

COROLLARY TO A GHS INEQUALITY

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Received 25 July 1974

For two Ising spin systems linked by a single ferromagnetic bond and satisfying the hypothesis of the Griffiths-Hurst-Sherman inequality, increasing any member of a subset of bonds in system 2 leads to a decrease of the pair-spin covariance in system 1.

Introduction. In the process of establishing the concavity of the magnetization as a function of magnetic field for Ising spin systems with purely ferromagnetic, binary interactions, Griffiths, Hurst and Sherman [1] developed a powerful inequality which implies that the covariance between two spins is a non-increasing function of the magnetic field on any spin. An alternative proof of the inequality has recently been given by Lebowitz [2].

A corollary to the GHS result is the following: consider two, independent Ising spin systems satisfying the hypothesis of the GHS inequality. Connect the two systems by a single ferromagnetic bond from, say, spin σ_{p-1} in system 1 to spin σ_p in system 2. Then the effect of increasing any bond J_{pq} between σ_p and any spin σ_q in system 2 is to decrease (not increase) the covariance $\text{cov}(\sigma_r, \sigma_s)$ between any two spins σ_r, σ_s in system 1.

When combined with a Griffiths inequality [3], this corollary shows that^{†1} $\text{cov}(\sigma_r, \sigma_s)$ is independent of J_{pq} for zero external field and temperatures above which the averages $\langle \sigma_r \rangle, \langle \sigma_s \rangle$ are zero.

Attempts to extend the result to systems linked by more than one ferromagnetic bond have led to a counter-example. In an intuitive sense one could say that the covariance "flowing" from one system via one bond may "return" via another bond.

We expect (but have not proved) that the corollary is true for any bond in system 2; i.e., it is not restricted to bonds emanating from σ_p ^{‡2}.

Corollary and proof. Consider two sets N_n ($n = 1, 2$) of positive integers:

$$N_1 = \{1, 2, \dots, p-1\}, \quad N_2 = \{p, p+1, \dots, M\} \quad (1)$$

where $3 \leq p < M$.

Associate a spin Hamiltonian \mathcal{H}_n with set N_n :

$$\mathcal{H}_n = - \sum_{i < j \in N_n} J_{ij} \sigma_i \sigma_j - \sum_{i \in N_n} h_i \sigma_i \quad (2)$$

where $J_{ij}, h_i \geq 0, \sigma_i = \pm 1$. The Hamiltonians $\mathcal{H}_1, \mathcal{H}_2$ refer to two independent Ising spin systems each with binary, ferromagnetic interactions and external field parameters h_i .

Now link the systems with a single, ferromagnetic bond $J^{(12)} \geq 0$ connecting spins σ_{p-1} and σ_p . The interaction Hamiltonian

$$\mathcal{H}_{12} = -J^{(12)} \sigma_{p-1} \sigma_p \quad (3)$$

and the Hamiltonian of the combined system is

$$\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_{12}. \quad (4)$$

The canonical ensemble average $\langle Q \rangle$ of any function Q on the 2^M spin configurations σ is defined by

$$\langle Q \rangle = \sum_{\sigma} Q \exp(-\beta \mathcal{H}) / \sum_{\sigma} \exp(-\beta \mathcal{H}) \quad (5)$$

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^{†1} If the spins in system 2 are in zero field, it may be verified directly that the canonical ensemble average of any function of the configurations of the spins in system 1, only, is independent of all bond strengths relating to pairs of spins in system 2.

^{‡2} In commenting on the preprint of this letter, J. Groeneveld and, subsequently, Masuo Suzuki [4] kindly provided elegant proofs of this conjecture.

where the absolute temperature $T = (k_B \beta)^{-1}$.

For $i \neq j$ the covariance of any two spins σ_i, σ_j is defined by

$$\text{cov}(\sigma_i, \sigma_j) = \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle. \quad (6)$$

Corollary. For $r, s \in N_1, q \in N_2, q > p$:

$$\partial \text{cov}(\sigma_r, \sigma_s) / \partial J_{pq} \leq 0. \quad (7)$$

Before considering the proof recall that Griffiths' inequality [3] requires that $\partial \langle \sigma_r \sigma_s \rangle / \partial J_{pq} \geq 0$ for all $T > 0$ and $h_i \geq 0$. But for $T > T_c$ and all $h_i = 0$, $\langle \sigma_r \sigma_s \rangle = \text{cov}(\sigma_r, \sigma_s)$; therefore, one consequence⁺¹ of the corollary is

$$\partial \text{cov}(\sigma_r, \sigma_s) / \partial J_{pq} = 0 \quad (8)$$

for all $T > T_c \geq 0, h_i = 0, r, s \in N_1, q \in N_2$.

Proof. In terms of new spin variables t_k , where $t_k = \pm 1$ for $k = 1, 2, \dots, M$ write the transformation

$$\begin{aligned} \sigma_i &= t_i \quad (i = 1, 2, \dots, p) \\ &= t_i t_p \quad (i = p+1, p+2, \dots, M). \end{aligned} \quad (9)$$

The \mathcal{H}_1 and \mathcal{H}_{12} are unchanged except for the replacement of each σ_i by t_i ; whereas,

$$\mathcal{H}_2 = - \sum_{i < j \in N_2} \bar{J}_{ij} t_i t_j - \sum_{j \in N_2} \bar{h}_j t_j \quad (10)$$

where

$$\begin{aligned} \bar{J}_{ij} &= J_{ij} \quad (p+1 \leq i < j \leq M) \\ &= h_j \quad (p = i < j \leq M) \end{aligned} \quad (11)$$

$$\begin{aligned} \bar{h}_j &= J_{pj} \quad (j = p+1, \dots, M) \\ &= h_p \quad (j = p). \end{aligned} \quad (12)$$

The transformed Hamiltonian \mathcal{H} is thus of the form for which the GHS inequality obtains and the J_{pj} , for $j = p+1, \dots, M$, play the role of external field parameters. Furthermore, for $r, s \in N_1$

$$\text{cov}(\sigma_r, \sigma_s) = \text{cov}(t_r, t_s) \quad (13)$$

consequently

$$\partial \text{cov}(\sigma_r, \sigma_s) / \partial J_{pq} = \partial \text{cov}(t_r, t_s) / \partial \bar{h}_q \leq 0. \quad (14)$$

I would like to thank J. Groeneveld and Th.W. Ruijgrok for their kind and constructive interest in this work and for the hospitality of the Instituut.

References

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