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LETTER TO THE EDITOR

Statistical errors of coëfficients of experimental angular distributions expanded into spherical harmonics

For comparison of experimentally determined angular distributions with theory an expansion into spherical harmonics may be useful. The experimental data generally are subject to statistical errors introduced by the limited number of events observed at any particular angle. The question then arises as to the order of magnitude of the statistical errors of the coëfficients obtained by numerical integration from such an experimental angular distribution. A computation of these errors will be given in this note.

Let us suppose that $I_1 \ldots I_k \ldots I_n$ events have been observed at angles $\vartheta_1 \ldots \vartheta_k \ldots \vartheta_n$ corresponding to the cosines $\mu_1 \ldots \mu_k \ldots \mu_n$ ($\mu = \cos\vartheta$) spread out at equal intervals through the region $-1 < \mu < 1$. The angular distribution $I(\mu)$ is expanded into spherical harmonics:

$$I(\mu) = \Sigma_{i=0}^{\infty} a_i P_i(\mu),$$

the coëfficients a_i being obtained by numerical integration from:

$$a_i = (i + \tfrac{1}{2}) \int_{-1}^{1} I(\mu) P_i(\mu) d\mu$$

or, if written as a summation:

$$a_i = (i + \tfrac{1}{2}) \Delta\mu \, \Sigma_{k=1}^{n} I_k P_i(\mu_k).$$

If errors are of a purely statistical nature the error $\tilde{I}_k$ in I_k is given by:

$$\tilde{I}_k = (I_k)^{\frac{1}{2}}.$$

The statistical error $\tilde{a}_i$ in a_i then becomes:

$$\tilde{a}_i = (i + \tfrac{1}{2}) \Delta\mu \, [\Sigma_{k=1}^{n} I_k \{P_i(\mu)\}^2]^{\frac{1}{2}}$$

which may be replaced by an integral:

$$\tilde{a}_i = (i + \tfrac{1}{2}) (\Delta\mu)^{\frac{1}{2}} \left[\int_{-1}^{1} I(\mu) \{P_i(\mu)\}^2 d\mu \right]^{\frac{1}{2}}.$$

The square of the Legendre polynomial may be transformed into a finite sum of Legendre polynomials [1]):

$$\{P_i(\mu)\}^2 = \Sigma_{r=0}^{i} (A_r^2 A_{i-r}/A_{i+r}) \{(4r+1)/(2i+2r+1)\} P_{2r}(\mu)$$

with $A_r = 1.3.5 \ldots (2r-1)/r!$ and $A_0 = 1$.

By substituting this into the integral we obtain:

$$\tilde{a}_i = (i + \tfrac{1}{2}) (\Delta\mu)^{\frac{1}{2}} \{ \Sigma_{r=0}^{i} 2a_{2r} A_r^2 A_{i-r}/A_{i+r} (2i + 2r + 1)\}^{\frac{1}{2}}.$$

This can be simplified somewhat by introducing the total number N of observed events:

$$N = \Sigma_{k=1}^{n} I_k = 2a_0/\Delta\mu,$$

and by taking all coëfficients and errors of coëfficients relative to a_0. We then get finally:

$$\tilde{a}_i/a_0 = \{(2i + 1)/N^{\frac{1}{2}}\} [\Sigma_{r=0}^{i} \{A_r^2 A_{i-r}/A_{i+r} (2i + 2r + 1)\} (a_{2r}/a_0)]^{\frac{1}{2}}.$$

For the first four coëfficients this expression for the statistical errors becomes:

$$\tilde{a}_0/a_0 = (1/N)^{\frac{1}{2}},$$
$$\tilde{a}_1/a_0 = (3/N)^{\frac{1}{2}} (1 + 2a_2/5a_0)^{\frac{1}{2}},$$
$$\tilde{a}_2/a_0 = (5/N)^{\frac{1}{2}} (1 + 2a_2/7a_0 + 2a_4/7a_0)^{\frac{1}{2}},$$
$$\tilde{a}_3/a_0 = (7/N)^{\frac{1}{2}} (1 + 4a_2/15a_0 + 2a_4/11a_0 + 100a_6/429a_0)^{\frac{1}{2}}.$$

As an example we apply these results to the angular distribution of protons from the $C^{13}(d, p)C^{14}$ reaction [2]) measured at an effective bombarding energy of 370 keV. The total number of counted proton tracks amounted to $N = 21{,}379$, while the even coëfficients were given by: $a_2/a_0 = 0.75$, $a_4/a_0 = 0.08$.

This gives: $\tilde{a}_0/a_0 = 0.007$, $\tilde{a}_1/a_0 = 0.013$, $\tilde{a}_2/a_0 = 0.017$ and $\tilde{a}_3/a_0 = 0.020$.

It must be remarked that measurements were made at equal ϑ-intervals, instead of at equal $\cos\vartheta$-intervals as was supposed in the calculations given above, but it seems improbable that this difference has a large effect on the final result.

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REFERENCES

1) Whittaker, E. T. and Watson, G. N., "Modern Analysis", p. 331, Cambridge University Press 4th Ed. 1946.
2) Koudijs, Endt, Van der Hart and Palmen, Physica **18** (1952) 415.