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1960

Physica 26
Congress on Many
Particle Problems, Utrecht
S 200–203

TWO REMARKS ON THE THEORY OF THE FERMI GAS

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1. *The Cooper pair problem and the applicability of perturbation theory to the groundstate of an imperfect Fermi gas.* In connection with the groundstate theory of a Fermi gas with forces Cooper [1]) has considered a simplified problem dealing with a pair of particles of opposite momenta $\boldsymbol{k}$ and $-\boldsymbol{k}$, in mutual interaction and in presence of an ideal Fermi gas in its groundstate. The particles in the pair do not interact with those of the gas, but the exclusion principle is supposed to hold for all particles present, so that the range of variation of $\boldsymbol{k}$ is limited to $k = |\boldsymbol{k}| > k_F$, k_F being the Fermi momentum. $\Phi_{\boldsymbol{k}}$ being the pair state of momenta $\boldsymbol{k}, -\boldsymbol{k}$, the kinetic energy operator K is simply

$$K\Phi_{\boldsymbol{k}} = \varepsilon_k \Phi_{\boldsymbol{k}}, \quad \varepsilon_k = k^2/m.$$

The interaction between the two particles in the pair is represented by an operator V, with matrix elements

$$\langle \Phi_{\boldsymbol{k}} | V | \Phi_{\boldsymbol{k}'} \rangle = V(\boldsymbol{k}, \boldsymbol{k}').$$

Both $\boldsymbol{k}$ and $\boldsymbol{k}'$ are restricted to the region outside the Fermi sphere (k and $k' > k_F$).

The remarkable finding of Cooper is that for $V(\boldsymbol{k}, \boldsymbol{k}')$ negative the pair problem of hamiltonian

$$H = K + V \tag{1}$$

has a bound state of energy lower than the lower limit ε_{k_F} of the continuum, irrespective of the strength of V (Cooper has established this result for $V(\boldsymbol{k}, \boldsymbol{k}')$ constant in a shell $k_F < k$, $k' < k_F + \omega$ and vanishing outside). This result is a consequence of the sharp edge at k_F in the momentum distribution of the ideal gas. It suggests that for an imperfect Fermi gas with attractive forces one has the following features

i) the groundstate ψ_0 is not obtainable from perturbation theory

ii) in the groundstate there is no discontinuity in the momentum distribution

$$n_k = \langle \psi_0 | a_{\boldsymbol{k}}^* a_{\boldsymbol{k}} | \psi_0 \rangle$$

($a_{\boldsymbol{k}}^*$ and $a_{\boldsymbol{k}}$ are the usual creation and destruction operators for particles of momentum $\boldsymbol{k}$)

iii) the groundstate of a gas of N particles, with N even, can be approximated by a self-consistent superposition of $N/2$ bound pairs of the type met by Cooper. The lowest excited state is separated from the groundstate by an energy gap.

The approximate theories developed by Bardeen, Cooper and Schrieffer [2]) and by Bogoliubov [3]) for Fermi gases with forces embody these three properties.

The aim of the present communication is to mention a result which, in our view, strongly indicates that properties i) and ii) above must hold for arbitrary sign of the interaction. If this conjecture is valid, it will be necessary to develop a non-perturbative treatment of the imperfect Fermi gas for general interaction. This treatment will have to be essentially different from the Bardeen-Cooper-Schrieffer-Bogoliubov method which is known to give for repulsive forces a groundstate identical to the groundstate of the ideal gas.

Our result is the following. Consider for the pair hamiltonian (1) the scattering state $\Psi_{\boldsymbol{k}}$ corresponding to the unperturbed pair state $\Phi_{\boldsymbol{k}}$ ($k > k_F$). Consider further any state vector $\Phi = \int c_{\boldsymbol{k}} \Phi_{\boldsymbol{k}} \mathrm{d}\boldsymbol{k}$ with $c_{\boldsymbol{k}}$ regular on the Fermi surface $k = k_F$. Then for $\boldsymbol{k} \to \boldsymbol{k}_0$ with $|\boldsymbol{k}_0| = k_F$ one has

$$\lim \langle \Phi | \Psi_{\boldsymbol{k}} \rangle = 0 \tag{2}$$

whenever the interaction V, assumed spherically symmetric, affects pair states of all angular momenta (otherwise the result holds only for Φ composed of those angular momenta which are present in V). The result (2) is again due to the sharp edge of the momentum distribution at k_F. It can be regarded as expressing the impossibility of accommodating pairs $\Phi_{\boldsymbol{k}}$ with $\boldsymbol{k}$ very close to the Fermi surface if the latter is sharp.

2. *The convergence of a simple example of linked cluster expansion.* We consider an ideal Fermi gas of infinite extension, in equilibrium at temperature T and density ρ. We take a finite volume ω inside the gas and study the probability distribution $p_\omega(n)$ of the number n of particles inside ω. We consider in particular the characteristic function

$$F_\omega(\lambda) = \Sigma_{n=0}^{\infty}\, p_\omega(n) \mathrm{e}^{\lambda n} = \langle \exp[\lambda \textstyle\int_\omega \psi^*(\boldsymbol{r}) \psi(\boldsymbol{r}) \mathrm{d}\boldsymbol{r}] \rangle_T \tag{3}$$

where $\langle \ldots \rangle_T$ denotes the equilibrium expectation value and $\psi(\boldsymbol{r})$ the quantized wave function verifying the usual anticommutation rules

$$[\psi(\boldsymbol{r}), \psi(\boldsymbol{r}')]_+ = 0, \ [\psi(\boldsymbol{r}), \psi^*(\boldsymbol{r}')]_+ = \delta(\boldsymbol{r} - \boldsymbol{r}').$$

The behaviour of (3) for large ω, $\omega\rho \gg 1$ is best analyzed by reducing the exponential to normal products

$$\exp[\lambda \int_\omega \psi^*(\boldsymbol{r})\psi(\boldsymbol{r})\mathrm{d}\boldsymbol{r}] =$$

$$\Sigma_{\nu=0}^{\infty} (\mathrm{e}^\lambda - 1)^\nu(\nu!)^{-1} \int_\omega \mathrm{d}\boldsymbol{r}_1 \ldots \int_\omega \mathrm{d}\boldsymbol{r}_\nu \psi^*(\boldsymbol{r}_1)\ldots\psi^*(\boldsymbol{r}_\nu)\psi(\boldsymbol{r}_\nu)\ldots\psi(\boldsymbol{r}_1)$$

and carrying out a diagram analysis of the expectation value in (3), in exactly the same way as is by now familiar in the theory of many-body systems with interaction. For $\omega\rho \gg 1$ the contribution of a given diagram is of order $(\omega\rho)^n$, where n is the number of its connected components. For arbitrary $\omega\rho$ one finds the exact formula

$$F_\omega(\lambda) = \exp[- \Sigma_{\nu=1}^{\infty} \nu^{-1}(1 - \mathrm{e}^\lambda)^\nu G_\nu]$$

where G_ν is the contribution of the only connected diagram of order ν

$$G_1 = \omega g(0)$$

$$G_\nu = \int_\omega \mathrm{d}\boldsymbol{r}_1 \ldots \int_\omega \mathrm{d}\boldsymbol{r}_\nu\, g(\boldsymbol{r}_1 - \boldsymbol{r}_2)\ldots g(\boldsymbol{r}_{\nu-1} - \boldsymbol{r}_\nu) g(\boldsymbol{r}_\nu - \boldsymbol{r}_1)$$

with

$$g(\boldsymbol{r}) = \langle\psi^*(\boldsymbol{r} + \boldsymbol{r}')\psi(\boldsymbol{r}')\rangle_T.$$

Every G_ν is proportional to $\omega\rho$ for $\omega\rho \gg 1$. The analogy of these results with the case of many-body systems with interaction is striking; ω here plays the part there taken by the total volume, and $\mu = \mathrm{e}^\lambda - 1$ plays a role comparable to the strength parameter of the interaction. Our aim is now to remark that the convergence properties of the "linked cluster series"

$$\Sigma_{\nu=1}^{\infty} \nu^{-1}(- \mu)^\nu G_\nu$$

are much poorer than those of the quantity $F_\omega(\lambda)$ itself, or in other words that for our simple problem the separation of diagrams in connected components artificially reduces the convergence radius. That a similar situation holds in quantum field theory had been noticed long ago by Caianiello [4]. Its occurrence for many body systems may have important methodological consequences if, as is sometimes suspected, linked cluster expansions turn out to be divergent.

Our assertion is established as follows. Consider the eigenvalue problem

$$\int_\omega g(\boldsymbol{r} - \boldsymbol{r}')f_\alpha(\boldsymbol{r}')\mathrm{d}\boldsymbol{r}' = l_\alpha f_\alpha(\boldsymbol{r})$$

Its eigenvalues verify

$$0 < l_\alpha < 1, \quad \Sigma_\alpha l_\alpha \text{ convergent.}$$

One can easily prove

$$G_\nu = \Sigma_\alpha l_\alpha^\nu.$$

Hence

$$\sum_{\nu=1}^{\infty} \nu^{-1}(-\mu)^{\nu} G_{\nu} = -\sum_{\alpha} \log(1 + \mu l_{\alpha}) \tag{4}$$

and

$$F_{\omega}(\lambda) = \prod_{\alpha} (1 + \mu l_{\alpha}).$$

Whereas (5) is an entire function of μ, the linked cluster series (4) has a finite radius of convergence.

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