

## s-PARTICLE DOUBLETS IN CERTAIN LIGHT NUCLEI

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**Abstract:** The splitting of an s-particle doublet in  $B^{11}$  is examined to see what information it gives about the effective 1p-2s shell model interaction and it is concluded that the small splitting is explainable by an interaction whose spin dependence is primarily of  $\mathbf{s}_1 \cdot \mathbf{s}_2 \mathbf{t}_1 \cdot \mathbf{t}_2$  nature. Certain other s-doublets of the "hole-particle" type are considered too and their small splitting appears to show that the spin dependence of the effective n-p interaction between inequivalent nucleons is weaker than normally assumed. A very simple procedure is used to evaluate the interaction energies.

## 1. Introduction

In a recent  $B^{10}(d, p)$  experiment by Bilaniuk and Hensel <sup>1)</sup> a pair of single-particle s levels were observed at 9.19 and 9.28 MeV as well as a single-particle d level at 8.93 MeV which has a measurable s admixture. The spins of the three levels are believed <sup>2)</sup> †† to be as in fig. 1.

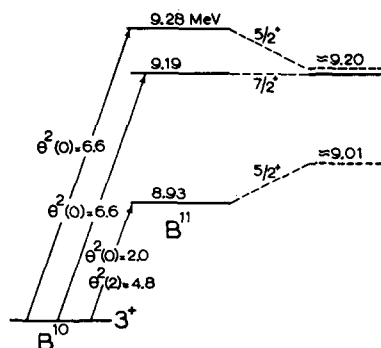


Fig. 1. The  $B^{11}$  single-particle levels with the observed  $l = 0$  and  $l = 2$  reduced widths (in an arbitrary unit) and the probable spins. The energies at the right are the "unperturbed" energies, the repulsive effect between the two  $\frac{5}{2}^+$  levels being subtracted out.

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†† Our conclusions would be essentially unaltered by a change in the assignments (all three levels must be  $\frac{5}{2}^+$ ,  $\frac{7}{2}^+$ ) as long as the upper two are assigned different spins.

By a "single particle" state we mean a state which is well described by a 2s or 1d particle coupled to the  $B^{10}$  ground state, this "weak coupling" model being that originally discussed by Lane <sup>3)</sup>. As far as the s-levels are concerned we would expect this to be a very favourable case for the weak-coupling model since no excited  $B^{10}$  state below 5.5 MeV can contribute to the  $\frac{7}{2}^+$  level and none below 3.6 MeV to the  $\frac{5}{2}^+$  level. Besides this we shall see that the interaction energy of an s particle with a  $T = 0$  core can be written in a very direct and simple way and it then becomes worthwhile to consider whether the small observed doublet splitting tells us anything about the shell model 1p-2s interaction. We begin by considering the three  $B^{11}$  levels as an isolated system and afterwards consider in a simple way the effects which core excitation and single-particle s-d mixing can have on the validity of the weak-coupling model. We shall pay a little attention also to a few other interesting s-particle doublets.

## 2. The s-Doublet Splitting in $B^{11}$

The observed splitting is 90 keV but the two  $\frac{5}{2}^+$  levels have been spread apart from their zero-order position by interaction between them. We need the zero-order splitting to compare directly with the calculated doublet splitting for the levels  $\{\phi_0 \times 2s\}$  (where  $\phi_0$  is the ground state of  $B^{10}$ ). If we write  $\Psi_1, \Psi_2$  for the  $\frac{5}{2}^+$  levels at 8.93 and 9.28 MeV respectively we have

$$\begin{aligned}\Psi_1 &= \alpha\{\phi_0 \times d\} + \beta\{\phi_0 \times s\}, \\ \Psi_2 &= -\beta\{\phi_0 \times d\} + \alpha\{\phi_0 \times s\}.\end{aligned}\tag{1}$$

Now directly from the  $l = 0$  relative reduced widths <sup>4)</sup> for the  $\frac{5}{2}^+$  states (as shown in fig. 1) we have  $|\alpha| = (6.6/8.6)^{\frac{1}{2}} = 0.88$ ,  $|\beta| = (2.0/8.6)^{\frac{1}{2}} = 0.48$ . The two wave functions together with the two known energy levels define the  $\frac{5}{2}^+$  Hamiltonian matrix in the  $\{\phi_0 \times l\}$  representation to be

$$H \approx \begin{pmatrix} 9.01 & \pm 0.15 \\ \pm 0.15 & 9.20 \end{pmatrix} \text{ MeV}.\tag{2}$$

If we had instead compared the widths for the two lowest levels (the middle one being a pure s level) we would have found slightly different values <sup>†</sup>; in fact making a very generous allowance for the uncertainty in the wave functions ( $\alpha^2 = 0.77 \pm 0.1$ ) we conclude that, for the zero-order s-doublet splitting  $\Delta$ , defined as  $E_{\frac{7}{2}}^{(0)} - E_{\frac{5}{2}}^{(0)}$ , and the interaction matrix element between the two  $\frac{5}{2}^+$  states, we have

$$\begin{aligned}-45 \text{ keV} &\leq \Delta \leq 25 \text{ keV}, \\ |\langle \{\phi_0 \times s\} H \{\phi_0 \times d\} \rangle| &\approx 140 \pm 20 \text{ keV}.\end{aligned}\tag{3}$$

<sup>†</sup> The corresponding 20 % discrepancy in the two determinations of the single-particle s width is within the range of fluctuation found for reduced widths in general. Note too that it is experimentally impossible to detect the  $l = 2$  admixture in the s levels.

We now calculate this splitting taking as a model that the wave functions are those of a 2s particle coupled to the ground state of  $B^{10}$  which in turn is described as belonging to the  $s^4p^6$  configuration (or more simply  $p^6$  since the  $s^4$  shell is inert) and more specifically as the particular state found by means of a standard intermediate-coupling calculation. Physically then one assumes, as is justified by the  $B^{10}$  spectrum, that the spin-orbit interaction is so strong in  $B^{10}$  that the s-particle interaction cannot uncouple the orbital and spin angular momenta of the core. The doublet splitting then arises from a difference in the orientations of the s-particle with respect to the core.

For cases such as presently interest us, the interaction matrix element is given by

$$\langle \{\phi_a \times l_1\} H \{\phi_b \times l'_1\} \rangle = \langle \phi_a H \phi_a \rangle \delta_{ab} \delta_{l_1 l'_1} \\ + \langle \phi_a(1 \dots n) \times l_1(n+1) | \sum_{i=1}^n H_{i,n+1} (1 - P_{i,n+1}) | \phi_b(1 \dots n) \times l'_1(n+1) \rangle. \quad (4)$$

Here  $\phi_a, \phi_b$  are two orthogonal states of the  $n$ -particle system and it is understood that the coupled particle  $l_1$  (or  $l'_1$ ) is not equivalent to any particle in  $\phi_a$  or  $\phi_b$ .  $\{\phi_a \times l\}$  is a completely antisymmetric state, the  $\times$  denoting vector coupling. If  $a = b$  and  $l_1 = l'_1$ , the first term on the right is simply the internal energy of the core ( $H$  of course being the  $n$ -particle Hamiltonian) and the second term, the interaction energy of the particle with the core, produces the splitting. For the rest of this section we shall be concerned with this interaction energy where the coupled particle is an s particle ( $l_1 = l'_1 = 0$ ).

We evaluate the energy by a simple direct procedure. If  $\phi$  belongs to  $l^n$ , closed shells being ignored, and if, as is conventionally assumed, the interaction is a central one  $\dagger$ , we can use the fact that the two-particle system  $\{l \times s\}$  has only four multiplets and thus, representing the  $l$ -s interaction by four independent two-body operators we can write by introducing the spin and isobaric spin operators ( $s^2 \equiv t^2 \equiv \frac{3}{4}$ ),

$$H_{12}(1 - P_{12}) = B_0 + B_s s_1 \cdot s_2 + B_t t_1 \cdot t_2 + B_{st} s_1 \cdot s_2 t_1 \cdot t_2. \quad (5)$$

We evaluate the quantities  $B$  by using

$$\langle (l \times s) | H_{12}(1 - P_{12}) | (l \times s) \rangle = \langle \{l \times s\} | H_{12} | \{l \times s\} \rangle,$$

which is simply the two-body multiplet energy. Writing this as  $^{2T+1, 2S+1}E$  we have four equations for the coefficients and solving these gives

$$\begin{pmatrix} 16B_0 \\ 4B_s \\ 4B_t \\ B_{st} \end{pmatrix} = \begin{pmatrix} 1 & 3 & 3 & 9 \\ -1 & 1 & -3 & 3 \\ -1 & -3 & 1 & 3 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} ^{11}E \\ ^{13}E \\ ^{31}E \\ ^{33}E \end{pmatrix}. \quad (6)$$

$\dagger$  We can of course ignore the single-particle spin-orbit interaction and consider only the two-body interaction. It will appear later that the central-force assumption is not essential for the  $B^{11}$  case.

The first three terms of eq. (5) lead to an interaction operator in eq. (4) which can be summed over the core particles. For the special cases where  $\langle \phi \mathbf{t}_i \phi \rangle$  or  $\langle \phi \mathbf{s}_i \phi \rangle$  is independent of  $i$ , the fourth term can be summed too and we have then

$$\sum_{i=1}^n H_{i,n+1}(1-P_{i,n+1}) \equiv nB_0 + B_s \mathbf{S}_1 \cdot \mathbf{s}_{n+1} + B_t \mathbf{T}_1 \cdot \mathbf{t}_{n+1} + \frac{B_{st}}{n} (\mathbf{S}_1 \cdot \mathbf{s}_{n+1})(\mathbf{T}_1 \cdot \mathbf{t}_{n+1}), \quad (7)$$

where  $\mathbf{S}_1$  and  $\mathbf{T}_1$  are spin and isobaric spin operators for the core. The principal cases in which eq. (7) is valid are those in which  $T_1 = 0$  (the  $B_t, B_{st}$  terms then do not contribute),  $S_1 = 0$ ,  $T_1 = \frac{1}{2}n$ ,  $S_1 = \frac{1}{2}n$  and of course the trivial ones where  $n = 1$  or where  $B_{st} = 0$ . From now on we deal only with such cases.

The matrix element can now be evaluated by a Landé type argument; since the angular momentum coupling is defined by the scheme

$$[(\mathbf{L}_1 \times \mathbf{S}_1)_{J_1} \times \mathbf{s}]_J, \quad \mathbf{S}_1 \equiv \frac{(\mathbf{S}_1 \cdot \mathbf{J}_1) \mathbf{J}_1}{J_1(J_1+1)}$$

and then since

$$\begin{aligned} \mathbf{S}_1 \cdot \mathbf{s}_{n+1} &\equiv \frac{(\mathbf{J}_1^2 + \mathbf{S}_1^2 - \mathbf{L}_1^2)(\mathbf{J}^2 - \mathbf{J}_1^2 - \mathbf{s}_{n+1}^2)}{4J_1(J_1+1)}, \\ \mathbf{T}_1 \cdot \mathbf{t}_{n+1} &\equiv \frac{1}{2}(\mathbf{T}^2 - \mathbf{T}_1^2 - \mathbf{t}_{n+1}^2), \end{aligned} \quad (8)$$

the interaction energy becomes

$$\begin{aligned} nB_0 + \frac{1}{2}B_t \{T(T+1) - T_1(T_1+1) - \frac{3}{4}\} \\ + \frac{1}{4} \left[ B_s + \frac{B_{st}}{2n} \{T(T+1) - T_1(T_1+1) - \frac{3}{4}\} \right] \\ \cdot [J(J+1) - J_1(J_1+1) - \frac{3}{4}] \left[ 1 - \frac{\langle \mathbf{L}_1^2 - \mathbf{S}_1^2 \rangle}{J_1(J_1+1)} \right], \end{aligned} \quad (9)$$

and of course  $\{T(T+1) - T_1(T_1+1) - \frac{3}{4}\} = T_1, -(T_1+1)$  for  $T = T_1 + \frac{1}{2}, T_1 - \frac{1}{2}$  respectively, and similarly for the  $J$  expression. In eq. (9),  $\langle \mathbf{L}_1^2 - \mathbf{S}_1^2 \rangle$  is the expectation value in the core state of the indicated operator.

For the case  $T_1 = 0$  the doublet splitting  $\Delta \equiv E_{J>} - E_{J<}$  is a significant quantity and we have

$$\Delta = \frac{(2J_1+1)}{4} B_s \left\{ 1 - \frac{\langle \mathbf{L}_1^2 - \mathbf{S}_1^2 \rangle}{J_1(J_1+1)} \right\}, \quad (10)$$

while rewriting the interaction energy for this case gives

$$E_{J>} = nB_0 + \frac{J_1 \Delta}{(2J_1+1)}, \quad E_{J<} = nB_0 - \frac{(J_1+1) \Delta}{(2J_1+1)}. \quad (11)$$

The expectation value  $\langle \mathbf{L}_1^2 - \mathbf{S}_1^2 \rangle$  is trivial to calculate if the core function is available in an  $LS$  representation. Using functions previously derived (defined by a Rosenfeld <sup>5)</sup> interaction with the Slater-integral ratio  $L/K = 6$ ) we find for the  $B^{10}$  ground state  $\langle \mathbf{L}_1^2 - \mathbf{S}_1^2 \rangle / [J_1(J_1 + 1)] = 0.49, 0.51, \frac{1}{3}$  for  $\zeta = 3.8, 5.7, \infty$  where  $\zeta \equiv a/K$  is the usual spin-orbit parameter and the value  $\infty$  defines  $jj$  coupling. The result is rather insensitive to the spin-orbit parameter as it should be in any practical case when  $J_1$  is large. Taking the  $jj$  value we then conclude that

$$A = \frac{7}{6}B_s = -\frac{7}{24}\{^{11}E - ^{13}E + 3\ ^{31}E - 3\ ^{33}E\}, \quad (12)$$

and inversely the experimental splitting determines a very small (essentially zero) value for this combination of the two-body  $1p-2s$  multiplet energies. The splitting can of course be written simply in terms of the  $jj$  coupling two-body energies but this we leave for the next section.

If now we ignore the exchange interaction in eq. (4) (the exchange integral should be an order of magnitude smaller than the direct one) we see that the splitting will be proportional to the  $\mathbf{s}_1 \cdot \mathbf{s}_2$  part of the interaction and will therefore vanish for an interaction such as Rosenfeld's <sup>5)</sup> which is proportional to  $\mathbf{t}_1 \cdot \mathbf{t}_2$ ; the other interaction which has been conventionally used in p-shell spectroscopy, that of Inglis <sup>6)</sup>, should produce a large splitting. Since the  $p-s$  energies calculated with a conventional interaction span an energy region of 10 MeV we expect then that a Rosenfeld-type interaction should give a "small" splitting, a few hundred kilovolts, while an Inglis-type interaction should give a "large" splitting of a few MeV. This is exemplified in table 1, which gives the energies for a Gaussian interaction.

TABLE 1

The  $p-s$  multiplet energies and interaction constants (in MeV) for a Gaussian interaction with  $L/K = 6.6$  and  $K = -1.1$  MeV.

	$^{11}E$	$^{13}E$	$^{31}E$	$^{33}E$	$B_0$	$B_s$	$B_t$	$B_{tt}$
Rosenfeld	7.5	-5.5	-3.3	1.4	-0.41	0.27	2.5	17.7
Inglis	4.1	-5.5	-3.3	2.5	0.01	1.9	4.1	15.4

The experimental result of course favours the Rosenfeld-type interaction but still the 300 keV value calculated is much larger than the "very small" value experimentally found and we might ask whether a different interaction still compatible with the many  $p$  shell results could indeed give the observed zero splitting. For this purpose we observe that for a given radial dependence (adequately specified by the conventional  $L/K$  for  $p^2$  and  $I_e/I_d$ , the ratio of the exchange and direct integrals, for  $ps$ ) we can express the  $ps$  multiplet energies in terms of those for  $p^2$ . This simple operation then gives for  $B_s = 0$  the

following equation for the  $p^2$  multiplet energies  $^{2T+1, 2S+1}L$ :

$$[3 \text{ } ^{33}\text{P} - ^{11}\text{P}] = \frac{(1-3K/L)(1+I_e/I_d)}{(1+2K/L)(1-I_e/I_d)} [3 \text{ } ^{31}\text{S} - ^{13}\text{S}]. \quad (13)$$

The Gaussian interaction which we use as an example gives  $K/L = 0.15$ ,  $I_e/I_d = 0.14$  and any other reasonable radial dependence will give about the same results. Then since one knows that  $^{31}\text{S} \approx -5.5$  MeV,  $^{13}\text{S} \approx -9$  MeV, the condition is that  $[3 \times ^{33}\text{P} - ^{11}\text{P}] \approx -4.5$  MeV. This may be satisfied for example by  $^{33}\text{P} = 0$ ,  $^{11}\text{P} = 4.5$  MeV without disturbing any of the  $p$  shell results.

We conclude from all this that there may be a good reason for the  $B^{11}$  doublet splitting to be *small* (namely the interaction having a  $\mathbf{t}_1 \cdot \mathbf{t}_2$  dependence) but that there seems no general reason why it has the *very small* value found, though the latter value is quite compatible with the  $p$  shell results. In fact, because of departures from the simple model (which we consider in the next section) it may be that the very small value has no particular significance. Besides this, of course, there seems no *a priori* reason why the effective central interaction valid for  $p^2$  should be satisfactory for  $ps$ , though whether it is or not is a question of interest.

We obviously expect to find further  $s$ -doublets with "small" (though not necessarily "very small") splittings at higher excitations in  $B^{11}$  (according to eq. (11) since  $\Delta \approx 0$  their position should in principle simply mirror the  $T = 0$  spectrum of  $B^{10}$ ), and similarly in other odd-mass nuclei throughout the  $p$  shell as discussed by Lane <sup>3</sup>). However, many of the possible cases which one thinks of are far less favourable than the present one because there are other low-lying core states which can contribute. As far as the absolute energy of the  $s$  levels is concerned we observe that the binding energy of the  $s$  levels to a  $T_1 = 0$  core is simply  $nB_0$ . Then proceeding in the same way as Unna and Talmi <sup>7</sup>) but restricting ourselves to levels with a  $T_1 = 0$  core and to cases where the excitation corrections should not be too large we could in principle determine  $B_0$  from the known  $s$  level positions. Of course the  $s$ -particle binding does not display the characteristic difference found between the binding energy of a  $p$  particle to an  $A = 4m$  and  $A = (4m+2)p^n$  core. The fact that the lowest  $s$  level is at 3 MeV in  $C^{13}$  while the  $B^{11}$  doublet is at 9 MeV simply reflects this difference.

### 3. Some Corrections

In this section we first consider briefly the circumstances under which we should expect the simple model used above to be satisfactory. We do this by calculating some corrections to it. These are of two types corresponding either to a mixing of the core states (in which, in the above case, the  $B^{10}$  core would be excited) or to a mixing of the single particle states (in which the  $s$  particle would

be promoted to d). One such matrix element of the second class has already been empirically determined to be about 140 keV and we shall calculate it using explicit interactions. It is very convenient and obviously satisfactory to calculate the corrections in  $jj$  coupling and we first derive the  $jj$  analogues of equations (9—11).

The two-particle system  $\{j \times s_{\frac{1}{2}}\}$  has four states and consequently, when dealing with the interaction of  $s_{\frac{1}{2}}$  with  $j^n$ , we can write,

$$H_{12}(1-P_{12}) = B_0 + B_j \mathbf{j}_1 \cdot \mathbf{j}_2 + B_t \mathbf{t}_1 \cdot \mathbf{t}_2 + B_{jt}(\mathbf{j}_1 \cdot \mathbf{j}_2)(\mathbf{t}_1 \cdot \mathbf{t}_2) \quad (14)$$

and then for the several cases  $T_1 = 0, J_1 = 0, T_1 = \frac{1}{2}n, J_1 = nj, n = 1, B_{jt} = 0$  we have for the interaction energy and doublet splitting

$$E_J = nB_0 + \frac{1}{2}B_t\{T(T+1) - T_1(T_1+1) - \frac{3}{4}\} \\ + \frac{1}{2} \left[ B_j + \frac{1}{2n} B_{jt}\{T(T+1) - T_1(T_1+1) - \frac{3}{4}\} \right] [J(J+1) - J_1J_1+1 - \frac{3}{4}], \quad (15)$$

$$\Delta = \frac{1}{2}(2J_1+1)B_j,$$

which then gives for the  $B^{11}$  doublet  $\Delta = \frac{7}{2}B_j$ .

The constants  $B$  are given in terms of the two-body  $jj$  energies  $E_{TJ}(T = 0, 1; J = j \pm \frac{1}{2})$  by

$$(4j+2) \begin{pmatrix} B_0 \\ B_j \\ B_t \\ B_{jt} \end{pmatrix} = \begin{pmatrix} \frac{3}{2}(j+1) & \frac{3}{2}j & \frac{1}{2}(j+1) & \frac{1}{2}j \\ 3 & -3 & 1 & -1 \\ 2(j+1) & 2j & -2(j+1) & -2j \\ 4 & -4 & -4 & 4 \end{pmatrix} \begin{pmatrix} E_{1>} \\ E_{1<} \\ E_{0>} \\ E_{0<} \end{pmatrix}. \quad (16)$$

It is not necessary to assume here a central interaction since, unlike the  $LS$  case, every interaction is diagonal in the two-particle states. We can infer from this, since  $jj$  coupling is not a bad approximation for  $B^{10}$ , that our calculated  $B^{11}$  doublet splitting is likewise insensitive to the central-interaction assumption provided we express it in terms of the two-body  $jj$  energies. If we do have a central interaction given by eq. (5) the two sets of interaction constants of eqs. (5) and (14) are related by observing <sup>†</sup> that, if  $l, j$  are specified for a particle, a Landé argument gives

$$\mathbf{s} \equiv \frac{(\mathbf{s} \cdot \mathbf{j})\mathbf{j}}{j(j+1)} \equiv \frac{(-1)^{j-l-\frac{1}{2}}}{(2l+1)} \mathbf{j}. \quad (17)$$

<sup>†</sup> Or alternately from

$$E_{T, J=l(j, s_{\frac{1}{2}})} = \frac{1}{2}(2l+1)^{-1}\{(2j+1)^{2T+1, 1}L + (4l-2j+1)^{2T+1, 3}L\}, \\ E_{T, J \neq l(j, s_{\frac{1}{2}})} = {}^{2T+1, 3}L.$$

Then for two particles  $l_j$  and  $s_{\frac{1}{2}}$ ,  $(\mathbf{s}_1 \cdot \mathbf{s}_2) \equiv (2l+1)^{-1}(-1)^{j-l-\frac{1}{2}}(\mathbf{j}_1 \cdot \mathbf{j}_2)$  and thus we have †

$$\frac{B_j}{B_s} = \frac{B_{jt}}{B_{st}} = (2l+1)^{-1}(-1)^{j-l-\frac{1}{2}}, \quad (18)$$

while  $B_0$ ,  $B_t$  have the same significance in the two forms. The equality of the two forms found for the  $B^{11}$  doublet splitting ( $\Delta = \frac{7}{6}B_s = \frac{7}{2}B_j$ ) supplies an example of eq. (18).

Considering now the core-excitation corrections, and restricting ourselves to excitations inside the same configuration as the principal core state, we see that the first three terms of eq. (14) have no off-diagonal matrix elements at all and thus produce no first-order mixing of the core states. The fourth term of eq. (14) cannot admix two  $T_1 = 0$  states. Thus, ignoring for the moment the single-particle s—d mixing, we conclude that the simple model of an s particle coupled to a  $T_1 = 0$  core might be very good unless there is, not too far away, a  $T'_1 = 1$ ,  $J'_1 = J_1$ ,  $J_1 \pm 1$  core state which may be admixed. In the case of the  $B^{11}$  doublet built on the  $B^{10}$  ground state there is no such state.

The doublet founded on the first excited  $B^{10}$  state ( $T, J = 0, 1$ ) would be a quite different case since a  $(1, 0)$  state lies one MeV away. A more interesting quite analogous case would be the  $N^{15}$  doublet involving the  $N^{14}$  ground state where once again an excited  $(T_1 J_1) = (1, 0)$  state can admix with the  $(0, 1)$  ground state. A simple calculation gives for the matrix element connecting the two states  $\{s_{\frac{1}{2}} \times (p_{\frac{1}{2}})_{T=0, J=1}^2\}$  and  $\{s_{\frac{1}{2}} \times (p_{\frac{1}{2}})_{T=1, J=0}^2\}$  the value  $\frac{3}{8}B_{jt}$  and relating this to  $B_{st}$  by eq. (18) and using table 1 we have that our Gaussian interaction of either exchange nature would produce a value for  $-\frac{1}{8}B_{st}$  of about 2 MeV. Since the two core states are only 2.4 MeV apart (which becomes 2.4 MeV +  $B_j - B_t$  when the s-particle interaction is added in) we obviously have the possibility of large admixtures and large distortion of the s-doublets splitting †† in  $N^{15}$ .

As far as the second type of admixing is concerned (the single particle s—d mixing), we guess that it is not usually important for the lower states. This is certainly demonstrated by the  $N^{15}$  calculations<sup>8)</sup> and also is consistent with the very small value (140 keV) found for the matrix element “measured” in the  $B^{11}$  experiment. A calculation of that matrix element, by the way, produces, with the Gaussian interaction the considerably larger value 800 keV for either exchange mixture. The reason for the disagreement is unclear but the fact that the measured matrix element is smaller reinforces the belief that s—d admixtures are in general small.

† For a coupled  $p_{\frac{1}{2}}$  particle ( $l_j$  and  $p_{\frac{1}{2}}$ ) the ratio is  $-\frac{1}{3}(2l+1)^{-1}(-1)^{j-l-\frac{1}{2}}$ .

†† Moreover it is very probably not reasonable to consider only the two  $(p_{\frac{1}{2}})^2$  core states, as is shown by the elaborate calculations of Halbert<sup>8)</sup>. A simple and reasonable treatment would use the *LS* form for the interaction and consider the first *three* core states, also however taking account of the effects due to the center-of-mass motion. The  $N^{15}$  example has been considered also by Unna and Talmi<sup>7)</sup> but using only the lowest *jj* configurations.



#### 4. Hole-Particle s-Doublets

Doublets of this type occur in  $N^{16}$  (involving the first and third excited states) and as ground-state doublets in  $Al^{28}$  and  $P^{32}$ , the doublet splittings being respectively 270, -31, 77 keV. In each case there is strong evidence that the states are well described in  $jj$  coupling, the configurations being  $\{j^{-1} \times s\}_{T=1}$  or  $\{j \times s^{-1}\}_{T=1}$  with  $j \equiv p_{\frac{1}{2}}, d_{\frac{3}{2}},$  and  $d_{\frac{5}{2}}$  respectively. The  $N^{16}$  levels have been studied in detail by Elliott and Flowers<sup>9)</sup> using an explicit interaction and, among many other pieces of data, have been considered by Unna and Talmi<sup>7)</sup> in their determination of the  $p_{\frac{1}{2}}-s_{\frac{1}{2}}$  interaction constants. The other doublets have been considered by Mayer and by Inglis as reported in Inglis' review article<sup>6)</sup>.

As they stand our equations (15) do not apply in this case but on the other hand it is obvious, from the tensorial character of the separate terms, that if the particle-particle interaction is given by (14) with  $n = 1$ , then to within a multiple of  $B_0$ , the hole-particle interaction is given by the same expression with the sign of  $B_{jt}$  changed<sup>†</sup>. Thus

$$\Delta(j^{\pm 1} \times s) = \frac{1}{2}(2j+1)(B_j \pm \frac{1}{4}B_{jt}). \quad (19)$$

The  $\{p_{\frac{1}{2}} \times s_{\frac{1}{2}}\}$  doublet may be identified<sup>††</sup> in the  $A = 14$  nuclei and has  $\Delta \approx -650$  keV. Comparing the  $\{p_{\frac{1}{2}} \times s\}$  and  $\{p_{\frac{1}{2}}^{-1} \times s\}$  doublets then shows that  $B_{jt} < 0$  and  $|B_{jt}| > 4B_j$ ; indeed taking both splittings seriously would give  $B_j \approx -200$  keV,  $B_{jt} \approx -1800$  keV. If instead we determine the constants from the four  $\{p_{\frac{1}{2}} \times s_{\frac{1}{2}}\}$  energies<sup>10)</sup> we have  $B_j \approx -280$  keV,  $B_{jt} \approx -1400$  keV and, for  $N^{16}$ ,  $\Delta \approx +70$  keV. The constants which follow from the work of Unna and Talmi<sup>7)</sup> are  $B_j = -270$  keV,  $B_{jt} = -2060$  keV which fit both splittings quite well and should of course be regarded as the best values.

<sup>†</sup> Alternatively observe<sup>6)</sup> that the interaction which produces the splitting is simply the  $n-p$  interaction which is half the sum of the  $T = 0$  and  $T = 1$  interactions. Since then  $\mathbf{t}_1 \cdot \mathbf{t}_2 \rightarrow -\frac{1}{4}$  we have from eq. (14) that

$$H_{12}(1-P_{12}) \rightarrow H_{12}^{n,p} = (B_0 - \frac{1}{4}B_t) + (B_j - \frac{1}{4}B_{jt})\frac{1}{2}\mathbf{j}_1 \cdot \mathbf{j}_2;$$

this of course to be used without antisymmetrization between neutrons and protons. Eq. (19) now follows directly from this as do the more general equations for the interaction between an  $s_{\frac{1}{2}}$  neutron and a  $(j^m)_{J_1}$  proton core (the  $j$ -neutron subshell being filled), namely

$$\begin{aligned} E_J &= m(B_0 - \frac{1}{4}B_t) + \frac{1}{2}(B_j - \frac{1}{4}B_{jt})[J(J+1) - J_1(J_1+1) - \frac{3}{4}], \\ \Delta &= \frac{1}{2}(2J_1+1)(B_j - \frac{1}{4}B_{jt}). \end{aligned}$$

Of course  $\Delta[j(p) \times s_{\frac{1}{2}}(n)] = \Delta[j^{-1}(p) \times s_{\frac{1}{2}}(n)]$  (and it would be of interest to see whether  $P^{34}$  has the same ground-state doublet as  $P^{32}$ ). If in the equations for  $E_J$ , we change the signs of  $B_t$ ,  $B_{jt}$  we have the interaction of an  $s_{\frac{1}{2}}$  neutron with an open *neutron* subshell, both proton subshells  $j$ ,  $s$  being filled (or both empty). Finally the equations have obvious  $LS$  analogues.

<sup>††</sup> We follow here the assignments made by Warburton and Pinkston<sup>10)</sup> which differ slightly from those used by Unna and Talmi<sup>7)</sup>.

Our explicit interactions produce values for  $B_{jt}$  (see table 1 and eq. (18)) which have the correct sign but which are too large by a factor 3. They would produce an  $N^{16}$  splitting  $\approx 1400$  keV (Rosenfeld) or 640 keV (Inglis); these numbers would become 1000 keV and 500 keV with a Yukawa radial dependence. This appears to indicate that the shell model  $1p-2s$  interaction has a weaker spin dependent  $n-p$  part than does the corresponding  $p^2$  interaction. The same conclusion will come from the  $Al^{28}$  and  $P^{32}$  doublets.

The  $B_s \approx 0$  value implied by the  $B^{11}$  doublet splitting would, if the interaction were central, imply that the  $N^{14}$  and  $N^{16}\{p_{\frac{1}{2}}^{\pm 1} \times s_{\frac{1}{2}}\}$  splittings would be equal and opposite but it is hard to guess the significance of the deviation from this result which is observed. On the other hand if we knew the  $\{p_{\frac{3}{2}}^{-1} \times s_{\frac{1}{2}}\}$  levels of  $N^{16}$  we might be able to decide (if the levels were not too broad) whether we have an effective *central* interaction; such an interaction would imply, via eq. (18),

$$\Delta(p_{\frac{3}{2}}^{-1} \times s_{\frac{1}{2}}) = -2\Delta(p_{\frac{1}{2}}^{-1} \times s_{\frac{1}{2}}), \quad (20)$$

and any major deviation not ascribable to interaction with nearby levels could be used to argue that the effective interaction has a non-central part.

The conventional interactions will give, for  $Al^{28}$  and  $P^{32}$ , doublet splittings of the correct sign but with far too large values. As discussed by Inglis<sup>6)</sup> the Majorana interaction, the dominant part of a conventional interaction, does not contribute to an  $n-p$  s-doublet<sup>†</sup> splitting but this is not sufficient to explain the smallness of the observed splittings. Formally, writing the interaction as  $[A_W + A_B B_{12} + A_H H_{12} + A_M M_{12}]V(r_{12})$  where  $B$ ,  $H$ ,  $M$  are the exchange operators (and  $H \equiv (-1)^{T+1}$ ), we find

$$\Delta = (-1)^{j-l-\frac{1}{2}} \frac{(2j+1)}{(2l+1)} \{A_B I_d - A_H I_e\}, \quad (21)$$

where  $I_d$ ,  $I_e$  are the direct and exchange integrals for  $V(r)$ . The interaction of table 1 gives  $I_d = -3.7$  MeV,  $I_e = -0.85$  MeV if we maintain the same  $r_1$  value (the single-particle radial function having the factor  $\exp(-r^2/r_1^2)$ ) and then the calculated splittings, for  $Al^{28}$  and  $P^{32}$  respectively, are  $-1800$ ,  $+1200$  keV (Rosenfeld) or  $-900$ ,  $+600$  keV (Inglis). One actually expects that  $r_1$  might decrease by 15 %, and there is evidence from the equivalent-particle spectra that the range of the effective interaction might increase by as much as 50 %, its strength at the same time decreasing. But these effects can scarcely change the splittings by as much as a factor two; in any case a large overall reduction in the radial integrals would be inconsistent with the  $\{d_{\frac{1}{2}} \times s_{\frac{1}{2}}\}_{T=1}$  doublet<sup>††</sup> splitting which is apparently<sup>11)</sup> larger than 1.2 MeV, and with the

<sup>†</sup> A descriptive alternative name for doublets, such as the hole-particle doublets with maximum isobaric spin, in which the splitting is due to the  $n-p$  interaction.

<sup>††</sup> It would be worthwhile to locate the members of this doublet by the  $Si^{29}(d, p)$  experiment.

“absolute”  $n-p$  energies determined <sup>12)</sup> for the  $\{d_{\frac{3}{2}} \times s_{\frac{1}{2}}\}$  nuclei to be about  $-1$  MeV.

It seems then that the smallness of the  $n-p$  s-doublet splittings indicates a cancellation between the two terms of eq. (19) (or (21)) which is not given by the conventional interactions. Moreover if in eq. (21) we put  $\Delta \approx 0$ ,  $I_e/I_d = \frac{1}{4}$  and rewrite the equation in terms of the  $p^2$  energies with  $L/K = 6$  we find

$$(^{33}\text{P} - ^{11}\text{P}) \approx 1.6 (^{31}\text{S} - ^{13}\text{S}) \approx 6 \text{ MeV}. \quad (26)$$

This result is not compatible with binding-energy data in the  $p$  shell which demands  $^{33}\text{P} \approx 0$  nor for that matter with the low-lying level structure of  $\text{Li}^8$  which requires that the  $^{11}\text{P}$  multiplet be at least  $2-3$  MeV above the  $^{33}\text{P}$ .

One tends to conclude then that no central interaction compatible with the  $p$ -shell data will explain the smallness of the  $n-p$  splittings and that a different approach is required to explain them. Finally we note that a central interaction would give  $\Delta(\text{Al}^{28}) = -\frac{3}{2}\Delta(\text{P}^{32})$  and this is satisfied in the sense that both splittings have an essentially zero value.

## 5. Final Remarks

Using very simple procedures we have looked into the information which certain s-doublet splittings give about the shell model interaction. It seems that  $\text{B}^{11}$  doublet splitting, which measures the average spin-dependent interaction between two nucleons (with a  $(2T+1)$  weighting) is easily explainable by an interaction which also fits the many  $p$ -shell results. On the other hand the hole-particle splittings, which measure the spin-dependent  $n-p$  interaction, are too small to be thus explainable. Further information about levels of  $\text{N}^{16}$  (as well as  $^{\dagger} \text{B}^{12}$ ) and  $\text{Si}^{30}$  would be helpful.

From the formal standpoint the reason why a simple technique has sufficed to evaluate the interaction energy is that, except for the double vector  $\mathbf{s}_i \cdot \mathbf{t}_i$ , the interaction operator is expressible in terms of simple vectors; this simplicity would be lost if the smaller of the two interacting  $j$ 's were greater than  $\frac{1}{2}$ , for then of course tensor interactions of higher order would enter and the simple vector procedure would produce only sum rules for the energies; more important is the fact that a calculation would require more information about the core state. Note too that the interaction of a *group* of  $s$  particles with a core, as considered for example by Unna and Talmi <sup>7)</sup>, can also be handled by the direct procedure.

One correction which has been ignored is that connected with the fact

<sup>†</sup> We expect, from eq. (18) that  $\Delta(p_{\frac{3}{2}}^{-1} \times s_{\frac{1}{2}}) < 0$  and thus that the 1.67 MeV level in  $\text{B}^{12}$  (which is the first negative-parity level and has a large  $l = 0$  reduced width to  $\text{B}^{11}$ ) should be  $2^-$ . Our further expectation that  $\Delta(p_{\frac{3}{2}}^{-1} \times p_{\frac{1}{2}}) > 0$  is satisfied since the  $\text{B}^{12}$  ground-state spin is  $1^+$ . (However, the  $\text{B}^{12}$  spectrum would not determine a value of this splitting satisfactory for use in eq. (20) because, as shown by the probable existence of more than two strong  $l = 1$  transitions in  $\text{B}^{11}(d, p)\text{B}^{12}$   $jj$  coupling is not a sufficiently accurate approximation.) Observe too that the  $\text{Y}^{90}$  ground state doublet (apparently <sup>13)</sup> a  $\{p_{\frac{1}{2}} \times d_{\frac{3}{2}}\}$   $n-p$  doublet) has  $\Delta > 0$  as we would expect. The level ordering of all the  $n-p$  doublets we have considered is in agreement with Nordheim's rules.

discussed by Elliott and Skyrme<sup>14</sup>), that the shell model functions for excited configurations may contain a part in which the centre of mass of the system has an orbital angular momentum. The way which they suggest to eliminate this difficulty, and which has been used on a few occasions, is to consider the entire function space of the states which are degenerate in the harmonic oscillator sense and to use the subspace of this in which the centre of mass is in an  $S$  state. It is physically not clear that this is always the most reasonable procedure<sup>15</sup>); for example if there is a strong spin-orbit coupling<sup>†</sup> it may be more economical of energy to bring the center of mass to rest by admixing in low states of an even more highly excited configuration. Besides this the procedure is opposed in spirit to the weak coupling model; it is clear that, when we have a particle weakly coupled to a core which is hard to excite, it should not be necessary to reconstruct the core function using a different set of basic states than are used when the particle is not present. It is possible in fact to consider a model in which the added particle is described with respect to the centre of mass of the core (instead of the entire system) and in which the summed interactions of the added particle with the core particles provided a one-body potential for the added particle. This is not quite the same as the conventional shell model but it seems easy in this case to treat the centre of mass problem and it is obvious that the formal procedures used above for handling the spin and isobaric spin dependence of the interaction can be used without change. It seems improbable then that the conclusions which we have drawn about the nature of the interaction will be modified by a reasonable treatment of the centre of mass effects though a closer study would seem to be desirable.

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*Notes added in proof:* (1) Professor D. H. Wilkinson, in a private communication, suggests that the 8.93 MeV level has negative parity, this result following from an analysis of a low-energy ( $d, p$ ) experiment. If this be so the off-diagonal matrix element of eq. (3) should be ignored as well as the repulsive effect between the two levels shown in fig. 1, the  $s$ -doublet splitting of eq. (3) becoming then  $-90$  keV. However, all the arguments and conclusions about the interaction would be unchanged.

(2) The  $n$ - $p$   $s$ -doublet splitting in certain heavier nuclei has recently been calculated by F. C. Barker (private communication and Nuclear Physics, in press), using an interaction (given by Elliott and Flowers<sup>9</sup>) which has a weakly spin-dependent  $n$ - $p$  part. These calculations, which give small splittings

<sup>†</sup> Or if, for some other reason, the spacing between single-particle levels is less than that corresponding to their harmonic oscillator excitation.

in apparent agreement with experiment, reinforce the conclusion that the n—p interaction between inequivalent nucleons is indeed weakly spin-dependent, at least if one of the nucleons is in an s-orbit. The interaction used would, however, not fit the  $p^4$  spectrum of  $\text{Li}^8$ .

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