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INFRARED PHENOMENA IN QUANTUM ELECTRODYNAMICS

I. THE PHYSICAL ONE-ELECTRON STATES IN THE INFRARED REGION

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Synopsis

In view of remaining obscurities and difficulties in existing treatments of the infrared divergences in quantum electrodynamics this problem has been considered anew. The approximate model introduced in 1937 by Bloch and Nordsieck is rediscussed. It is explicitly shown to be a good substitute for the complete theory as long as one restricts oneself to infrared radiation. A non-covariant diagram technique is used to prove that neglect of recoil and pair effects is indeed allowed in the infrared radiation range. The effects of vacuum polarization and charge renormalization require special attention. They are treated in second order with the regularization method of Pauli and Villars.

1. *Introduction.* The present investigation is devoted to quantum electrodynamics in the electromagnetic wavelength range large compared to the Compton wavelength $\hbar/mc$ of the electron. This range is often called the infrared (i.r.) region. As is well known special difficulties are arising in the i.r. region as a consequence of the vanishing rest mass of the electromagnetic quanta (photons). In particular they lead to divergent integrals in the perturbation expansion of radiative corrections originating from the i.r. region. In contradistinction to the divergences coming from the ultraviolet (short) wavelength range the i.r. divergences cannot be eliminated by means of the well known renormalization method of quantum electrodynamics.

The fundamental contribution to the solution of the i.r. difficulties was made by Bloch and Nordsieck [1]). They treated a simplified model of electrodynamics in which pair effects and recoil of the electron are neglected and noticed that this model is a good substitute for the complete theory as long as one restricts oneself to i.r. radiation. This model is exactly solvable and leads to the insight that scattering processes are necessarily accompanied by the emission of unobservably weak radiation consisting of infinitely many quanta of very small total energy. This observation gives the key to the solution of the i.r. problem. The treatment of Bloch and Nordsieck, however, is far from complete. For instance it is a priori not clear to which extent the conclusions of these authors can be applied to the complete theory in which hard (short wavelength) radiation plays a role too.

A serious attempt to obtain a rigorous solution of all i.r. difficulties was undertaken by Jauch and Rohrlich [2]). In a relativistic covariant treatment they extend the work of Bloch and Nordsieck in the sense that radiative effects of all wavelengths are considered simultaneously. Feynman diagrams play a fundamental role in the analysis. By means of a very useful but not quite clearly defined concept of 'basic diagram', Jauch and Rohrlich readily obtain a generalization of Bloch and Nordsieck's results. Although this work seemed at first sight to exhaust the i.r. problem it soon appeared that several questions required further study. The limits of validity of Jauch and Rohrlich's results were questioned by Lomon [3]), while Nakanishi [4]) noted that diagrams in which soft (i.e. very low frequency) photons are attached to internal lines, were disregarded without justification. Yennie and Suura [5]) used arguments similar to Jauch and Rohrlich's and showed more clearly the origin of the cancellation of i.r. divergences originating from soft real and virtual photons. None of these papers can be regarded as clarifying fully the complicated i.r. phenomena of electrodynamics *).

Aside from these different questions and investigations there is still another problem which concerns *Compton scattering* in the non relativistic limit (soft photons scattered on slow electrons). What does one find if one calculates for this process radiative corrections up to a given order? According to a theorem of Thirring [6]) all observable radiative corrections should vanish in the non relativistic limit, their only effect being the replacement of mass and charge by their renormalized values in the second order cross section formula (usually called the *Thomson formula*). However, when a finite order calculation is made the result turns out to be i.r. divergent in clear violation of the above theorem. As an example we quote Brown and Feynman's [7]) calculation of the cross section in sixth order in the charge. Obviously the theorem of Thirring should be amended so as to take soft photon effects properly into account.

In view of the afore mentioned difficulties it was considered worthwhile to devote a new detailed investigation to the problem of the i.r. divergences in quantum electrodynamics. The method to be followed is closely inspired on the pioneering work of Bloch and Nordsieck but aims at including the whole photon spectrum. In line with Bloch and Nordsieck a non-covariant treatment will be adopted. The perturbation expansions will be based on the non-covariant perturbation formalism of Van Hove [8]) and Hugenholtz [9]). This formalism turns out to be well suited for the treatment of the problem in question. This manifests itself most clearly in the non-covariant

*) Both D. R. Yennie and E. L. Lomon have been engaged lately in a new study of the i.r. difficulties in quantum electrodynamics (private communication to L. Van Hove).

diagram technique, which enables us to split off i.r. divergent effects directly. In addition our work provides a concrete application of this formalism to an interesting aspect of low energy quantum electrodynamics. As is usually done in i.r. studies we concentrate entirely on the transversely polarized components of the electromagnetic field. The main lines of this investigation will be as follows. We consider first electrodynamics with a very low cut off ($\hbar\bar{\omega} \ll mc^2$) and we ask how well this system is approximated by the model of Bloch and Nordsieck. One verifies in succession that recoil and pair effects can be neglected in the low frequency region. Pair effects form the least trivial and most interesting point in this connection, due to the fact that even for a very low photon cut off they give rise to vacuum polarization and charge renormalization. The treatment of electrodynamics with a low cut off will be the main object of the present article. In the following article we then study within the framework of full quantum electrodynamics two concrete scattering processes with all i.r. aspects connected to them. We select *bremsstrahlung* and *Compton scattering*, the latter process because of its relation to the theorem of Thirring [6]) mentioned before. In the treatment of these scattering processes we shall make extensive use of diagrams in the non-covariant formalism and shall often rely on the arguments of the present paper.

We start in the next section with the treatment of the Bloch-Nordsieck model. The corresponding eigenvalue problem is solved by a method different from Bloch and Nordsieck's, and more directly susceptible of extension to the complete theory. We give explicit expressions for the physical one-electron states expanded in unperturbed states. In section 3 we start the discussion of the range of validity of the model by investigating the nature and order of magnitude of all effects neglected. The complications due to the self-energy of the photon are treated separately in section 4, which makes essential use of the regularization method of Pauli and Villars [10]).

2. *The Bloch-Nordsieck model.* Bloch and Nordsieck studied the interaction between a classical current distribution and the transverse electromagnetic field with the aid of the hamiltonian *)

$$H_B = (\boldsymbol{\mu}, \boldsymbol{p} - \Sigma_{s\lambda}\, \boldsymbol{a}_{s\lambda}\, [P_{s\lambda} \cos(\boldsymbol{k}_s, \boldsymbol{r}) + Q_{s\lambda} \sin(\boldsymbol{k}_s, \boldsymbol{r})]) + \\ + m(1 - \mu^2)^{\frac{1}{2}} + \tfrac{1}{2} \Sigma_{s\lambda}\, (P_{s\lambda}{}^2 + Q_{s\lambda}{}^2)\, \omega_s, \qquad (2.1)$$

where $\boldsymbol{a}_{s\lambda} = 2e(\pi/\Omega\omega_s)^{\frac{1}{2}}\; \boldsymbol{\varepsilon}_{s\lambda}$ and $(\boldsymbol{\varepsilon}_{s\lambda},\, \boldsymbol{k}_s) = 0$. The electromagnetic wave vector $\boldsymbol{k}_s$ runs over the points of an infinite cubic lattice in momentum space with a lattice spacing $2\pi\Omega^{-\frac{1}{3}}$; Ω is the volume of the cube in which the system is enclosed with the usual periodic boundary conditions; Σ_λ is a summation over the two transverse polarizations of the electromagnetic

*) We use units so that $\hbar = c = 1$.

waves, $\boldsymbol{\varepsilon}_{s\lambda}$ being a unit vector in the direction of polarization; $\boldsymbol{p} = -i\partial/\partial \boldsymbol{r}$; $P_{s\lambda} = -i\partial/\partial Q_{s\lambda}$; $\omega_s = |\boldsymbol{k}_s|$ is the circular frequency of the wave with propagation vector $\boldsymbol{k}_s$; $\boldsymbol{\mu}$ is the velocity vector replacing the Dirac $\boldsymbol{\alpha}$ matrices. The difference with complete quantum electrodynamics is the replacement of the Dirac matrix fourvector $(\boldsymbol{\alpha}, \beta)$ by $(\boldsymbol{\mu}, (1 - \mu^2)^{\frac{1}{2}})$. As noticed by Bloch and Nordsieck and confirmed below such a replacement is allowed if only photons of low energy are considered in the interaction; we therefore expect to obtain from a study of (2.1) valuable information on the infrared difficulties of quantum electrodynamics.

The eigenvalue problem

$$H_B f(\boldsymbol{r}, Q_{s\lambda}) = E f(\boldsymbol{r}, Q_{s\lambda}) \tag{2.2}$$

is exactly solvable. Bloch and Nordsieck obtain the solution by means of a suitable canonical transformation. Here another method is followed, more convenient for later extension to full electrodynamics. We first introduce creation and destruction operators for the electromagnetic and electron fields. The creation operator $c_{s\lambda}{}^*$ and the destruction operator $c_{s\lambda}$ for a photon with propagation vector $\boldsymbol{k}_s$ and polarization λ are given by

$$-\,ic_{s\lambda}{}^* = 2^{-\frac{1}{2}}(P_{s\lambda} + iQ_{s\lambda}) \tag{2.3a}$$

$$+\,ic_{s\lambda} = 2^{-\frac{1}{2}}(P_{s\lambda} - iQ_{s\lambda}). \tag{2.3b}$$

It is easily verified that

$$[c_{s\lambda}, c_{s'\lambda'}{}^*] = \delta_{ss'}\,\delta_{\lambda\lambda'};\ [c_{s\lambda}, c_{s'\lambda'}] = [c_{s\lambda}{}^*, c^*{}_{s'\lambda'}] = 0. \tag{2.4}$$

We write $H_B = H_B{}^{0\prime} + V_B{}'$ with

$$H_B{}^{0\prime} = -\,i(\boldsymbol{\mu}, \partial/\partial\boldsymbol{r}) + m(1 - \mu^2)^{\frac{1}{2}} + \textstyle\sum_{s\lambda}\omega_s c_{s\lambda}{}^* c_{s\lambda} \tag{2.5a}$$

$$V_B{}' = 2^{-\frac{1}{2}}i(\boldsymbol{\mu}, \textstyle\sum_{s\lambda}\boldsymbol{a}_{s\lambda}[c_{s\lambda}{}^*\, e^{-i(\boldsymbol{k}_s, \boldsymbol{r})} - c_{s\lambda}\, e^{i(\boldsymbol{k}_s, \boldsymbol{r})}]). \tag{2.5b}$$

In (2.5*a*) we omitted the zero-point energy of the electromagnetic field $\frac{1}{2}\sum_{s\lambda}\omega_s$.

The general solution of

$$[-\,i(\boldsymbol{\mu}, \partial/\partial\boldsymbol{r}) + m(1 - \mu^2)^{\frac{1}{2}}]\,\psi(\boldsymbol{r}) = E\psi(\boldsymbol{r}) \tag{2.6}$$

with periodic boundary conditions on the surface of the cube with volume Ω is

$$\psi(\boldsymbol{r}) = \textstyle\sum_{\boldsymbol{p}} \Omega^{-\frac{1}{2}} b(\boldsymbol{p})\, e^{i(\boldsymbol{p}, \boldsymbol{r})}, \tag{2.7}$$

the sum running over the same momentum values as $\boldsymbol{k}_s$. By taking the coefficients $b(\boldsymbol{p})$, $b^*(\boldsymbol{p})$ as operators with the usual anti-commutation rules

$$\{b(\boldsymbol{p}), b^*(\boldsymbol{p}')\} = \delta_{\boldsymbol{p}\boldsymbol{p}'};\ \{b(\boldsymbol{p}), b(\boldsymbol{p}')\} = \{b^*(\boldsymbol{p}), b(\boldsymbol{p}')\} = 0 \tag{2.8}$$

and by putting

$$[c_{s\lambda}, b(\boldsymbol{p})] = [c_{s\lambda}, b^*(\boldsymbol{p})] = 0;\ [c_{s\lambda}{}^*, b(\boldsymbol{p})] = [c_{s\lambda}{}^*, b^*(\boldsymbol{p})] = 0, \tag{2.9}$$

we get after this second quantization of the classical current field

$$H_B{}^{0''} = \Sigma_p\{(\boldsymbol{\mu}, \boldsymbol{p}) + m(1 - \mu^2)^{\frac{1}{2}}\} b^*(\boldsymbol{p})\, b(\boldsymbol{p}) + \Sigma_{s\lambda}\, \omega_s c_{s\lambda}{}^* c_{s\lambda} \tag{2.10a}$$

$$V_B'' = \Sigma_{ps\lambda}\, 2^{-\frac{1}{2}} i(\boldsymbol{\mu}, \boldsymbol{a}_{s\lambda})\, b^*(\boldsymbol{p})\, b(\boldsymbol{p} + \boldsymbol{k}_s)\{c_{s\lambda}{}^* - c_{-s\lambda}\}, \tag{2.10b}$$

where $c_{-s\lambda}$ destroys a photon with propagation vector $-\boldsymbol{k}_s$. As verified later the electron self-energy due to the interaction with the electromagnetic field is

$$\delta E'(\boldsymbol{p}) = -\tfrac{1}{2}\Sigma_{s\lambda}(\boldsymbol{\mu}, \boldsymbol{a}_{s\lambda})^2/(\omega_s - (\boldsymbol{\mu}, \boldsymbol{k}_s)). \tag{2.11}$$

Therefore we rewrite finally H_B as $H_B = H_B{}^0 + V_B$ with

$$H_B{}^0 = H_B{}^{0''} + \Sigma_p\, \delta E'(\boldsymbol{p})\, b^*(\boldsymbol{p})\, b(\boldsymbol{p}) \tag{2.12a}$$

$$V_B = V_B'' - \Sigma_p\, \delta E'(\boldsymbol{p})\, b^*(\boldsymbol{p})\, b(\boldsymbol{p}). \tag{2,12b}$$

Let $|\boldsymbol{p};\ \{m_{s\lambda}\}\rangle = b^*(\boldsymbol{p})\ \Pi_{s\lambda}(m_{s\lambda}!)^{-\frac{1}{2}} c^{*m_{s\lambda}}_{s\lambda}\, |0\rangle$ be the normalized one-electron solutions of

$$(H_B{}^0 - E)\, |\alpha\rangle = 0, \tag{2.13}$$

$|0\rangle$ being the normalized unperturbed vacuum state ($\langle 0|0\rangle = 1$). The photon numbers $m_{s\lambda}$ are non negative integers. The normalized one-electron solutions of

$$(H_B - E)\, |\alpha) = 0 \tag{2.14}$$

can be written formally as [11])

$$\begin{aligned} |\boldsymbol{p};\{m_{s\lambda}\})_\mu = N^{\frac{1}{2}}_{\boldsymbol{p};\{m_{s\lambda}\}}\, [|\boldsymbol{p};\{m_{s\lambda}\}\rangle + \Sigma_{\{n_{s\lambda}\}}\langle \boldsymbol{p} + \Sigma_{s\lambda}(m_{s\lambda} - n_{s\lambda})\boldsymbol{k}_s;\{n_{s\lambda}\}|\\ \{\Sigma_{l=1}^{\infty}(-D_B{}^0(E'(\boldsymbol{p}) + \Sigma_{s\lambda}\,\omega_s m_{s\lambda})V_B)^l\}_{n.i.}\, |\boldsymbol{p};\{m_{s\lambda}\}\rangle\\ |\boldsymbol{p} + \Sigma_{s\lambda}(m_{s\lambda} - n_{s\lambda})\boldsymbol{k}_s;\{n_{s\lambda}\}\rangle \end{aligned} \tag{2.15}$$

where $D_B{}^0(z) = (H_B{}^0 - z)^{-1}$ and $N_{\boldsymbol{p};\{m_{s\lambda}\}}$ is a normalization constant; further $E'(\boldsymbol{p}) = (\boldsymbol{\mu}, \boldsymbol{p}) + m(1 - \mu^2)^{\frac{1}{2}} - \delta E'(\boldsymbol{p})$. By *n.i.* is meant that all intermediate states and the final state are non-initial, i.e. different from the initial state $|\boldsymbol{p};\{m_{s\lambda}\}\rangle$. This formula is valid for arbitrary interaction V_B.

The coefficients (i.e. the matrix elements) of the unperturbed (bare particle) states in (2.15) are easily found with the aid of diagrams. In these diagrams an electron is represented by a full line, a photon by a dotted one. The operator V_B'' acts in the vertices, the operator $-\Sigma_p\, \delta E'(\boldsymbol{p})b^*(\boldsymbol{p})b(\boldsymbol{p})$ acts in the points marked with a cross and further denoted as δE-points, and finally the operator $D_B{}^0(z)$ must be taken in all intermediate states as well as in the final state which is on the left of the diagram. The diagrams are read from right to left.

It can be proved that in the matrix elements of (2.15) all contributions of diagrams with internal photon lines and/or δE-corrections exactly cancel each other.

The proof runs as follows: Start from a certain diagram M, contributing to some matrix element in (2.15), in which internal photon lines and δE-points may occur.

Insert in M in all possible ways one internal photon line or one δE-point. In this way one gets a set of diagrams representing the whole second order correction to M. We prove that this correction vanishes. Let the diagram M be represented by fig. 1 in which each point is either a vertex or a δE-point. We omit all photon lines in M because they are irrelevant for the argument.

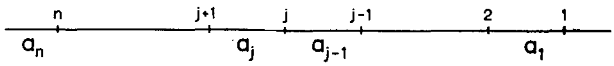

Fig. 1. Diagram M (the photon lines are not drawn). The points are either vertices or δE-points. The a's denote the energy denominators.

Let a_i be the i^{th} energy denominator in M (going from right to left). Now consider the diagrams of fig. 2, which form a subclass of the set mentioned above.

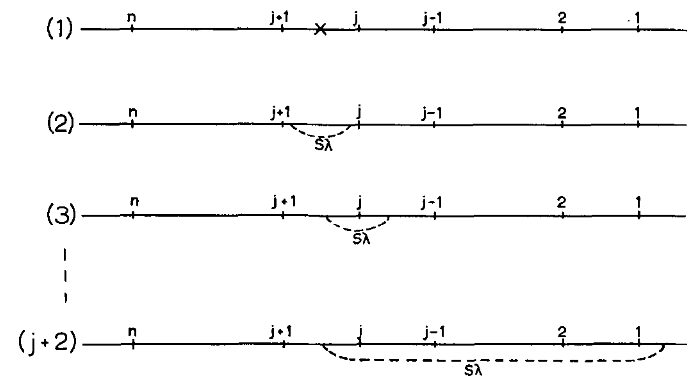

Fig. 2. Subclass of diagrams contributing to the second order correction to diagram M.

Write $\omega_s - (\boldsymbol{\mu}, \boldsymbol{k}_s) = b$, where s refers to the photon line in the figure. If we omit all summations and also the factor $\prod_{i=1}^{n} L_i(\boldsymbol{\mu}, \boldsymbol{a}_{s\lambda})^2/2$ where

$$L_i = \begin{cases} \pm\, 2^{-\frac{1}{2}}\, i(\boldsymbol{\mu}, \boldsymbol{a}_{s_i\lambda_i}) \text{ if the } i^{th} \text{ point is a vertex} \\ \frac{1}{2}\sum_{s'\lambda'} (\boldsymbol{\mu}, \boldsymbol{a}_{s'\lambda'})^2/(\omega_{s'} - (\boldsymbol{\mu}, \boldsymbol{k}_{s'})) \text{ if the } i^{th} \text{ point is a } \delta E\text{-point,} \end{cases}$$

the diagrams (2), (3) ... $(j+2)$ contribute together

$$\frac{1}{\prod_{i=1}^{n} a_i}\left[\frac{1}{a_j(a_j+b)} + \frac{a_j}{a_j(a_j+b)(a_{j-1}+b)} + \ldots + \frac{a_j a_{j-1} \ldots a_1}{a_j(a_j+b)(a_{j-1}+b) \ldots (a_1+b)b}\right] = \frac{1}{\prod_{i=1}^{n} a_i} \cdot \frac{1}{a_j b}.$$

However in view of (2.11) the diagram (1) contributes

$$-\frac{1}{\prod_{i=1}^{n} a_i} \cdot \frac{1}{a_j b}.$$

So the total contribution of the diagrams in fig. 2 vanishes and as this result is independent of j the whole second order correction to M vanishes. From this we conclude that we can forget all diagrams containing internal photon lines and/or δE-points.

This conclusion allows us to find in closed form the explicit expression of the matrix elements in (2.15), for the remaining number of diagrams contributing to them is finite. One obtains

$$|\boldsymbol{p}; \{m_{s\lambda}\})_\mu = \bar{N}_\mu^{\frac{1}{2}} \Sigma_{\{n_{s\lambda}\}} \Pi_{s\lambda}(-2^{-\frac{1}{2}}i\,\sigma_{\mu s\lambda})^{|n_{s\lambda}-m_{s\lambda}|}\,(m_{s\lambda}!\,n_{s\lambda}!)^{\frac{1}{2}}$$
$$\Sigma_{\zeta=0}^{s\lambda}(-\tfrac{1}{2}\sigma^2_{\mu s\lambda})^\zeta/(l_{s\lambda}-\zeta)!\,\zeta!(|n_{s\lambda}-m_{s\lambda}|+\zeta)!|\,\boldsymbol{p}+\Sigma_{s\lambda}(m_{s\lambda}-n_{s\lambda})\boldsymbol{k}_s; \{n_{s\lambda}\}\rangle, \quad (2.16)$$

where $\Sigma_{\{n_{s\lambda}\}}$ is a summation over all possible sets of photons;

$l_{s\lambda} = \min.(n_{s\lambda}, m_{s\lambda})$ and

$$\sigma_{\mu s\lambda} = (\boldsymbol{\mu}, \boldsymbol{a}_{s\lambda})/(\omega_s - (\boldsymbol{\mu}, \boldsymbol{k}_s)) \quad (2.16a)$$

$$\bar{N}_\mu = \exp[-\tfrac{1}{2}\Sigma_{s\lambda}\sigma^2_{\mu s\lambda}]. \quad (2.16b)$$

If $\Omega \to \infty$, the sum in the exponential becomes an integral divergent in the infrared as well as in the ultraviolet region.

In particular, for the one-electron state one finds

$$|\boldsymbol{p}; \{0\})_\mu = \bar{O}_p\,|0\rangle \quad (2.17)$$

$$\bar{O}_p = \bar{N}_\mu^{\frac{1}{2}} \Sigma_{\{n_{s\lambda}\}} \Pi_{s\lambda}(-2^{-\frac{1}{2}}\,i\sigma_{\mu s\lambda})^{n_{s\lambda}}(n_{s\lambda}!)^{-1} b^*(\boldsymbol{p} - \Sigma_{s\lambda} n_{s\lambda}\boldsymbol{k}_s)\, c^{*\,n_{s\lambda}}_{s\lambda}. \quad (2.18)$$

Note that the above derivation confirms the value (2.11) of the electron self-energy. (2.16) is the result of Bloch and Nordsieck. From now on we only use solutions (2.16) in which $\boldsymbol{p}$ and $\boldsymbol{\mu}$ are related by $\boldsymbol{p} = m\boldsymbol{\mu}|(1 - \mu^2)^{\frac{1}{2}}$, this meaning that the electron has velocity $\boldsymbol{\mu}$.

3. *Range of validity of the Bloch-Nordsieck model.* In quantum electrodynamics, if the electromagnetic field is restricted to the transverse components, the hamiltonian has the form

$$H = (\boldsymbol{\alpha}, \boldsymbol{p} - \Sigma_{s\lambda}\,\boldsymbol{a}_{s\lambda}\,[P_{s\lambda}\cos(\boldsymbol{k}_s, \boldsymbol{r}) + Q_{s\lambda}\sin(\boldsymbol{k}_s, \boldsymbol{r})]) + \beta m +$$
$$+ \tfrac{1}{2}\Sigma_{s\lambda}(P_{s\lambda}^2 + Q_{s\lambda}^2)\omega_s \quad (3.1)$$

which becomes after second quantization

$$H = H_0 + V, \quad (3.2)$$

with

$$H_0 = \Sigma_{pr}\,\varepsilon(\boldsymbol{p})[b_r^*(\boldsymbol{p})\,b_r(\boldsymbol{p}) + d_r^*(\boldsymbol{p})\,d_r(\boldsymbol{p})] + \Sigma_{s\lambda}\,\omega_s\,c_{s\lambda}^* c_{s\lambda}, \quad (3.2a)$$

$$V = \Sigma_{prr's\lambda}\,im(2\varepsilon(\boldsymbol{p})\,\varepsilon(\boldsymbol{p} + \boldsymbol{k}_s))^{-\frac{1}{2}}[b_r^*(\boldsymbol{p})w_r^*(\boldsymbol{p}) + \mathrm{d}_r(-\boldsymbol{p})\,v_r^*(-\boldsymbol{p})]$$
$$(\boldsymbol{\alpha}, \boldsymbol{a}_{s\lambda})[b_{r'}(\boldsymbol{p}+\boldsymbol{k}_s)\,w_{r'}(\boldsymbol{p}+\boldsymbol{k}_s) + d_{r'}^*(-\boldsymbol{p}-\boldsymbol{k}_s)v_{r'}(-\boldsymbol{p}-\boldsymbol{k}_s)][c_{s\lambda}^* - c_{-s\lambda}]. \quad (3.2b)$$

Most of the notation has been defined previously. α, β are the Dirac matrices and $\varepsilon(\boldsymbol{p}) = (m^2 + \boldsymbol{p}^2)^{\frac{1}{2}}$ is the unrenormalized electron energy. The operators $d_r(\boldsymbol{p})$, $d_r^*(\boldsymbol{p})$ destroy and create the positrons. The indices r, r' indicate the spin orientation of electrons and positrons. The spinor notation is the same as in reference 12.

The physical state of the electron with momentum $\boldsymbol{p}$ and spin r can formally be written as [11])

$$|\boldsymbol{p}_r; \{0\}) = O_{\boldsymbol{p}r} |0) \tag{3.3}$$

$$O_{\boldsymbol{p}r} = N_p^{\frac{1}{2}}[b_r^*(\boldsymbol{p}) + \Sigma_\alpha \Sigma_\delta \langle\alpha|\{\Sigma_{j=1}^{\infty}(- D^0(E(\boldsymbol{p}))V)^j_{n.i.}\}_\delta |\boldsymbol{p}_r;\{0\}\rangle A^*(\alpha)] \tag{3.4}$$

where $|0)$ is the normalized physical vacuum state; Σ_δ is a summation over connected diagrams; the index $n.i.$ has the same meaning as in (2.15); $|\alpha\rangle$ are the normalized eigenstates of H_0; the operator $A^*(\alpha)$ creates $|\alpha\rangle$ from the bare vacuum $|0\rangle$; $D^0(E(\boldsymbol{p})) = (H_0 - E(\boldsymbol{p}))^{-1}$; $E(\boldsymbol{p})$ is the difference in energy between the states $|\boldsymbol{p}_r; \{0\})$ and $|0)$; N_p is a normalization constant, given by

$$N_p = (1 + ((\partial/\partial l)\, G_{\boldsymbol{p}}(l))_{l=E(\boldsymbol{p})})^{-1} \tag{3.5}$$

where

$$G_\alpha(l) = \langle\alpha|[- V + VD(l)V - VD(l)VD(l)V + \ldots]_{idc} |\alpha\rangle. \tag{3.6}$$

In (3.6) $D(l)$ is the diagonal part of the resolvent operator $R(l) = (H - l)^{-1}$; the subscript idc means that only contributions of irreducible connected diagrams should be taken *).

Let us introduce for the photons a very low cut off $\bar{\omega}$, verifying $\bar{\omega} \ll m$. Then it is our contention that the physical one-electron state (3.3) can be approximated by $\bar{O}_{\boldsymbol{p}r} |0)$, where $\bar{O}_{\boldsymbol{p}r}$ is, except for the spin, the same operator as in (2.18), i.e.

$$\bar{O}_{\boldsymbol{p}r} = \bar{N}_p^{\frac{1}{2}} \Sigma_{\{n_{s\lambda}\}} \Pi_{s\lambda}(- 2^{-\frac{1}{2}} i\sigma_{\mu s\lambda})^{n_{s\lambda}}(n_{s\lambda}!)^{-1} b_r^*(\boldsymbol{p} - \Sigma_{s\lambda} n_{s\lambda}\boldsymbol{k}_s)\, c^{*}{}_{s\lambda}^{n_{s\lambda}}. \tag{3.7}$$

The difference between $O_{\boldsymbol{p}r}$ and $\bar{O}_{\boldsymbol{p}r}$ mathematically originates from two facts. The class of diagrams contributing to (3.4) is much larger than the corresponding class for (3.7); this is due to pair creation and annihilation terms. Moreover, identical diagrams give different contributions, the difference being the neglect of the electron's recoil in (3.7). So we have to show that pair effects as well as recoil effects become unimportant if $\bar{\omega}$ is small enough.

Let us rewrite (3.4) and (3.7) in abbreviated form

$$\begin{aligned} O_{\boldsymbol{p}r} &= N_p^{\frac{1}{2}}[b_r^*(\boldsymbol{p}) + \Sigma_\alpha C_\alpha A^*(\alpha)] \\ \bar{O}_{\boldsymbol{p}r} &= \bar{N}_p^{\frac{1}{2}}[b_r^*(\boldsymbol{p}) + \Sigma_\alpha \bar{C}_\alpha A^*(\alpha)]. \end{aligned} \tag{3.8}$$

We want to show

$$\Sigma_\alpha |C_\alpha - \bar{C}_\alpha|^2 = O(e^2\bar{\omega}/m)\, \Sigma_\alpha |\bar{C}_\alpha|^2 \tag{3.9}$$

if $\bar{\omega}/m \ll 1$ and $e^2\bar{\omega}/m \ll 1$. The charge e is the unrenormalized one, but our statement is only true if the renormalized charge e' is equal to the unrenormalized one, as we shall see in section 4. In practice, since $e^{-2} \simeq 137$ the condition $e^2\bar{\omega}/m \ll 1$ follows from $\bar{\omega}/m \ll 1$.

*) For the concept of irreducibility here used see Hugenholtz [9]), p. 494.

Let us first consider diagrams without pairs . The lowest order diagram which contributes to the left-hand sum in (3.9) is given in fig. 3.
This contribution is $\frac{1}{2}\sum_{s\lambda}^{\varpi}\{(\boldsymbol{\mu}, \boldsymbol{a}_{s\lambda})^2/(\omega_s - (\boldsymbol{\mu}, \boldsymbol{k}_s))^2\}O(\omega_s/E(\boldsymbol{p})) =$
$= O(e^2\bar{\omega}/m)$ if $\bar{\omega}/m \ll 1$. The lowest order corrections (without pairs) to fig. 3 are represented by fig. 4.

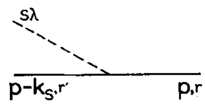

Fig. 3. Lowest order diagram contributing to $\Sigma_\alpha |C_\alpha - \bar{C}_\alpha|^2$.

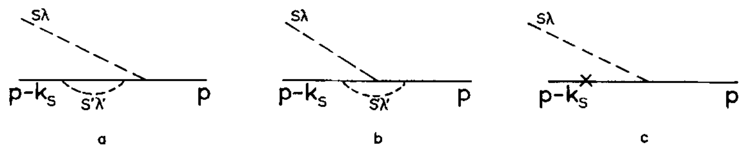

Fig. 4. All second order corrections to fig. 3 without pairs.

As we found in the preceding section these diagrams give a vanishing contribution to the coefficients $\bar{C}_\alpha$. When calculating the coefficients C_α, the contribution of the δE-point (cross) in fig. 4 is no longer given by (2.11) but by the electron self-energy in the full theory with cut off $\bar{\omega}$. We here include that part of the self-energy which originates from the diagram in fig. 5*a* (the part originating from fig. 5*b* is included later on).

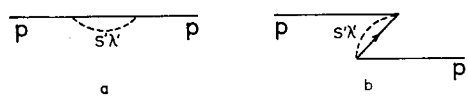

Fig. 5. Second order electron self-energy diagrams.

By so doing one finds for the second order correction to fig. 3 a contribution of relative order $e^2\bar{\omega}/m$ if $\bar{\omega}/m \ll 1$; higher order corrections to fig. 3 are found to be of the same order. When discussing diagrams without virtual photons or δE-points, with more than one external photon line, one finds similarly that the contribution to $\sum_\alpha |C_\alpha - \bar{C}_\alpha|^2$ equals
$O(e^2\bar{\omega}/m)\{\frac{1}{2}\sum_{s\lambda}^{\varpi}\sigma^2{}_{\mu s\lambda}\}^{n-1}/(n-1)!$ if n is the number of external photon lines and if $\bar{\omega}/m \ll 1$. Radiative corrections to these diagrams are again of order $e^2\bar{\omega}/m$. Restricting us to diagrams without pairs we find in this way

$$\sum_\alpha |C_\alpha - \bar{C}_\alpha|^2 = O(e^2\bar{\omega}/m)\exp[\tfrac{1}{2}\sum_{s\lambda}^{\varpi}\sigma^2{}_{\mu s\lambda}], \tag{3.10}$$

where

$$\exp[\tfrac{1}{2}\sum_{s\lambda}^{\varpi}\sigma^2{}_{\mu s\lambda}] = \bar{N}_p{}^{-1} = \sum_\alpha |\bar{C}_\alpha|^2. \tag{3.11}$$

Now consider diagrams with pairs. The contribution to $\sum_\alpha |C_\alpha - \bar{C}_\alpha|^2$ originating from the diagram of fig. 6*a* turned out to be $O(e^2\bar{\omega}/m)\,\frac{1}{2}\sum_{s\lambda}^{\varpi}\sigma^2{}_{\mu s\lambda}$.

This result is not affected by the diagram in fig. 6*b*. Although the vertex factors of fig. 6*b* are of order 1 in contradistinction to those in fig. 6*a*, which are of order $\mu = v/c$, the fact that the latter diagram has one small denominator less dominates in such a way that its contribution can be neglected compared to the contribution of fig. 6*a*. One easily extends this result to diagrams with more external photon lines; quite generally, as long as there are no virtual photons or δE-points one can neglect all diagrams in which the electron line is not straight, i.e. in which the initial electron annihilates with a positron.

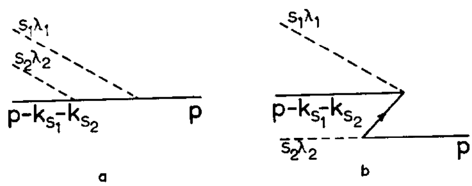

Fig. 6. Second order diagrams contributing to $\Sigma_\alpha|C_\alpha - \bar{C}_\alpha|^2$.

The situation is different when virtual effects are present. Fig. 7*a* gives an example of a diagram in which pair effects give rise to an additional small energy denominator. This diagram is however compensated in the limit $\omega_s \to 0$ by the diagram of fig. 7*b* where the cross contributes the part of the self-energy corresponding to the diagram in fig. 5*b*.

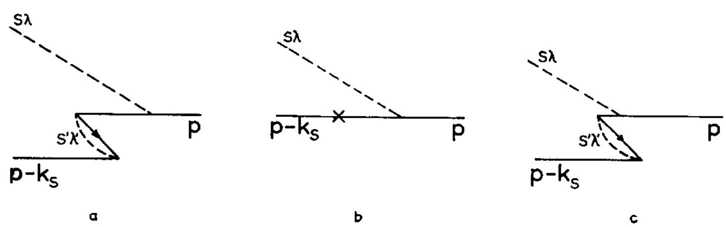

Fig. 7. Second order corrections to fig. 3 with pairs.

This is true except for contributions of relative order $e^2\bar{\omega}/m$. The diagram in fig. 7*c* and similar diagrams can be neglected because of two large energy denominators. They give rise to corrections to the diagram of fig. 3 which are of relative order $e^2\bar{\omega}/m$.

Finally we have to deal with diagrams in which the photon self-energy plays a role, i.e. diagrams where photons create pairs. In fig. 8 for instance all second order corrections to fig. 3 involving such effects are represented.

In fig. 9 the lowest order diagrams are represented with pairs in the final state. It will be shown in the next section that the error made by neglecting them is of order $e^2\bar{\omega}/m$ if the renormalized charge is equal to the unrenormalized one, a result which can be extended to higher orders.

At first sight the diagrams now considered seem to give rise to highly divergent integrals over $\boldsymbol{k}_s$ in the infrared region; for instance in fig. 9*a* the contribution to $\sum_\alpha |C_\alpha - \bar{C}_\alpha|^2$ is proportional to $\int_0^\infty \mathrm{d}\omega_s/\omega_s^2$. In addition the integration over $\boldsymbol{p}'$ becomes linearly divergent for large $\boldsymbol{p}'$. In the case of fig. 8*a* the integration over the virtual pair turns out to be even quadratically

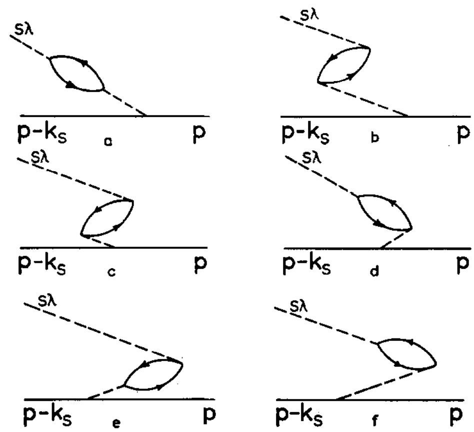

Fig. 8. Second order corrections to fig. 3; only corrections in the photon line are considered.

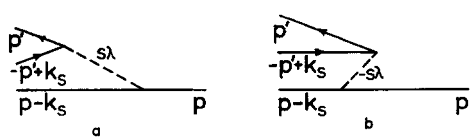

Fig. 9. Lowest order diagrams contributing to matrix elements with pairs in the final state.

divergent. The problems arising here are all connected with the photon self-energy. We must deal with them in some detail in order to complete our proof of (3.9). At the same time we shall point out and correct an error in the discussion of the physical photon state by Frazer and Van Hove in their paper on the stationary states of interacting fields [11]).

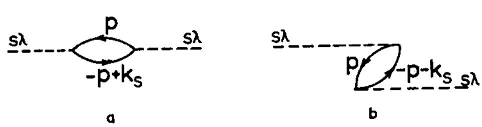

Fig. 10. Second order diagrams contributing to $G_{s\lambda}(l)$.

4. *The dressed photon. Photon self-energy. Charge renormalization.* The function $G_{s\lambda}(l)$ for the one-photon state $|0; s\lambda\rangle$, defined in analogy with (3.6) enables us to compute the photon self-energy $\delta\omega_s = -(G_{s\lambda}(l))_{l=\omega_s}$ ($\omega_s =$ observed frequency), and the normalization constant $N_{s\lambda}$ defined in analogy

with (3.5). We do this to second order. There are two second order diagrams contributing to $G_{s\lambda}(l)$ (fig. 10*ab*).
Their joined contribution turns out to be

$$G_{s\lambda}^{(2)}(l) = \Sigma_{prr'} \left[\frac{m^2}{2E(\boldsymbol{p})E(-\boldsymbol{p}+\boldsymbol{k}_s)} \cdot \right.$$
$$\cdot \frac{v_r^*(-\boldsymbol{p}+\boldsymbol{k}_s)(\boldsymbol{\alpha},\boldsymbol{a}_{s\lambda})w_{r'}(\boldsymbol{p})w_{r'}^*(\boldsymbol{p})(\boldsymbol{\alpha},\boldsymbol{a}_{s\lambda})v_r(-\boldsymbol{p}+\boldsymbol{k}_s)}{E(\boldsymbol{p})+E(-\boldsymbol{p}+\boldsymbol{k}_s)-l} +$$
$$+ \frac{m^2}{2E(\boldsymbol{p})\,E(-\boldsymbol{p}-\boldsymbol{k}_s)} \cdot$$
$$\left. \cdot \frac{v_r^*(-\boldsymbol{p}-\boldsymbol{k}_s)(\boldsymbol{\alpha},\boldsymbol{a}_{s\lambda})w_{r'}(\boldsymbol{p})w_{r'}^*(\boldsymbol{p})(\boldsymbol{\alpha},\boldsymbol{a}_{s\lambda})\,v(-\boldsymbol{p}-\boldsymbol{k}_s)}{E(\boldsymbol{p})+E(-\boldsymbol{p}-\boldsymbol{k}_s)+2\omega_s-l} \right]. \quad (4.1)$$

After computing the spinor sums one finds *)

$$G_s^{(2)}(l) = e^2\omega_s^{-1}\,[f(\boldsymbol{k}_s, l) + f(\boldsymbol{k}_s, l-2\omega_s)], \quad (4.2)$$

where

$$f(\boldsymbol{k}_s, l) = 2\pi\Omega^{-1}\,\Sigma_p \frac{m^2 + E(\boldsymbol{p})\,E(\boldsymbol{p}-\boldsymbol{k}_s) - (\boldsymbol{p},\boldsymbol{k}_s) + (\boldsymbol{p},\boldsymbol{k}_s)^2/\omega_s^2}{E(\boldsymbol{p})\,E(\boldsymbol{p}-\boldsymbol{k}_s)(E(\boldsymbol{p})+E(\boldsymbol{p}-\boldsymbol{k}_s)-l)}. \quad (4.3)$$

The summation over $\boldsymbol{p}$ is quadratically divergent at high $\boldsymbol{p}$ values. We treat the ultraviolet divergences of this type with the regularization prescription of Pauli and Villars [10]), consisting in the introduction, in a Lorentz and gauge invariant manner, of a sufficiently large number of auxiliary Fermi fields with complex coupling constants.

The regularized function $f_R(\boldsymbol{k}_s, l)$ is defined by

$$f_R(\boldsymbol{k}_s, l) = 2\pi\Omega^{-1}\,\Sigma_p\,\Sigma_i\,c_i \frac{M_i^2 + E_i(\boldsymbol{p})\,E_i(\boldsymbol{p}-\boldsymbol{k}_s) + p_{k_s}{}^2 - (\boldsymbol{p},\boldsymbol{k}_s)}{E_i(\boldsymbol{p})\,E_i(\boldsymbol{p}-\boldsymbol{k}_s)(E_i(\boldsymbol{p})+E_i(\boldsymbol{p}-\boldsymbol{k}_s)-l)} \quad (4.4)$$

with the conditions

$$\Sigma_i c_i = 0;\ \Sigma_i c_i M_i^2 = 0, \quad (4.5)$$

where

$M_0 = m$; $c_0 = 1$; $M_i > 0$; $E_i(\boldsymbol{p}) = (\boldsymbol{p}^2 + M_i^2)^{\frac{1}{2}}$; $p_{k_s} = (\boldsymbol{p},\boldsymbol{k}_s)/\omega_s$.

The auxiliary fields have masses M_i and coupling constants $c_i^{\frac{1}{2}}e$ $(i = 1,2, \ldots)$. The formal regularization prescription is to use f_R instead of f and take the limit $M_i \to \infty$ $(i > 0)$ under the restrictions (4.5) The conditions (4.5) are exactly those needed to ensure convergence of $(G_s^{(2)}(l))_{l=\omega_s}$. With the regularization prescription one finds zero for the photon self-energy **),

*) As expected $G_{s\lambda}^{(2)}(l)$ is independent of the polarization index λ. We omit the argument λ in the following.

**) See for instance Gupta's calculation [13]), where an equivalent regularization procedure is used.

in agreement with the fact that the unrenormalized and renormalized photon mass must be equal and zero. When ω_s and l are small compared with m we write for (4.4) under the conditions (4.5)

$$f_R(\boldsymbol{k}_s, l) = 2\pi\Omega^{-1}\Sigma_{\boldsymbol{p}}\,\Sigma_i\, c_i(2E_i^3(\boldsymbol{p}))^{-1}[M_i^2 + E_i^2(\boldsymbol{p}) + p_{k_s}^2 - 2(\boldsymbol{p}, \boldsymbol{k}_s) + \\ + \tfrac{1}{2}\omega_s^2(1 - p_{k_s}^2/E_i^2(\boldsymbol{p}))][1 + l/2E_i(\boldsymbol{p}) + 3(\boldsymbol{p}, \boldsymbol{k}_s)/2E_i^2(\boldsymbol{p}) - \tfrac{1}{2}\omega_s^2(3/2E_i^2(\boldsymbol{p}) - \\ - 5p_{k_s}^2/\mathrm{E}_i^4(\boldsymbol{p})) + l^2/4E_i^2(\boldsymbol{p}) + l(\boldsymbol{p}, \boldsymbol{k}_s)/E_i^3(\boldsymbol{p}) + \ldots]. \qquad (4.6)$$

Since this expression depends only on the length ω_s of $\boldsymbol{k}_s$ we write $f_R(\boldsymbol{k}_s, l) = f_R(\omega_s, l)$. Further properties of this function are easily found from (4.6)

$$f_R(0, 0) = 0 \qquad (4.7)$$

$$((\partial/\partial l)\, f_R(0, l))_{l=0} = -\tfrac{1}{8}\Sigma_i\, c_i M_i \qquad (4.8)$$

$$((\partial/\partial\omega_s)\, f_R(\omega_s, 0))_{\omega_s=0} = 0 \qquad (4.9)$$

$$((\partial^2/\partial l^2)\, f_R(0, l))_{l=0} = (1/3\pi)\,\Sigma_i\, c_i \ln M_0/M_i \qquad (4.10)$$

$$((\partial/\partial\omega_s^2)\, f_R(\omega_s, 0))_{\omega_s=0} = -(1/6\pi)\,\Sigma_i\, c_i \ln M_0/M_i. \qquad (4.11)$$

In this way we obtain the expansion

$$f_R(\omega_s, l) = f_R(0, 0) + (l/1!)((\partial/\partial l)\, f_R(0, l))_{l=0} + (l^2/2!)((\partial^2/\partial l^2)\, f_R(0, l))_{l=0} + \\ + (\omega_s^2/1!)(\partial/\partial\omega_s^2)\, f_R(\omega_s, 0))_{\omega_s=0} + \ldots = \\ = -\tfrac{1}{8}\, l\,\Sigma_i\, c_i M_i + (\omega_s^2 - l^2)(1/6\pi)\,\Sigma_i\, c_i \ln M_i/M_0 + \ldots \qquad (4.12)$$

Among the unwritten terms in (4.12) those containing auxiliary masses and coupling constants vanish in the limit $M_i \to \infty$ $(i > 0)$. Only terms without auxiliary constants remain. The function $G_{Rs}^{(2)}(l)$ defined by the regularized righthand side of (4.2) is found to be for small ω_s and l

$$G_{Rs}^{(2)}(l) = e^2\omega_s^{-1}\{\tfrac{1}{4}(\omega_s - l)\,\Sigma_i\, c_i M_i - (\omega_s - l)^2(1/3\pi)\,\Sigma_i\, c_i \ln M_i/M_0 + \ldots\}. \qquad (4.13)$$

The unwritten terms in the curly bracket of (4.13) have the order of magnitude

$$(\omega_s - l)\, m[\mathrm{O}(\omega_s^2/m^2) + \mathrm{O}(l\omega_s/m^2) + \mathrm{O}(l^2/m^2)].$$

They depend on ω_s and l, not only on the difference $(\omega_s - l)$. The regularized normalization constant N_{Rs} is for small ω_s, to second order

$$N_{Rs}^2 = (1 + ((\partial/\partial l)\, G_{Rs}^2(l))_{l=\omega_s})^{-1} = 1 + \tfrac{1}{4}e^2\omega_s^{-1}\,\Sigma_i\, c_i M_i + \mathrm{O}\,(e^2\omega_s/m). \qquad (4.14)$$

The expression (4.14) diverges for $\omega_s \to 0$ unless we put as an additional condition

$$\Sigma_i\, c_i M_i = 0. \qquad (4.15)$$

When this is done we find

$$\lim_{\omega_s \to 0} N_{Rs}^{(2)} \equiv N_{R0}^{(2)} = 1. \qquad (4.16)$$

The renormalization of the electric charge in the present non-covariant formulation will now be studied by considering the process of bremsstrahlung with the emission of a single soft photon. Virtual photon corrections on the electron line do not contribute to the charge renormalization as follows from the Ward identity [14] (which is also found to hold in the non-covariant formalism). We have to consider only the virtual pair corrections to the real photon line and we do this to second order. For low photon energy the charge renormalization will follow. We use the following expression of the S-matrix *).

$$\langle\alpha'|S|\alpha\rangle = \delta_{\mathrm{kr}}(\alpha, \alpha') - 2\pi i\delta(E(\alpha') - E(\alpha))\, N_{\alpha'}{}^{\frac{1}{2}}\, N_{\alpha}{}^{\frac{1}{2}}$$
$$\Sigma_\delta\, \langle\alpha'|\{V - VR(E(\alpha) + i0)V\}_\delta|\alpha\rangle, \tag{4.17}$$

where $N_{\alpha'}$ and N_α are defined in analogy with (3.5); $E(\alpha)$ is the difference in energy between the physical state $|\alpha)$ and the physical vacuum $|0)$; R is the resolvent and Σ_δ extends over diagrams δ which have the following properties; they have no ground state components, they are not entirely composed of one-particle components, nor is any of the smaller diagrams obtained by erasing in some δ all what is on the left or on the right of an intermediate state.

Consider the diagram of fig. 11, contributing to

$$\langle(\boldsymbol{p}' - \boldsymbol{k}_s)_{r'};\, s\lambda|S|\boldsymbol{p}_r;\{0\}\rangle,$$

where the action of an external field W is indicated by the dot on the electron line.

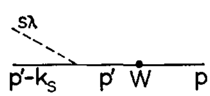

Fig. 11. Bremsstrahlung with the emission of one soft photon.

As $\omega_s \ll m$ we can calculate the diagram in the Bloch-Nordsieck approximation,

$$\langle(\boldsymbol{p} - \boldsymbol{k}_s)_{r'};\, s\lambda\, |S_{(11)}|\, \boldsymbol{p}_r;\, \{0\}\rangle = -\, 2\pi i\delta(E(\boldsymbol{p}' - \boldsymbol{k}_s) + \omega_s - E(\boldsymbol{p}))$$
$$N_p{}^{\frac{1}{2}} N_{p'}{}^{\frac{1}{2}} N_{Rs}{}^{\frac{1}{2}} W(\boldsymbol{p}', r'; \boldsymbol{p}, r).\, i(\boldsymbol{\mu}', \boldsymbol{a}_{s\lambda})/(\omega_s - (\boldsymbol{\mu}', \boldsymbol{k}_s)) = AN_{Rs}, \tag{4.18}$$

where $\boldsymbol{\mu}' = \boldsymbol{p}'/E(\boldsymbol{p}')$.

When we correct the photon line in fig. 11 to second order, we have to deal with the ten diagrams of fig. 12.

With the aid of the function $f_R(\omega_s, l)$ expanded as in (4.12) we find for their total contribution

$$\langle(\boldsymbol{p}' - \boldsymbol{k}_s)_{r'};\, s\lambda\, |S_{(12)}\, |\boldsymbol{p}_r;\, \{0\}\rangle = AN_{Rs}{}^{\frac{1}{2}}\, [-\, \tfrac{1}{8}e^2\omega_s{}^{-1}\, \Sigma_i\, c_i M_i\, +$$
$$+\, e^2(1/3\pi)\, \Sigma_i\, c_i \ln M_i/M_0 + \mathrm{O}(e^2\omega_s/m)], \tag{4.19}$$

*) See Frazer and Van Hove [11]) (4.20).

where A is the quantity defined in (4.18). (4.18) and (4.19) together give us in second order in e

$$\langle(\boldsymbol{p}' - \boldsymbol{k}_s)_{r'}; s\lambda\, |S_{(11)+(12)}|\, \boldsymbol{p}_r;\{0\}\rangle = AN_{Rs}{}^{(2)\frac{1}{2}}[1 - \tfrac{1}{8}e^2\omega_s{}^{-1} \textstyle\sum^i c_i M_i + $$
$$+ e^2(1/3\pi) \textstyle\sum_i c_i \ln M_i/M_0 + \mathrm{O}(e^2\omega_s/m)] =$$
$$= A[1 + e^2(1/3\pi) \textstyle\sum_i c_i \ln M_i/M_0 + \mathrm{O}(e^2\omega_s/m)], \quad (4.20)$$

where we used (4.14).

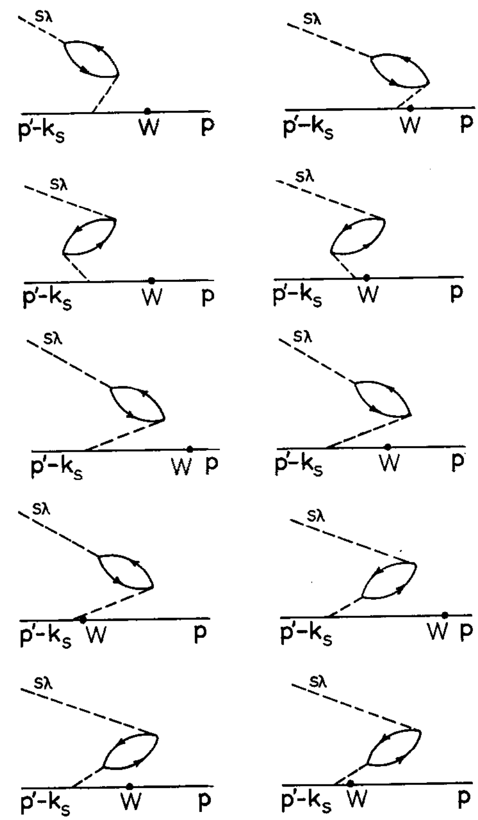

Fig. 12. Second order corrections to fig. 11 in the photon line.

According to renormalization theory the only effect of the corrections in fig. 12 for $\omega_s \to 0$ is to renormalize the charge. In this way we find the renormalized charge e' in second order *)

$$e' = e(1 + e^2(1/3\pi) \textstyle\sum_i c_i \ln M_i/M_0). \quad (4.21)$$

This result agrees with the covariant treatment of Pauli and Villars [10]). In view of (4.14) and (4.21) the assertion by Frazer and Van Hove [11]) that $N_{photon} = Z_3$, where Z_3 is the Z-constant of the photon in Dyson's

*) Note that there is no need to use the additional condition (4.15) in deriving (4.21).

notation [15]) (and therefore $Z_3^{\frac{1}{2}}$ the charge renormalization) is hereby shown to be incorrect. The argument given in their paper fails because it disregards the quadratic divergence in the photon self-energy and the need to apply the regularization prescription.

Returning to the problem discussed in section 3 we are now able to show that the diagrams of fig. 8 and fig. 9 only give in (3.9) corrections of relative order $O(e^2\bar{\omega}/m)$, under the condition that the renormalized and unrenormalized charges are put equal. In second order this amounts to putting

$$\sum_i c_i \ln M_i/M_0 = 0. \tag{4.22}$$

Indeed, when calculating the contribution C of fig. 3 + fig. 8 to the matrix element $\langle(\boldsymbol{p} - \boldsymbol{k}_s)_r;\, s\lambda|\{\sum_{j=1}^{\infty}(-\, D^0(E(\boldsymbol{p}))V)^j{}_{n\cdot i}\}|\, \boldsymbol{p}_r;\{0\}\rangle$ (see (3.4)), we find for $\omega_s \ll m$ by making extensive use of the function $f_R(\omega_s, l)$,

$$C = [-\, 2^{-\frac{1}{2}}i(\boldsymbol{\mu}, \boldsymbol{a}_{s\lambda})/(\omega_s - (\boldsymbol{\mu}, \boldsymbol{k}_s))][1 + e^2[\tfrac{1}{8}\omega_s{}^{-1} \sum_i c_i M_i + \\ + (1/3\pi) \sum_i c_i \ln M_i/M_0 + O(\omega_s/m)]]. \tag{4.23}$$

In this calculation the terms of order ω_s/m in the electron recoil are already neglected. When applying (4.15) and putting $e' = e$ (i.e. applying (4.22)) we find for C

$$C = [-\, 2^{-\frac{1}{2}}i(\boldsymbol{\mu}, \boldsymbol{a}_{s\lambda})/(\omega_s - (\boldsymbol{\mu}, \boldsymbol{k}_s))][1 + O(e^2\omega_s/m]. \tag{4.24}$$

The diagrams of fig. 9 are treated in a somewhat different way. One readily shows that under the conditions (4.15) and (4.22) the sum over $\boldsymbol{p}'$ and $s\lambda$ of the absolute squared contribution of fig. 9*ab* is of order $e^2\bar{\omega}/m$. The calculation is again based on extensive consideration of the function $f_R(\omega_s, l)$. As higher order corrections in the photon lines of fig. 3 can be treated in a similar way *), we can consider the proof of (3.9) to be complete.

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REFERENCES

1) Bloch, F., Nordsieck, A., Phys. Rev. **52** (1937) 54.
2) Jauch, J. M., Rohrlich, F., Helv. Phys. Acta **27** (1954) 613; The theory of photons and electrons. Addison-Wesley Publ. Comp., Inc. pp. 390–405.
3) Lomon, E. L., Nucl. Phys. **1** (1956) 101.
4) Nakanishi, N., Progr. Theor. Phys. **19** (1958) 159.
5) Yennie, D. R., Suura, H., Phys. Rev. **105** (1957) 1378.

*) For a treatment of regularization to higher order see Bogoliubov and Shirkov [16]), pp. 364 and foll.

6) Thirring, W., Phil. Mag. **41** (1950) 1193.
7) Brown, L. M., Feynman, R. P., Phys. Rev. **85** (1952) 231.
8) Van Hove, L., Physica **21** (1955) 901; **22** (1956) 343.
9) Hugenholtz, N. M., Physica **23** (1957) 481.
10) Pauli, W., Villars, F., Rev. Mod. Phys. **21** (1949) 434.
11) Frazer, W. R., Van Hove, L., Physica **24** (1958) 137.
12) Schweber, S., Bethe, H. A., De Hoffmann, F., Mesons and Fields. Row, Peterson and Comp.
13) Gupta, S. N., Proc. Phys. Soc. A **66** (1953) 137.
14) Ward, J. C., Phys. Rev. **78** (1950) 182.
15) Dyson, F. J., Phys. Rev. **75** (1949) 1736.
16) Bogoliubov, N. N., Shirkov, D. V., Introduction to the theory of quantized fields. Interscience Publ. Inc.