

## NOTE

Observations on the Icosahedron in Euclid's *Elements*

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Proposition 16 of Book XIII of Euclid's *Elements* [Heath 1926 3, 481–486] contains a beautiful construction of a regular icosahedron  $SBCDEFMNPQRU$  (Fig. 1). In the following description of the essentials of the construction and proof we ignore the fact that the icosahedron has to be inscribed in a given sphere. Consider a circle with center  $A$ , and write  $h$ ,  $p$ , and  $d$  for the sides of the inscribed regular pentagon, hexagon, and decagon in this circle. Let  $BGCHDJKEF$  be an inscribed regular decagon. Consider a new plane parallel to the plane of the circle and at a distance  $h$  from it, and let  $MNPQR$  and  $T$  be the perpendicular projections of the pentagon  $GHJKL$  and the center  $A$  on the new plane. Define  $S$  and  $U$  on  $AT$  extended such that  $AS = TU = d$ . Then the sides  $BC$ ,  $CD$ ,  $DE$ ,  $EF$ , and  $FB$  and  $MN$ ,  $NP$ ,  $PQ$ ,  $QR$ , and  $RM$  have length  $p$  by construction. By the theorem of Pythagoras all other sides of the solid  $SBCDEFMNPQRU$  are equal to  $\sqrt{h^2 + d^2} = p$ , so that the solid is a regular icosahedron. The last part of the argument is based on the identity

$$p^2 = h^2 + d^2, \quad (1)$$

which is proved by means of a complicated planimetric reasoning in a separate proposition (*Elements* XIII, 10 [Heath 1926 3, 457–460]). In 1917 Eva Sachs [1917, 89–107] showed that both propositions (10 and 16) are due to Theaetetus (died 369 BC).

Sachs also discussed the question of how the construction of the icosahedron could have been found. She argued that Theaetetus discovered (1) in connection with the investigations of the icosahedron, and that the planimetric proof was constructed afterward. This raises the question, how can (1) be discovered in a regular icosahedron? According to Sachs [1917, 103], Theaetetus could have guessed that  $AJPT$  is a square, or that the triangles  $BMG$  and  $SBA$  are congruent; either guess is sufficient for the purpose. These guesses were supposedly motivated by intuition, or by measurements in an accurately drawn figure. Sachs' suggestions have received attention in the modern literature [Dijksterhuis 1930, 263; Neuenschwander 1974–1975, 106; Mueller 1981, 258, 304 (note 8)], and the plausibility of the two guesses has been judged differently.

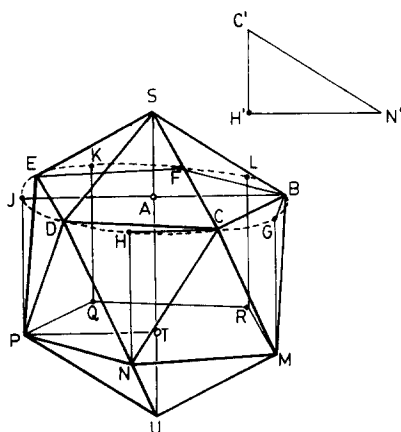


FIG. 1.

It seems not to have been noticed thus far that (1) can be derived in the icosahedron by means of theoretical deductions without guesses, or in other words, by way of a rigorous analysis in the Greek sense of the word. We assume that  $SBCDEFMN PQRU$  is a regular icosahedron, and we compare triangles  $SAB$  and  $CHN$ . Angles  $A$  and  $H$  are right angles, and we have  $SA \parallel HN$  and  $AB \parallel HC$ , as a little consideration shows. Hence the planes of the triangles are parallel. An argument of symmetry shows that  $BS$  is perpendicular to plane  $NCK$ . We now rotate triangle  $CHN$  counterclockwise over a right angle about an arbitrary point in its plane. Let the new position be  $C'H'N'$ . It is now immediately clear that  $C'H' \parallel SA$ ,  $H'N' \parallel AB$ , and  $C'N' \parallel SB$ . Hence the triangles  $SAB$  and  $C'H'N'$  are similar. Since  $SB = C'N'$ , the triangles must be congruent, so that  $HN = H'N' = AB = h$  and  $SA = C'H' = CH = d$ . The identity  $p^2 = h^2 + d^2$  follows by the theorem of Pythagoras applied to the right triangle  $SAB$  or  $CHN$ , q.e.d. The reader may argue that this reasoning involves techniques, such as "rotation," which are not generally used in the Greek mathematical treatises that have been transmitted to us. However, one could rephrase the argument in terms of congruent triangles, and one could choose the center of rotation conveniently. We may also recall in this connection the essential role of motions in the construction of two mean proportionals of Theaetetus' contemporary Archytas of Tarentum. Thus Theaetetus *could* have found the basic idea of his construction of the icosahedron by means of a rigorous analysis, as described above.

### REFERENCES

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