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SYMPLECTIC SYMMETRY IN THE NUCLEAR SHELL MODEL

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Abstract: The nature of the general two-particle interaction which is compatible with symplectic symmetry in the jj coupling shell model is investigated. The essential result is that, to within an additive constant and an additive multiple of T^2, the interaction should have the form of a sum of scalar products of single-particle tensors which have odd rank in the single-particle j space An example of an interaction satisfying these conditions is a central interaction with $\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2$ exchange nature The condition for good symplectic symmetry is expressed also as a set of linear constraints on the two-particle energies, again as constraints on the particle-hole energies and finally in terms of the relationship between the particle-particle and particle-hole spectra When we deal with identical particles only, the conditions for good symplectic symmetry (or seniority) are greatly relaxed and in particular are satisfied for a short-range (δ-function) interaction, as shown earlier by Racah and Talmi.

1. Introduction

The theory of seniority in many-particle spectroscopy which was invented by Racah [1]) in 1943 was later [2]) reformulated by him in terms of group theory. Racah's principal interest in these papers was in atomic spectroscopy in an LS representation (l^n) and the corresponding group was the orthogonal group in $(2l+1)$ dimensions. The later paper did point out however that the symplectic group would naturally enter in a jj representation.

For the more complicated nuclear case a classification of j^n according to symplectic symmetry was made by Flowers [3]), and in later papers Edmonds and Flowers [4]) calculated fractional parentage coefficients and applied their results to calculate spectra for a variety of cases. It was remarked by these authors that symplectic symmetry appeared to be rather "good" for the interactions and cases which they considered but little formal discussion was given about the conditions under which an interaction between nucleons is compatible with symplectic symmetry. Racah and Talmi [5]) investigated, for identical particles only, the closely related question of the pairing property of nuclear interactions and some of the results of the present paper represent an extension of theirs

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The purpose of this paper is to investigate rather generally the conditions for good symplectic symmetry. The principal outcome will be that symplectic symmetry will be good if the two-particle interaction (apart from an interaction behaving like an additive arbitrary constant and arbitrary multiple of T^2) behaves in the combined orbital+spin space of one particle as a sum of tensors of odd rank only. This result, which is not surprising in view of some theorems of Racah [1]) concerning odd-rank tensors and the results of Racah and Talmi [5]), will be translated into a set of conditions on the two-particle energies and also into a relationship between the particle-particle and particle-hole spectra for equivalent particles. Our results for the most part will apply to configurations containing both neutrons and protons but we shall also pay special attention to the case of identical particles only. It will turn out that there are not enough experimental data to decide to what extent symplectic symmetry is actually good but we shall raise a few questions about spectra, the answers to which would be helpful.

The procedure which we shall use is quite elementary (and results quite simple) but we give them in some detail because they will later be found useful in dealing with inequivalent-nucleon configurations and for other purposes too. Before proceeding to this we remind the reader how the symplectic group enters into spectroscopy [3]).

A function in the vector space j^n may be taken to be a sum of products of n single-particle functions ψ_j^m each associated with a different particle number. The exclusion principle requirement (if the particles are identical, or if we use an isobaric-spin formalism for neutrons and protons) is that the function must be antisymmetric in all n particles. For mixed protons and neutrons, when we do not use a T-formalism (and we shall not), both one- and two-columned Young diagrams defining the representation of S_n are allowed, a single such diagram defining states of definite T. It is more useful not to consider the permutation group S_n explicitly but rather the group defined by unitary transformations in the $(2j+1)$-dimensional space of the single-particle functions ψ_j^m, the unitary group U_{2j+1}. It is well known that, for the product functions which we have, a function belonging to an irreducible representation of U_{2j+1} at the same time belongs to an irreducible representation of S_n and indeed the two representations may be labelled by the same Young diagram. A subset of the unitary transformations is of special interest, namely that subset which leaves invariant the $J = 0$ state of two particles; this subset is in fact the subgroup Sp_{2j+1} of the unitary group and since, in turn, the rotation group R_3 is a subgroup of the symplectic group we can classify states simultaneously by T (group U_{2j+1}), by symplectic symmetry† σ (group Sp_{2j+1}) and by J (group R_3).

† σ defines two parameters which we may take as the seniority (s) and reduced isobaric spin (t) of Edmonds and Flowers For identical particles (or states with $T = \frac{1}{2}n$) there is only one significant parameter, the seniority

In what follows we shall make very little use of group theory beyond indentifying certain operators as infinitesimal operators of the pertinent groups.

2. The Two-Body Interaction in Terms of Unit Tensors

Two-particle wave functions $\Psi_J(j^2)$ are not of course adequate to give a representation of an interaction operator H_{12} because in most cases H_{12} will connect states of different configurations. If we are resolved to consider only the states j^2 it is convenient to replace the physical interaction H_{12} by an interaction which gives the same results inside j^2 but which does not connect with other configurations This is simply done by introducing Racah's unit tensor operators $u^r(i)$ [6]) which we may define by †

$$u_q^r(i)\psi_j^m = [j]^{-\frac{1}{2}} C^{jrj}_{m,q} \psi_j^{m+q}; \tag{1}$$

u^r operates on the angular and spin part of the function (not on the radial part) and, as shown by eq. (1), does not change the j value. The double-barred matrix element of Racah [6]) is then simply $\langle j||u^r||j\rangle = 1$ or more completely $\langle j'||u^r||j''\rangle = \delta_{jj'}\delta_{jj''}$ (where the u^r are understood to affect only functions with angular momentum j and might be more completely labelled as ${}^{(j)}u^r$; for a quite complete notation we should specify also the orbital angular momentum). For $r = 0$, 1 we have simply

$$u^0 = [j]^{-\frac{1}{2}}, \quad u_q^1 = \{j(j+1)[j]\}^{-\frac{1}{2}} \mathbf{J}_q,$$

where $\mathbf{J}$ is the angular momentum operator. The higher-rank tensors could be constructed from $\mathbf{J}$; for example $u^2 \propto (\mathbf{J}\times\mathbf{J})^2$, $u^3 \propto ((\mathbf{J}\times\mathbf{J})^2\times\mathbf{J})^3$ and so on; but this representation is not very useful. From eq. (1) it also follows that $(u^r(i)\cdot u^r(i)) = [j]^{-1}$.

If now we are dealing with a j^2 configuration any two-particle interaction which conserves the total angular momentum can be written as [5])

$$H_{12} \equiv \sum_{r=0}^{2j} [r]\alpha_r (u^r(1)\cdot u^r(2)) = \sum_r H_{12}^r \tag{2}$$

provided we make an appropriate choice of the $(2j+1)$ interaction constants α_r. Questions which immediately come to mind now are:

(1) How are the α_r to be determined if we are given the 2-body energies $E_J^{(2)}$?

(2) What are the α_r for the interactions of common interest?

(3) What are the conditions on α_r and consequently on $E_J^{(2)}$ which will

† Notation q is the "m" value and r the tensorial rank, C is a Clebsch-Gordan coefficient and $[j] = (2j+1)$. We shall also write $[ab \quad] = (2a+1)(2b+1)$. , $2\varepsilon_a = 1+(-1)^a$, $2\varepsilon_{ab} = 1+(-1)^{a+b+}$ The scalar product of two tensor operators will be written as $(A^r \; B^r)$ and the general tensor product as $(A^r\times B^s)^t$, note then that $(A^r\cdot B^r) = (-1)^r[r]^{\frac{1}{2}}(A^r\times B^r)^0$

ensure that the interaction is compatible with symplectic symmetry in the j^n case?

The first of these is trivial. Simply evaluating the matrix element of H_{12} in the j^2 states gives immediately

$$E_J^{(2)} = (-1)^{J+1} \sum_r [r] W(jjjj; rJ)\alpha_r, \tag{3}$$

and solving this backwards, using the unitarity condition on the Racah coefficients, we have for the constants

$$\alpha_r = \sum_J (-1)^{J+1} [J] W(jjjj; rJ) E_J^{(2)}. \tag{4}$$

We see that $(-1)^J E_J^{(2)}$ and the interaction constants α_r are in a sense simply Racah transforms † of each other.

The second question is somewhat outside the range of the present paper and it will be adequate for our present purposes to consider two special cases of a central interaction. Consider first a Wigner interaction $H_{12} = J(r_{12})$. We could apply eq. (4) to the calculated energies or could proceed directly as follows:

$$J(r_{12}) = \sum_k J_k(r_1, r_2)(C^k(1) \cdot C^k(2))$$

where $C_q^k = \{4\pi/[k]\}^{\frac{1}{2}} Y_k^q$, and, on evaluating the radial integral, $J_k(r_1, r_2)$ becomes $\varepsilon_k F^k$, the usual Slater integral Since

$$\langle j||C^k||j\rangle = (-1)^{k+\frac{1}{2}-j}[j][k]^{-\frac{1}{2}} C^{jjk}_{\frac{1}{2},-\frac{1}{2}} \equiv (-1)^{k+\frac{1}{2}-j} D_{jjk},$$

we have

$$J(r_{12}) \equiv \sum_k \varepsilon_k (D_{jjk})^2 F^k (u^k(1) \cdot u^k(2))$$

and thus for the Wigner interaction

$$\alpha_r = \varepsilon_r [r]^{-1} (D_{jjr})^2 F^r \tag{5}$$

We observe the important fact that a Wigner interaction involves only tensors of *even rank* A minor point is that for interactions $V(r)$ which are everywhere positive (or negative) the α_r are all positive (or negative).

A little more effort will show that an interaction $\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 J(r_{12})$ has tensors of *ocd rank only* [5]) and once again, if $J(r_{12}) \geqq 0$ for all r_{12}, then the α_r are all positive. We find in fact for the $\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2$ interaction

$$\alpha_r = 6[j]^2 \sum_k \varepsilon_k (D_{llk})^2 \begin{Bmatrix} l & \frac{1}{2} & j \\ l & \frac{1}{2} & j \\ k & 1 & r \end{Bmatrix}^2 F^k \tag{6}$$

† Many of the results of this paper could be derived most simply by using an explicit formalism for such transforms

where $D_{llk} = [l][k]^{-\frac{1}{2}}C^{llk}_{0,0}$, the redundant ε_k is written for emphasis, and the vanishing of α_r for even r follows from the vanishing of the $9j$-symbol when $k+1+r$ is odd.

The other two central interactions, which we might write † as $J(r_{12})M_{12}$ and $\sigma_1 \cdot \sigma_2 J(r_{12})M_{12}$, where M_{12} is the space exchange operator, have more complicated forms, involving tensors of both even and odd orders, and it is not worth our while to write them here. The same is true of the two-body spin-orbit forces (vector forces) and the tensor forces whose forms have considerable complexity.

Before proceeding to the many-particle case we record the combination and commutation rules for the single-particle tensors. These results follow directly and easily from the defining eq. (1). We have

$$u_p^r(i)u_q^s(i) = (-1)^{r+s}\sum_t [t]^{\frac{1}{2}} Y_{rst} C^{rst}_{p,q} u^t_{p+q}(i), \tag{7}$$

$$\left(u^r(i)\times u^s(i)_m{}^t\right) = (-1)^{r+s}[t]^{\frac{1}{2}} Y_{rst} u_m{}^t(i), \tag{8}$$

$$[U_p{}^r, U_q{}^s] = 2(-1)^{r+s}\sum_t [t]^{\frac{1}{2}} \varepsilon_{rst1} C^{rst}_{p,q} Y_{rst} U^t_{p+q}, \tag{9}$$

$$(U^r\times U^s)_m{}^t-(-1)^{r+s-t}(U^s\times U^r)_m{}^t = 2(-1)^{r+s}[t]^{\frac{1}{2}}\varepsilon_{rst1} Y_{rst} U_m{}^t, \tag{10}$$

$$\left(u^r(i)\times u^s(i)\right)_m{}^t-(u^s(i)\times u^r(i)_m{}^t = 0 \tag{11}$$

Here we have introduced the symbol

$$Y_{rst} = (-1)^t W(jsjr; jt), \tag{12}$$

which is completely symmetrical in all three indices. Eqs. (9), (10) are valid for the single-particle tensors $u^r(i)$ but, as we shall immediately see, for a more general case too, and we have indicated this by writing U^r instead of $u^r(i)$. Eqs. (8), (10) are entirely equivalent respectively to (7) and (9) and follow from them by introducing the definition of the tensor product of two tensor operators. The reader should note carefully that when a commutator is written in tensor-product form as in eq. (10) the phase factor enters automatically into the second term on the left-hand side. It comes from the fact that, in the Condon and Shortley [7]) phase convention which we use, the function describing the vector-coupling of angular momenta j_1, j_2 to a resultant J undergoes a phase change $(-1)^{j_1+j_2-J}$ when the ordering of the angular momenta is changed Observe too that the commutator of two tensors U^r, U^s vanishes for arbitrary s if $r = 0, 1$ (as is *a priori* obvious); moreover the commutator of two odd-rank tensors is itself of odd rank Note also that the expression occurring in eq. (11) is a commutator or anticommutator according as $r+s+t$ is even or odd

† Remember that we do not have an explicit isobaric spin formalism and must therefore not introduce $\tau_1 \cdot \tau_2$

3. Many-Particle Tensors

We introduce the symmetrical tensors which will be adequate to deal with any charge-independent † interaction

$$U^r = \sum_{i=1}^{n} u^r(i). \tag{13}$$

A bilinear form involving these tensors consists of both a one- (or zero) and a two-particle operator and the combination rule is not all identical with the $n = 1$ case. One result, important to us, is

$$A^t_{rs} = -A^t_{sr} \equiv (U^r \times U^s)^t - (U^s \times U^r)^t = 2\varepsilon_{rst1} \sum_{i,\, j(i \neq j)} (u^r(i) \times u^s(j))^t. \tag{14}$$

From this one can see that for given t the quantities A^t_{rs} for different pairs of (r, s) values are linearly independent (consider each pair of particles (i, j) separately). The commutation rules on the other hand are the same for all n; i.e. eqs. (9) and (10) apply equally well to the single-particle and many-particle tensors.

Since a Hamiltonian involving only two-particle interactions is essentially equivalent to a bilinear form in the U's it is worthwhile investigating further commutators, namely certain of those involving triple and quadruple combinations of U's. For the first of these commutators which we evaluate in the usual way we find ††

$$[(U^r \cdot U^r), U^s] = 2(-1)^s [s]^{-\frac{1}{2}} \sum_t [t] \varepsilon_{rst1} Y_{rst} A^s_{rt}. \tag{15}$$

A significant question is now: *What is the general bilinear form* $\sum_r [r]\beta_r (U^r \cdot U^r)$ *which commutes with* U^s. Remembering that

$$(U^r \cdot U^r) = (-1)^r [r]^{\frac{1}{2}} (U^r \times U^r)^0,$$

we have for the commutator

$$[\sum_r [r]\beta_r (U^r \cdot U^r), U^s] = 2[s]^{-\frac{1}{2}} \sum_{t,\, r} (-1)^{t+r} [tr] \beta_r \varepsilon_{rst1} Y_{rst} A^s_{rt}. \tag{16}$$

This commutator will vanish for arbitrary s if β_r vanishes except for $r = 0$, 1 (and of course for all β_r if $s = 0, 1$). These results are trivial. It will vanish too if $\beta_r = 1$ since it is then a sum over two indices of a function antisymmetric in the indices. We shall see that this demonstrates only the charge-independence of the interaction operators constructed with symmetric tensors. For arbitrary s the linear independence of the A^t_{rs} ensures that there is no other solution. The

† For a charge-dependent interaction we would need as well the tensors which transform by a two-rowed Young diagram but it would in most cases then be simpler to introduce explicitly the isobaric spin

†† Here and elsewhere (e g. in eq (13)) we do not write the "m" value which of course is the same for each term in the equation

non-trivial case important for consideration of symplectic symmetry is the case $s =$ odd; then noting that $(-1)^t \equiv (-1)^r$ so that $(-1)^r = \varepsilon_{rt}$ we see that $\beta_r = (-1)^r$ is also a solution (or of course $\beta_r = \varepsilon_r$ or ε_{r1}). In summary then we have

$$[\sum_r [r]\beta_r(U^r \cdot U^r), U^s] = 0 \quad \text{if} \begin{cases} s = 0,1; \\ \beta_r = 1; \\ s = \text{odd and} \\ \beta_r = (-1)^r, \quad \text{or} \quad \varepsilon_r, \quad \text{or} \quad \varepsilon_{r1}. \end{cases} \tag{17}$$

Next we evaluate the commutator for two scalar products This follows easily from the preceding one and we find

$$[(U^r \cdot U^r),\ (U^s \cdot U^s)] = -2 \sum_t [t]\varepsilon_{rst1} Y_{rst}\{(rts)^0 - (trs)^0 + (srt)^0 - (str)^0\} \tag{18}$$

where

$$(rts)^0 = ((U^r \times U^t)^s \times U^s)^0. \tag{19}$$

The quantity appearing inside {} in eq. (18) is antisymmetric in all three indices (r, s, t). Temporarily labelling it as A_{rst} we have, since the quantity on the left of eq. (18) is an (rs) commutator,

$$A_{rst} = -A_{srt}. \tag{20}$$

But interchanging (r, t) we have

$$A_{tsr} = (trs)^0 - (rts)^0 + (str)^0 - (srt)^0 = -A_{rst} \tag{21}$$

and these two equations demonstrate the complete antisymmetry. Besides this the distinct components A_{rst} are linearly independent; we see this as before by considering the three-body part of A_{rst} with definite particle numbers i, j, k.

We can now answer the following question: *What are the conditions on α_r, β_s so that*

$$[\tfrac{1}{2}\sum_r [r]\alpha_r(U^r \cdot U^r), \sum_s [s]\beta_s(U^s \cdot U^s)] = 0?$$

We have for this commutator the value

$$-\sum_{r,s,t} [rst]\alpha_r\beta_s\varepsilon_{rst1} Y_{rst} A_{rst}$$

and, remembering that Y_{rst} is symmetric while A_{rst} is antisymmetric and linearly independent, we have as the condition that $\alpha_r\beta_s$ must vanish when antisymmetrized with respect to the three indices r, s, t provided that $r+s+t =$ odd and (r, s, t) form a triangle. We may write this as

$$\Delta_{rst}\varepsilon_{rst1}\{\alpha_r(\beta_s - \beta_t) + \alpha_s(\beta_t - \beta_r) + \alpha_t(\beta_r - \beta_s)\} = 0 \tag{22}$$

where $\Delta_{rst} = 1$ if (rst) form a triangle, zero otherwise.

At this stage we should make some contact with the theory of symplectic symmetry. Directly from their definition (1) the single-particle unit tensors are seen to form a set of infinitesimal operators of the unitary group U_{2j+1} and since the many-particle tensors have the same commutation rules so also have they. The odd-rank tensors then, according to eq. (9), are the operators of a subgroup which is of course the symplectic group. The bilinear form $\sum[r](U^r \cdot U^r)$ which commutes with every U^s is the Casimir operator of the unitary group; it is related to the T^2 operator (in the configuration j^n) as follows [4]):

$$\sum[r](U^r \cdot U^r) \equiv -2T^2+\tfrac{1}{2}n(4j+6-n). \tag{23}$$

The bilinear form $2\sum_{r\ \mathrm{odd}}[r](U^r \cdot U^r)$ which commutes with odd rank tensors is the Casimir operator for the symplectic group; its eigenvalues, a general expression for which is given by Racah [8]), serve to label the various irreducible representations of this group or, as we shall say, serve to specify the symplectic symmetry. These facts are of course entirely well known; they are discussed quite throughly by Edmonds and Flowers [4]) and also by Racah [2, 8]).

Returning now to the commutator of two bilinear forms we may regard the α-form as defining an interaction † and the β-form as specifying an operator whose eigenvalues label the wave-functions. For the trivial cases $\beta_s = \delta_{s0}, \delta_{s1}, 1$ we derive from eq. (23) that the commutator vanishes for any α-form (the third of these demonstrates, via eq. (23), the charge independence to which we have earlier referred). The form $\beta_s = \varepsilon_{s1}$ will define the symplectic Casimir operator and the α-forms commuting with it are the interactions which allow symplectic symmetry. Since $\varepsilon_{s1}-\varepsilon_{r1} \equiv (-1)^s\varepsilon_r$, eq. (23) becomes (no summation being understood)

$$\Delta_{rst}\varepsilon_{rst1}\{(-1)^s\varepsilon_r\alpha_r+(-1)^t\varepsilon_s\alpha_s+(-1)^r\varepsilon_t\alpha_t\} = 0, \tag{24}$$

which is automatically satisfied unless two of the indices are even and the other odd. In the latter case, we must also exclude a zero value for the even index (since if one index is zero the other two must be equal). The equation now becomes

$$\alpha_r-\alpha_s = 0 \quad \text{for} \quad r, s \equiv \text{even}, \quad \neq 0 \tag{25}$$

The general bilinear operator then which preserves symplectic symmetry has the form

$$\alpha_0(U^0 \cdot U^0)+\alpha \sum_r [r](U^r \cdot U^r)+\tfrac{1}{2} \sum_{r\ \mathrm{odd}} [r]\alpha_r(U^r \cdot U^r) \tag{26}$$

which we might describe as *a constant+multiple of T^2+an operator with odd-*

† If the two-particle interaction is that of eq. (2) the interaction operator for n particles is

$$\tfrac{1}{2}\Sigma[r]\alpha_r(U^r \cdot U^r)-\tfrac{1}{2}n[j]^{-1}\Sigma[r]\alpha_r$$

It is because of this that we have included the factor $\frac{1}{2}$ in defining the α-form The last term (constant for given n) is of no interest to us at present and we shall usually ignore it *rank scalar products only* The first two parts are trivial; the symplectic classification operates inside the classification by isobaric spin (the symplectic group is a subgroup of the other). It is the third term which is of real interest We shall in fact often speak of the general symplectic-symmetry-preserving interaction as simply one which involves odd-rank scalars only, thereby ignoring the first two parts We note incidentally the fact that no Wigner interaction can preserve symplectic symmetry while every $\sigma_1 \cdot \sigma_2$ interaction does so. In section 5 we shall comment on the relationship between these results and those of Racah and Talmi [5]) for identical particles.

4. Conditions on the Two-Body Energies

The conditions (25) impose $(j-\frac{3}{2})$ linear conditions on the two-body energies $E_J^{(2)}$. Thus symplectic symmetry is always good for $j = \frac{3}{2}$, we have one condition for $j = \frac{5}{2}$, and so on. From eq. (4) we see that the conditions are

$$\sum_J (-1)^J [J] W(jjjj; rJ) E_J^{(2)} = -\alpha \quad \text{for} \quad r \quad \text{even}, \quad \neq 0. \tag{27}$$

We may write these in homogeneous form as

$$\Delta^{(i)}(E_J) = 0 \quad (i = 1, \quad .., (j-\tfrac{3}{2})), \tag{28}$$

where the $\Delta^{(i)}$ are linearly independent † linear combination of $E_J^{(2)}$ *which do not involve* E_0 (formally because $W(jjjj : r0)$ is independent of r for even r, more fundamentally because the $J = 0$ two-particle state belongs by itself to an irreducible representation of Sp_{2j+1}). Moreover if we write $\Delta^{(i)} = \sum_J \Delta_J{}^i E_J{}^{(2)}$ we have

$$\sum_J \Delta_J{}^i = \sum_{J\ \text{even}} \Delta_J{}^i = \sum_{J\ \text{odd}} \Delta_J{}^i = \sum_J \Delta_J{}^i J(J+1) = 0,$$

since $E_J^{(2)}$ = constant, and $E_J^{(2)}$ = constant for J even, $\neq 0$, and $E_J^{(2)}$ = a multiple of $J(J+1)$, all define interactions which preserve symplectic symmetry Besides this we see at once that in a symplectic symmetry representation any off-diagonal matrix element of the Hamiltonian, expressed in terms of the two-body energies, must be a linear combination †† of the $\Delta^{(i)}$ (for it must vanish when $\Delta^{(i)} \to 0$). As a special case we see that for $j = \frac{5}{2}$ there is a fixed ratio, independent of the interaction, between any two off-diagonal matrix elements

Eqs. (27) do not at all imply that if symplectic symmetry is good the states

† Linear independence follows from the fact that the equations

$$\sum_{r\ \text{even} \neq 0} \beta_r W(jjjj, rJ) = 0 \text{ for all } J$$

implies $\beta_r = 0$ (multiply by $[J]W(jjjj, r'J)$ and sum over J)

†† This fact could be very helpful in evaluating fraction parentage coefficients

with a definite T and σ are degenerate. An inverse statement is however true; separate degeneracy of the two-particle even $J(\neq 0)$ states and the odd J states would imply good symplectic symmetry, the interaction in this case being not only an operator which commutes with the symplectic Casimir operator but indeed a multiple of it.

To apply eq. (27) as they stand we should know all the two-particle (or two hole) energies both for $T = 0$ and $T = 1$. If we knew all but v of them, where $v < j-\frac{3}{2}$, we could eliminate the unknown energies and produce $j-\frac{3}{2}-v$ constraints involving the known energies which would be necessary but not sufficient for good symplectic symmetry. The only case where enough data are available and where there is at least some chance that jj coupling is respectable is Al^{26} ($j = \frac{5}{2}$) We have here only one $\Delta^{(1)} = \Delta$ say and (arbitrarily choosing the normalization)

$$\Delta = E_2 - E_4 + \tfrac{1}{45}(36E_1 - 91E_3 + 55E_5). \tag{29}$$

A reasonable identification of the $(d_{\frac{5}{2}})^2$ levels is [9] (the energies are in MeV)

$$E_5 = 0, \quad E_0 = 0\ 23, \quad E_3 = 0\ 42, \quad E_2 \approx 2, \quad E_4 \gtrsim 5, \quad E_1 = ?$$

The lower limit for E_4 comes from the Mg^{26} spectrum; the uncertainty in E_2 reflects the fact that there are several 2^+ levels, without doubt belonging to rather mixed configurations; it appears that the first $J = 1$ level may be at ≈ 1.0 MeV. These numbers give

$$\Delta \approx -3.8 + 0.8E_1 \xrightarrow{E_1 \to 1\cdot 0} -3 \text{ MeV}. \tag{30}$$

Increasing E_4 enlarges Δ; to make $\Delta \approx 0$ we would need $E_1 \approx 5$ MeV. A 3 MeV Δ-value is not small compared with the energy spacings in the $d_{\frac{5}{2}}$ shell (our normalization of Δ has been reasonably chosen to make the energy coefficients around unity) and we can expect that it would not be compatible with good symplectic symmetry.

Consider the first two $J = \frac{5}{2}$ levels of Al^{25} at 0 and 1 81 MeV (there is probably no other below 4.5 MeV). There are two possible $(d_{\frac{5}{2}})^{-3}$ states which, in Flowers' classification [3]), have $(s, t) = (1, \frac{1}{2}), (3, \frac{1}{2})$. In this representation one finds (e.g. by using the c.f.p. tables) that the off-diagonal matrix element is $\frac{3}{28}\sqrt{10}\Delta$ and using the two-body energies above this has the value -1 MeV. The Hamiltonian matrix, in the symplectic symmetry representation, would have the form

$$H \approx \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \text{MeV}. \tag{31}$$

(since this has eigenvalues 0, 2.0 MeV). The eigenfunctions then would be eqaul mixtures of the two symplectic symmetries.

We tend to conclude from this that symplectic symmetry is quite probably not good in the $d_{\frac{5}{2}}$ shell. The conclusion is not certain, however; the same values of the two-body energies used above do not produce the degeneracy of eq. (31) between the two zero-order states. We have in fact for the zero-order energy difference

$$\langle H\rangle_{s=1}-\langle H\rangle_{s=3} = \tfrac{1}{1260}\{1470E_0-1065E_2-405E_4-162E_1-553E_3+715E_5\}, \tag{32}$$

and, for any reasonable value of E_1, this has a value ≈ -3.4 MeV instead of the zero value demanded by eq (31). This large discrepancy indicates that jj coupling is a bad approximation or at the very least that the obvious identification of the $(d_{\frac{5}{2}})^{-2}$ levels is badly in error. So far as symplectic symmetry is concerned, there is still the possibility that the residual shell model interaction is of the odd-rank tensor type, the above disagreement then resulting from the departure from jj coupling.

For the higher shells, essentially nothing is known [10]) about the level structure of Sc^{42} or $Co^{54}(f_{\frac{7}{2}})^2$ while for the $g_{\frac{9}{2}}$ shell Nb^{82} is an altogether improbable nucleus. We turn therefore to a different method for testing the symplectic-symmetry hypothesis

An essential feature which distinguishes between an even-rank and an odd-rank symmetrical tensor is the behaviour of its matrix elements when we proceed from particles to holes Specifically we have Racah's theorem [6])

$$\langle (j^n)_\alpha||U^r||(j^n)_\beta\rangle = (-)^{r+1}\langle (j^{N-n})_{\alpha_c}||U^r||(j^{N-n})_{\beta_c}\rangle \tag{33}$$

where N is the number of particles in a closed shell and α_c, β_c describe the states which are complementary to α, β. In the present instance, since we do not use an isobaric-spin formalism, $N = (2j+1)$ = numoer of particles in a proton or neutron shell. To apply eq. (33) we must then consider neutrons and protons separately but we encounter here the difficulty that charge-independence will not in general permit the angular momenta of the separate groups to be specified (nor indeed can the separate seniorities usually be fixed even when the interaction preserves symplectic symmetry). These difficulties can be solved in may ways but at present we avoid them entirely by considering only the spectra for two particles, two holes, and a hole and a particle where, as is trivially obvious, the difficulties do not arise.

It will do us no good, except as a check, to compare spectra for two particles and two holes since these are identical (formally because the interaction is bilinear in the symmetrical tensors and the sign change of (33) is lost). For particle-hole system on the other hand it should be clear from eq. (33) (the reader can easily supply a formal proof or else see below) that for an odd-rank-tensor interaction the particle-particle and particle-hole spectra are identical while for an even-rank tensor interaction one is inverted with respect to the other Formally, if we write

$$E_J = E_J^{(\text{even})} + E_J^{(\text{odd})} \tag{34}$$

where $E_J^{(\text{even})}$ is that part of the energy due to the even-rank interaction, we have

$$E_J^{(\text{even})}(j^{-1}, j) = -E_J^{(\text{even})}(j^2), \quad E_J^{(\text{odd})}(j^{-1}, j) = +E_J^{(\text{odd})}(j^2) \tag{35}$$

Writing the energies in terms of the interaction constants by means of eq. (3) we have

$$E_J(j^{-1}, j) = (-1)^J \sum_r (-1)^r [r] W(jjjj; rJ)\alpha_r \tag{36}$$

and inversely

$$\alpha_r = (-1)^r \sum_J (-1)^J [J] W(jjjj, rJ) E_J(j^{-1}, j). \tag{37}$$

We observe parenthetically that writing the interaction constants of eq. (36) in terms of $E_J(j^2)$ by eq. (4) will give Pandya's result [11]) connecting the two energy-level systems, viz.

$$E_J(j^{-1}, j) = -\sum_{J'} [J'] W(jjjj; JJ') E_{J'}(j^2), \tag{38}$$

and conversely we could have derived eqs. (34)—(37) by starting with this.

We finally combine eqs. (4) and (37) to obtain

$$\alpha_r = -\sum_J (-1)^J [J] W(jjjj; rJ)\{E_J(j^2) - (-1)^r E_J(j^{-1}, j)\} \tag{39}$$

from which we see that identity of the spectra implies that the interaction constants vanish for even r. Thus the identity of the two spectra is both a necessary and a sufficient condition for the corresponding interaction to be of odd rank only.

Our general operator which preserves symplectic symmetry admits also an additive constant which we can obviously ignore and an additive multiple of T^2. Since the T values are not the same for the two systems [$J = 0$, even ($\neq 0$), odd, has $T = 1, 1, 0$, respectively, for j^2 and $T = \frac{1}{2}N, \frac{1}{2}N-1, \frac{1}{2}N-1$ for $j^{-1}j$] the effect of this operator will be to shift the three groups with respect to each other †. If it should turn out that the separate spectra for even $J(\neq 0)$ and for odd J are identical in the two systems the two separations between the groups are fixed by one parameter (the strength of the T^2 term). We can now state the symplectic symmetry condition in the following way: for the inter-

† Formally, by using

$$\sum_x [x] W(aaaa, xy) = 1, \quad \sum_x (-1)^x [x] W(aaaa, xy) = (-1)^{2a}[a]\delta_{y0}$$

and eq (39), we can see that the shift of the even spectra with respect to the odd has the expected effect on α_r Remember too (see the footnote between eqs (23) and (24)) that our odd-rank interaction contains a constant term proportional to n, which we ignore action to produce good symplectic symmetry it is necessary and sufficient that the particle-particle and particle-hole spectra for the odd-J states be identical and similarly with the even $J(\neq 0)$ spectra.

Eq. (37) could be used in the same way as eq. (4) if the appropriate spectrum was available. For the $f_{\frac{7}{2}}$ and $g_{\frac{9}{2}}$ shells the nuclei would be Sc^{48} and Nb^{90} but little or no information is available. If we knew the spectra for both a particle-particle and the particle-hole system we would first check their relationship by eq. (38). If (38) were not satisfied we would be sure that either the coupling scheme was incorrect or that the effective interaction was not the same for both nuclei. If the equation were satisfied we would apply eq. (4) to the particle-particle system and settle things that way.

In a different manner however we can now use a small amount of data to investigate symplectic symmetry. For example consider Sc^{48}. The ground state [10]) has $J = 6$ or 7. The $J = 6$ value would require (if symplectic symmetry were good) that the lowest even J ($\neq 0$) state of Sc^{42} be also $J = 6$ but this of course would be the analogue of the first excited state of Ca^{42}. It is in fact well known that the $J = 6$ state of Ca^{42} lies several MeV above the first-excited $J = 2$ state and so we can conclude immediately that a $J = 6$ value for Sc^{48} would imply a large breakdown of symplectic symmetry. A $J = 7$ value on the other hand would merely imply that, if symplectic symmetry is good, the lowest odd-J state of Sc^{42} would also be $J = 7$. For the $g_{\frac{9}{2}}$ shell the particle-hole nucleus is Nb^{90}. Nb^{82} for $(g_{\frac{9}{2}})^2$ is not accessible but the even J spectra could be compared with that of Mo^{92}. It is clear that in general we expect the lowest J even ($\neq 0$) state for a particle and equivalent hole to be $J = 2$

It is obvious that if some levels of the particle-particle system and some others of the particle-hole system were known the information could be combined by the proper use of eqs. (4) and (37). Also one is not restricted to comparison of simple spectra But we give no results for more general cases because there seem once again to be few pertinent data.

Of considerably greater interest and with far more experimental possibilities is the comparison, for example, of the spectrum of a particle and an inequivalent group with that of the particle and the group complement. There are a great many different kinds of cases here (the particle and the group may be identical or not, if non-identical the subshell equivalent but non-identical with the particle may be filled or empty, and so on) and it seems better therefore to postpone discussion of the inequivalent case until a later paper.

5. The Identical-Particle Case

We have defined an interaction preserving symplectic symmetry as being one which commutes with the symplectic Casimir operator. We can be certain that an interaction which does not so commute will have *some* matrix elements connecting different symplectic symmetries. We must notice, however, that only those states can physically occur whose permutation symmetry is described by one- or two-columned Young diagrams and therefore there emerges immediately the possibility that our definition of a "good" operator has been too demanding. Indeed if we deal only with identical nucleons (defined by one-columned diagrams) it is obvious that this is so, for in this case the conditions, expressed in terms of the two-body energies, cannot involve the odd-J states (only accessible for non-identical particles) while the conditions of constraint of eq (27) involve both even-J and odd-J states. An immediate outcome of this consideration is that, as shown first by Racah and Talmi [5]), a delta-function force does give good seniority † even though no Wigner interaction preserves symplectic symmetry in general. For in the short range limit for identical particles we have $\delta(r_{ij}) \equiv -\frac{1}{3}\sigma_i \cdot \sigma_j \delta(r_{ij})$ and as shown above the $\sigma_1 \cdot \sigma_2$ interaction does preserve the seniority.

We must therefore find a modified set of conditions for dealing with the identical-particle case and indeed we must even reconsider the more general case of two-columned representations ($T < T_{\max}$). Let us first demonstrate that we have not made any very serious error in dealing with the latter case.

Consider the states of three particles (j^3) with $J = j$. Let the number of such states with $T = \frac{1}{2}, \frac{3}{2}$, be respectively $N_{\frac{1}{2}}$, $N_{\frac{3}{2}}$. We have ††

$$
\begin{aligned}
N_{\frac{1}{2}} &= \tfrac{1}{3}[j] - \frac{2}{3\sqrt{3}} \sin \tfrac{4}{3}\pi(j+\tfrac{1}{2}) \\
&= 1, 1, 2, 3, 3, 4, 5, 5, 6 \ldots \quad \text{for} \quad j = \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}, \ldots \\
N_{\frac{3}{2}} &= \tfrac{1}{2}[j] - N_{\frac{1}{2}} \\
&= 0, 1, 1, 1, 2, 2, 2, 3, 3, 3, 4 \ldots \quad \text{for} \quad j = \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}, \ldots
\end{aligned} \tag{40}
$$

For either T value there is a unique state with seniority 1 [Flowers' $(s, t) = (1, \frac{1}{2})$ state]. The $(N_T - 1)$ other states then have different symplectic symmetries than $(1, \frac{1}{2})$ and the requirement that the matrix element connecting each of these states with the seniority 1 state should vanish is a system of $(N_T - 1)$ linearly independent ††† linear constraints on the two-body energies. For $T = \frac{1}{2}$ we have $(N_{\frac{1}{2}} - 1) = 0, 0, 1, 2, 2, 3, \ldots$ for $j = \frac{1}{2}, \frac{3}{2}, \ldots$ while we have earlier imposed 0, 0, 1, 2, 3, 4, ... conditions. It follows then that for $j \leqq \frac{7}{2}$ the full set

† We shall speak of "seniority" instead of the more cumbersome "symplectic symmetry" when dealing with identical particles

†† See for example ref [12]), correcting there an obvious misprint

††† The matrix element has the form

$$3 \sum_{J'} \langle J\sigma | J' \rangle \langle J\sigma' | J' \rangle E_{J'}^{(2)}$$

where $\langle J\sigma | J' \rangle$ is a fractional parentage coefficient, $J = j$, $\sigma \equiv (1, \frac{1}{2})$ and σ' describes the other state. The linear independence of the restraints for fixed σ, different σ', now follows from the linear independence of the c f p considered as vectors, which in turn reflects the linear independence of the σ' states of eq. (28) is necessary; for $j = \frac{9}{2}, \frac{11}{2}$ there is a possibility that we have imposed one condition too many and for $j > \frac{11}{2}$ more than one condition. It scarely seems worthwhile now to investigate this further.

For identical particles we have $(N_{\frac{3}{2}}-1) = -1, 0, 0, 0, 1, 1, 1, 2, \ldots$ and there is thus the possibility that seniority is always good (for two-body interactions) if $j \leqq \frac{7}{2}$; in point of fact it is already well known that this is the case †. We will now show that the $(N_{\frac{3}{2}}-1)$ is for all j (except of course $j = \frac{1}{2}$) exactly the number of linearly independent conditions rather than the larger number implied by eq. (28) and we shall exhibit them explicitly. Our procedure will be as follows: we first find what conditions are imposed on the even-J two-body energies above by the set of equations (28) These conditions are then insufficient for good symplectic symmetry but they are sufficient for good seniority Next by simple counting we establish that the number of such conditions is $N_{\frac{3}{2}}-1$. There are many ways of doing this: what follows is a particularly convenient one. We apply the identity

$$(-1)^{r+J} W(jjjj; rJ) = -\frac{\delta_{rJ}}{[r]} + 2 \sum_{J'} \varepsilon_{J'} [J'] W(jjjj; rJ') W(jjjj; JJ') \tag{41}$$

to eq. (36) and have then for the particle-hole energy (additive constants and multiples of T^2 are of course of no consequence here)

$$E_J(j^{-1}, j) = -\alpha_J + 2 \sum \varepsilon_{J'} [rJ'] W(jjjj, rJ') W(jjjj; JJ') \alpha_r, \tag{42}$$

and using eq. (3) we finally obtain the curious result

$$E_J(j^{-1}, j) = -\alpha_J - 2 \sum_{J' \text{ even}} [J'] W(jjjj; JJ') E_{J'}^{(2)}. \tag{43}$$

But if symplectic symmetry is good, then for even $J (\neq 0)$, $\alpha_J = \alpha$, $E_J(j^{-1}, j) = E_J^{(2)}$ and moreover the $J' = 0$ term on the right is independent of J. Thus we have as the conditions imposed on the even-J energies by the symplectic symmetry requirement

$$\sum_{J' \text{ even}, \neq 0} \{[J'] W(jjjj, JJ') + \tfrac{1}{2} \delta_{JJ'}\} E_{J'}^{(2)} = \text{constant for all even } J, \neq 0, \tag{44}$$

and writing this in terms of the interaction constants by eqs (3) and (41) we find the completely parallel equation

$$\sum_{r' \text{ even}, \neq 0} \{[r'] W(jjjj; rr') + \tfrac{1}{2} \delta_{rr'}\} \alpha_{r'} = \text{constant} \quad \text{for all even } r, \neq 0. \tag{45}$$

In each case the restriction that the summation parameter must not equal zero is redundant The remarkable correspondence between the two equations

† For $j \leqq \frac{7}{2}$ seniority must be good because the states are uniquely defined by their J value For $j = \frac{9}{2}$ it has been noted by Schwartz and de Shalit [18]), in their consideration it arises from the fact that the only states not uniquely defined by J are certain states of the half-filled shell

could in fact have been directly inferred from the properties of the Racah transforms mentioned above. The reader should remember that (44) and (45) are necessary but not sufficient for good symplectic symmetry; they are sufficient, but we have not yet proved that they are necessary, for good seniority.

The final step involves a problem of recognition. The $(j-\frac{3}{2})$ equations (44) are not linearly independent but they are closely related to the matrix element between the seniority 1 three-particle state with $J = j$ and the seniority 3 states, the form of which is given in the footnote following eq. (40) We have in fact [13]) for the unnormalized c.f p (J' of course being even)

$$\begin{aligned} \langle J = j, s = 1|J'\rangle &= \delta_{J'0} - \frac{2[J']}{[j]}, \\ \langle J = j, s = 3|J'\rangle &= (1-\delta_{J'0})[J']^{\frac{1}{2}}\left\{ W(jjjj;\, J'x) + \frac{\delta_{J'x}}{2[J']} - \frac{2}{(2j-1[j]}\right\}. \end{aligned} \tag{46}$$

In the $s = 3$ c.f.p , x is even $\neq 0$ and is otherwise arbitrary We are free of course to take only $(N_{\frac{3}{2}}-1)$ values of x; the c f.p. sets for other x depend linearly on these. For a given x the matrix element is, to within a multiplicative constant,

$$\sum_{J' \text{ even}, \neq 0} \{[J']W(jjjj;J'x) + \tfrac{1}{2}\delta_{J'x}\}E_{J'}^{(2)} - \frac{2}{(2j-1)[j]} \sum_{J' \text{ even}, \neq 0} [J']E_{J'}^{(2)},$$

and the condition for the vanishing of all the $\langle s = 3|s = 1\rangle$ matrix elements exactly reproduces eq. (44), since the last term in the above expression is x-independent The constant on the right-hand side of eq. (44) is identified here. One could verify that the value is correct by multiplying (44) by $[J]$ and summing over even $J \neq 0$.

We have now shown quite directly that eqs (44) or (45) form a set of necessary and sufficient conditions for good seniority. They imply no constraint for $j \leqq \frac{5}{2}$, are satisfied automatically for $j = \frac{7}{2}$ and imply one condition for $j = \frac{9}{2}, \frac{11}{2}, \frac{13}{2}$ and this condition is satisfied for short-range (δ-function) forces. For $j = \frac{9}{2}$ the explicit condition is

$$65E_2 - 315E_4 + 403E_6 - 153E_8 = 0 \tag{47}$$

and all interaction matrix elements in the $j = \frac{9}{2}$ shell which are non-diagonal in seniority are proportional to the expression occurring here.

An experimental check would be possible if the $(g_{\frac{9}{2}}{}^2)$ levels of Mo^{92} were available but of course they are not. Remembering that our method of deriving the constraint shows that if seniority is good for three particles it is good for the whole subshell, we see that we could investigate seniority via a three-particle system (or a more complicated one) One possibility here would be to use the well-known rule of Racah [1]) that odd-rank tensors are diagonal in seniority (which follows from our eq. (17)); this would imply for example that, if seniority were good, the M1 or M3 electromagnetic transitions from $(g_{\frac{9}{2}})^3$ excited states of Tc^{93} to the ground state would be forbidden [14]).

We finally comment on the description of the general interaction which preserves good seniority. The interaction constants are as defined by eq. (45), one solution of which is simply the odd-rank interaction of eq. (25). Eq. (45) admits, however, a far wider class of solutions, as is physically obvious anyway, since good seniority imposes a restriction only on the like-particle forces. Let us remember, however, that, when dealing with identical particles only, we have more interaction constants than energy levels and thus the identical-particle wave functions and energies are invariant under a class of transformations of the α_r. The explicit transformations are (where A_x is arbitrary),

$$\alpha_r \rightarrow \alpha_r + \sum_{x \text{ odd}} A_x W(jjjj, rx), \tag{48}$$

since, from eq. (4), the A_x interaction gives zero energies for identical-particle states Defining interactions which are connected by such a transformation as "equivalent interactions", we now observe that any seniority-preserving interaction is, to within an additive constant, equivalent to an odd-rank interaction; this follows from the fact that if we specify a set of even-J energies which satisfy the seniority conditions (44) we can always find a supplementary set of odd-J energies such that the total set satisfies the symplectic symmetry equations (27). One such set in fact will be

$$E_J^{(2)} = -\tfrac{3}{2} \sum_{J' \text{ even}} [J'] W(jjjj; JJ') E_{J'}^{(2)} \text{ for } J \text{ odd}, \tag{49}$$

as may be verified directly by insertion into eq. (27), making use of eq. (41). It is clear of course that this set is not unique (as shown for $j = \frac{5}{2}$ by inspection of eq. (29)) but this is of no present consequence. The point is that the symplectic symmetry equations (27) imply for the even-J energies the equations (44) and imply nothing else about them. We have indeed implicitly used this fact in deciding above that equations (44) were *sufficient* for good seniority.

The general interaction now which preserves the seniority is, apart of course from a constant and multiple of T^2, an odd-rank interaction plus an interaction which vanishes for identical-particle states. The result that odd-rank tensor interactions preserve the seniority is not at all new, since this was shown by Racah and Talmi [5]). They showed moreover that such interactions have the physically important pairing property defined by their eq. (4). We now see that *only* odd-rank interactions (or interactions equivalent to these) preserve the seniority and, from their work, that any seniority-preserving interaction has the pairing property.

A final word concerns the recent paper of Helmers [15]), who investigated, for identical particles and Wigner interactions only, the conditions under which there is "symplectic invariance" (which we would prefer to call "symplectic degeneracy") of the energy matrix. The result here of course is that, to within an additive constant, the interaction must be a multiple of the symplectic Casimir operator or equivalent to a multiple of it.

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