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ON THE INFLUENCE OF FRICTION ON THE MOTION OF A TOP

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Synopsis

A simple calculation gives, to a first approximation, the influence of sliding friction on the motion of a top, spinning on a horizontal plane.

For a spherical body with axially symmetric mass distribution two effect are found:

1. The centre of gravity will be lifted if it lies eccentric in the sphere;
2. the axis with the greatest moment of inertia moves towards the vertical, if the principal momenta are unequal.

The equilibrium condition for the combination of the two effects is derived.

The rising of the tippe top is ascribed to sliding friction.

An ordinary top, describing its common regular precession, will usually roll; rolling friction contributes to the rising of the top.

The curious behavior of a little toy, called "tippe top" or "toupie magique" has led us to a study of the influence of frictional forces on the motion of a top. The study was limited to a spherical body with axially symmetric mass distribution, spinning on a horizontal plane, and to the disturbance of regular motions by friction. It deals with two kinds of regular motions and so explains the difference between the rising of the tippe top and the well known rising of ordinary tops. We hope that this simple treatment may suffice to give a physical understanding of the main things that happen, in avoidance of the cumbersome mathematics inherent to a description of the general motion.

The following symbols will be used:

C = centre of the sphere,

r = radius of the sphere,

B = centre of mass; BC is the axis of symmetry,

c = distance BC,
A = point of support,
XYZ = cartesian coordinates, fixed in space, with vertical Z axis,

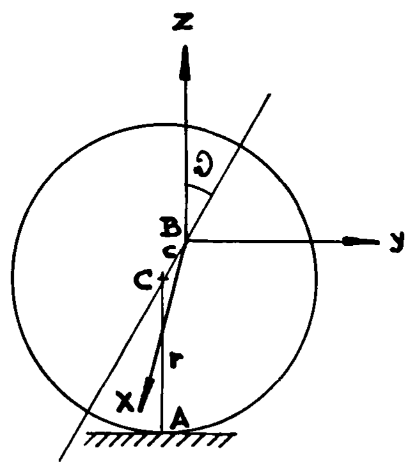

FIG. 1

$\vartheta\varphi\psi$ = eulerian angles;
$\vartheta = 0$ when B lies vertically above C,
$\varphi = 0$ when C lies in the left half of the YZ plane,
$\psi = 0$ when a fixed point of the top passes the vertical plane through BC,
$\mathbf{L}$ = angular momentum,
$\boldsymbol{\omega}$ = instantaneous angular velocity,
I = inertia tensor,
I_3 = moment of inertia about BC,
I_1 = moment of inertia about an axis through B, $\perp BC$,
M = mass of the top,
$\mathbf{G}$ = weight of the top,
$\mathbf{F}$ = frictional force in A,
$\mathbf{N}(G)$ = torque of the vertical reaction in A,
$\mathbf{N}(F)$ = torque of the friction, (moments are referred to B),
$\mathbf{v}(A)$ = horizontal velocity of A,
$\mathbf{v}(B)$ = horizontal velocity of B,
$\mathbf{v}(\omega)$ = horizontal velocity of A, referred to B,
h = index for horizontal components,

t = the time; at $t = 0$, B coïncides with the origin of the XYZ system and $\varphi = 0$.
Furthermore we shall use the abbreviations:
$e = c/r$ and $E = (I_3 - I_1)/I_3$.

A few remarks on the undisturbed motions may serve as an introduction.

1. *Sliding without friction* (ϑ constant). An extensive discussion of the general motion of a top, spinning on a perfectly smooth horizontal plane, is given by Klein and Sommerfeld[1]); they show that ϑ will oscillate in this case between two extreme values, depending on the initial rotation of the top. As mentioned already we shall limit our considerations to the steady motion — the regular precession — obtained by requiring ϑ = constant.

The vertical reaction in A gives no torque along the vertical axis or the axis of symmetry, hence the components of the angular momentum along these two axes must be constant:

$$(I_1 \sin^2 \vartheta + I_3 \cos^2 \vartheta)\, \dot{\varphi} + I_3\, \dot{\psi} \cos \vartheta = a$$

and

$$I_3\, (\dot{\psi} + \dot{\varphi} \cos \vartheta) = b.$$

As now:

$$a = I_1\, \dot{\varphi} \sin^2 \vartheta + b \cos \vartheta,$$

$\dot{\varphi}$ and $\dot{\psi}$ must also be constant.

A relation between $\dot{\varphi}$ and $\dot{\psi}$ is easily obtained by means of the following reasoning. At $t = 0$, L_h is directed along the Y axis.

$$L_y = I_3\, \dot{\psi} \sin \vartheta + (I_3 - I_1)\, \dot{\varphi} \cos \vartheta \sin \vartheta.$$

L_h will rotate with the vertical plane through BC, so that

$$\dot{L}_h = \dot{\varphi}\, L_h,$$

$\dot{L}_h$ rotating $\frac{1}{2}\pi$ radians ahead of L_h. From $\mathbf{N} = \dot{\mathbf{L}}$, we therefore have at $t = 0$:

$$(I_3 - I_1) \cos \vartheta\, \dot{\varphi}^2 + I_3\, \dot{\psi}\dot{\varphi} - Gc = 0.$$

This equation defines the minimum value of $\dot{\psi}$ for keeping ϑ constant, by means of:

$$I_3\, \dot{\psi}^2 + 4Gc(I_3 - I_1) \cos \vartheta \geq 0$$

and gives for every $\dot{\psi}$, satisfying this inequality, two possible precession velocities:

$$\dot{\varphi}_{1,2} = \frac{-\, I_3\, \dot{\psi} \pm \sqrt{I_3^2\, \dot{\psi}^2 + 4Gc(I_3 - I_1) \cos \vartheta}}{2(I_3 - I_1) \cos \vartheta},$$

or, for large ψ:

$$\dot\varphi_1 = -\dot\psi/E \cos\vartheta \text{ (fast precession),}$$

and
$$\dot\varphi_2 = Gc/I_3\dot\psi \text{ (slow precession).}$$

For later use we note that the point of contact slides in a direction, perpendicular to the vertical plane through BC. At $t = 0$ we have in the fast precession:

$$v_x(\omega) = (\dot\varphi c - \dot\psi r)\sin\vartheta = r\,\dot\varphi(e + E\cos\vartheta)\sin\vartheta$$

2. *Steady rolling*. Again we can find two types of steady motion. Putting ϑ, $\dot\varphi$ and $\dot\psi$ constant, we have at $t = 0$:

$$L_h = L_y = I_3\,\dot\psi\sin\vartheta + (I_3 - I_1)\,\dot\varphi\cos\vartheta\sin\vartheta$$

and
$$N_h = N_x = -I_3\,\dot\psi\dot\varphi\sin\vartheta - (I_3 - I_1)\dot\varphi^2\cos\vartheta\sin\vartheta.$$

The point A is instantaneously at rest, B moves in a circle with angular velocity $\dot\varphi$ and linear velocity:

$$v_x(B) = (\dot\psi r - \dot\varphi c)\sin\vartheta,$$

so that
$$\dot v_y(B) = \dot\varphi(\dot\psi r - \dot\varphi c)\sin\vartheta.$$

The horizontal reaction:

$$F = F_y = M\dot\varphi(\dot\psi r - \dot\varphi c)\sin\vartheta,$$

together with gravity, gives a second expression for N_h. Equating, we find:

$$\{(I_3 - I_1)\cos\vartheta - Mc(r + c\cos\vartheta)\}\,\dot\varphi^2 + \\ + \{I_3 + M\,r(r + c\cos\vartheta)\}\,\dot\psi\dot\varphi - Gc = 0,$$

which again determines a minimum value for $\dot\psi$ and gives two possible precession velocities, approximated for large $\dot\psi$ by:

$$\dot\varphi_1 = -\frac{\{I_3 + M\,r(r + c\cos\vartheta)\}\,\dot\psi}{(I_3 - I_1)\cos\vartheta - Mc\,(r + c\cos\vartheta)} \text{ (fast precession),}$$

$$\dot\varphi_2 = Gc/\{I_3 + M\,r(r + c\cos\vartheta)\}\,\dot\psi \quad \text{(slow precession).}$$

It will prove to be useful to calculate the frictional force, necessary to maintain this motion.

For the fast precession:

$$-F_y = M\,I_3\,r\dot\varphi^2\sin\vartheta(e + E\cos\vartheta)/\{I_3 + M\,r(r + c\cos\vartheta)\}.$$

For the slow precession:

$$F_y = \frac{MGc}{I_3 + M\,r(r + c\cos\vartheta)} \cdot \left[r - \frac{Gc^2}{\{I_3 + M\,r(r + c\cos\vartheta)\}\,\dot{\psi}^2} \right].$$

We have seen so far that two kinds of steady motions exist, which, after Fokker[2]), we called fast and slow precession. The difference between the two can be made clear by comparison of the motions of the angular momentum vector. Taking the equations for sliding without friction, we see that in the first case **L** describes a small cone about the vertical. The ratio of $\dot{\psi}$ and $\dot{\varphi}$ approaches for great $\dot{\varphi}$ a constant which makes L_h vanish. In the second case, $\dot{\psi}$ is the main component of the rotation, $\dot{\varphi}$ being inversely proportional to $\dot{\psi}$, and the angular momentum is closely following the axis of symmetry.

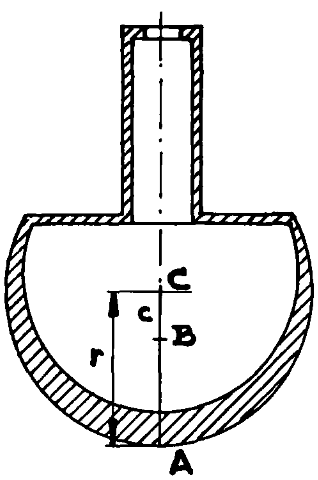

FIG. 2

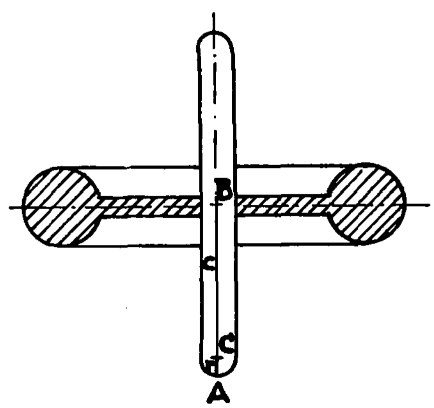

FIG. 3

In the literature about the heavy top, the latter type is usually called regular precession. But in the general motion there appear nutations and they can, to a first approximation, be described as a fast precession, superposed on the slow one. The tippe top, after being started to rotate about its vertical axis of symmetry, will exhibit a fast precession — disturbed, as we shall see, by sliding friction — to which conversely a slow precession can be added.

3. *Conditions for rolling.* To illustrate our calculations we take two tops, the tippe top with $c \ll r$ and an ordinary top with $c \gg r$ (cf. fig. 2 and 3). We shall henceforth express fast precessions in terms of $\dot{\varphi}$, limiting for convenience $\dot{\varphi}$ to positive values, and slow precessions in terms of $\dot{\psi}$, limiting $\dot{\psi}$ to positive values.

The tippe top consists, apart from the stem, of a hollow sphere with slightly eccentric centre of gravity and slightly different moments of inertia. Let $r = 2$ cm, $e = -0.1$, $I_3 = \frac{2}{3}Mr^2$, $I_1 > I_3$, $E = -0.1$.

Then for rolling in fast precession we find:

$$F = 0.8\,(0.1 + 0.1 \cos\vartheta)\, M\dot{\varphi}^2 \sin\vartheta,$$

so that, if $\dot{\varphi} = 100$ and the maximal friction $= 0.3$ Mg, the top will start to slide if $\sin\vartheta$ becomes greater than 0.2.

For slow precession: $F = 0.03\, G\,\{2 - 6/\dot{\psi}^2\}$, which reduces to:

$$F = 0.06\, G.$$

As under normal conditions the coefficient of friction is much greater then 0,06, the top will usually roll.

In an ordinary top, the moment of inertia about the axis of symmetry is usually made as great as possible in relation to the other principal momenta. But in a symmetrical top E can not exceed $\frac{1}{2}$. Now c is the length of the stem on which the top spins; if c is of the same order of magnitude as the radius of the ring which is supposed to contain the greater part of the mass, we can have for example $I_3 = M\,c^2$, $E = 0.4$. Let further $r = \varrho c$, and $\varrho \ll 1$.

Then rolling in fast precession requires:

$$F = M\,c\dot{\varphi}^2 \sin\vartheta,$$

or, if $c = 2$ cm, $\dot{\varphi} = 100$ and $F \leq 0.3$ Mg:

$$\sin\vartheta \leq 0.015.$$

The condition for rolling in slow precession is:

$$F = G\,\{\varrho - g/c\dot{\psi}^2\}.$$

From it, we see that if the coefficient of friction surpasses the greater one of ϱ and $g/c\dot{\psi}^2$, the top is prevented from sliding. If $\varrho = 0.1$, $c = 2$ cm and $\dot{\psi} = 50$, this condition will be amply satisfied.

4. *Sliding friction.* Having seen that in a fast precession sliding is likely to occur, we will now consider the disturbance of a steady

motion of this kind, caused by sliding friction. Although the angular velocities can no more be uniform, energy being dissipated, we shall consider $\dot{\psi}$, $\dot{\psi}$ and $\dot{\vartheta}$ as constants, of which $\dot{\vartheta}$ is small. Thus a first approximation of the exact solution is found from which the assumptions about $\dot{\varphi}$, $\dot{\psi}$ and $\dot{\vartheta}$ can be verified.

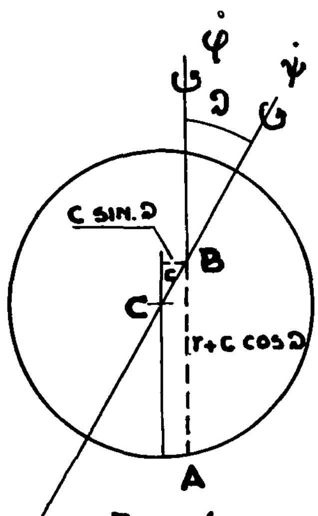

FIG. 4

To give a first idea of the procedure we mention — neglecting $\mathbf{v}(B)$ — that the point of contact in fig. 1 slides along the X-axis, so that the moment of friction is directed along the Y-axis; the angular momentum therefore has a component rotating $\frac{1}{2}\pi$ radians behind $N_h(F)$, by which the angular velocity has a component $\dot{\vartheta}$.

For a closer description of the phenomenon we take into account the velocity of B, caused by $\mathbf{F}$, and the angle, α, between $\mathbf{F}$ and the X-axis. Equations are computed at $t = 0$ (cf. fig. 4 and 5).

From fig. 4 we see:

$$v_x(\omega) = c\ (\dot{\varphi} + \dot{\psi} \cos \vartheta) \sin \vartheta - (r + c \cos \vartheta)\ \dot{\psi} \sin \vartheta,$$

$$v_y(\omega) = -(r + c \cos \vartheta)\ \dot{\vartheta}.$$

From $\mathbf{F} = M\ \dot{\mathbf{v}}(B)$:

$$v_x(B) = -F \sin \alpha / M\dot{\varphi},$$

$$v_y(B) = F \cos \alpha / M\dot{\varphi}.$$

($F = f\,\mathrm{Mg}$; f is a constant).

The sliding of the point of contact is therefore:

$$v_x(A) = -(F \sin \alpha/M\dot{\varphi}) + c(\dot{\varphi} + \dot{\psi} \cos \vartheta) \sin \vartheta - (r + c \cos \vartheta)\, \dot{\psi} \sin \vartheta, \quad (1)$$

$$v_y(A) = (F \cos \alpha/M\dot{\varphi}) - (r + c \cos \vartheta)\, \dot{\varphi}. \quad (2)$$

From $\mathbf{N} = \dot{\mathbf{L}}$, we have:

$$L_x = F(r + c \cos \vartheta) \cos \alpha/\dot{\varphi}$$

$$L_y = \{F(r + c \cos \vartheta) \sin \alpha + Gc \sin \vartheta\}/\dot{\varphi}.$$

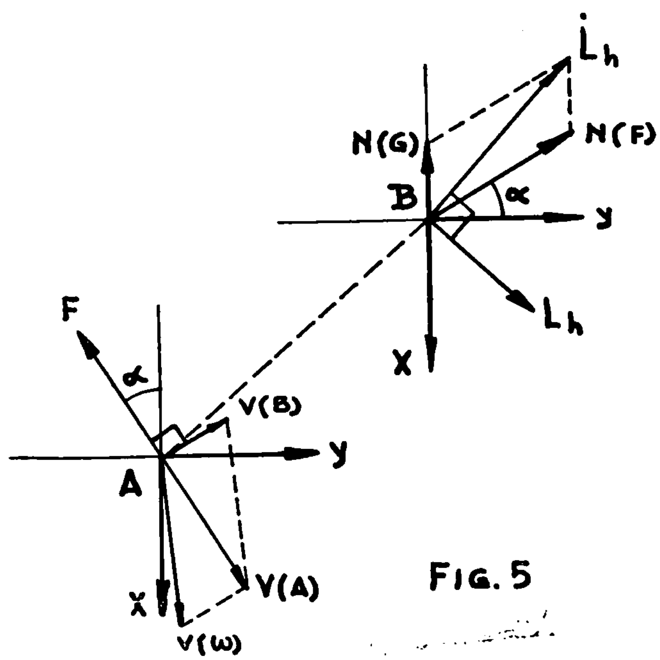

FIG. 5

Whereas, from $\mathbf{L} = I . \boldsymbol{\omega}$:

$$L_x = I_1 \dot{\vartheta},$$

$$L_y = I_3 \dot{\psi} \sin \vartheta + (I_3 - I_1)\, \dot{\varphi} \cos \vartheta \sin \vartheta,$$

so that:
$$\dot{\vartheta} = -F\, (r + c \cos \vartheta) \cos \alpha/I_1\, \dot{\varphi} \quad (3)$$

$$\dot{\psi} = \frac{-(I_3 - I_1)\, \dot{\varphi}^2 \cos \vartheta \sin \vartheta + F(r + c \cos \vartheta) \sin \alpha + Gc \sin \vartheta}{I_3\, \dot{\varphi} \sin \vartheta} \quad (4)$$

Finally, $\mathbf{F}$ shall be directed opposite to $\mathbf{v}(A)$:

$$v_x(A) \sin \alpha = v_y(A) \cos \alpha$$

In order to distinguish between the effects due to the eccentricity of B and the inequality of I_1 and I_3, we consider the special cases $I_1 = I_3$ and $c = 0$ separately.

a. $c \neq 0, I_1 = I_3 = I$, reduces the equations (1) — (5) to:

$$\dot{\vartheta} = - F(r + c \cos \vartheta) \cos \alpha / I\dot{\varphi} \tag{3}$$

$$\sin \alpha = \frac{F}{Mc \sin \vartheta} \cdot \frac{I + M(r + c \cos \vartheta)^2}{I\dot{\varphi}(\dot{\varphi} + \dot{\psi} \cos \vartheta) - G(r + c \cos \vartheta)} . \tag{6}$$

$$\dot{\psi} = \{F(r + c \cos \vartheta) \sin \alpha + Gc \sin \vartheta\} / I\dot{\varphi} \sin \vartheta. \tag{7}$$

Although of course $\dot{\psi}$ and $\sin \alpha$ can easily be eliminated from (6) and (7), we prefere to keep the relations in this form because (7) shows that, if $\dot{\varphi}$ is great, $\dot{\psi}$ becomes negligible. Thus, for $\dot{\varphi}^2 \gg 3g/2r$, $c \ll r$, $I = \frac{2}{3} M r^2$ and normal values of F, i.e. f not much greater than 1, we obtain:

$$\dot{\vartheta} = - 3fg \cos \alpha / 2r\dot{\varphi}, \tag{8}$$

$$\sin \alpha = 5fg/2\, er\, \dot{\varphi}^2 \sin \vartheta. \tag{9}$$

To find the sign of $\cos \alpha$ in (8), we notice that $0 < \alpha < \pi/2$ (if $\dot{\varphi}$ and c are positive). This is seen by considering the influence of a change in f on the angle α. Making f small, we see (cf. fig. 5) $\mathbf{v}(B)$ vanish and $\mathbf{v}(\omega)$ move to the X-axis ($\dot{\vartheta}$ vanishing also), so that α approaches 0. If f is made greater, α grows untill $\mathbf{v}(B) = - \mathbf{v}(\omega)$, in which case the top rolls; as we saw, $\mathbf{F}$ is directed then along the $-Y$-axis.

Equation (8) therefore shows that ϑ *is changing so, that the centre of gravity is lifted.*

The maximum value of $\dot{\vartheta}$ for a given $\dot{\varphi}$ is obtained if $\sin \alpha = \frac{1}{2}\sqrt{2}$. For $\vartheta = \pi/2$, this condition yields:

$$- \dot{\vartheta} \leq 0.3\, e\dot{\varphi}; \text{ maximum obtained if } f = \frac{\sqrt{2}\, er\dot{\varphi}^2}{5g} . \tag{10}$$

$\dot{\vartheta}$ is therefore relatively small. It can be shown, that $\ddot{\varphi}$ and $\ddot{\psi}$ are also small, which justifies our approximation.

b. $c = 0, I_1 \neq I_3$, gives instead of equations (1) — (5):

$$\dot{\vartheta} = - \frac{Fgr \cos \alpha}{I_1\dot{\varphi}}, \tag{11}$$

$$\tfrac{1}{2} Er\dot{\varphi} \sin 2\vartheta \sin \alpha = \frac{Fr^2 (I_1 \sin^2 \vartheta + I_3 \cos^2 \vartheta)}{I_1 I_3 \dot{\varphi}} + \frac{F}{M\dot{\varphi}} . \tag{12}$$

If $E \ll 1$ and $I = \frac{2}{3}Mr^2$, we can write for (11) and (12)

$$\dot{\vartheta} = -3fg \cos \alpha / 2r\dot{\psi}, \tag{13}$$

$$\sin \alpha = 5fg/Er\dot{\varphi}^2 \sin 2\vartheta \tag{14}$$

A disucission about the sign of cos ϑ, similar to that, we gave for $c \neq 0$, will show that *ϑ changes so, that the axis with the greatest moment of inertia turns towards the vertical.*

Again we can find maximum conditions: For $\vartheta = \pi/4$:

$$|\dot{\vartheta}| \leq 0.15\, E\dot{\varphi};\ \text{maximum obtained if}\ f = \frac{\sqrt{2}\, Er\dot{\varphi}^2}{10g}. \tag{15}$$

c. If both $c \neq 0$ and $I_1 \neq I_3$, the question is, which of the two effects, described above, will win.

There will be an equilibrium if the sliding of the point of contact vanishes. By putting either $F = 0$ in the equations for rolling or $v(\omega) = 0$ in the equations for sliding, we find the conditions:

$$\sin \vartheta = 0 \quad \text{or} \quad \cos \vartheta = -\frac{e}{E}.$$

Whether the so found equilibrium values of ϑ are stable or not, appears from the sign of $\dot{\vartheta}$ at interjacent values of ϑ.

This, however, is again determined by the direction of $\mathbf{v}(\omega)$ in the undisturbed motion; the sign of $\dot{\vartheta}$ being opposite to the sign of $v_x(\omega)$. Remembering that in the case of sliding without friction, for great $\dot{\varphi}$:

$$v_x(\omega) = \dot{\varphi}(e + E \cos \vartheta)\, r \sin \vartheta,$$

we conclude:

1°. If $E < -|e|$, $\vartheta = 0$ and $\vartheta = \pi$ are unstable; $\vartheta = \arccos\left(-\frac{e}{E}\right)$ is stable.

2°. If $-|e| < E < |e|$, $\vartheta = \pi$ is unstable; $\vartheta = 0$ is stable.

3°. If $|e| < E$, $\vartheta =$ are cos $(-e/E)$ is unstable; $\vartheta = 0$ and $\vartheta = \pi$ are stable.

The tippe top is an example of the third kind, both e and E are negative. But before ϑ has reached its stable value, the stem touches the supporting plane. The friction arising from the sliding of the end of the stem over the plane then takes the part of the friction in A, so that the rising of the top will continue. Attaching a small weight to the stem, so that e decreases and E increases, one can easily demonstrate the stable ϑ. An example of the second kind — communicated by Fokker[3]) — is a sphere consisting of two halves of different specific weight.

A demonstration of the fact expressed by (10) can be given by putting up the tippe top alternatively on a hard table and a sheet of abrasive paper. Another demonstration — again due to Fokker[3]) — is also easily prepared. Putting up the top on a sheet of tracing-paper one obtains a line, written on the top, showing the velocities $\dot{\vartheta}$, $\dot{\psi}$ and because of the impossibility to avoid any initial horizontal component of $\mathbf{L}$ also $\dot{\varphi}$, relative to each other.

Adding a constant horizontal component to $\mathbf{L}$ in the fast precession of the tippe top results in an oscillation of ϑ with the period of φ. The axis of symmetry rotates about $\mathbf{L}$ and over one period there remains an average value of the torque of gravity which gives rise to a slow precession. (Thus L_h can only be considered as constant during the relatively short period $2\pi/\dot{\varphi}$). Furthermore, L_h gives a constant component of $\mathbf{v}(\omega)$, which because of the sliding friction will be compensated by an opposite constant velocity of B. The experiment with the tracing-paper will demonstrate the correlation between $\mathbf{v}(B)$ and the oscillations of ϑ.

In an ordinary top, e is much greater than E, so that at fast precession the sliding friction will always cause the centre of gravity to rise. As we saw, sliding occurs already at very small values of ϑ. Although the superposition of a fast precession — nutation — to the slow one will certainly complicate the calculations, it seems justified to state, that this influence of the sliding friction limits the amplitude of the nutations. In other words, the axis of symmetry will closely follow the angular momentum in its slow precession.

5. *Influence of rolling friction*. As we saw, the tippe top in fast precession will roll if ϑ does not exceed a certain value. The torque of rolling friction is then directed opposite to the horizontal component of $\boldsymbol{\omega}$, which in the undisturbed motion was directed along the Y-axis:

$$\omega_y = \dot{\psi} \sin \vartheta = -\tfrac{2}{5} (E \cos \vartheta - \tfrac{3}{2} e)\, \dot{\varphi} \sin \vartheta.$$

Rolling friction will cause a change in ϑ:

$$\dot{\vartheta} = -N_y/I_1 \dot{\varphi},$$

$\dot{\vartheta}$ having the same sign as ω_y.

Rolling friction will therefore tend to change ϑ so, that the centre of gravity falls, or that the axis with the greatest moment of inertia turns towards the vertical. Which one of these two effects will win, depends in this case on the relation between E and $\frac{3}{2}e$.

It seems interesting to notice that Klein and Sommerfeld [1]) found similar effects for the influence of air-resistance on the motion of a top.

In the case of an ordinary top in slow precession the angular momentum, under the influence of the torque of rolling friction, moves towards the vertical and as we saw, it will be followed by the axis of symmetry of the top. An experiment, described by Fokker [2]), as well as our calculations, shows that the top will usually not slide; it seems that the influence of rolling friction, however, will make an important contribution to the rising of the top.

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3) Fokker, A. D., private communication.