kind of algorithms are based to a complete structural characterisation of the corresponding graph class. In this direction, an algorithm for the class of graphs with node search number ≤ 3 has been given in [11] (actually the algorithm in [11] concerns graphs with pathwidth ≤ 2 but can be easily transfered to the class of graphs with node search number ≤ 3). However, no "tailor-made" algorithms for the linear-width, the mixed search number, or the edge search number are known.

In this paper we give a linear time algorithm checking if a graph has linear-width ≤ 2 and, if so, outputs an edge ordering with optimal linear-width. Moreover, we prove a structural connection between linear-width and the three search parameter we mentioned before (this connection generalises the one proved in [3]). According to this result, our algorithm can be directly modified to one that checks whether the mixed or the edge search number of a graph is at most 2. and, if so, outputs an optimal search.

Our algorithm is based on a complete structural characterisation of the class of graphs with linear-width ≤ 2. Using this characterisation, we prove that $\mathrm{ob}(\mathcal{G}[\text{linear-width}, 2])$ consists of the 57 graphs depicted on Figures 6 and 7. Moreover, we prove that, for any k, there exists an injection from $\mathrm{ob}(\mathcal{G}[\mathrm{ms}, k])$ to $\mathrm{ob}(\mathcal{G}[\text{linear-width}, k])$. A direct consequence is that $\mathrm{ob}(\mathcal{G}[\mathrm{ms}, k])$ can be easily determined if we know $\mathrm{ob}(\mathcal{G}[\text{linear-width}, k])$. Applying this result for the case that $k = 2$ we can determine $\mathrm{ob}(\mathcal{G}[\mathrm{ms}, 2])$ and, in that way, reproduce the result of [34].

Finally, for any k, we determine all the trees in $\mathrm{ob}(\mathcal{G}[\text{linear-width}, k])$. More specifically, we prove that, for any k, there exist a bijection between the trees in $\mathrm{ob}(\mathcal{G}[\text{linear-width}, k])$ and the trees in $\mathrm{ob}(\mathcal{G}[\mathrm{ms}, k])$. Our results indicate that, for $k > 2$, a complete structural characterisation of the class of graphs with linear-width $\leq k$ is rather hard to be found even for small values of k.

The paper is organised as follows. In Section 2 we give some basic definitions and results concerning the structure of the graphs with linear-width ≤ 2. In Section 3 we present the main algorithm of this paper. In Section 4 we prove the correctness of the algorithm and the obstruction set. Section 5 is devoted to the relation between linear-width and the three variants of the graph searching game. Finally, in Section 6 we end up with some conclusions and open problems.

2 Definitions and preliminary results

We consider finite undirected graphs without loops or multiple edges unless otherwise is mentioned.

Let G be a graph. If $S \subseteq V(G)$, we call the graph $(S, \{\{v,u\} \in E(G) : v,u \in S\})$ the *subgraph of G induced by S* and we denote it as $G[S]$. Given two graphs G_1, G_2 we set $G_1 \cup G_2 = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$. Given a vertex $v \in V(G)$ we denote $G \Leftrightarrow v = G[V(G) \Leftrightarrow \{v\}]$. Also, if $e \in E(G)$ we set $G \Leftrightarrow e = (V(G), E(G) \Leftrightarrow \{e\})$. A *contraction* of an edge $\{u,v\}$ in G to v is the operation that removes u and makes v adjacent to all the vertices that were adjacent to u. We denote the result of the contraction of e as $G \dot{-} e$. For any edge set $E \subseteq E(G)$ we denote as $V(E)$ the set of vertices that are incident to edges of H (i.e. $V(E) = \cup_{e \in E} e$). We denote the degree of a vertex v with respect to some graph G as $d_G(v)$. We denote as $A(G)$ the set of articulation vertices of G (i.e. $A(G) = \{v \in G \mid G \Leftrightarrow v \text{ is disconnected}\}$). We call a vertex *pendant* if it has degree 1. We call an edge *pendant* if it contains a pendant vertex. We denote by $A^*(G)$ the vertices of $A(G)$ that are not adjacent with pendant vertices. We call a vertex v *almost pendant* if $v \in A(G) \Leftrightarrow A^*(G)$. Finally, we call an edge *almost pendant* if it is not pendant and one of its endpoints is *almost pendant.*

2.1 Minors: proper and rooted

We say that H is a *minor* of G (denoted by $H \preceq G$) if H can be obtained by a series of the following operations: vertex deletions, edge deletions, and edge contractions. We say that H is a proper *minor* of G (denoted by $H \prec G$) if $H \preceq G$ and H is not isomorphic to G. If $\mathcal{H}$ is a set of graphs such that some of them is a minor of G, then we denote it as $\mathcal{H} \sqsubseteq G$. If no element of H is a minor of G then we denote it as $\mathcal{H} \not\sqsubseteq G$. If in some graph G we distinguish some vertex v we call this graph v-rooted or, simply, v-graph (we also call v a *root* of G). Any v-graph that can be obtained by a v-rooted graph after a sequence of edge deletions or vertex/edge contractions that do not remove v is called v-minor of G and we denote it as $H \preceq_v G$ (from now on, whenever we mention a contraction in a rooted graph we will assume that the removed vertex is different than its root). Analogously to the non-rooted case, we define the relations "$\prec_v$", "$\sqsubseteq_v$", and "$\not\sqsubseteq_v$".

2.2 Linear-width

We define linear-width is as follows. Let G be a graph and $l = (e_1, \ldots, e_{|E(G)|})$ be a linear ordering of $E(G)$. We set $\delta_l(e_i) = V(\{e_1, \ldots, e_i\}) \cap V(\{e_{i+1}, \ldots, e_{|E(G)|}\})$ (i.e. $\delta_l(e_i)$ is the set of vertices in $V(G)$ that are incident to an edge in $\{e_1, \ldots, e_i\}$ and also to an edge in $\{e_{i+1}, \ldots, e_{|E(G)|}\}$.

We set linear-width$(l) = \max_{1\le i\le |E(G)|-1}\{|\delta_l(e_i)|\}$. The linear-width of a graph is the minimum linear-width over all the orderings of $E(G)$ (if $|E(G)| \le 1$ then linear-width$(G) = 0$). If $l = (e_1, \ldots, e_{|E(G)|})$, we set $l^{-1} = (e_{|E(G)|}, \ldots, e_1)$. Clearly, linear-width$(l)$ = linear-width(l^{-1}).

Lemma 1 *The class of graphs with bounded* linear-width *is closed under taking of minors.*

Proof. Let G be a graph having an edge ordering l with linear-width equal to k. It is enough to prove that for any $v \in V(G), e \in E(G)$, graphs $G \Leftrightarrow v, G \Leftrightarrow e, G \dot{-} e$ have linear-width $\le k$. Let l be an edge ordering of G. It is easy to see that, if we remove from l all the edges not existing any more in $G \Leftrightarrow v$ or $G \Leftrightarrow e$, we have again an edge ordering of linear-width $\le k$. Suppose now that $G' = G \dot{-} e$. Let $e = \{v, u\}$ and assume that the contraction removes u. We now remove edge e from l and then replace u with v in any edge containing u (if during this operation appears an edge that is already in the ordering, then we remove it). It is now easy to see that the linear-width of the new ordering is no more than k. □

We denote by $\mathcal{L}_2$ the set consisting of the graphs depicted in Figures 6 and 7. The following lemma is a consequence of Lemma 1 and the fact that all the graphs in $\mathcal{L}_2$ have linear-width more than two.

Lemma 2 *Let G be a graph that $\mathcal{L}_2 \sqsubseteq G$. Then,* linear-width$(G) > 2$.

After a careful inspection, one can verify that set $\mathcal{L}_2$ is a minor minimal set, i.e. no graph in $\mathcal{L}_2$ is a proper minor of an other member of $\mathcal{L}_2$. Therefore, we have the following.

Lemma 3 $\mathcal{L}_2 \subseteq$ ob($\mathcal{G}$[linear-width, 2]).

In the next two sections, we will prove that every graph with linear-width more than two contains a graph in $\mathcal{L}_2$ as a minor and therefore $\mathcal{L}_2$ is the obstruction set for the class of graphs with linear-width ≤ 2.

2.3 Small, long, and weak edges

A v-graph G is called a *v-wing* if $\mathcal{D} \sqsubseteq_v G$ (graphs in $\mathcal{D}$ are depicted in Figure 1). A v-hair is a v-graph G that is isomorphic with $\mathcal{A}_1$ and $d_G(v) = 1$ (graph $\mathcal{A}_1$ is depicted in Figure 1). Let G be a graph. If $\{C_1^v, \ldots, C_\rho^v\}$ is the set of the connected components of $G[V(G) \Leftrightarrow \{v\}]$, we set $\mathcal{D}(G, v) = \{D_1^v, \ldots, D_\rho^v\}$ where $D_i^v = G[V(C_i^v) \cup \{v\}], 1 \le i \le \rho$. For any vertex $v \in A^*(G)$, we define $\alpha(G, v)$ as the number of v-wings in $\mathcal{G}(G, v)$. We call a pendant edge $e = \{v, u\} \in E(G)$ *small* if $d_G(v) \ge 3$. We also call a pendant vertex *small* if it is incident to a small edge. We also call $e = \{v, u\}$ *long* if $d_G(v) = d_G(u) = 2$.

Lemma 4 *Let H be a graph with* linear-width $\le k$. *The following hold.*

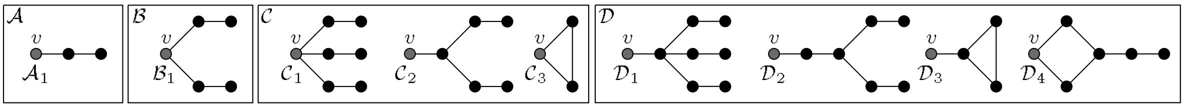

Figure 1: The classes of rooted graphs $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$.

i. *Let v, v' be vertices such that $v \in V(H), d_G(v) \geq 2$, and $v' \notin V(H)$. If $H' = (V(H) \cup \{v'\}, E(H) \cup \{\{v, v'\}\})$, then* linear-width$(H') \leq k$ *(notice that v' is a small vertex of H').*
ii. *Let v be a vertex that is adjacent only with vertices w and u in H. Let also $H' = (V(H) \cup \{u'\}, E(H) \cup \{\{u, u'\}, \{u', v\}\} \Leftrightarrow \{\{v, u\}\})$. Then,* linear-width$(H') \leq k$.

Proof. Let $l = (e_1, \ldots, e_r)$ be an edge ordering of H where linear-width$(l) = k$.
i. Notice that, as $d_H(v) \geq 2$, l contains at least one edge e_i with $v \in \delta_l(e_i)$. It is now easy to see that $l' = (e_1, \ldots, e_i, \{v, v'\}, e_{i+1}, \ldots, e_r)$ is an edge ordering of G' with linear-width $\leq k$.
ii. W.l.o.g. we assume that $\{v, u\}$ comes before $\{w, v\}$ in l (if not, we choose l^{-1}). Let also $e_i = \{v, u\}, 1 \leq i \leq r$. Now observe that $l' = (e_1, \ldots, e_{i-1}, \{u, u'\}, \{u', v\}, e_{i+1}, \ldots, e_r)$ is an edge ordering of G' with linear-width $\leq k$. □

Lemma 5 *Let G be a graph. Then, there exist a graph G' such that $G' \preceq G$, G' does not contain any long or small hairs, and* linear-width$(G) \leq k \Leftrightarrow$ linear-width$(G') \leq k$. *Moreover, if l' is an edge ordering of G' with* linear-width $\leq k$, *one can construct an edge ordering of G with* linear-width $\leq k$ *in $O(|E(G)|)$ time.*

Proof. Let G' be the graph that is obtained if we apply the following operation on G as long as it is possible:

- If e is a long edge or a small hair in G, then set $G \leftarrow G \dot{-} e$.

Clearly, if we have an edge ordering of G' with linear-width at most k, we can construct an edge ordering of G with linear-width at most k undoing the above sequence of contractions. Since we need $O(1)$ time for each contraction, the rebuilding process needs $O(|E(G)|)$ time. What remains is to prove that linear-width$(G) \leq k \Leftrightarrow$ linear-width$(G') \leq k$. The "$\Rightarrow$" direction follows immediately from Lemma 1. The "$\Leftarrow$" direction follows if we apply inductively Lemma 4 on the number of the edges contracted. □

Let G be an outerplanar graph. In this paper, we will denote a face F of a planar embedding of G as the graph induced by the vertices that are incident to F (certainly, such a graph is always a cycle). We call the edges of the outer face of G *outer* edges and all the others *inner*. We denote the set of the outer (inner) edges of an outerplanar graphs as out(G) (inn(G)). For two vertices x, y, we say that $x \sim y$ if $\{x, y\} \in$ out(G). We call a face F *simplicial* if it contains at most one inner edge. Let F be a simplicial face. The edges F that belong in out(G) are called *simplicial*. The vertices of F that are not incident to the unique edge of F that is in $\text{inn}(G)$ are called *simplicial*. If an edge (vertex) of F is not simplicial then we call it *critical*. (if G is a cycle or a single edge, all its edges are simplicial and outer). The set of simplicial faces of G is denoted by $\mathcal{S}(G)$. We say that a biconnected component of an outerplanar graph G is a *bolbe* if it is not a pendant or an almost pendant edge. We denote as $\mathcal{B}(G)$ the set of all the bolbes of G. For an example of the given definitions see Figure 2.

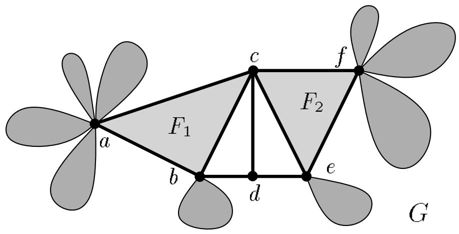

Figure 2: $B = G[\{a, b, c, d, e, f\}]$ is a bolbe of an outerplanar graph G. The outer edges of B are $\{a, c\}, \{c, f\}, \{f, e\}, \{e, d\}, \{d, b\}, \{b, a\}$. The inner edges of B are $\{b, c\}, \{d, c\}, \{e, c\}$. B contains two simplicial faces F_1, F_2 where $F_1 = B[\{a, b, c\}]$, $F_2 = B[\{c, e, f\}]$. F_1 (F_2) contains only a (f) as simplicial vertex and the simplicial edges of F_1 (F_2) are $\{a, b\}, \{a, c\}$ ($\{c, f\}, \{e, f\}$). The critical vertices of F_1 (F_2) are b, c (c, e) and the critical edge of F_1 (F_2) is $\{b, c\}$ ($\{c, e\}$).

We call an outer edge $\{x, y\}$ *weak* if $\{x, y\} \cap A(G) = \emptyset$ and there exist a vertex z such that if $E = \{\{z, x\}, \{y, x\}\}$ then $\text{inn}(G) \cap E \neq \emptyset$ and $\text{out}(G) \cap E = \emptyset$ (i.e. one edge in E is inner and z is not adjacent to x or x through an outer edge). Notice that the bolbe depicted in Figure 2 does not contain any weak edges.

Lemma 6 *Let e be a weak edge of an outerplanar graph G. Then* $\text{linear-width}(G \dot{-} e) \leq 2 \Rightarrow \text{linear-width}(G) \leq 2$.

Proof. Let $e = \{x, y\}$ and suppose that $\{x, z\}$ is an inner edge of G. Let H be the result of the contraction of e to x. We observe that $\{x, z\}$ is an inner edge of H and $x \notin A(H)$. Let B be the unique biconnected component of H that contains $\{x, z\}$. Clearly $B[V(B) \Leftrightarrow \{x, z\}]$ has exactly two connected components C_1, C_2. Let $B_i = B[V(C_i) \cup \{x, z\}] \Leftrightarrow \{x, z\}, i = 1, 2$. Suppose that $l' = (e_1, \ldots, e_i, \ldots, e_r)$ is an edge ordering of H where $e_i = \{x, z\}$ and $\text{linear-width}(l) = 2$ (notice that it is impossible $\text{linear-width}(G') < 2$). Let e_j be the first edge of B appearing in l. Clearly $i > j$ because, otherwise, $\delta_l(e_{j+1}) \geq 3$. W.l.o.g. we assume that $e_j \in E(B_1)$. Let also e_h be the first edge of B_2 appearing in l. We claim that $h > i$. Suppose in contrary that e_i comes after e_h in l. Notice that $\delta_l(e_{h-1}) \subseteq V(B_1)$, and $\delta_l(e_{h-1}) \geq 2$. Moreover, as e_h is the first edge of B_2 in l we have that $\delta_l(e_h) = \delta_l(e_{j-1}) \cup \{x\}, x \in V(C_2)$ and this means that $\delta_l(e_h) \geq 3$ a contradiction. Let now $e_{h'}$ be the last edge in B_1 appearing in l. Applying the same arguments on l^{-1} ($e_{h'}$ is the first edge in B_1 appearing in l^{-1}), we can prove that $h' < i$. We now have that $h' = i \Leftrightarrow 1$, $h = i + 1$ and thus $\delta_l(e_{i-1}) = \delta_l(e_i) = \{x, z\}$. We now set $l_1 = (e_1, \ldots, e_{i-1})$, $l_2 = (\{x, z\}, \{x, y\}, \{y, z\})$,and $l_3 = (e_{i+1}, \ldots, e_r)$. It is now easy to see that $l_+ = l_1 \oplus l_2 \oplus l_3$ is an edge ordering of $G^+ = (V(G), E(G) \cup \{\{x, z\}, \{y, z\}\})$ where linear-width $= 2$ (notice that $\{y, z\}$ is not necessarily an edge of G). Finally notice that $G \preceq G^+$. From Lemma 1, we have that linear-width$(G) = 2$. □

Lemma 7 *Let G be an outerplanar graph. Then, there exist a graph G' such that $G' \preceq G$, G' does not not contain any weak edge, and* linear-width$(G) \leq 2 \Leftrightarrow$ linear-width$(G') \leq 2$. *Moreover, if l' is an edge ordering of G' with* linear-width ≤ 2, *one can construct an edge ordering of G with* linear-width ≤ 2 *in $O(|V(E)|)$ time.*

Proof. The proof is similar to the one of Lemma 5 with the difference that we now apply inductively Lemma 6 (G' is constructed if we perform contractions of weak edges as long as this is possible). □

2.4 Smooth graphs and wings

We call a graph G *smooth* if each of the following conditions is satisfied.

(sm–i) G does not contain small or long edges,

(sm–ii) G is outerplanar and does not contain weak edges,

(sm–iii) for any bolbe B of G, $|\mathcal{S}(B)| \leq 2$,

(sm–iv) for any vertex v, $\alpha(G, v) \leq 2$.

Lemma 8 *Let G be a graph satisfying conditions* **(sm–i)** *and* **(sm–ii)** *above but not* **(sm–iii)** *or* **(sm–iv)**. *Then $\{4K_3\} \cup \mathcal{L}_2^1 \sqsubseteq G$.*

Proof. Let B be a bolbe containing at least tree simplicial faces F_1, F_2, and F_3. Then, if we first contract all the edges not having both their endpoints in $F_1 \cup F_2 \cup F_3$, and then contract all the long edges, we obtain $4K_3$. Let now $v \in V(G)$ such that $\mathcal{D}(G, v)$ contains at least three v-wings W_1, W_2, W_3. This means that $\mathcal{D} \sqsubseteq_v W_i, i = 1, 2, 3$. It is now enough to observe that $W_1 \cup W_2 \cup W_3$ is a subgraph of G containing one of the graphs in $\mathcal{L}_2^1$ as a minor. □

In Figure 3 we show graphs Z_0, Z_1^1, Z_1^2.

Lemma 9 *Let G be a smooth v-graph such that $v \notin A(G)$. Then, either G is a v-wing or $G \preceq_v Z_0$.*

Proof. Let $\mathcal{D} \not\sqsubseteq G$. We will prove that $G \preceq Z_0$. Clearly, we can assume that G is not a v-hair. We distinguish the following cases. (The graphs $\mathcal{C}_i, i = 1, \ldots, 4$ and $\mathcal{D}_i, i = 1, \ldots, 4$ that are used in the case analysis below are depicted in Figure 1.)

Case a. $d_G(v) = 1$. Let u be the single neighbour of v. Notice that $d_G(u) \leq 3$, otherwise, $\mathcal{C}_1 \preceq_u G \Leftrightarrow v$ and thus $\mathcal{D}_1 \preceq_v G$. Also, $d_G(u) \geq 3$, otherwise, $\mathcal{C}_2 \preceq_u G \Leftrightarrow v$ and thus $\mathcal{D}_2 \preceq_v G$. Moreover, $G \Leftrightarrow v$ must be a u-tree, otherwise, $\mathcal{C}_3 \preceq_u G \Leftrightarrow v$ and thus $\mathcal{D}_3 \preceq_v G$. Let $\mathcal{D}(G \Leftrightarrow v, u) = \{G_1, G_2\}$. Clearly, both $G_i, i = 1, 2$ are u-hairs as, otherwise, $\{\mathcal{C}_2, \mathcal{C}_3\} \sqsubseteq_u G \Leftrightarrow v$ and thus $\{\mathcal{D}_2, \mathcal{D}_3\} \sqsubseteq_u G$. We conclude that G is isomorphic with $\mathcal{C}_1 \preceq_v Z_0$.

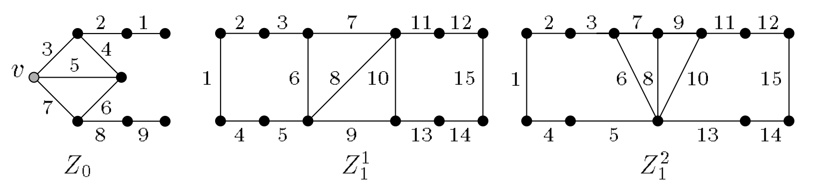

Figure 3: The graphs Z_0, Z_1^1, Z_1^2.

Case b. $d_G(v) \geq 2$. As G is outerplanar, one can easily see that G contains exactly one biconnected component B that is not a single edge and $v \in V(B)$ (otherwise $v \in A(G)$). From **(sm–iii)**, B has two simplicial faces F_1, F_2. Notice now that if $A = A(G) \cap V(B)$ then $|A| \leq 2$ otherwise $\mathcal{D}_1 \preceq_v G$ (contract all the edges in $E(B)$). Let now v_1, v_2 be the two vertices of $V(B)$ such that $\{v, v_1\}, \{v, v_2\} \in \text{out}(B)$. Notice that if $x \in A$, then $x \in \{v_1, v_2\}$ otherwise, $\mathcal{D}_4 \preceq_v G$. Moreover, $|\mathcal{D}(G, x)| = 2$ (otherwise, $\mathcal{D}_2 \preceq_v G$) and the graph in $\mathcal{D}(G, x)$ not containing v as a vertex must be a x-hair (otherwise, $\{\mathcal{D}_2, \mathcal{D}_3\} \sqsubseteq_v G$. Observe now that v is incident to all the inner edges of B, otherwise, $\mathcal{D}_4 \preceq_v G$. Finally, notice that $|V(B) \Leftrightarrow \{v, v_1, v_2\}| \leq 1$ as G does not contain long or weak edges. Summing up all the previous observations we can easily see that $G \preceq_v Z_0$. □

Let $l = (e_1, \ldots, e_{|E(G)|})$ be an edge ordering of a v-rooted graph G. We call such an ordering *simple* or *v-simple* if there exist an ordering $l = (e_1, \ldots, e_{|E(G)|})$ of its edges such that, $\forall i, 1 \leq i \leq |E(G)|$ $|\delta_l(e_i) \cup \{v\}| \leq 2$.

Lemma 10 *Let G be a smooth graph where for some $v \in A(G)$ $\alpha(G, v) = 0$. Then there exist a* v-simple *edge ordering of G.*

Proof. Clearly, any graph in $\mathcal{D}(G, v)$ is a minor of Z_0. The numbering depicted in Figure 3 gives a v-simple ordering for Z_0. Using this, it is not hard to find a v-simple ordering for any of its minors. If now $l_1, \ldots, l_r$ are v-simple orderings for the graphs in $\mathcal{D}(G, v)$, then $l = l_1 \oplus \cdots \oplus l_r$ is a v-simple ordering of G. □

3 An algorithm for linear-width

It is easy to verify that $\text{ob}(\mathcal{G}[\text{linear-width}, 0]) = \{\mathcal{A}_1\}$ and $\text{ob}(\mathcal{G}[\text{linear-width}, 1]) = \{\mathcal{C}_1, \mathcal{C}_3\}$ (graphs $\mathcal{A}_1$, $\mathcal{C}_1$ and $\mathcal{C}_3$ are depicted in Figure 1). Using this fact, one can easily construct an algorithm that decides whether linear-width$(G) \leq 1$ and, if so, outputs an edge ordering of minimum linear-width. In this section we will present an algorithm, that, given a graph G, decides whether linear-width$(G) \leq 2$ and, in such a case, outputs an edge ordering of linear-width ≤ 2. Before we present the algorithm we first need a series of definitions and lemmata about the structure of the graphs with linear-width ≤ 2. The main structural lemma, supporting the correctness of the algorithm, is presented in the next section.

3.1 Doors and passages

Let B be a bolbe of a smooth graph G. We set $R(B) = A(G) \cap V(B)$. For any $v \in R(B)$ we set $EX(B,v) = \{G_i^v \mid G_i^v \in \mathcal{D}(G,v), V(G_i^v) \cap V(B) = \{v\}\}$, $EX(B) = \{H \mid \exists v \in R(B)$ such that $H \in EX(B,v)\}$, and $B_v = \cup_{G_i^v \in EX(B,v)} G_i^v$. For example for the bolbe $B = G[\{a,b,c,d,e,f\}]$ depicted in Figure 4 we have that $R(B) = \{a,b,e,f\}$, $EX(B,a) = \{A,C,D,E\}$, $EX(B,b) = \{F\}$, $EX(B,e) = \{H\}$, $EX(B,f) = \{I,J,K\}$, $EX(B) = \{A,C,D,E,F,H,I,J,K\}$, $B_a = A \cup C \cup D \cup E$, $B_b = F$, $B_e = H$, and $B_f = I \cup J \cup K$.

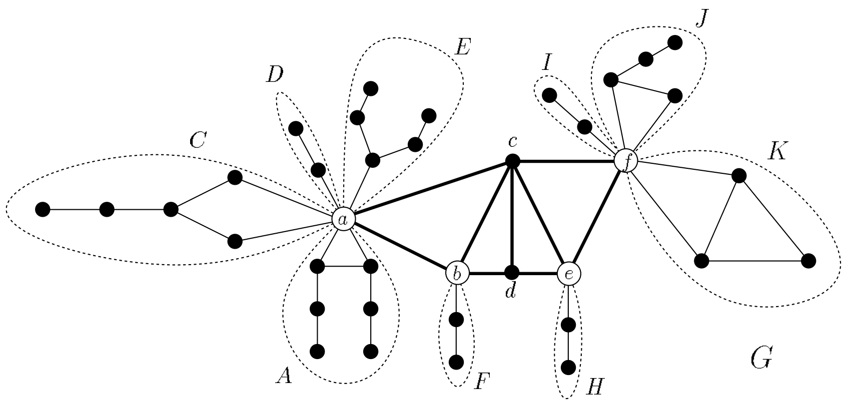

Figure 4: An example of a smooth graph G containing a bolbe $B = G[\{a,b,c,d,e,f\}]$.

We denote the null graph as $\mathcal{O}$ (i.e. $\mathcal{O} = (\emptyset, \emptyset)$). Let $Q = (R, H, I)$ be a triple consisting of a vertex set R and two graphs H, I. We say that such a triple is a *door* of B if one of the following hold.

a) $R = \emptyset$, $H = I = \mathcal{O}$. In such a case we call the door *empty*.

b) $R = \{v\} \subseteq R(B)$, $I = \mathcal{O}$, and $H = \cup_{G_i^v \in \mathcal{E}} G_i^v$ where $\mathcal{E}$ is a subset of $EX(v,G)$ that contains at most one v-wing. We call v a *passage* of the door.

c) $R = \{v,u\} \subseteq R(B)$, $v \sim u$, I is a u-hair in $EX(G,u)$ and $H = \cup_{G_i^v \in \mathcal{E}} G_i^v$ where $\mathcal{E}$ is a subset of $EX(v,G)$ that contains at most one v-wing. We call v a *passage* of the door.

As an example we mention that the triples $(\emptyset, \mathcal{O}, \mathcal{O})$, $(\{b\}, F, \mathcal{O})$, $(\{f, e\}, I, H)$, $(\{f, e\}, K, H)$, $(\{f, e\}, I \cup K, H)$, $(\{a, b\}, F, D)$, $(\{a, b\}, D, F)$, $(\{e, f\}, H, I)$, $(\{a, b\}, A \cup C \cup D \cup E, F)$, and $(\{f, e\}, I \cup K, H)$ are some of the doors of the bolbe B depicted in Figure 4,

Let B be a bolbe of a smooth graph G Let F_1, F_2 be the simplicial faces of B (if B has only one simplicial face, we have $F_1 = F_2$ – if $|V(B)| = 2$ we set $F_1 = F_2 = V(B)$). Let $Q_i = (R_i, H_i, I_i), i = 1, 2$ be two doors of B. We say that the pair $\mathcal{P} = (Q_1, Q_2)$ *opens* B if

(op–i). $G = B \cup H_1 \cup I_1 \cup H_2 \cup I_2$.

(op–ii). $R_i \subseteq F_i, i = 1, 2$,

(op–iii). $|V(H_1) \cap V(H_2)| \leq 1$ (i.e. if both $H_i, i = 1, 2$ are nonempty then they are different members of $EX(B)$).

(op–iv). $|V(I_1) \cap V(I_2)| \leq 1$ (i.e. if both $I_i, i = 1, 2$ are nonempty then they are different members of $EX(B)$).

(op–v). If $Q_i, i = 1, 2$ have the same passage v, then $\alpha(H_i, v) = 1, i = 1, 2$.

(op–vi). If B is not an edge and $R_i, i = 1, 2$ induce edges in B, then these edges are different.

We call $\mathcal{P} = ((R_i, H_i, I_i), i = 1, 2)$ an *opening pair* of B. B is *open* when it is opened by some pair of doors. We call a vertex v a *passage* of an opening pair if v is a passage of its doors. For example, a pair opening the bolbe B depicted in Figure 4 is $((\{a, b\}, A \cup C \cup D \cup E, F), (\{f, e\}, I \cup J \cup K, H))$ and the corresponding passages the are a and f. Notice that it is possible a bolbe to be opened by more than one pairs. It is easy to see that, if we know whether each rooted graph in $EX(B)$ is a wing, or a hair, or none of them, we can assign to B an opening pair (if exist) in $O(|EX(B)|)$ time. This observation will appear to be useful for proving the linearity of the algorithm $\mathsf{LW2}(G)$ that we will present in Theorem 1.

Lemma 11 *Any open bolbe B of a smooth graph G is a proper minor of one of the graphs Z_1^1, Z_1^2 depicted in Figure 3.*

Proof. Let $R_i, i = 1, 2$ be the vertex sets of the opening pair of B. The case where $\mathrm{F}_s(B) \leq 1$ is simple as in such a case G is either an edge or a cycle of at most 6 edges (notice that, since B is open, $|R(B)| \leq 4$). Suppose now that F_1, F_2 are the simplicial faces of B and assume w.l.o.g. that $R_i \subseteq F_i, i = 1, 2$. Since $F_i, i = 1, 2$ does not contain long edges, it is a cycle containing at most 6 vertices. Taking now in mind that the critical edge of F_i has at most one vertex in common with R_i, one can easily see that B is always a proper minor of Z_1^1 or Z_1^2. □

Lemma 12 *Let $((R_i, H_i, I_i), i = 1, 2)$ be the pair of doors opening a bolbe B of a smooth graph. Then, there exist an edge ordering of B with* linear-width ≤ 2 *and with the property that R_1 (R_2) is a subset of its first (last) edge.*

Proof. From Lemma 11 we have that $B \prec Z_1^1$ or $B \prec Z_1^2$. Using now the orderings depicted in Figure 3 for Z_1^1 and Z_1^2 as a starting point, one can easily construct a suitable edge ordering for any of their minors. □

Actually, the forms that open bolbes of smooth graphs can have are not many. Using Lemma 11 as a starting point, one can easily determine all of them. A back up of these graphs and the corresponding orderings (according to Lemma 12) can be useful for the implementation of the algorithm LW2(G) that we present in Theorem 1. The same remark holds for the graphs mentioned in Lemma 9.

3.2 Finding a starting bolbe

We plan to prove that any open smooth graph G has an edge ordering with linear-width ≤ 2 (Theorem 15). In this direction, Lemmata 10 and 12 show how to construct two different types of edge ordering for the graphs that are consisting G. What we now need is to merge all these orderings into an edge ordering of the hole graph. For this purpose we need to distinguish which parts of an open smooth graph require each type of ordering.

Given a v-graph G where $v \notin A(G)$, we define the v-bolbe of G as the unique bolbe of G containing v as a vertex. Let $\mathcal{P}$ be a pair opening a bolbe B. If the two doors in $\mathcal{P}$ are non empty and have the same passage, then we call P *marginal.*

Notice that if a bolbe is opened by a (non)-marginal pair then all the pairs opening it are (non)-marginal. Using this remark we can define that a bolbe B is *marginal* if it is opened by a marginal pair, otherwise we call it non-marginal.

Lemma 13 *Let G be a graph containing a marginal bolbe B. Let v be the unique passage of a marginal pair $\mathcal{P}$ opening B and G' be the graph in $\mathcal{D}(G, v)$ whose v-bolbe is B. Then then following hold.*
i. *G' is not a v-wing,*
ii. *$\mathcal{D}(G, v)$ contains two v-graphs whose v-bolbes are non-marginal.*

Proof. From **(op–v)** we have that $\mathcal{D}(G, v)$ contains two v-wings G_1, G_2 that are different from G'.
i. As, from **(sm–iv)**, $\alpha(G, v) \leq 2$, G' cannot be a v-wing.
ii. Let now B_i be the v-bolbe of $G_i, i = 1, 2$. Then $B_i, i = 1, 2$ is non-marginal, otherwise, applying 13.i on B_i we have that G_i is not a wing, a contradiction. □

Lemma 14 *Let G be a connected open smooth graph. Then one of the following holds.*

(a) There exist a vertex $v \in A^*(G)$ such that $\alpha(G, v) = 0$.

(b) G has a non-marginal bolbe B opened by a non-marginal pair $\mathcal{P}$ where $\mathcal{P}$ has either an empty door or a passage $v \in V(B)$ such that $\alpha(G, v) = 1$. We call such a bolbe starting *bolbe.*

Proof. From **(sm–iv)** we have $\forall v \in A(G)$ $\alpha(G, v) \leq 2$. Assume now that that (a) does not hold. Then, $\forall v \in A(G)$ $\alpha(G, v) = 1$ or 2. We also assume that G contains at least one bolbe, otherwise, (a) holds. Finally, we can assume that for any bolbe of G the pairs opening it contain only non-empty doors as otherwise such a pair is clearly non-marginal and (b) holds. Let $\mathcal{W}$ be the set of the non-marginal bolbes of G. From Lemma 13.ii, we have that $\mathcal{W} \neq \emptyset$.

In what follows, we will prove that there exist a bolbe $B \in \mathcal{W}$ opened by a non-marginal pair $\mathcal{P}$ that contains a passage v where $\alpha(G, v) = 1$. If $\mathcal{W}$ contains only one bolbe B, then it is trivial to see that for any passage v of a pair opening B, $\alpha(G, v) = 1$. We now assume that $|\mathcal{W}| \geq 2$. Suppose, towards a contradiction, that any pair opening a bolbe $B \in \mathcal{W}$ contains two passages v_1, v_2 such that $\alpha(G, v_1) = \alpha(G, v_2) = 2$. Notice that, in such a case, if (Q_1, Q_2) is a pair opening a bolbe $B \in \mathcal{W}$ then, apart from (Q_2, Q_1), there exist no other pair opening B. As (Q_1, Q_2) and (Q_2, Q_1) has the same passages we can call them *passages* of B. Let A be the vertices of $A^*(G)$ that are also passages of bolbes in $\mathcal{W}$. Clearly, any bolbe of $\mathcal{W}$ contains two vertices of A as passages. Let now $v \in A$. Let also G_v^1, G_v^2 be the two v-wings in $\mathcal{D}(G, v)$ and B_1, B_2 be the v-bolbes of G_v^1 and G_v^2 respectively. Clearly, $B_1, B_2 \in \mathcal{W}$ and therefore, each vertex in A is the common vertex of two different bolbes in $\mathcal{W}$. We construct now G' as follows: first remove from G all the vertices not belonging to graphs in $\mathcal{W}$ and then, for each bolbe $B \in \mathcal{W}$, contract all the edges in $E(B)$ except from one (in case $|\mathcal{W}| = 2$, we allow multiple edges in G'). It is not hard to see that $V(G') = A$ and $E(G')$ consists of the edges that where excepted. Moreover, notice that each vertex in G' has degree exactly 2 and therefore G' is a cycle, a contradiction as the vertices in A should be articulation vertices of G' as well. □

3.3 Constructing an edge ordering

In this subsection we present the way to merge the edge orderings of the trivial bolbes, the non-trivial bolbes, and the hairs of an open smooth graph. The non-trivial bolbes will form the main axis of the hole ordering.

Lemma 15 *Let G be an graph that is smooth and open. Then, there exist an edge ordering of G with* linear-width *at most 2.*

Proof. We can assume that G is connected, (otherwise we apply the proof on each of the connected components of G). Using Lemma 14, we can assume that 14.(b) holds as, otherwise, the result follows immediately from Lemma 10.

We apply the following procedure on G. For an example see Figure 5.

1. Let B_1 be a starting bolbe of G and $(Q_1, Q_2) = (R_j^1, H_j^1, I_j^1), j = 1, 2$ be the non-marginal pair opening B_1. If both Q_1, Q_2 are non-empty we can assume that their passages are v_1^1 and v_2^1 respectively. (Notice that, since (Q_1, Q_2) is non-marginal, $v_1^1 \neq v_2^1$.) If only one, say Q_2, of Q_1, Q_2 has a passage, then we consider that v_2^1 is the passage of Q_2 and v_1^1 is some simplicial vertex of the simplicial face of B corresponding to Q_1 (if B is a cycle or a single edge, then we can choose as v_1^1 any vertex of B that is not v_2^1). (If none of Q_1, Q_2 has a passage then $G = B_1$ and the required edge ordering can be constructed according to Lemma 12.)

2. Let G_1 be the unique v-wing in $\mathcal{D}(G, v)$ (in case $\alpha(G, v) = 0$ the result follows immediately

from Lemma 10).
3. Set $H_0 = \{H^1_1\}$ (clearly, as $\alpha(G, v^1_1) = 1$, H_0 is not a v^1_1-wing).
4. Set $i = 1$.
5. If $R^i_2 = \emptyset$, then set $H_i = \mathcal{O}$, $\rho = i$, and **stop**.
6. If H^i_2 is not a v^i_2-wing, then we set $H_i = H^i_2$, $\rho = i$ and **stop**.
7. If H^i_2 is a v^i_2-wing, then let G_{i+1} be the unique member of $\mathcal{D}(H^i_2, v^i_2)$ that is a v^i_2-wing.
8. Set $H_i = \cup_{H \in \mathcal{D}(H^i_2, v^i_2) - \{G_{i+1}\}} H$ (i.e. H_i contains all the others). Notice also that $\alpha(H_i) = 0$.
9. Set $i = i + 1$.
10. Let B_i be the v^{i-1}_2-bolbe of G_i. Let also $(Q^i_1, Q^i_2) = ((R^i_j, H^i_j, I^i_j), j = 1, 2)$ be a pair opening B_i. Let also v^i_1, v^i_2 be the corresponding passages. Clearly, v^{i-1}_2 is one, say v^i_1, of v^i_1, v^i_2. Recall that G_i is a v^i_1-wing. Therefore, from Lemma 13.i, (Q^i_1, Q^i_2) is a non-marginal pair and thus, $v^i_1 \neq v^i_2$.
11. Goto to step **5**.

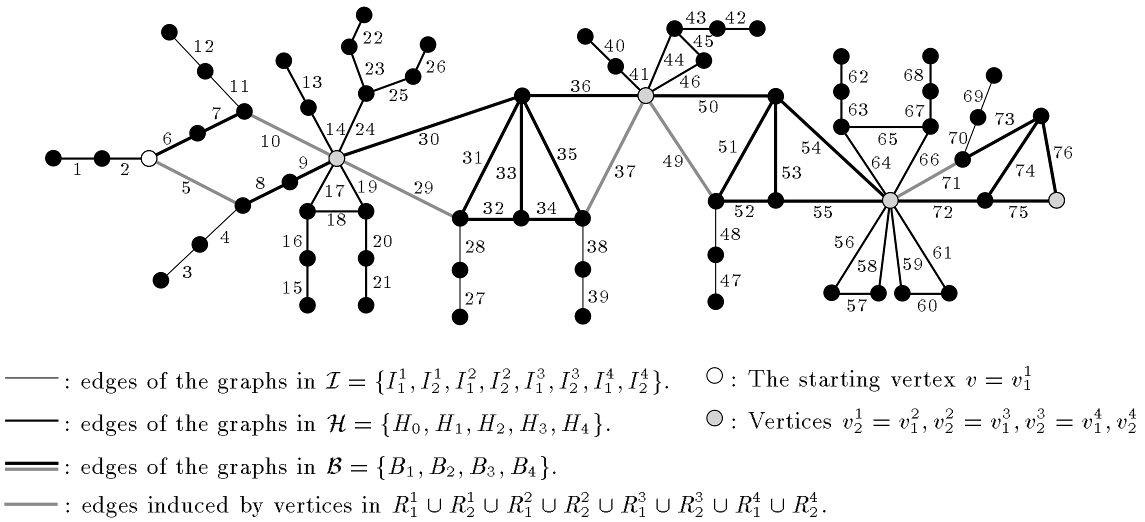

Figure 5: An example of an edge ordering with linear-width ≤ 2.

Clearly in each repetition of loop **5**–**10** the graph G_{i+1} produced has always less vertices than G_i. Therefore, the procedure will stop after producing the sequence of graph sequences $\mathcal{H} = \{H_0, \ldots, H_\rho\}$, $\mathcal{I} = \{I^1_1, I^1_2, \ldots, I^\rho_1, I^\rho_2\}$ and $\mathcal{B} = \{B_1, \ldots, B_\rho\}$. As any member of $B_i \in \mathcal{B}$ is a bolbe, we can apply Lemma 12 to get, for any $i = 1, \ldots, \rho$, an edge ordering $l^{\mathcal{B}}_i$ of B_i with linear-width$(l) \leq 2$ and with the property that if e^i_1 and $e^i_{|B_i|}$ are the first and last edges of f $l^{\mathcal{B}}_i$, then $R^i_1 \subseteq e^i_1$ and $R^i_2 \subseteq e^i_{|E(B_i)|}$. Notice now that any non null member I^i_j of $\mathcal{I}, 1 \leq i \leq \rho, j = 1, 2$ is a u^i_j-hair $(\{u^i_j, x^i_j, y^i_j\}, \{\{u^i_j, x^i_j\}, \{x^i_j, y^i_j\}\})$ where u^i_j is the unique element of $R^i_j \Leftrightarrow \{v^i_1\}$. For $i = 1, \ldots \rho$ and $j = 1, 2$ we define $l^{\mathcal{I}}_{j,i} = (\{u^i_j, x^i_j\}, \{x^i_j, y^i_j\})$. Let now H_i be a member of

$\mathcal{H}, 0 \leq i \leq \rho$. Recall that H_i is not a v_2^i-wing. From Lemma 10, we have that there exist a v_2^i-weak edge ordering $l_i^{\mathcal{H}}$ of $E(H_i)$.

Notice now that $G = (\cup_{i=1,\dots,\rho} B_i) \bigcup (\cup_{i=0,\dots,\rho} H_i) \bigcup (\cup_{i=1,\dots,\rho} I_1^i) \bigcup (\cup_{i=1,\dots,\rho} \mathcal{I}_2^i)$ and that if $l = l_0^{\mathcal{H}} \oplus (l_{1,1}^{\mathcal{I}})^{-1} \oplus l_1^{\mathcal{B}} \oplus l_{2,1}^{\mathcal{I}} \oplus l_1^{\mathcal{H}} \oplus (l_{1,2}^{\mathcal{I}})^{-1} \oplus l_2^{\mathcal{B}} \oplus l_{2,2}^{\mathcal{I}} \oplus l_2^{\mathcal{H}} \oplus \cdots \oplus (l_{1,\rho}^{\mathcal{I}})^{-1} \oplus l_\rho^{\mathcal{B}} \oplus l_{2,\rho}^{\mathcal{I}} \oplus l_\rho^{\mathcal{H}}$, then l is an edge ordering of G with linear-width ≤ 2. □

3.4 The algorithm

Algorithm LW2(G)
Input: A graph G
Output: If linear-width$(G) \leq 2$, the algorithm outputs an edge ordering of G with linear-width ≤ 2. If not the algorithm reports "linear-width$(G) > 2$".

1. Let G^1 be a graph such that $G^1 \preceq G$, linear-width(G^1) = linear-width(G), and G^1 does not have small or long edges.

2. If G^1 is not outerplanar, then Return "linear-width$(G) > 2$" and **stop**.

3. Let G^2 be a graph such that $G^2 \preceq G^1$, linear-width $(G^1) \leq 2 \Leftrightarrow$ linear-width(G^2), and G^2 does not have weak edges.

4. If $\exists_{B \in \mathcal{B}(G^2)} |\mathcal{S}(B)| \geq 3$ then Return "linear-width$(G) > 2$" and **stop**.

5. If $\exists_{v \in A(G^2)} \alpha(G^2, v) \geq 3$ then **stop**. (Notice that if the algorithm does not stop, then G is smooth.)

6. If $\exists_{B \in \mathcal{B}(G^2)}$ B is not open, then Return "linear-width$(G) > 2$" and **stop**. (Notice that if the algorithm does not stop, then G^2 is smooth and open.)

7. If $\exists_{v \in A^*(G)} \alpha(G, v) = 0$ then construct an ordering l of G^2 according to Lemma 10 and goto step **10**.

8. Find a starting bolbe of each of the connected components of G^2 (these bolbes exist because of Lemma 14).

9. For each connected component of G^2, construct an edge ordering l^2 with linear-width ≤ 2, using the procedure of the proof of Lemma 15. Finally, merge the edge orderings found to an edge ordering of G^2.

10. Construct an edge ordering l^1 of G^1 with linear-width ≤ 2.

11. Construct an edge ordering l of G with linear-width ≤ 2.

12. Return l and **stop**.

Theorem 1 *Algorithm* LW2(G) *is linear on* $|V(G)|$ *and outputs, if it exists, an edge ordering of* G *with* linear-width ≤ 2.

Proof. We first prove that LW2(G) needs $O(|V(G)|)$ time. Steps **1** and **3** can be done in linear time because of Lemmata 5 and 7 (take in mind that any outerplanar graph G has $O(|V(G)|)$ edges). Clearly, step **2** can be done in linear time. Moreover, it is possible in linear time to compute all the biconnected components of G^2 and, thus, step **4** needs $O(|V(G^2)|)$ time. Notice that it is possible to check in constant time whether a graph is v-minor of a graph with constant size. Therefore, according to Lemma 9 checking whether a graph is a v-wing or not requires constant time. Moreover, it is not hard to see that for any outerplanar graph $\sum_{v\in V(G)}|\mathcal{D}(G,v)| = O(|V(G)|)$ and $\sum_{B\in\mathcal{B}(G)}|EX(B)| = O(|V(G)|)$. Using the above observations and Lemma 11, one can easily verify that each of steps **5**–**9** can be done in $O(|V(G^2)|)$ time. Finally, the fact that steps **10** and **11** can be performed in linear time, follows directly from Lemmata 7 and 5.

What remains now is to prove that algorithm $\mathsf{LW2}(G)$ is correct. Notice that, if for some input G the algorithm enters step **7**, then G^2 is smooth and open. Therefore, linear-width$(G^2) \leq 2$ and thus the required ordering can be correctly constructed according to Lemma 15. Suppose now that for some input G the algorithm never enters at step **7**. We claim that, then, linear-width$(G) > 2$. In what follows we prove that $\mathcal{L}_2 \sqsubseteq G$. The claim then will be a direct consequence of Lemma 2.

Suppose first that $\mathsf{LW2}(G)$ stops at step **2**. Then G^1 is not outerplanar and $\mathcal{L}_2 \sqsubseteq \{K_{2,3}, K_4\} \sqsubseteq G^1 \preceq G$ (it is known that any non outerplanar contains either $K_{2,3}$ or K_4 as a minor). If now $\mathsf{LW2}(G)$ stops at steps **4** or **5**, the result follows directly from the fact that $G^2 \preceq G^1 \preceq G$ and from Lemma 8. Finally, if the algorithm stops at step **6**, this means that G is smooth and contains a bolbe B that is not open. The result now follows from the fact that $G^2 \preceq G^1 \preceq G$ and Lemma 16 (Lemma 16 will be presented in the next section). □

We remark that the main algorithm of this section can be easily parallelised. A parallel version of $\mathsf{LW2}(G)$ would require $O(\log|V(G)|\log^*|V(G)|)$ time and $O(|V(G)|)$ operations on an EREW PRAM and $O(\log|V(G)|)$ time and $O(|V(G)|)$ operations on a CRCW PRAM. We do not proceed with a detailed elaboration of the parallel case as it is easy and based on standard techniques.

4 Computing the obstruction set

In this section we will prove the basic structural lemma of this paper. Moreover, we will examine the case where multiple edges are considered.

4.1 The main lemma

The proof of the main lemma is based in an exhaustive case analysis of all the possible ways the graphs in $G[V(G) \Leftrightarrow V(B)]$ can be attached on B. We will show that either an opening pair exist or some graph in $\mathcal{L}_2^3 \cup \cdots \cup \mathcal{L}_2^9$ is a minor of G.

Lemma 16 *Let B be a bolbe of a smooth graph G. Then, either B is open or $\mathcal{L}_2^3 \cup \cdots \cup \mathcal{L}_2^9 \sqsubseteq G$.*

Proof. We assume that $\mathcal{L}_2^3 \cup \cdots \cup \mathcal{L}_2^9 \not\sqsubseteq G$. We will construct a pair of doors $\mathcal{P} = ((R_i, H_i, I_i), i = 1,2)$ opening B. We examine first the case where there exists a vertex $v \in R(B)$ such that

$\mathcal{D}(B_v, v)$ contains two v-wings G_1, G_2. Let G_3 be the graph in $\mathcal{D}(G, v)$ whose v-bolbe is B. Clearly, as $\alpha(G, v) \leq 2$ (recall that G is smooth), this v-graph is not a v-wing and using Lemma 9 we can see that G_3 is a biconnected v-minor of Z_0. Notice that $R(B) \leq 3$ and if $x \in R(G) \Leftrightarrow \{v\}$, then $x \sim v$. Let $H_1 = \cup(EX(B, v) \Leftrightarrow \{G_2\})$ and $H_2 = G_2$. If $R(B) \Leftrightarrow \{v\} = \emptyset$, then set $R_i = \emptyset, I_i = \mathcal{O}, i = 1, 2$. If $R(B) \Leftrightarrow \{v\} = \{x\}$ and G_3 is isomorphic to $\mathcal{C}_2$ (graph $\mathcal{C}_2$ is depicted in Figure 1) we set $R_1 = R_2 = \{v, x\}$, I_1 is the one of the two x-hairs of $\mathcal{D}(G_3, x)$ and I_2 the other. If $R(B) \Leftrightarrow \{v\} = \{x\}$ and G_3 is isomorphic to $\mathcal{C}_2$ we set $R_1 = \{v, x\}, I_1 = B_x, R_2 = \{v\}, I_2 = \mathcal{O}$. Finally, if $R(B) \Leftrightarrow \{v\} = \{x_1, x_2\}$, then set $R_i = \{v, x_i\}, I_i = B_{x_i}, i = 1, 2$. It is now easy to observe that $((R_i, H_i, I_i), i = 1, 2)$ opens B.

We assume now that $\forall v \in R(B)$ $EX(B, v)$ contains at most one v-wing. We define a function $\phi : R(B) \to \{0, 1, 2\}$ where for any $v \in R(B)$, $\phi(v) = 0$ if B_v is a v-hair, $\phi(v) = 1$ if B_v consists of two u-hairs (i.e. is isomorphic to graph $\mathcal{B}_1$ depicted in Figure 1), and $\phi(v) = 2$ in any other case (i.e. contains some graph in $\mathcal{C}$ as a v-minor – $\mathcal{C}$ is depicted in Figure 1). We call the value of $\phi(v)$ *strength* of v. Notice now the following.

(e–i) Any vertex in $R(B)$ belongs in some simplicial face, otherwise, $A_1^+ \preceq B$.
(e–ii) $|R(B)| \leq 4$ otherwise, $5A_1 \preceq G$.
(e–iii) $R(B)$ contains at most two vertices with strength 2, otherwise, $\mathcal{L}_2^3 \sqsubseteq G$.
(e–iv) If $|R(B)| = 3$ then $\exists v, u \in R(B)$ $v \sim u$, otherwise $3A_1^+ \preceq G$.
(e–v) If $|R(B)| = 4$ then $\exists v, u, w, x \in R(B)$ $v \sim u$ and $w \sim x$, otherwise $3A_1^+ \preceq G$.
Suppose now that $R(G) \leq 2$. If $R(B) = \emptyset$ then set $R_i = \emptyset, H_i = I_i = \mathcal{O}, i = 1, 2$. If $R(B) = \{v\}$ then set $R_1 = \{v\}, H_1 = B_v, R_2 = \emptyset, H_2 = I_1 = I_2 = \mathcal{O}$. If $R(B) = \{v_1, v_2\}$ then set $R_i = \{v_i\}, H_i = B_{v_i}, I_i = \emptyset, i = 1, 2$. Since in any of the above cases $((R_i, H_i, I_i), i = 1, 2)$ opens B, we may assume that $R(B) \geq 3$ (and thus $\mathcal{S}(B) \geq 1$). From the smoothness of G we have that $\mathcal{S}(G) \leq 2$. The proof proceeds with the following case analysis.

We examine first the case where $\mathcal{S}(B) = 1$ (Notice that, in this case B is a cycle).

a. $R(B) = \{v, u, w\}$. From **(e–iv)** we assume that $v \sim u$.
a.I. For at least one, say u, of v, u, $\phi(u) = 0$. Then, $\mathcal{P} = ((\{v, u\}, B_v, B_u), (\{w\}, B_w, \mathcal{O})$.
a.II. $\phi(v) = \phi(u) = 1$. Then, for one of v, u, say v, $v \sim w$ (otherwise $A_1 2B_1 \preceq G$). We set $\mathcal{P} = ((\{u, v\}, B_u, I_1), (\{w, v\}, B_w, I_2))$ where I_1 is the one of the two v-hairs of B_v and I_2 is the other.
a.III. $\phi(v) = 2, \phi(u) = 1$. If $\phi(w) = 0$, then for one of v, u, say v, $v \sim w$ (otherwise $\{A_1 2B_1, A_1 B_1 C_3\} \sqsubseteq G$) and we set $\mathcal{P} = ((\{u\}, B_u, \mathcal{O}), (\{v, w\}, B_v, B_w))$. If $\phi(w) \geq 1$, then $w \sim u$, otherwise, either $\mathcal{L}_2^8 \sqsubseteq G$ or $\mathcal{L}_2^9 \sqsubseteq G$. We set $\mathcal{P} = ((\{w, u\}, B_w, I_1), (\{v, u\}, B_v, I_2))$ where I_1 is one of the two u-hairs of B_u and I_2 the other.
a.IV. $\phi(v) = \phi(u) = 2$. Then, $\phi(w) = 0$ and for one of v, u, say v, $w \sim v$, otherwise, either $\mathcal{L}_2^8 \sqsubseteq G$ or $\mathcal{L}_2^3 \sqsubseteq G$ or $\mathcal{L}_2^9 \sqsubseteq G$. We set $\mathcal{P} = ((\{u\}, B_u, \mathcal{O}), (\{v, w\}, B_v, B_w))$.
b. $R(B) = \{v, u, w, x\}$. From **(e–v)** we assume that $v \sim u$ and $w \sim x$. Let N be the set of neighbours of v and u in B.

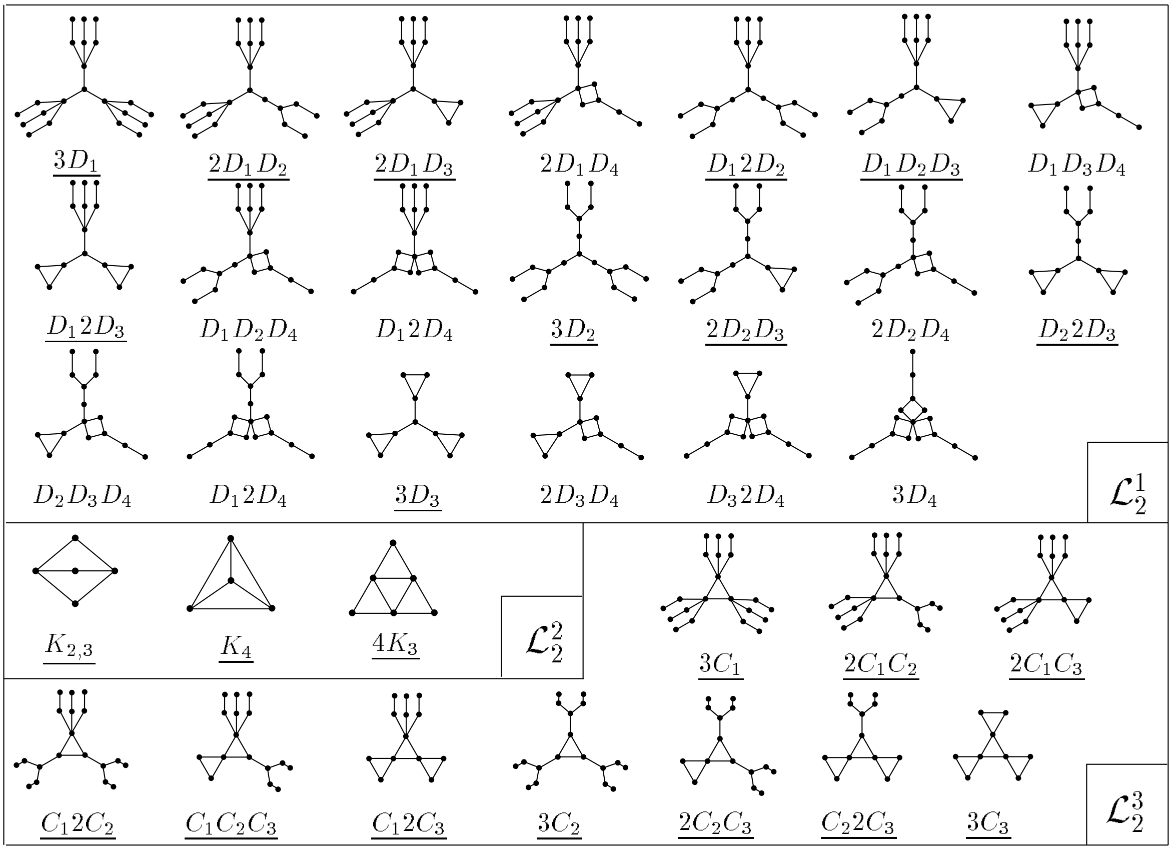

Figure 6: The sets $\mathcal{L}_2^1, \mathcal{L}_2^2$, and $\mathcal{L}_2^3$.

b.I. $|N \cap \{w, x\}| \leq 1$. Notice that for at least one of v, u, say u, $\phi(u) = 0$ (otherwise, $\mathcal{L}_2^8 \sqsubseteq G$) and for at least one of w, x, say x, $\phi(x) = 0$ (otherwise, $\mathcal{L}_2^8 \sqsubseteq G$). We set $\mathcal{P} = ((\{v, u\}, B_v, B_u), (\{w, x\}, B_w, B_x))$.
b.II. $N \cap \{w, x\} = \{w, x\}$. If at least two vertices, say u, w, in $\{v, u, x, w\}$, have strength 0, then we set $\mathcal{P} = (\{v, u\}, B_v, B_u), (\{v, w\}, B_x, B_w)$. If at least 3 vertices in $\{v, u, x, w\}$ have strength ≥ 1, then it is easy to see that, either $\mathcal{L}_2^6 \sqsubseteq G$ or $\mathcal{L}_2^3 \sqsubseteq G$.

It remains to examine the case where $\mathcal{S}(B) = 2$. We set $\{F_1, F_2\} = \mathcal{S}(B)$. Clearly, $|F_1 \cap F_2| \leq 2$. We call a vertex *crucial* if it is a critical vertex of both simplicial faces of B (i.e. belongs in $F_1 \cap F_2$). We notice first the following fact.

(e–vi) Any face can contain at most 2 non crucial vertices v, u that belong in $R(B)$ (otherwise, $2A_1 \preceq G$). Moreover $v \sim u$ (otherwise, $2A_1 \preceq G$) and for one of them, say u, $\phi(u) = 0$ (otherwise, $\mathcal{L}_2^5 \sqsubseteq G$).

Case 1. $F_1 \cap F_2 = \emptyset$. Notice that all vertices in $R(B)$ are non crucial.

1.a. $R(B) = \{v, u, w\}$. From **(e–vi)** we may assume that $v, u \in F_1, w \in F_2, v \sim u, \phi(u) = 0$ and we set $\mathcal{P} = ((\{v, u\}, B_v, B_u), (\{w\}, B_w, \mathcal{O}))$.
1.b. $R(B) = \{v_1, u_1, v_2, u_2\}$. From **(e–vi)** we may assume that $v_i, u_i \in F_i, v_i \sim u_i, \phi(u_i) = 0, i = 1, 2$ and we set $\mathcal{P} = ((\{v_i, u_i\}, B_{v_i}, B_{u_i}), i = 1, 2)$.

Case 2. $F_1 \cap F_2 = \{v\}$. We assume that $v \in R(B)$ (i.e. v is crucial) as, otherwise, Case 2 is reduced to Case 1.
2.i. $\phi(v) = 0$.
2.i.a. $R(B) = \{v, u, w\}$. There are two cases.
2.i.a.I. u, w belong to the same simplicial face, say F_1. As u, w are non crucial, from **(e–vi)**, we can assume that $u \sim w$ and $\phi(w) = 0$. We set $\mathcal{P} = ((\{v\}, B_v, \mathcal{O}), (\{u, w\}, B_u, B_w))$.
2.i.a.II. u, w belong to different simplicial faces. Then, for at least one of u, w, say u, we have that $u \sim v$ (otherwise $\{3A_1, 3A_1^+\} \sqsubseteq G$) and we set $\mathcal{P} = ((\{u, v\}, B_u, B_v), (\{w\}, B_w, \mathcal{O}))$.
2.i.b. $R(B) = \{v, u, w, x\}$. As all the vertices u, w, x are non crucial, from **(e–vi)**, we can assume that $u, w \in F_1$, $x \in F_2$, $u \sim w$, and $\phi(u) = 0$. Notice also that $x \sim v$, otherwise, $\{3A_1^+, 3A_1\} \sqsubseteq G$. We set $\mathcal{P} = ((\{w, u\}, B_w, B_u), (\{x, v\}, B_x, B_v))$.
2.ii. $\phi(v) = 1$.
2.ii.a. $R(B) = \{v, u, w\}$. There are two cases.
2.ii.a.I. u, w belong to the same simplicial face. Similar to Case 2.i.a.I.
2.ii.a.II. u, w belong to different simplicial faces. Then, for at least one of u, w, say u, we have that $u \sim v$ (otherwise $\{3A_1, 3A_1^+\} \sqsubseteq G$). If $\phi(u) = 0$, we set $\mathcal{P} = ((\{v, u\}, B_v, B_u), (\{w\}, B_w, \mathcal{O}))$. If $\phi(u) > 0$, then $w \sim v$ (otherwise $\{A_1 2B_1 A_1 B_1 C_3\} \sqsubseteq G$). We can now set $\mathcal{P} = ((\{u, v\}, B_u, I_1)$, $(\{w, v\}, B_w, I_2))$ where I_1 is one of the two v-hairs of B_v and I_2 is the other.
2.ii.b. $R(B) = \{v, u, w, x\}$. Using **(e–vi)**, we can assume that $w, x \in F_1$, $y \in F_2$, $w \sim x$, and $\phi(x) = 0$. We also notice that $y \sim v$ (otherwise, $3A_1^+$) and $\phi(y) = 0$ (otherwise, $\{A_1 B_1 C_3, A_1 2B_1\} \sqsubseteq G$). We set $\mathcal{P} = ((\{w, x\}, B_w, B_x), (\{v, y\}, B_v, B_y))$.
2.iii. $\phi(v) = 2$.
2.iii.a. $R(B) = \{v, u, w\}$. There are two cases.
2.iii.a.I. u, w belong to the same simplicial face. Similar to Case 2.i.a.I.
2.iii.a.II. u, w belong to different simplicial faces. Then, for at least one of u, w, say u, we have that $u \sim v$ (otherwise, $\{3A_1, 3A_1^+\} \sqsubseteq G$). If $\phi(u) = 0$, set $\mathcal{P} = ((\{v, u\}, B_v, B_u), (\{w\}, B_w, \mathcal{O}))$. If $\phi(u) > 0$, then $w \sim v$ (otherwise, $\mathcal{L}_2^8 \sqsubseteq G$) and $\phi(w) = 0$ (otherwise $\mathcal{L}_2^9 \sqsubseteq G$). We set $\mathcal{P} = ((\{u\}, B_u, \mathcal{O}), (\{v, w\}, B_v, B_w))$.
2.iii.b. $R(B) = \{v, u, w, x\}$. From **(e–vi)** we can assume that $u, w \in F_1, u \sim w$, and $\phi(w) = 0$. We also notice that $x \sim v$ (otherwise, $3A_1^+ \preceq G$) and that $\phi(x) = 0$ (otherwise, $\mathcal{L}_2^8 \sqsubseteq G$). We set $\mathcal{P} = ((\{u, w\}, B_u, B_w), (\{v, x\}, B_v, B_x))$.

Case 3. $F_1 \cap F_2 = \{v, u\}$. Notice that if $|R(B) \cap F_1 \cap F_2| = 0$ then Case 3 is reduced to Case 1. Also, if $|R(B) \cap F_1 \cap F_2| = 1$ then Case 3 is mainly the same with Case 2 (the only difference is that set $\{3A_1, 3A_1^+\}$ could be replaced by $\{3A_1^+\}$). We now assume that $|R(B) \cap F_1 \cap F_2| = 2$.

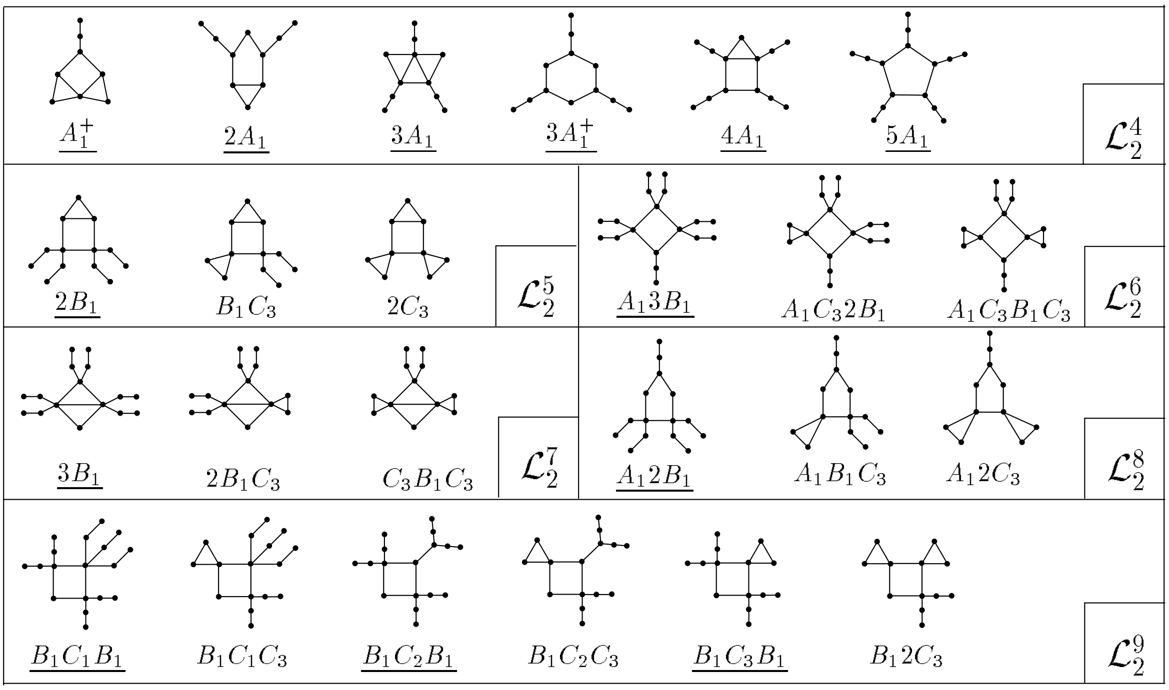

Figure 7: The sets $\mathcal{L}_2^4, \mathcal{L}_2^5, \mathcal{L}_2^6, \mathcal{L}_2^7, \mathcal{L}_2^8$, and $\mathcal{L}_2^9$.

3.i. $\phi(v) = 0$ and $\phi(u) = 0$.

3.i.a. $R(B) = \{v, u, w\}$. From **(e–iv)**, we can assume that $w \sim v$ and set $\mathcal{P} = ((\{w, v\}, B_w, B_v), (\{u\}, B_u, \mathcal{O}))$.

3.i.b. $R(B) = \{v, u, w, x\}$. From **(e–v)**, we can assume that $w \sim v$ and $x \sim u$. Also, w and x must belong into different simplicial faces, otherwise $4A_1 \preceq G$. We can now set $\mathcal{P} = ((\{v, w\}, B_w, B_v), (\{u, x\}, B_x, B_u))$.

3.ii. $\phi(v) = 0$ and $\phi(u) > 0$.

3.ii.a. $R(B) = \{v, u, w\}$. If $\phi(w) > 0$, then $w \sim v$ (otherwise, $\{3A_1^+\} \cup \mathcal{L}_2^8 \preceq G$) and we set $\mathcal{P} = ((\{w, v\}, B_w, B_v), (\{u\}, B_u, \mathcal{O}))$. If $\phi(w) = 0$ then either $w \sim v$ or $w \sim u$, otherwise, $3A_1^+ \preceq G$. We may assume that $w \sim v$ and set $\mathcal{P} = ((\{v, w\}, B_v, B_w), (\{u\}, B_u, \mathcal{O}))$.

3.ii.b. $R(B) = \{v, u, w, x\}$. We distinguish the following cases.

3.ii.b.I. $\phi(w) = \phi(x) = 0$. From **(e–v)** we may assume that $w \sim v$, and $x \sim u$. Also w, x must belong to different simplicial faces (otherwise $4A_1 \preceq G$). We can now set $\mathcal{P} = ((\{v, w\}, B_v, B_w), (\{u, x\}, B_u, B_x))$.

3.ii.b.II. $\phi(w) = 0$ and $\phi(x) > 0$. In this case, $v \sim x$, $u \sim w$, and w and x must belong into different simplicial faces (in any other case $\{4A_1, 3A_1^+\} \sqsubseteq G$). We can now set $\mathcal{P} = ((\{x, v\}, B_x, B_v), (\{u, w\}, B_u, B_w))$.

3.ii.b.III. If both $\phi(w), \phi(x) \geq 1$ then $\{4A_1\} \cup \mathcal{L}_2^6 \sqsubseteq G$.

3.iii. $\phi(v) > 0$ and $\phi(u) > 0$.
3.iii.a. $R(B) = \{v, u, w\}$. Clearly $\phi(w) = 0$, otherwise $\mathcal{L}_2^7 \sqsubseteq G$. Also, either $v \sim w$ or $w \sim u$ (otherwise $3A_1^+ \preceq G$). We can assume that $v \sim w$ and set $\mathcal{P} = ((\{v, w\}, B_v, B_w), (\{u\}, B_u, \mathcal{O}))$.
3.iii.b. $R(B) = \{v, u, w, x\}$. From **(e–v)** we may assume that $w \sim v$, $\phi(w) = 0$, $x \sim u$, and $\phi(x) = 0$. Also w and x belong to different simplicial faces (otherwise, $4A_1 \preceq G$. We set $\mathcal{P} = ((\{v, w\}, B_v, B_w), (\{u, x\}, B_u, B_x))$. □

Following the case analysis of the above the proof one can easily modify algorithm LW2 so that, in case linear-width$(G) > 2$, it outputs the forbidden minor G contains. Notice that LW2(G) is based only on the structural characterisation of $\mathcal{G}$(linear-width, 2) given in Lemma 15 and does not involve at all the case analysis of the proof of Lemma 16 above. The following theorem gives a complete structural characterisation of the class of graphs with linear-width ≤ 2.

Theorem 2 *$\mathcal{L}_2$ is the obstruction set for the class of graphs with* linear-width ≤ 2 *i.e.* $\mathcal{L}_2 = \text{ob}(\mathcal{G}[\text{linear-width}, 2])$.

Proof. By Lemma 3, it is enough to prove that any graph with linear-width more than 2 contains at least one of the graphs in $\mathcal{L}_2$ as a minor. Suppose now that linear-width$(G) > 2$. It is easy to see that, if G is not smooth, then $\mathcal{L}_2^1 \cup \mathcal{L}_2^2 \sqsubseteq G$ (use Lemma 8). If now G is smooth then it cannot be open otherwise, from Lemma 15, linear-width ≤ 2. Therefore, it contains a bolbe that is not open. From Lemma 16 we have that $\mathcal{L}_2^3 \cup \cdots \cup \mathcal{L}_2^9 \sqsubseteq G$. □

4.2 The case of multiple edges

During the presentation of the proof and the algorithm of sections 3 and 4, we assumed that the graphs cannot contain loops or multiple edges. We have to mention that it is possible to obtain the same results without this restriction. The only essential difference is that that graphs $\mathcal{C}_3$ and $\mathcal{D}_3$ should be replaced with graphs $\mathcal{C}_3'$ and $\mathcal{D}_3'$ depicted in Figure 8. This would result to a different obstruction set. This obstruction set can be constructed from $\mathcal{L}_2$ if for any graph $G \in \mathcal{L}_2$ we apply the following two operations as long as it is possible:

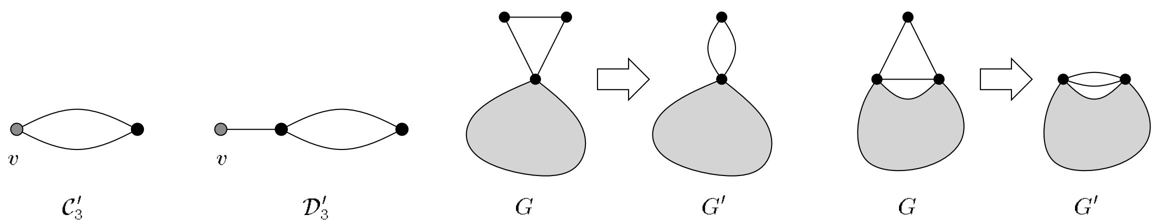

Figure 8: The graphs $\mathcal{C}_3'$ and $\mathcal{D}_3'$ and the transformations for the case of multiple edges.

a. If G has a biconnected component that is a triangle, replace this triangle by $\mathcal{C}_3'$

b. If G has a simplicial face F containing only one simplicial vertex that is not an articulation vertex, remove this vertex (along with the two edges containing it) and introduce a new edge connecting the critical vertices of F.

We avoid examining the case of multiple edges in detail as it would be a tedious resumption of what we have already presented.

5 Linear-width and search parameters

In this section we give the definitions of edge searching, node searching, and mixed searching and we prove that the problem of computing the corresponding graph parameters can be reduced to the one of linear-width.

5.1 Mixed search and other variants

A *mixed searching game* is defined in terms of a graph representing a system of tunnels where an agile and omniscient fugitive with unbounded speed is hidden (alternatively, we can formulate the same problem considering that the tunnels are contaminated by some poisonous gas). The object of the game is to *clear* all edges, using one or more *searchers*. An edge of the graph is cleared if one of the following cases occur.

A: *both of its endpoints are occupied by a searcher,*

B: *a searcher slides along it*, i.e., a searcher is moved from one endpoint of the edge to the other endpoint.

A search is a sequence containing some of the following moves. $a(v)$: placing a new searcher on v, $b(v)$: deleting a searcher from v, $c(v, u)$: sliding a searcher on v along $\{v, u\}$ and placing it on u.

The object of a mixed search is to clear all edges using a search. The search number of a search is the maximum number of searchers on the graph during any move. The mixed search number, $\mathrm{ms}(G)$, of a graph G is the minimum search number over all the possible searches of it. A move causes *recontamination* of an edge if it causes the appearance of a path from an uncleared edge to this edge not containing any searchers on its vertices or its edges. (Recontaminated edges must be cleared again.) A search without recontamination is called *monotone*.

The *node (edge) search number*, $\mathrm{ns}(G)$ ($\mathrm{es}(G)$) is defined similarly to the mixed search number with the difference that an edge can be cleared only if **A** (**B**) happens.

The following results were proved by Bienstock and Seymour in [3] (see also [35]).

Theorem 3 *For any graph G the following hold:*
i. *If* $\mathrm{ms}(G) \leq k$ *then there exist a monotone mixed search in G using $\leq k$ searchers.*
ii. $\text{linear-width}(G) \leq \mathrm{ms}(G)$.
iii. *If G does not contain pendant vertices, then* $\text{linear-width}(G) = \mathrm{ms}(G)$.

iv. *If G^e is the graph occurring from G after subdividing each of it edges, then* $\mathrm{es}(G) = \mathrm{ms}(G^e)$.
vi. *If G^n is the graph occurring if we replace every edge in G with two edges in parallel, then* $\mathrm{ns}(G) = \mathrm{ms}(G^n)$.

We mention that the mixed search number is equivalent with the parameter of proper-pathwidth defined by Takahashi, Ueno, and Kajitani in [33, 35]. It is also known that the node search number is equal to the pathwidth, the interval thickness, and the vertex separation number (see [21, 22, 25, 19, 12]).

5.2 The relation between linear-width and mixed search

A pendant vertex is called *fully* pendant when it is adjacent with an almost pendant vertex. Any edge containing a fully pendant vertex is called *fully pendant*. Clearly, a pendant edge is fully pendant iff it is not small. Let G be a graph. We denote as ΛG the graph obtained from G if for any pendant vertex we introduce one new vertex and an edge connecting them (formally, if $P = \{p_1, \ldots, p_r\}$, is the set of pendant vertices of G, then $\Lambda G = (V(G) \cup \{p'_1, \ldots, p'_r\}, E(G) \cup \{\{p_1, p'_1\}, \ldots, \{p_r, p'_r\}\}$ where $\{p'_1, \ldots, p'_r\} \cap V(G) = \emptyset$). We denote as $\Lambda^{-1}G$ the graph obtained if we remove all the pendant vertices. Observe that if a graph does not contain small edges, the graphs $\Lambda\Lambda^{-1}G$, $\Lambda^{-1}\Lambda G$, and G are isomorphic. For an example of operations Λ and Λ^{-1} see Figure 9.

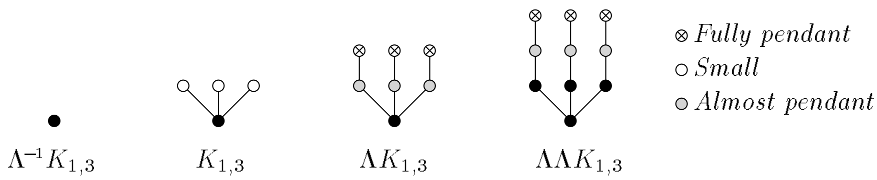

Figure 9: The graphs $\Lambda^{-1}K_{1,3}$, $K_{1,3}$, $\Lambda K_{1,3}$ and $\Lambda\Lambda K_{1,3}$.

Clearly, any pendant edge of G becomes almost pendant in ΛG and any almost pendant edge in G becomes pendant in $\Lambda^{-1}G$.

We will need the following easy result (for the proofs see e.g. in [35]).

Lemma 17 *For any graph G the following hold.*
i. *If v is a fully pendant vertex v in G then* $\mathrm{ms}(G) = \mathrm{ms}(G \Leftrightarrow v)$.
ii. *If e is a long edge in G then* $\mathrm{ms}(G) = \mathrm{ms}(G \dot{-} e)$.

Theorem 4 *Let G be a graph. Then,* $\mathrm{ms}(G) = \text{linear-width}(\Lambda G)$ *and* $\text{linear-width}(G) = \mathrm{ms}(\Lambda^{-1}G)$.

Proof. We prove first that the first equality implies the second. We denote as G^s the graph obtained from G if we remove all the small vertices. Applying inductively Lemma 4.i on the number of small vertices of G we can prove that linear-width(G) = linear-width(G^s). Since G^s has not small edges G^s is isomorphic with $\Lambda\Lambda^{-1}G^s$ and finally, we have that linear-width(G^s) = linear-width($\Lambda\Lambda^{-1}G^s$). The first equality now implies that linear-width($\Lambda\Lambda^{-1}G^s$) = ms($\Lambda^{-1}G^s$). Observe now that $\Lambda^{-1}G^s$ is isomorphic to $\Lambda^{-1}G$ and therefore linear-width(G) = ms($\Lambda^{-1}G$) as required. What remains now is to prove the first equality.

Let $E = \{g_1, \ldots, g_n\}$ be the set of pendant edges of G and let $g_i = \{x_i, y_i\}, 1 \leq i \leq n$ where $d_G(y_i) = 1$, $1 \leq i \leq n$. Let also $E' = (g_1, g_1', \ldots, g_n, g_n') \subseteq E(\Lambda G)$ where $g_i = \{y_i, y_i'\}$ is an almost pendant edge of ΛG and $g_i' = \{y_i, y_i'\}$ is a fully pendant edge of ΛG for $i = 1, \ldots, n$. Let $l = (e_1, \ldots, e_r)$ be an edge ordering of ΛG with linear-width $= k$.

For any $i = 1, \ldots, n$ we apply the following operation: if $g_i = e_j$ and $g_i' = e_h$ in l we replace l by the sequence $(e_1, \ldots, e_{\min\{j,h\}-1}, g_i, g_i', e_{\min\{j,h\}+1}, \ldots, e_{\max\{j,h\}-1}, e_{\max\{j,h\}+1}, \ldots, e_r)$ (i.e., we remove g_i and g_i' and place first g_i and then g_i' in the position where one of them appears for the first time). Notice that the above reordering operation does not increase the linear width of the ordering. Therefore, we end up with an edge ordering l^* that has linear-width $\leq k$ and where every edge $\{y_i, y_i'\}$ appears always immediately after $\{x_i, y_i\}$.

Let l' be an ordering of $E(G)$, obtained from l^* by replacing every occurrence of successively an edge $\{x_i, y_i\}$ and the edge $\{y_i, y_i'\}$ by an occurrence of the edge $\{x_i, y_i\}$. We claim that there exist a monotone mixed-search of G using $\leq k$ searchers, such that the edges of G are cleared in the order of l'. We prove the claim with induction. Suppose that we there exist a sequence of search moves that clears the the first i edges of l' (and not any other) in the order that they appear in l'. We denote this edge set as E_i. Let also e_j be the ith edge of l'. If e_{j+1} is missing from l' then set $h = j + 1$ otherwise set $h = j$. Notice that no vertex in $\{y_1, \ldots, y_n, y_1', \ldots, y_n'\}$ belongs to $\delta_{l^*}(e_h)$. Moreover, all the vertices of $\delta_{l^*}(e_h)$ are occupied by a searcher in G as they are incident both to a clear edge (an edge in E_i) and to a contaminated edge (an edge in $E(G) \Leftrightarrow E_i$). Clearly, if we now remove all the other searchers, no recontamination will occurs. In case $|e_{h+1} \cup \delta_{l^*}(e_h)| \leq k$, we place new searchers on the endpoints of e_{h+1} and clear it. We can now assume that $|e_{h+1} \cup \delta_{l^*}(e_h)| > k$ and, as $\delta_{l^*}(e_h) \leq k$, not both of the endpoints of e_{h+1} are guarded (i.e. they are occupied by some searcher).

Let v be an unguarded endpoint of e_{h+1} in G. As v is unguarded either it is incident only with contaminated edges or only with clear edges in G. The second case is impossible as e_{h+1} is contaminated. If v is incident only to e_{h+1} in G, this means that v has degree 1 in G and therefore has degree 2 in ΛG. It is now clear that, in any case, v has degree ≥ 2 in ΛG and therefore $v \in \delta_{l^*}(e_{h+1})$. If both of the endpoints of e_{h+1} are unguarded then we have $e_i \in \delta_{l^*}(e_{h+1}) \Leftrightarrow \delta_{l^*}(e_h)$ which means that $|\delta_{l^*}(e_h)| \leq k \Leftrightarrow 2$, a contradiction to the assumption that $|e_{h+1} \cup \delta_{l^*}(e_h)| > k$. Therefore, one of the endpoints of e_{h+1}, say u is guarded and one, say v is unguarded.

Suppose now that u is incident to a contaminated edge different from e_{h+1}. This means that $\delta_{l^*}(e_{h+1}) = \delta_{l^*}(e_h) \cup \{v\}$ and as, $\delta_{l^*}(e_h) \cup \{v\} = \delta_{l^*}(e_h) \cup \{v, u\}$, we have that $k \geq |\delta_{l^*}(e_{h+1})| = |\delta_{l^*}(e_h) \cup \{v, u\}| \geq k + 1$ a contradiction. Therefore, u is incident only with clear edges in G

and thus, we can clear e_{h+1} by sliding the searcher guarding u along e_{h+1} to v (i.e. applying $c(u, v)$), without causing any recontamination. This completes the proof of the fact that $\text{ms}(G) \leq \text{linear-width}(\Lambda G)$.

Suppose now that there exist a mixed search for G that uses k searchers. From Theorem 3.ii we have that $\text{linear-width}(\Lambda G) \leq \text{ms}(\Lambda G)$. Therefore, it is enough to prove that $\text{ms}(\Lambda G) = \text{ms}(G)$. This fact follows from Lemma 17.i by induction on the number of fully pendant vertices of ΛG. □

Notice that Theorem 4 is an extension of Theorem 3.iii. We summarise the consequences of Theorem 4 into the following theorem.

Theorem 5 i. *The problem of computing* linear-width *is NP-complete.*
ii. *There exist an algorithm that given a tree T computes* $\text{linear-width}(T)$ *in $O(|V(T)|)$ time.*
iii. *One can construct an algorithm that, given a graph G, checks whether G has mixed (edge) search number at most 2 and, if so, outputs a mixed (edge) search strategy that uses the minimum number of searchers.*

Proof. i. The NP-completeness of linear-width follows directly from Theorem 4 and the fact that computing $\text{ms}(G)$ is an NP-hard problem [35].
ii. The existence of an algorihtm computing linear-width of trees is a consequence of Theorem 4 and the fact that there exist an algorithm that given a tree T computes $\text{ms}(T)$ in $O(|V(T)|)$ time (see [35]).
iii. The result is trivial in case $\text{ms}(G) \leq 1$ ($\text{es}(G) \leq 1$). Using now Theorems 4 and 3.iv we have that, in order to check whether $\text{ms}(G) \leq 2$ ($\text{es}(G) \leq 2$), it is enough to apply $\mathsf{LW2}(\Lambda G)$ ($\mathsf{LW2}(\Lambda G^e)$). If this is the case, $\mathsf{LW2}(\Lambda G)$ ($\mathsf{LW2}(\Lambda G^e)$) will output an edge ordering of ΛG (ΛG^e). It is not hard to see that, following the machinery of the proof of Theorem 4, this edge ordering can be transformed to a mixed (edge) search in linear time. □

5.3 The acyclic minor minimal graphs with linear-width $> k$

It is easy to verify that $\text{ob}(\mathcal{G}[\text{ms}, 1]) = \{K_3, K_{1,3}\}$. $\text{ob}(\mathcal{G}[\text{ms}, 2])$ has been determined by Takahashi, Ueno, and Kajitani in [34] and consists of 36 graphs.

The following lemma is an immediate corollary of Lemmata 4 and 17.

Lemma 18 i. *No graph in* $\text{ob}(\mathcal{G}[\text{ms}, k])$ *contains edges that are fully pendant or long.*
ii. *No graph in* $\text{ob}(\mathcal{G}[\text{linear-width}, k])$ *contains edges that are small or long.*

Lemma 19 *Let $k \geq 1$. Then, $G \in \text{ob}(\mathcal{G}[\text{ms}, k]) \Rightarrow \Lambda G \in \text{ob}(\mathcal{G}[\text{linear-width}, k])$.*

Proof. Suppose that G is a graph in $\text{ob}(\mathcal{G}[\text{ms}, k])$. Clearly $\text{ms}(G) > k$ and, as G is minor minimal, $\forall_{H \prec G}\ \text{ms}(H) \leq k$. From Theorem 4, $\text{linear-width}(\Lambda G) > k$. Suppose, towards a contradiction, that $\Lambda G \notin \mathrm{ob}(\mathcal{G}[\text{linear-width}, k])$ and thus, there exist some edge $e \in E(\Lambda G)$ such that either linear-width$(\Lambda G \Leftrightarrow e) > k$ or linear-width$(\Lambda G \dot{-} e) > k$. In any case, we will find a proper minor of G that has linear-width $> k$. Notice that ΛG does not contain small or long edges. We examine three cases:

Case 1. $e = \{u, u'\}$ is a fully pendant edge of ΛG. We observe that one of u, u', say u' is a fully pendant vertex in G. Notice also that $\Lambda G \Leftrightarrow \{u, u'\}$ and $\Lambda G \dot{-} \{u, u'\}$ are isomorphic. For reasons of simplicity, we will denote both of them as $\Lambda G \Leftrightarrow \{u, u'\}$. Let v be the, unique, neighbour of u in $\Lambda G \Leftrightarrow \{u, u'\}$. As ΛG does not contain long edges, $\{v, u\}$ is the unique small edge in $\Lambda G \Leftrightarrow \{u, u'\}$ and from Lemma 4.i we have that linear-width$((\Lambda G \Leftrightarrow \{u, u'\}) \Leftrightarrow \{v, u\}) > k$. Notice that, From Theorem 4, $\mathrm{ms}(\Lambda^{-1}((\Lambda G \Leftrightarrow \{u, u'\}) \Leftrightarrow \{v, u\})) > k$. As $\Lambda^{-1}((\Lambda G \Leftrightarrow \{u, u'\}) \Leftrightarrow \{v, u\}) \prec G$, we have a contradiction.

Case 2. $e = \{v, u\}$ is an almost pendant edge of ΛG. The case where $\mathrm{lw}(G \dot{-} \{v, u\}) > k$ is similar to Case 1. We assume that $\mathrm{lw}(\Lambda G \Leftrightarrow \{v, u\}) > k$. Let u be the almost pendant vertex of $\{v, u\}$ and u' be the fully pendant vertex of ΛG that is adjacent to u. Clearly, vertices u', u induce one, of the two connected components of $\Lambda G \Leftrightarrow \{v, u\}$. We denote the other as H. Clearly, linear-width$(H) > k$ and, from Theorem 4, we have that $\mathrm{ms}(\Lambda^{-1} H) > k$. As $\Lambda^{-1} H \prec G$, we have a contradiction.

Case 3. $e = \{x, y\}$ is not a fully or an almost pendant edge of ΛG. Moreover, if linear-width$(\Lambda G \Leftrightarrow e) > k$ or linear-width$(\Lambda G \dot{-} e) > k$ then, from Theorem 4 we have that either $\mathrm{ms}(\Lambda G \Leftrightarrow e) > k$ or $\mathrm{ms}(\Lambda G \dot{-} e) > k$. Using now the fact that e is not a fully or an almost pendant edge of ΛG, one can easily see that $\Lambda^{-1}(\Lambda G \Leftrightarrow e)$ $(\Lambda^{-1}(\Lambda G \dot{-} e))$ is isomorphic to a minor of $G \Leftrightarrow e$ $(G \dot{-} e)$, a contradiction. □

According to Lemma 19, Λ is an injection from $\mathrm{ob}(\mathcal{G}[\mathrm{ms}, k])$ to $\mathrm{ob}(\mathcal{G}[\text{linear-width}, k])$. Using this fact, it is easy to determine $\mathrm{ob}(\mathcal{G}[\mathrm{ms}, k])$ if we know $\mathrm{ob}(\mathcal{G}[\text{linear-width}, k])$. Indeed, if we apply Λ^{-1} on all the graphs in $\mathrm{ob}(\mathcal{G}[\text{linear-width}, k])$, we will obtain a set $\mathcal{M}$ of graphs containing $\mathrm{ob}(\mathcal{G}[\mathrm{ms}, k])$ as a subset. We can now obtain $\mathrm{ob}(\mathcal{G}[\mathrm{ms}, k])$ from $\mathcal{M}$ by discarding all the graphs having proper minors in $\mathcal{M}$ (i.e. we keep only the minor minimal elements).

Using the above methodology, we can directly verify the result of [34]. One can easily see that $\mathrm{ob}(\mathcal{G}[\mathrm{ms}, k])$ can be obtained if we apply Λ^{-1} on the 36 underlined graphs depicted in Figures 6 and 7.

We now denote as $\mathrm{aob}(\mathcal{G}[\text{linear-width}, k])$ $(\mathrm{aob}(\mathcal{G}[\mathrm{ms}, k]))$ the set consisting of the acyclic graphs in $\mathrm{ob}(\mathcal{G}[\text{linear-width}, k])$ $(\mathrm{ob}(\mathcal{G}[\mathrm{ms}, k]))$. Let $(G_i, i = 1, 2, 3)$ be a triple of v_i-graphs and let v be a vertex such that $v \notin V(G_1) \cup V(G_2) \cup V(G_3)$. We call the graph $G_1 \cup G_2 \cup G_3 \cup (\{v, v_1, v_2, v_3\}, \{\{v, v_1\}, \{v, v_2\}, \{v, v_3\}\})$ star-composition of $G_i, i = 1, 2, 3$.

The following has been proved in [33].

Theorem 6 *Let $k \geq 2$. A tree T is in $\mathrm{aob}(\mathcal{G}[\mathrm{ms}, k])$ iff T is a star decomposition of three graphs in $\mathrm{aob}(\mathcal{G}[\mathrm{ms}, k \Leftrightarrow 1])$.*

Notice that, as $\text{aob}(\mathcal{G}[\text{ms}, 1]) = \{K_{1,3}\}$, Theorem 6 explicitly defines $\mathcal{P}_k^a$ for any $k \geq 1$. The following theorem shows that $\text{aob}(\mathcal{G}[\text{ms}, k \Leftrightarrow 1])$ and $\text{aob}(\mathcal{G}[\text{linear-width}, k])$ are not very different.

Theorem 7 *Let T be tree and $k \geq 1$. Then, $T \in \text{aob}(\mathcal{G}[\text{ms}, k]) \Leftrightarrow \Lambda T \in \text{aob}(\mathcal{G}[\text{linear-width}, k])$.*

Proof. The "$\Rightarrow$" direction follows from Lemma 19. Let now $T \in \text{aob}(\mathcal{G}[\text{linear-width}, k])$. From Theorem 4 we have that $\text{ms}(\Lambda^{-1}T) > k$. Let $e \in E(\Lambda^{-1}T)$. We will prove that $\text{ms}(\Lambda^{-1}T \Leftrightarrow e) \leq k$ and $\text{ms}(\Lambda^{-1}T \dot{-} e) \leq k$. Suppose in contrary, that for some edge $e \in E(\Lambda^{-1}T)$ either $\text{ms}(\Lambda^{-1}T \Leftrightarrow e) > k$ or $\text{ms}(\Lambda^{-1}T \dot{-} e) > k$. We examine first the case where $e = \{v, u\}$ is a pendant edge of $\Lambda^{-1}T$. W.l.o.g we assume that $d_{\Lambda^{-1}T}(u) = 1$. From Lemma 18.ii we have that, in T, u is an almost pendant vertex adjacent to some pendant vertex u'. Moreover, $\{v, u\}$ is a small edge in $\Lambda^{-1}(T)$ and the removal or the contraction of it does not result to the appearance of a new pendant edge. One can now see that $\Lambda(\Lambda^{-1}T \Leftrightarrow \{u, v\})$ is isomorphic to $(T \Leftrightarrow \{v, u\}) \Leftrightarrow \{u, u'\} \prec T$, which is a contradiction, as, from Theorem 4, $\text{linear-width}(\Lambda(\Lambda^{-1}T \Leftrightarrow \{u, v\})) > k$ or $\text{linear-width}(\Lambda(\Lambda^{-1}T \dot{-} \{u, v\})) > k$. Suppose now that $e = \{v, u\}$ is not a pendant edge of $\Lambda^{-1}T$. We examine two cases.

Case 1. $\text{ms}(\Lambda^{-1}T \Leftrightarrow e) > k$. We notice first that $\Lambda^{-1}T \Leftrightarrow e$ consists of two connected components T_1, T_2. W.l.o.g. we may assume that $\text{ms}(T_1) > k$. From Theorem 4, we have that $\text{linear-width}(\Lambda T_1) > k$. It is now easy to see that that $\Lambda T_1 \prec T$, a contradiction.

Case 2. $\text{ms}(\Lambda^{-1}T \dot{-} e) > k$. We first claim that after the contraction of e no new pendant edge appears. Notice that the only case where the contraction of a non pendant edge e results to the appearance of a new pendant edge is the case where exist a vertex adjacent with both of the endpoints of e. As T is a tree, this case must be excluded and the claim holds. Using this claim, it is easy to see that $\Lambda(\Lambda^{-1}T \dot{-} e)$ is isomorphic with $T \dot{-} e \prec T$. We now have a contradiction as, from Theorem 4, $\text{linear-width}(\Lambda(\Lambda^{-1}T \dot{-} e)) > k$. □

From Theorem 7, we have that Λ is an bijection from $\text{aob}(\mathcal{G}[\text{ms}, k])$ to $\text{aob}(\mathcal{G}[\text{linear-width}, k])$. Using this fact and Theorem 6, we can determine all the acyclic graphs in $\text{ob}(\mathcal{G}[\text{linear-width}, k])$ for any $k \geq 1$. We can easily conclude to the following result.

Lemma 20 *If $T \in \text{aob}(\mathcal{G}[\text{linear-width}, k])$, then $|V(T)| = (3^{k+1} + 2 \cdot 3^k \Leftrightarrow 1)/2$. Moreover, $|\text{aob}(\mathcal{G}[\text{linear-width}, k]) \geq (k!)^2$.*

We mention that, according to [33], the cardinality of $\text{aob}(\mathcal{G}[\text{linear-width}, k])$, for $k = 1, 2, 3$, and 4 is 1, 4, 1,330, and 2,875,919,312,080 respectively.

6 Conclusions

Lemma 20 suggests that a complete structural characterisation of $\mathcal{G}[\text{linear-width}, k]$ is not easy to be found for $k > 2$. However, we believe that a more general version of the distinction between marginal and non marginal bolbes, that we followed in this paper, can be applied in the more general cases. Using Lemma 20, one can see that no graph in ob[linear-width, k] has more than $f(k) = (3^{k+1} + 2 \cdot 3^k \Leftrightarrow 1)/2$. Therefore, if we enumerate all the graphs having at most $f(k)$ vertices, and detect those that are minor minimal graphs that do not belong in $\mathcal{G}$[linear-width, k], we will end up with ob($\mathcal{G}$[linear-width, k]). Clearly, such a procedure is rather impractical because of the immense number of graphs that have to be checked. Clearly, one can make it more efficient by applying further restrictions on the graphs enumerated (for example, graphs in ob($\mathcal{G}$[linear-width, 3]) cannot have small or long vertices). These restrictions can be based on some partial characterisation of $\mathcal{G}$[linear-width, k] (for the case where $k = 2$, such a partial characterisation could be the one of smoothness that we defined in Subsection 2.4).

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