On Probabilistic Completeness and Expected Complexity of Probabilistic Path Planning

Petr Švestka
Department of Computer Science, Utrecht University
P.O.Box 80.089, 3508 TB Utrecht, the Netherlands
e-mail: petr@cs.ruu.nl

Abstract

The *probabilistic path planner* (*PPP*) is a general planning scheme that yields fast robot path planners for a wide variety of problems, involving, e.g., high degree of freedom articulated robots, nonholonomic robots, and multiple robots. In this paper we go into theoretical aspects regarding the behaviour of *PPP*. We formulate general properties that guarantee probabilistic completeness of *PPP*, and we show how these apply to various robot types. Furthermore, we present results which, under certain assumptions on the free configuration space, give estimates of the expected running times. For example, under one such assumption, we show that the expected running time of *PPP* grows only logarithmically with the complexity of the problem that it solves.

1 Introduction

Robot path planning, which asks for the computation of collision free paths in environments containing obstacles, has received a great deal of attention in the last decades [Lat91, HA92]. We consider the basic problem, where there is one robot present in a static and known environment, and the task is to compute a collision-free path describing a motion that brings the robot from its current position to some desired goal position. The space where the robot and the obstacles are physically present is called the *workspace* $\mathcal{W}$. The planning is mostly performed in another space, the *configuration space* $\mathcal{C}$. Each placement of the robot in $\mathcal{W}$ is mapped to a point in $\mathcal{C}$. The portion of $\mathcal{C}$ corresponding to collision-free placements of the robot is referred to as the *free-configuration space* $\mathcal{C}_f$.

This paper is on probabilistic completeness and expected running time of the *probabilistic path planner* (or *PPP*). *PPP* is a general planning scheme building probabilistic roadmaps by randomly selecting configurations from the free configuration space and interconnecting certain pairs by simple feasible paths. The method has been applied to a wide variety of robot path planning problems with remarkable success.

A first version of the planner for free-flying planar robots was described by Overmars in [Ove92] and subsequently expanded into a general learning approach, for various robot types, in [OŠ94]. Independently, "*PPP*-like" preprocessing schemes for holonomic robots where introduced in [KL94] and [HST94]. These schemes also build probabilistic roadmaps in the free C-space, but focus on the case robots with many degrees of freedom (dofs). In [KŠLO95] the ideas developed in [OŠ94] and [KL94] have been combined, resulting in a even more powerful planner for high-dof robots. Simultaneously, *PPP* has been applied to nonholonomic robots. Planners for car-like robots that can move both forwards and backwards as well as robots that can only move forwards are described in [Šve93, ŠO95b]. *PPP* applied to tractor-trailer robots is the topic of [ŠV95, SŠLO95]. Probabilistic completeness of the planners for nonholonomic robots is proven in [ŠO95b]. Recently some first results on the expected running times of *PPP*, under certain geometric assumptions on the free configuration space, have been obtained [KLMR95, KKL96, BKL$^+$95]. For a thorough survey of probabilistic path planning for holonomic robots we also refer to the thesis of Kavraki [Kav95]. Finally, extensions of *PPP* addressing multi-robot path planning problems have been presented in [ŠO95a, ŠO96].

Most of the above mentioned papers concentrate on experimental results. In this paper theory is presented regarding probabilistic completeness and expected running times. In Section 2 the *PPP*-scheme is described (in a simplified form). In Section 3 probabilistic completeness of the method is discussed, and proofs of probabilistic completeness are given for various (holonomic and nonholonomic) robots. Section 4 deals with analyses of expected running time. Results by Kavraki et al. [KLMR95, KKL96, BKL+95] are reviewed, and some new results are presented.

2 The probabilistic path planner PPP

For ease of presentation, we describe here *PPP* in its simplest form. Various refinements, adaptions, and extensions of the algorithm (some general, some robot specific) are possible. For this, as well as experimental results for various robots and problem settings, we refer to [ŠO95b, KŠLO95, Kav95, KL94, OŠ94, ŠO95c, Šve93, Van94, ŠO96, SŠLO96]. In this paper we want to keep the algorithm definition as simple as possible, to facilitate the presentations in the next two sections. We note however that the planners in the above mentioned papers are technically equivalent to the general algorithm presented below, and the results presented in this paper hold for them as well.

We assume that we are dealing with a robot $\mathcal{A}$, and that L is a *local planner* that constructs paths for $\mathcal{A}$. That is, L is a function that takes two argument configurations and returns a path connecting them, which is feasible for $\mathcal{A}$ in the absence of obstacles. Furthermore, we assume that $\mathcal{A}$ is present in a known environment. That is, for any configuration c, we can determine whether $\mathcal{A}$ placed at c intersects any obstacles.

PPP constructs a probabilistic roadmap, stored in a graph $G = (V, E)$. The nodes in V are free configurations, and the edges in E correspond to collision free paths constructed by the local planner L. We refer to these as *local paths*. Initially, V consists of just two nodes, a *start node* s and a *goal node* g, and E is empty. The further construction of the roadmap is performed incrementally in a probabilistic way. Repeatedly a random free configuration c is generated and added to V. For each such c we construct local paths connecting it to the previously added configurations using L, and for those local paths that are collision free, we add the corresponding edges to E. This is described more formally below:

PPP (simplified version):

Let s be the start configuration and g the goal configuration.
$V = \{s, g\}$, $E = \emptyset$
loop until s and g are graph-connected
  $c =$ a configuration randomly chosen from $\mathcal{C}_f$
  $V = V \cup \{c\}$
  forall $n \in V - \{c\}$
    if $L(c, n) \subset \mathcal{C}_f$ is collision-free **then** $E = E \cup \{(c, n)\}$

Once s and g are graph-connected, the path planning problem is solved, since a graph search in G will yield a graph path from s to g which can be transformed to a feasible path for the robot with the local planner L.

Whether the graph G is directed or undirected depends on the local planner. If it is *symmetric*, that is, for any pair of configurations (a, b) $L(a, b)$ equals $L(b, a)$ reversed, then an undirected graph is used. If the local planner does not have this symmetry property, then the graph is directed.

The major simplification of the presented algorithm with respect to that used in the above mentioned papers is that for a new node c connections are tried to all nodes of the graph. In the "real" implementations only nodes that lie relatively nearby to c are selected as nodes to which c

can be connected. This improves the performance of the planner. It is however of no influence on the probabilistic completeness and the asymptotic expected running time of the algorithm.

2.1 Local planners used for various robots

We now briefly describe the various local planners used for different robot types in previous applications of *PPP*.

Holonomic robots Free-flying robots and articulated robots fall into the class of holonomic robots. A very general local planner exists, that is directly applicable to all holonomic robots. Given two configurations, it connects them by a straight line segment in configuration space and checks this line segment for collision and joint limits (if any). We refer to this planner as *the general holonomic local planner* (See, e.g., [KŠLO95, ŠO95c] for more details). Clearly this local planner is symmetrical.

Car-like robots We consider two types of car-like robots, i.e., such that can drive both forwards and backwards, and such that can only drive forwards. We refer to the former as *general car-like robots*, and to the latter as *forward car-like robots*.

A *RTR path* is defined as the concatenation of a rotational path, a translational path, and another rotational path. In other words, it is the concatenation of two circular arcs and a straight line segment, with the latter in the middle. The *RTR local planner* constructs the shortest *RTR* path connecting its argument configurations. such paths are feasible for general car-like robots.

For forward car-like robots a very similar local planner is used, i.e., the *RTR forward local planner*. It constructs the shortest *RTR forward path* connecting its argument configurations. A RTR forward path is defined as a normal RTR path, but with the extra requirement that the arcs and line segments must describe forward motions of the robot.

Whereas the RTR local planner is symmetrical, the RTR forward local planner is not. For more details on these local planners we refer to [Šve93, ŠO95b].

Tractor-trailer robots For tractor-trailer robots, that is, (general) car-like robots pulling a number of trailers, we use the so-called *sinusoidal local planner*. This local planner has been developed by Sekhavat and Laumond [SL96]. It transforms its argument configurations to the chained form, and uses sinusoidal inputs [TMS93] to solve the resulting equations. The sinusoidal local planner is symmetrical.

3 Probabilistic completeness

In this section we discuss *probabilistic completeness* of *PPP*, and we prove this completeness for the specific planners defined by the local planners presented in the previous section.

A path planner is called *probabilistically complete* if, given a problem that is solvable in the open free configuration space, the probability that the planner solves the problem goes to 1 as the running time goes to infinity. Hence, a probabilistically complete path planner is guaranteed to solve such a problem, provided that it is executed for a sufficient amount of time. For ease of presentation we introduce some shorthand notations. We denote the version of *PPP* using undirected underlying graphs (respectively directed graphs) by PPP_u (respectively PPP_d). The notation $PPP_u(L)$ (respectively $PPP_d(L)$) is used for referring to PPP_u (respectively PPP_d) with a specific local planner L. We denote the free configuration space by $\mathcal{C}_f$, and, throughout this and the next section, we make the assumption that it is bounded.

We will show that with *PPP* one obtains a probabilistically complete planner for any robot that is *locally controllable* (see below), provided that one defines the local planner properly. If, in addition to the local controllability, the robot also has a symmetric control system then $PPP_u(L)$ is suitable, otherwise $PPP_d(L)$ must be used. In Section 3.1 we define a general property for local planners that is sufficient for probabilistic completeness of PPP, and we point out that, given the local controllability of the robot, a local planner satisfying this property always exists (but it must be found). We also present a relaxation of the property, that guarantees probabilistic completeness of $PPP_u(L)$ as well, for locally controllable robots with symmetric control systems. It is easier to create local planners that satisfy this relaxed property. All holonomic robots, as well as for example general car-like robots and tractor-trailer robots, fall into this class. Forward car-like robots however are not locally controllable (and neither symmetric). In Section 3.3 we show that all the planners described in the previous section are probabilistically complete.

First we describe the concept of local controllability (in the literature also referred to as small-time local controllability or local-local controllability), adopting the terminology introduced by Sussman [Sus83]. Given a robot $\mathcal{A}$, let $\Sigma_{\mathcal{A}}$ be its control system. That is, $\Sigma_{\mathcal{A}}$ describes the velocities that $\mathcal{A}$ can attain in configuration space. For a configuration c of a robot $\mathcal{A}$, the set of configurations that $\mathcal{A}$ can reach within time T is denoted by $A_{\Sigma_{\mathcal{A}}}(\leq T, c)$. $\mathcal{A}$ is defined to be *locally controllable* iff for any configuration $c \in \mathcal{C}$ $A_{\Sigma_{\mathcal{A}}}(\leq T, c)$ contains a neighbourhood of c (or, equivalently, a ball centred at c) for all $T > 0$. It is a well-known fact that, given a configuration c, the area a locally controllable robot $\mathcal{A}$ can reach without leaving the ϵ-ball around c (for any $\epsilon > 0$) is the entire open ϵ-ball around c.

3.1 The general local topology property

We assume now that robot $\mathcal{A}$ is locally controllable. For probabilistic completeness of PPP a local planner L is required that exploits the local controllability of $\mathcal{A}$. This will be the case if L has what we call the *general local topological property*, or *GLT-property*, as defined in Definition 2 using the notion of *ϵ-reachability* introduced by Definition 1. We denote the ball (in configuration space) of radius ϵ centred at configuration c by $B_\epsilon(c)$, and we denote the set of all such balls by $\mathcal{B}_\epsilon$.

Definition 1 *Let L be a local planner for $\mathcal{A}$. Furthermore let $\epsilon > 0$ and $c \in \mathcal{C}$ be given. The* ϵ-reachable area of c by L, *denoted by $R_{L,\epsilon}(c)$, is defined by*

$$R_{L,\epsilon}(c) = \{\tilde{c} \in B_\epsilon(c) | L(c, \tilde{c}) \textit{ is entirely contained in } B_\epsilon(c)\}$$

Definition 2 *Let L be a local planner for $\mathcal{A}$. We say L has the GLT-property iff*

$$\forall \epsilon > 0 : \exists \delta > 0 : \forall c \in \mathcal{C} : B_\delta(c) \subset R_{L,\epsilon}(c)$$

We refer to $B_\delta(c)$ as the ϵ-reachable δ-ball *of c.*

A local planner verifying the GLT-property, at least in theory, always exists, due to the robots local controllability. Theorem 1 now states that this property is sufficient to guarantee probabilistic completeness of PPP. That is, of $PPP_u(L)$ if L is symmetric, and of $PPP_d(L)$ otherwise.

Theorem 1 *If L is a local planner verifying the GLT-property, then* PPP(L) *is probabilistically complete.*

Proof
The theorem can be proven quite easily (for both $PPP_u(L)$ and $PPP_d(L)$). Assume L verifies the GLT-property. Given two configurations s and g, lying in the same connected component of the open free configuration space, take a path P that connects s and g and lies in the open free configuration space as well. Let ϵ be the configuration space clearance of P (that is, the minimal distance between P and a configuration space obstacle), and take $\delta > 0$ such that $\forall c \in \mathcal{C} : B_\delta(c) \subset R_{L,\frac{3}{4}\epsilon}(c)$. Then, consider a covering of P by balls $B_1, \ldots, B_k$ of radius $\frac{1}{4}\delta$, such that balls B_i and B_{i+1}, for $i \in \{1, \ldots, k-1\}$, partially overlap. Assume each such ball B_i contains a node v_i of G. Then, $|v_i - v_{i+1}| \leq \delta$, and each node v_i has a configuration space clearance of at least $\epsilon - \frac{1}{4}\delta \geq \frac{3}{4}\epsilon$ (since $\delta \leq \epsilon$). Hence, due to the definition of δ, we have

$$L(v_i, v_{i+1}) \subset B_{\frac{3}{4}\epsilon}(v_i) \subset \mathcal{C}_f$$

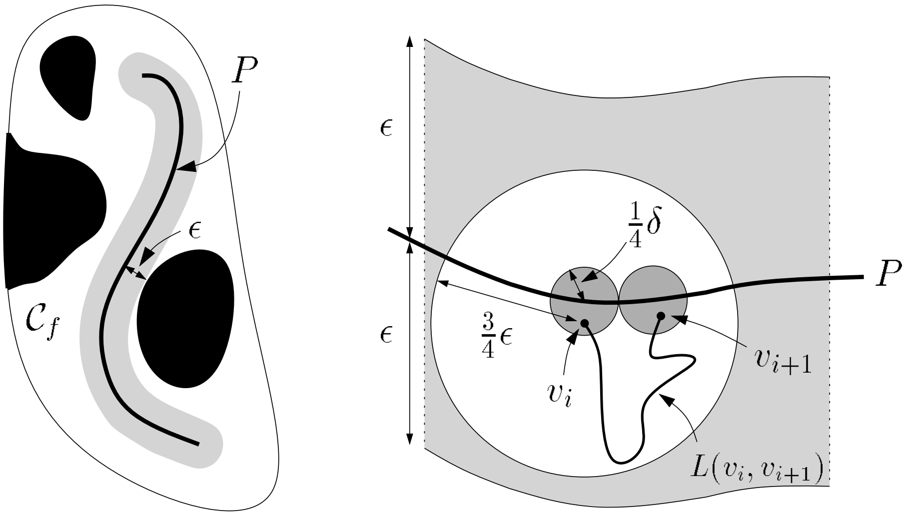

Figure 1: Path P has clearance $\epsilon > 0$. If $\delta > 0$ is chosen such that $\forall c \in \mathcal{C} : B_\delta(c) \subset R_{L,\frac{3}{4}\epsilon}(c)$, then we see that nodes in overlapping $\frac{1}{4}\delta$-balls, centred at configurations of P, can always be connected by the local planner.

It follows that if all the balls $B_1, \ldots, B_k$ contain a node of G, s and g will be graph-connected. Due to the random node adding, this is guaranteed to be the case within a finite amount of time. See also Figure 1. □

Clearly, given a locally controllable robot, the GLT-property is a proper criterion for choosing the local planner (sufficient conditions for local controllability of a robot are given in, e.g., [Sus87]). Path planning among obstacles for car-like robots using local planners with the GLT-property has also been studied by Laumond [LTJ90, JLT90].

3.2 The Lipschitz local topology property

Assume now that, in addition to the small-time local controllability, the robot also has a symmetric control system. We present a weaker local planner property which, for such robots, guarantees probabilistic completeness (of PPP_u) as well. This property has two advantages. Firstly, it is in general easier to verify for a given local planner, and, secondly, it allows for more local planners. (For example, up to now, we were unable to prove or disprove the GLT-property for the RTR local planner.) We refer to the weaker property as the *Lipschitz local topological property*, or the LTP-property. The basic relaxation in the LTP-property is that we no longer require the ϵ-reachable δ-ball of a configuration a to be centred around c. We do however make a certain requirement regarding the relationship between configurations and the corresponding ϵ-reachable δ-balls.

The LTP-property uses the notion of *Lipschitz continuity*. A function $f \in A \to B$ is called Lipschitz continuous if and only if there exists some constant d such that $\forall (a, b) \in A \times A$: $|f(a) - f(b)| \leq d|a - b|$. Such a constant is called a *Lipschitz constant* of the function f.

The LTP-property can now be defined as follows:

Definition 3 *Let L be a local method for a robot $\mathcal{A}$. L has the LTP-property, iff for every $\epsilon > 0$ there exists a $\delta_\epsilon > 0$ and a Lipschitz continuous function r_ϵ mapping configurations to δ_ϵ-balls, such that for each configuration $c \in \mathcal{C}$ the δ_ϵ-ball $r_\epsilon(c)$ is contained in c's epsilon-reachable area. Formally:*

$$\forall \epsilon > 0 : \exists \delta_\epsilon > 0, r_\epsilon \in \mathcal{C} \to \mathcal{B}_{\delta_\epsilon} : r_\epsilon \textit{ is Lipschitz continuous } \wedge \forall c \in \mathcal{C} : r_\epsilon(c) \subset R_{L,\epsilon}(c)$$

We refer to r_ϵ *as an* ϵ-reachability function *of* L, *and to* δ_ϵ *as* r_ϵ's radius.

We now prove that $PPP_u(L)$ is probabilistically complete if L is symmetric and has the LTP-property. The portion of $\mathcal{C}_f$ lying further than ϵ away from any configuration space obstacle (in configuration space, with respect to the Euclidean metric), is denoted by $\mathcal{C}_{f,\epsilon}$. A path lying in $\mathcal{C}_{f,\epsilon}$ is referred to as an *ϵ-free path*. Furthermore, given a graph $G = (V, E)$ computed by $PPP_u(L)$, we refer to the nodes lying in $\mathcal{C}_{f,\epsilon}$ as *ϵ-free nodes*, and we denote this set by V_ϵ.

Lemma 1 states that an ϵ-free node a which contains another node b in its ϵ-reachable area, must be graph-connected to this node.

Lemma 1 *Let* $G = (V, E)$ *be a graph computed by* $\mathrm{PPP}_u(L)$ *with* L *symmetric, and let* $\epsilon > 0$. *If* $a \in V_\epsilon$ *and* $b \in V \cap R_{L,\epsilon}(a)$, *then* a *and* b *are graph-connected in* G.

Proof
Given L's symmetry, the lemma follows directly from the definitions of $R_{L,\epsilon}(c)$ and PPP_u. ◻

The second lemma we need states that, again provided the used local method is symmetric and verifies the LTP-property, a certain node density guarantees ϵ-free nodes lying sufficiently near to each other to be graph-connected.

Lemma 2 *Let* $G = (V, E)$ *be a graph computed (at some point) by* $\mathrm{PPP}_u(L)$, *with* L *symmetric and verifying the LTP-property, and let* $\epsilon > 0$. *So* L *has an* ϵ*-reachability function* r_ϵ *with some radius* δ_ϵ, *and Lipschitz constant* d_ϵ. *Now if each free ball of radius* $\frac{\delta_\epsilon}{2}$ *(in* $\mathcal{C}_f$*) contains a node of* G, *then each pair of* ϵ*-free nodes which lie not more than* $\frac{\delta_\epsilon}{d_\epsilon}$ *apart will be graph-connected in* G. *Formally:*

$$\begin{array}{l} \forall B \in \mathcal{B}_{\frac{\delta_\epsilon}{2}} : B \subset \mathcal{C}_f \Rightarrow B \cap V \neq \emptyset \\ \quad \Longrightarrow \\ \forall a, b \in V_\epsilon : |a - b| \leq \frac{\delta_\epsilon}{d_\epsilon} \Rightarrow \textit{graph-connected}(a, b) \end{array}$$

Proof
Assume that

$$\forall B \in \mathcal{B}_{\frac{\delta_\epsilon}{2}} : B \subset \mathcal{C}_f \Rightarrow B \cap V \neq \emptyset$$

Let $\{a, b\} \subset V_\epsilon$ with $|a - b| \leq \frac{\delta_\epsilon}{d_\epsilon}$. The fact that d_ϵ is a Lipschitz constant of the Lipschitz continuous function r_ϵ, induces $|r_\epsilon(a) - r_\epsilon(b)| \leq \delta_\epsilon$. This means that the intersecting area of $r_\epsilon(a)$ and $r_\epsilon(b)$ contains a ball B of radius $\frac{\delta_\epsilon}{2}$, and this ball contains a node c. Because c lies in the ϵ-reachably areas of both a and b, it follows, by Lemma 1, that c is graph-connected to both a and b. Clearly, because G is undirected, this implies the graph-connectivity of a and b. See also Figure 2. ◻

With the above, we now prove the claim of probabilistic completeness.

Theorem 2 *If* L *is a symmetric local planner verifying the LTP-property, then* $\mathrm{PPP}_u(L)$ *is probabilistically complete.*

Proof
Let s and g be two configurations lying in the same connected component of the open free configuration space. Take $\epsilon > 0$ such that s and g lie in the same connected component of $\mathcal{C}_{f,\epsilon+\frac{\delta_\epsilon}{4d_\epsilon}}$ (where δ_ϵ is the radius of an ϵ-reachability function of L, and d_ϵ is a finite Lipschitz constant of

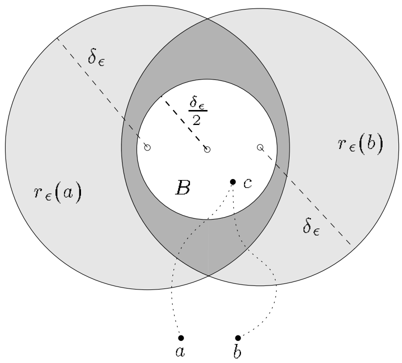

Figure 2: Intersecting δ-balls $r_\epsilon(a)$ and $r_\epsilon(b)$. Their intersection contains a $\frac{\delta_\epsilon}{2}$-ball B. The presence of a node c in B guarantees a and b to be graph-connected.

this function). In other words, there exists a $(\epsilon + \frac{\delta_\epsilon}{4d_\epsilon})$-free path $\mathcal{P}$ connecting configuration s to configuration g.

Let now $G = (V, E)$ be a graph, computed by $PPP_u(L)$, such that

$$\forall B \in \mathcal{B}_{\frac{\delta_\epsilon}{4d_\epsilon}} : B \subset \mathcal{C}_f \Rightarrow B \cap V \neq \emptyset$$

Due to the boundedness of $\mathcal{C}_f$, the probability of obtaining such a graph grows towards 1 when the running time grows towards infinity.

We now show that there exists a path P_G in G which connects the nodes a and b. Let $\{B_1, B_2, \ldots, B_n\}$ be a set of $\frac{\delta_\epsilon}{4d_\epsilon}$-balls (i.e., balls of radius $\frac{\delta_\epsilon}{4d_\epsilon}$) centred at configurations of $\mathcal{P}$, such that $B_1 \ni s$, $B_n \ni g$, and $\forall i \in \{1, \ldots, n-1\} : B_i \cap B_{i+1} \neq \emptyset$. See Figure 3. Each ball B_i is a $\frac{\delta_\epsilon}{4d_\epsilon}$-ball lying in free configuration space, so each B_i contains at least one node $c_i \in V$. Clearly

$$\forall i \in \{1, \ldots, n-1\} : |c_i - c_{i+1}| \leq \frac{\delta_\epsilon}{d_\epsilon}$$

Each c_i is an ϵ-free node, so, by Lemma 2, it follows that

$$\forall i \in \{1, \ldots, n-1\} : \textit{graph-connected}(c_i, c_{i+1})$$

Furthermore, $|s - c_1| \leq \frac{\delta_\epsilon}{d_\epsilon}$ and $|c_n - g| \leq \frac{\delta_\epsilon}{d_\epsilon}$ (and s and g are ϵ-free nodes), so, again by Lemma 2, it follows that

$$\textit{graph-connected}(s, c_1) \wedge \textit{graph-connected}(c_n, g)$$

Hence, node s is graph-connected to node g, which concludes the proof. □

3.3 Probabilistic completeness for specific robot classes

The local planners used for holonomic robots, general car-like robots, forward car-like robots, and tractor-trailer robots, as described in Section 2, guarantee probabilistic completeness.

3.3.1 Holonomic robots

The general holonomic local planner L constructs the straight line path (in configuration space) connecting its argument configurations. It immediately follows that $R_{\epsilon,L}(c) = B_\epsilon(c)$, for any configuration c and any $\epsilon > 0$. Hence, L verifies the GLT-property.

Theorem 3 PPP$_u(L)$, *with L being the general holonomic local planner, is probabilistically complete for all holonomic robots.*

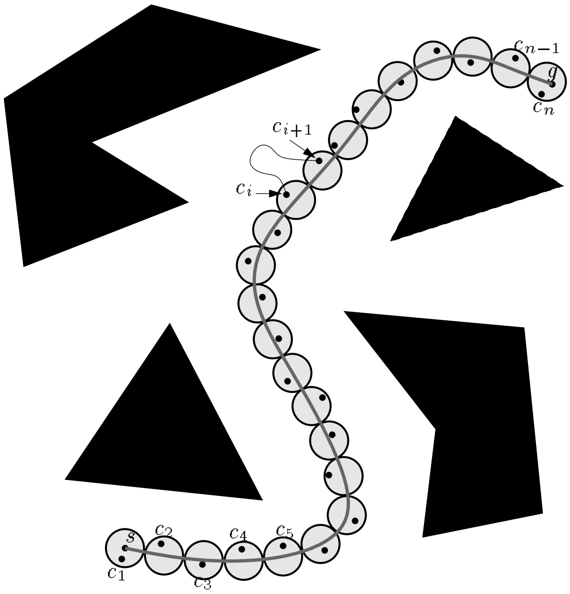

Figure 3: A "covering" of an $(\epsilon + \frac{\delta_\epsilon}{4d_\epsilon})$-free path by $\frac{\delta_\epsilon}{4d_\epsilon}$-balls, which contain the vertices of a graph path connecting s to g. Nodes c_i and c_{i+1} lying in "neighbouring" balls must be graph-connected.

3.3.2 Fully controllable nonholonomic robots

General car-like robots We now prove that the RTR local planner, as used for general car-like robots, has the LTP-property. Together with its symmetry this will prove probabilistic completeness of PPP_u.

Lemma 3 *The RTR local planner has the LTP-property.*

Proof
We assume, without loss of generality, that the minimal turning radius of the robot equals 1. Given a configuration c, we define r_c, respectively l_c, to be $(x + \sin\theta, y - \cos\theta)$, respectively $(x - \sin\theta, y + \cos\theta)$. These points are the centres of, what we call, c's rotational circles. Furthermore, we denote the configuration $(0, 0, 0)$ by c_0.

Let $\mathcal{D} = \{c \in \mathcal{C} \mid y_c > |x_c| \wedge |\theta| < \frac{1}{2}\pi \wedge |r_{c_0} - l_c| > 2\}$, and let $c \in \mathcal{D}$. We give an upper bound of the configuration space length of the RTR-path $R_1TR_2(c)$ which connects c_0 to c and consists of a (forward) rotational path (R_1) around r_{c_0}, a (backward) translational path (T), and rotational path (R_2) around l_c, as shown in Figure 4. Due to the definition of $\mathcal{D}$ we have the guarantee that such a path always exists. The configuration space length of a path (segment) P is denoted by $L_\mathcal{C}(P)$, and its workspace length by $L_\mathcal{W}(P)$.

$$\begin{aligned} L_C(R_1TR_2(c)) &= L_C(R_1) + L_C(T) + L_C(R_2) = \sqrt{2}L_W(R_1) + L_W(T) + \sqrt{2}L_W(R_2) \\ &= 2\sqrt{2}L_W(R_1) + L_W(T) + \sqrt{2}\theta_c \end{aligned}$$

It is easy to verify that $L_W(R_1) \leq \frac{1}{2}L_W(T)$, and hence

$$L_C(R_1TR_2) \leq (1 + \sqrt{2})L_W(T) + \sqrt{2}\theta_c \quad (1)$$

Now let I, A, and B be the points as shown in Figure 4. That is, $AB = T$ and I is its centre. Because $Al_cBr_{c_0}$ is a parallelogram, we have $I = (\frac{1}{2}(x_c - \sin\theta_c), \frac{1}{2}(y_c - 1 + \cos\theta_c))$. Since $l_c =$

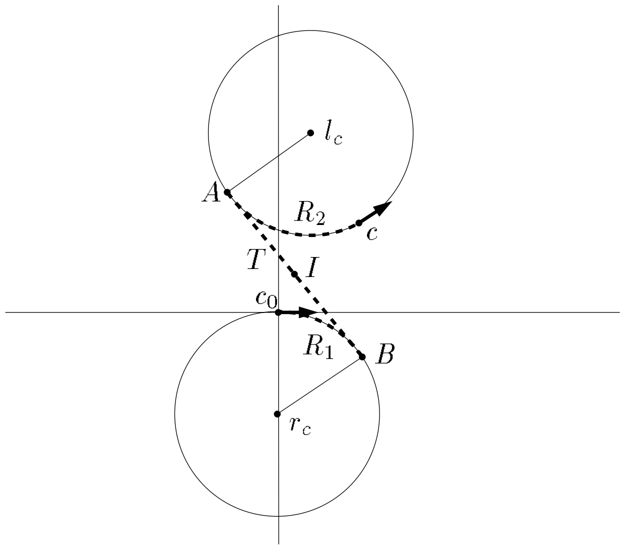

Figure 4: A R_1TR_2 path connecting configuration c_0 to configuration c.

$(x_c - \sin\theta_c, y_c + \cos\theta_c)$, it follows that $|l_c - I| = \sqrt{\frac{1}{4}(x_c - \sin\theta_c)^2 + \frac{1}{4}(y_c + \cos\theta_c + 1)^2}$. This gives

$$L_W(T) = 2\sqrt{|A-I|^2} = 2\sqrt{|L_c - I|^2 - 1} = 2\sqrt{\frac{1}{4}(x_c - \sin\theta_c)^2 + \frac{1}{4}(y_c + \cos\theta_c + 1)^2 - 1}$$

and hence, due to (1), $\hat{L}(c) = (2+2\sqrt{2})\sqrt{\frac{1}{4}(x_c - \sin\theta_c)^2 + \frac{1}{4}(y_c + \cos\theta_c + 1)^2 - 1} + \sqrt{2}\theta_c$ gives an upper-bound for $L_C(R_1TR_2(c))$. It is easily shown that $\hat{L}(c)$ is defined and continuous on $\mathcal{D}$.

Now let, for a given $\epsilon > 0$, configuration q_k be $(0, \frac{\epsilon}{k}, 0)$. Note that $q_k \in \mathcal{D}$ for any $k > 0$.

$$\hat{L}(q_k) = (2+2\sqrt{2})\sqrt{\frac{1}{4}\left(\frac{\epsilon}{k}+2\right)^2 - 1} = (2+2\sqrt{2})\sqrt{\frac{\epsilon^2}{k^2} + \frac{2\epsilon}{4k}}$$

Clearly, for any $\epsilon > 0$, when k goes to ∞ then $\hat{L}(q_k)$ goes to 0. Hence, given an arbitrary $\epsilon > 0$, if we choose k large enough we have $\hat{L}(q_k) < \epsilon$. Let k_ϵ be such a k. Because $\hat{L}$ is continuous on $\mathcal{D}$ and $\mathcal{D}$ is an open set, it follows that $\exists \delta > 0 : \forall \tilde{c} \in B_\delta(q_{k_\epsilon}) : \hat{L}(\tilde{c}) < \epsilon$. Let δ_ϵ be such a δ.

Now, because $\hat{L}(\tilde{c})$ gives an upper-bound on the length of a particular RTR path connecting c_0 to $\tilde{c}$, and the RTR local planner constructs the *shortest* RTR path connecting them, it follows that $B_{\delta_\epsilon}(q_{k_\epsilon}) \subset R_{L,\epsilon}(c_0)$, for L being the RTR local planner. Because we are free in choosing our coordinate frame as we wish, it follows that if we define

$$r_\epsilon = c \mapsto B_{\delta_\epsilon}((x_c, y_c, \theta_c) + \frac{\epsilon}{k_\epsilon}(\cos(\theta_c + \frac{1}{2}\pi), \sin(\theta_c + \frac{1}{2}\pi), 0))$$

then $\forall c \in \mathcal{C} : r_\epsilon(c) \subset R_{L,\epsilon}(c)$. Clearly r_ϵ is Lipschitz continuous. Hence, by Definition 3, it is a valid ϵ-reachability function (with radius δ_ϵ) of the RTR local planner, and the LTP-property holds. □

Theorem 4 PPP$_u(L)$, *with L being the RTR local planner, is probabilistically complete for general car-like robots.*

Proof
Follows directly from Theorem 2 and Lemma 3. ◻

Tractor-trailer robots Regarding tractor-trailer robots, Sekhavat and Laumond prove in [SL96] that the sinusoidal local planner, used for the tractor-trailer robots, verifies the GLT-property. Hence, for tractor-trailer robots we also have probabilistic completeness.

Theorem 5 $\mathrm{PPP}_u(L)$, *with* L *being the sinusoidal local planner, is probabilistically complete for tractor-trailer robots (with an arbitrary number of trailers).*

3.3.3 Forward car-like robots

As pointed out before, the theory of the previous sections applies only to robots that are locally controllable. If a robot does not have this property, a local planner verifying the GLT-property will not exist. A local planner verifying the weaker LTP-property may exist, but this planner will not be symmetric (this would imply the existence of a local planner verifying GTP).

Forward car-like robots are not locally controllable. One can nevertheless prove probabilistic completeness of $PPP_d(L)$, with L being the RTR forward local planner. That is, one can prove that, given two configurations s and g such that there exists a feasible path in the open free configuration space connecting them, $PPP_d(L)$ will surely solve the problem within finite time. The proof, which does not directly generalise to other cases, uses a property of RTR forward paths stated in Lemma 4. We say a path P lies within distance ϵ of a path Q, iff $\forall p \in P : \exists q \in Q : |p - q| \leq \epsilon$ (in configuration space).

Lemma 4 *Let* L *be the RTR forward local planner, and let* Q *be a RTR forward path connecting configurations* a *and* b *with a straight line path of non-zero length and both arc paths of total curvature less than* $\frac{1}{2}\pi$*. Then:*

$$\forall \epsilon > 0 : \exists \delta > 0 : \forall (\tilde{a}, \tilde{b}) \in B_\delta(a) \times B_\delta(b) : L(\tilde{a}, \tilde{b}) \textit{ lies within distance } \epsilon \textit{ of } Q$$

Intuitively, the lemma states that in most cases the path from a point close to a to a point close to b lies close to the path from a to b.

Theorem 6 $\mathrm{PPP}_d(L)$, *with* L *being the RTR forward local planner, is probabilistically complete for forward car-like robots.*

We give only a sketch of the proof here (See also Figure 5). Let L be the RTR forward local planner. Assume P_1 is a path in the open free configuration space connecting a (start) configuration s to a (goal) configuration g, that is feasible for our forward car-like robot $\mathcal{A}$. Then, one can prove, there exists also a feasible path P_2 in the open free configuration space, connecting s to g, that consists of (a finite number of) straight line segments and circular arcs, such that no two distinct arcs are adjacent and each arc has a total curvature of less than $\frac{1}{2}\pi$.[1]

Assume k is the number of arcs in P_2. Let $m_1 = s$, $m_k = g$, and $\{m_2, \ldots, m_{k-1}\}$ be points on P_2 such that m_i is the midpoint of the i-th arc of P_2 (that is, the unique point on the arc with equal distance to both end-points). Clearly, m_i is connected m_{i+1} by a RTR forward path with a straight line segment of non-zero length and both arc paths of total curvature less than $\frac{1}{2}\pi$ (for all $j \in \{1, \ldots, k-1\}$).

Let $\epsilon > 0$ be the clearance of P_2, and take $\delta > 0$ such that, for all $j \in \{1, \ldots, k-1\}$:

$$\forall (a, b) \in B_\delta(m_j) \times B_\delta(m_{j+1}) : L(a, b) \text{ lies within distance } \epsilon \text{ of } Q$$

[1] This does not necessarily hold if P_1 consists of just one or two circular arcs of maximal curvature. In this case however P_1 can be found directly with the RTR forward local planner.

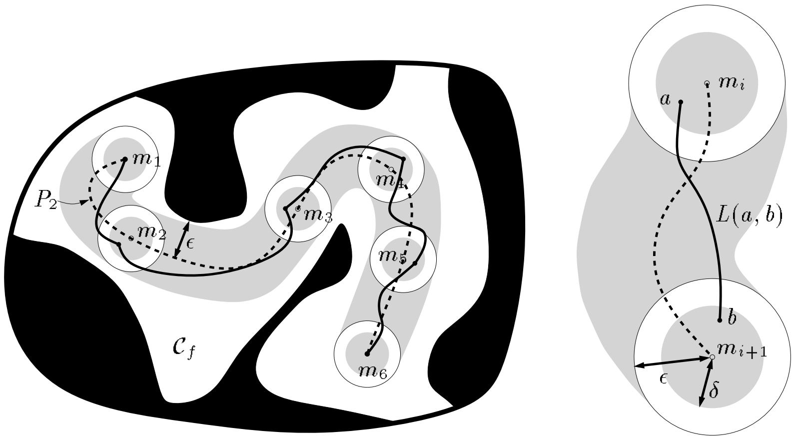

Figure 5: This figure illustrates the proof of Theorem 6. P_2 is a path, feasible for a forward car-like robot, of clearance $\epsilon > 0$. Centred at the configurations m_i are balls B_i of a radius $\delta > 0$, such that any pair of configurations $(a,b) \in B_i \times B_{i+1}$ is connected by the RTR forward local planner L with a path lying within distance ϵ of P_2, and hence lying in $\mathcal{C}_f$.

It follows from Lemma 4 that such a $\delta > 0$ always exists. When a node of G is present in every ball $B_\delta(m_j)$ for $2 \leq j < k$, G will contain a path connecting s to g. We know, due to the probabilistic nature of the node adding, that the probability of obtaining such a graph grows to 1 when the roadmap construction time goes to infinity.

4 On the expected complexity of probabilistic path planning

In the previous section we have formulated properties of local planners that guarantee probabilistic completeness of *PPP*. If these properties are satisfied, we know that as the running time of *PPP* goes to infinity, the probability of solving any solvable problem goes to 1. However, this gives no information about (expected) convergence time of the algorithm. In practice, one will not be satisfied with the guarantee that "eventually a path will be found". For real life applications, some estimate of the running time at beforehand is desirable.

Simulation results obtained by the application of *PPP* on certain "typical" problems can increase our trust in the planners performance and robustness, but they do not describe a formal relation between probabilities of failure and running times in general, and neither do they provide a theoretical explanation for the empirically observed success of the probabilistic planner. Recently some first theoretical results on expected running times of probabilistic planners have been obtained.

Kavraki et al. ([KLMR95, KKL96, BKL$^+$95]) show that, under certain geometric assumptions about the free configuration space $\mathcal{C}_f$, it is possible to establish a relation between the probability that probabilistic planners like *PPP* find paths solving particular problems, and their running times. They suggest two such assumptions, i.e., the *visibility volume assumption* and the *path clearance assumption*. We will discuss the path clearance assumption and present the main result obtained by Kavraki et al. Also, we will extend this result to the case of nonholonomic robots. Furthermore, we introduce a new assumption on the configuration space, called the *ϵ-complexity assumption*, under which it is possible to relate the success probabilities and running times of *PPP* to the complexity of the problems that are to be solved. Throughout this section we use the notations introduced in the previous section.

4.1 Some preliminaries

Definition 4 *Let $\mathcal{B}$ be a set of balls and V be a set of points in $\mathbb{R}^n$. We say the V* stabs *$\mathcal{B}$ if and only if*

$$\forall B \in \mathcal{B} : B \cap V \neq \emptyset$$

We denote this property by $\mathcal{S}(V, \mathcal{B})$.

Given a bounded object A in $\mathbb{R}^n$, we denote its volume by $\mathcal{V}(A)$. The volume of the unit ball in $\mathbb{R}^n$ we denote by $\mathcal{V}_n^1$. Lemma 5 gives a lower bound for the probability that a set of randomly picked points does not stab a given set of balls (in arbitrary dimension). It uses the well know fact that a ball of radius ϵ in $\mathbb{R}^n$ has a volume of $\mathcal{V}_n^1 \epsilon^n$.

Lemma 5 *Let A be a bounded object in $\mathbb{R}^n$. Let $\mathcal{B}$ be a set of balls of radius $\epsilon > 0$ lying (entirely) in A. Let V be a set of points randomly chosen from A. Then, the probability $Pr(\neg\mathcal{S}(V, B))$ that V does not stab $\mathcal{B}$ is at most*

$$|\mathcal{B}| \left(1 - \frac{\mathcal{V}_n^1}{\mathcal{V}(A)} \epsilon^n \right)^{|V|}$$

Proof

$\Pr(\neg\mathcal{S}(V, B)) = \Pr(\exists B \in \mathcal{B} : B \cap V = \emptyset) \leq \sum_{B \in \mathcal{B}} \Pr(B \cap V = \emptyset) = \sum_{B \in \mathcal{B}} \left(\frac{\mathcal{V}(A) - \mathcal{V}(B)}{\mathcal{V}(A)}\right)^{|V|} = \sum_{B \in \mathcal{B}} \left(1 - \frac{\mathcal{V}(B)}{\mathcal{V}(A)}\right)^{|V|} = |B| \left(1 - \frac{\mathcal{V}_n^1}{\mathcal{V}(A)} \epsilon^n\right)^{|V|}$ ⊔⊓

4.2 The path clearance assumption

In the path clearance assumption, it is assumed that there exists a collision-free path $\mathcal{P}$ between the start configuration s and goal configuration g, that has some clearance $\epsilon > 0$ with the configuration space obstacles. In [KKL96], Kavraki et al. study the dependence of the failure probability of PPP_u to connect s and g on (1) the length of $\mathcal{P}$, (2) the clearance ϵ, and (3) the number of nodes in the probabilistic roadmap G. Their main result is described by Theorem 7, which uses Lemma 6. We rephrase it a bit. For the original version we refer to [KKL96].

Lemma 6 *Let $\mathcal{A}$ be a holonomic robot, L the general holonomic local planner (for $\mathcal{A}$), and $G = (V, E)$ a graph constructed by $\mathrm{PPP}_u(L)$. Assume configurations s and g are connectable by a path $\mathcal{P}$ of length λ, that has a clearance $\epsilon > 0$ with the (configuration space) obstacles. Let $x_1, \ldots, x_k$ be configurations on P such that $\forall i \in \{1, \ldots, k-1\} : d(x_i, x_{i+1}) \leq \frac{1}{2}\epsilon$, $d(s, x_1) \leq \frac{1}{2}\epsilon$, and $d(x_k, g) \leq \frac{1}{2}\epsilon$. Then,*

$$S\left(V, \{B_{\frac{1}{2}\epsilon}(x_i) | 1 \leq i \leq k\}\right) \Rightarrow s \text{ and } g \text{ graph-connected in } G$$

Proof

See Figure 6. Assume $S\left(V, \{B_{\frac{1}{2}\epsilon}(x_i) | 1 \leq i \leq k\}\right)$. So every ball $\{B_{\frac{1}{2}\epsilon}(x_i) | 1 \leq i \leq k\}$ contains a node n_i. For any i, n_i lies within distance $\frac{1}{2}\epsilon$ of x_i. Since $d(x_i, x_{i+1}) \leq \frac{1}{2}\epsilon$, n_{i+1} cannot lie further than ϵ away from x_i. It follows that $L(n_i, n_{i+1})$ lies within distance ϵ of x_i, and hence lies in $\mathcal{C}_f$. So each n_i will be connected to n_{i+1}. With an analogous argument it follows that s is connected to x_1 and g to x_k. ⊔⊓

Theorem 7 ***[KKL96, BKL+95]*** *Let $\mathcal{A}$ be a holonomic robot, L the general holonomic local planner (for $\mathcal{A}$), and $G = (V, E)$ a graph constructed by $\mathrm{PPP}_u(L)$. Assume configurations s and g*

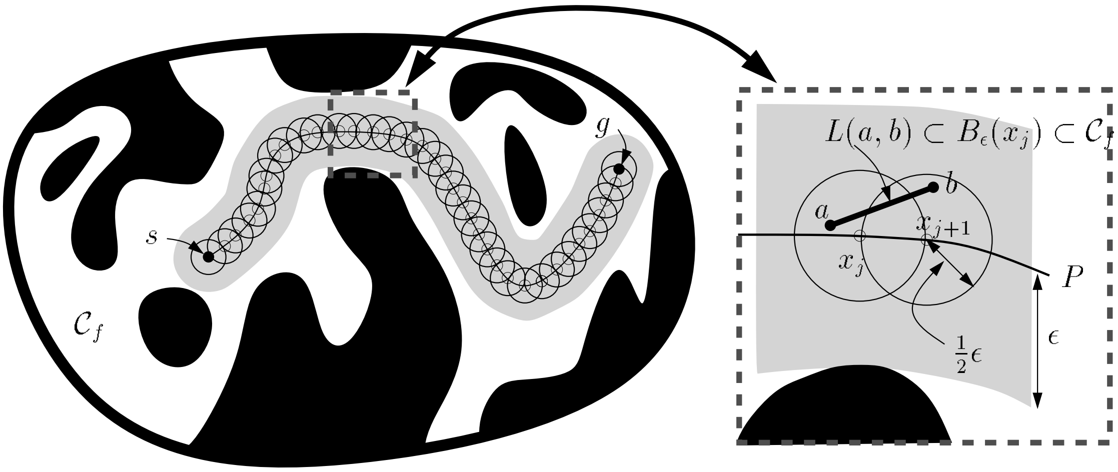

Figure 6: We see that configuration s is connectable to configuration g by a path P of clearance ϵ. Let $x_0 = s, x_1, \ldots, x_k = g$ be points on P, such that $|x_j - x_{j+1}| \leq \frac{1}{2}\epsilon$, for all j. If each ball $B_{\frac{1}{2}\epsilon}(x_j)$ contains a node of G, then s and g will be graph-connected.

are connectable by a path $\mathcal{P}$ of length λ, that has a clearance $\epsilon > 0$ with the (configuration space) obstacles. Then, the probability that s and g are not graph-connected in G is at most

$$\frac{2\lambda}{\epsilon}\left(1 - \frac{\mathcal{V}_n^1}{2^n \mathcal{V}(\mathcal{C}_f)}\epsilon^n\right)^{|V|}$$

Proof

Let $k = \frac{2\lambda}{\epsilon}$. Let $x_1, \ldots, x_k$ be configurations on P such that $\forall i \in \{1, \ldots, k-1\} : d(x_i, x_{i+1}) \leq \frac{1}{2}\epsilon$, $d(s, x_1) \leq \frac{1}{2}\epsilon$, and $d(x_k, g) \leq \frac{1}{2}\epsilon$. (it is obvious that such configurations must always exist). Let us denote the set $\{B_{\frac{1}{2}\epsilon}(x_1), \ldots, B_{\frac{1}{2}\epsilon}(x_k)\}$ by $\mathcal{B}$. Lemma 5 states that $\Pr(\neg S(V, \mathcal{B}))$ is at most

$$k\left(1 - \frac{\mathcal{V}_n^1}{2^n \mathcal{V}(\mathcal{C}_f)}\epsilon^n\right)^{|V|}$$

Lemma 6 concludes the proof. ◻

A number of important facts are implied by Theorem 7, which gives an upper bound for the failure probability $\mathcal{P}_\mathcal{F}$. For example, we see that $\mathcal{P}_\mathcal{F}$ is at most linear in the path length λ. Moreover, using the inequality $1 - x \leq e^{-x}$, it follows that the number of nodes required to be generated, in order for the planner to succeed with probability at least $1 - \mathcal{P}_\mathcal{F}$, is logarithmic in λ and $\frac{1}{\mathcal{P}_\mathcal{F}}$, and polynomial in $\frac{1}{\epsilon}$.

The analysis assumes the use of the general holonomic local planner (as described in Section 2). Hence, it is assumed that the robot is holonomic. An underlying assumption is namely that the ϵ-reachable area of any configuration c consists of the entire ϵ-ball B_ϵ(c), surrounding c. From a theoretical point of view, as pointed out in the previous section, for any locally controllable robot a local planner exists for which the ϵ-reachable area of any configuration c consists of the entire open ϵ-ball centred at c. With such a local planner the above result is directly applicable to such robots. However, it is not realistic to assume the "ϵ-ball reachability" for a nonholonomic local planner, since for most robots we are not able to construct such local planners, and, if we could, they would probably be vastly outperformed (in terms of computation time) by simple local planners verifying only weaker (but sufficient) topological properties, such as those presented in the previous section. However, the analyses presented in [KKL96, BKL+95] can be extended to the case where the local planner verifies only the GLT-property. In this way, we can give running time estimates for locally controllable nonholonomic robots that are realistic in the sense that we can actually build the planners that we analyse. Theorem 8, which uses Lemma 7, extends the result of Theorem 7 to locally controllable nonholonomic robots with local planners verifying the *GLT*-property.

Lemma 7 *Let $\mathcal{A}$ be a fully controllable robot, L a local planner for $\mathcal{A}$ verifying the* GLT*-property, and $G = (V, E)$ a graph constructed by* PPP(L)*. Assume configurations s and g are connectable by a path $\mathcal{P}$ of length λ, that has a clearance $\epsilon > 0$ with the (configuration space) obstacles. Take $\delta > 0$ such that*

$$\forall c \in \mathcal{C} : B_\delta(c) \subset R_{L,\frac{3}{4}\epsilon}(c)$$

Let $x_1, \ldots, x_k$ be configurations on P such that $\forall i \in \{1, \ldots, k-1\} : d(x_i, x_{i+1}) \leq \frac{1}{2}\delta$, $d(s, x_1) \leq \frac{1}{2}\delta$, and $d(x_k, g) \leq \frac{1}{2}\delta$. Then,

$$S\left(V, \{B_{\frac{1}{4}\delta}(x_i) | 1 \leq i \leq k\}\right) \Rightarrow s \text{ and } g \text{ graph-connected in } G$$

Proof
This lemma can be proven similar to Theorem 1. □

Theorem 8 *Let $\mathcal{A}$ be a fully controllable robot, L a local planner for $\mathcal{A}$ verifying the* GLT*-property, and $G = (V, E)$ a graph constructed by* PPP(L)*. Assume configurations s and g are connectable by a path $\mathcal{P}$ of length λ, that has a clearance $\epsilon > 0$ with the (configuration space) obstacles. Take $\delta > 0$ such that*

$$\forall c \in \mathcal{C} : B_\delta(c) \subset R_{L,\frac{3}{4}\epsilon}(c)$$

Then, the probability that s and g are not graph-connected in G is at most

$$\frac{2\lambda}{\delta}\left(1 - \frac{\mathcal{V}_n^1}{4^n \mathcal{V}(\mathcal{C}_f)}\delta^n\right)^{|V|}$$

Proof
Let $k = \frac{2\lambda}{\delta}$. Let $x_1, \ldots, x_k$ be configurations on P such that $\forall i \in \{1, \ldots, k-1\} : d(x_i, x_{i+1}) \leq \frac{1}{2}\delta$, $d(s, x_1) \leq \frac{1}{2}\delta$, and $d(x_k, g) \leq \frac{1}{2}\delta$. (its obvious that such configurations must always exists). Let us denote the set $\{B_{\frac{1}{4}\delta}(x_1), \ldots, B_{\frac{1}{4}\delta}(x_k)\}$ by $\mathcal{B}$. Lemma 5 states that $\Pr(\neg S(V, \mathcal{B}))$ is at most

$$k\left(1 - \frac{\mathcal{V}_n^1}{4^n \mathcal{V}(\mathcal{C}_f)}\delta^n\right)^{|V|}$$

Lemma 7 concludes the proof. □

Since, by definition of the *GLT*-property, δ is a constant with respect to ϵ, the dependencies implied by Theorem 7 hold for nonholonomic robots as well.

4.3 The ϵ-complexity assumption

A drawback of Theorem 7 and and Theorem 8 is that no relation is established between the failure probability and the *complexity* of a particular problem. In our opinion, to a considerable extent, the observed success of *PPP* lies in the fact that not the complexity of the configuration space, but the complexity of the resulting path defines the (expected) running time of *PPP*. For example,

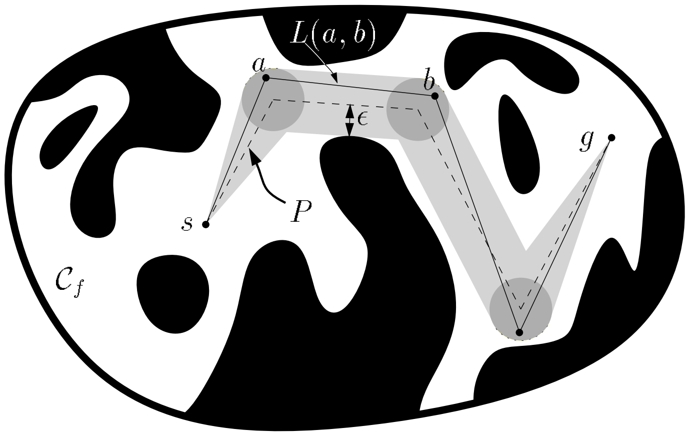

Figure 7: We see that configuration s is connectable to configuration g by a piecewise linear path P (dashed) of complexity 4 and clearance ϵ. If each of 3 dark grey balls (of radius ϵ, placed at the vertices of P) contains a node of G, then G contains a path, lying in the grey area, that connects s and g.

assume a particular problem is solvable by a path P of clearance $\epsilon > 0$, consisting of say 4 straight line segments. Consider three balls of radius ϵ, centred at the 3 inner vertices of P. Then, as is illustrated in Figure 7, it suffices that all the 3 balls contain a node of G to guarantee that the problem is solved. We see in this example that the failure probability in no way relates to the length of the path, and neither to the complexity of $\mathcal{C}_f$. The only relevant factors are the clearance and the complexity of the path. Definition 5 introduces the notion of *ϵ-complexity*, which captures this measure of problem complexity. We refer here to a path composed of k straight line segments as a piecewise linear path of complexity k.

Definition 5 *Given a holonomic robot and a particular path planning problem (s, g), let P be the lowest complexity piecewise-linear path connecting s and g, that has a configuration space clearance of $\epsilon > 0$. We define the ϵ-*complexity *of problem (s, g) as the complexity of P.*

Theorem 9 gives a result relating the failure probability of *PPP* to the *ϵ-complexity* of the problem to be solved. It applies only to holonomic robots and assumes the use of the general holonomic local planner. It is based on Lemma 8.

Lemma 8 *Let $\mathcal{A}$ be a holonomic robot, L the general holonomic local planner (for $\mathcal{A}$), and $G = (V, E)$ a graph constructed by $\mathrm{PPP}_u(L)$. Assume configurations s and g are connectable by a piecewise linear path $\mathcal{P}$, that has a clearance $\epsilon > 0$ with the (configuration space) obstacles. Let $x_1, x_2, \ldots, x_{k-1}$ be the internal vertices of P. Then,*

$$S(V, \{B_\epsilon(x_i) | 1 \leq i \leq k-1\}) \Rightarrow s \text{ and } g \text{ graph-connected in } G$$

Proof

Assume $S(V, \{B_\epsilon(x_i) | 1 \leq i \leq k-1\})$. That is, each ball $B_\epsilon(x_i)$ contains a node n_i. Let $L_{i,i+1}$ be the straight line segment (in configuration space) connecting x_i and x_{i+1} ($L_{i,i+1}$ is a subpath of $\mathcal{P}$ of complexity 1). Since n_i lies within distance ϵ of x_i, and n_{i+1} lies within distance ϵ of x_{i+1}, no point on the straight line segment $L(n_i, n_{i+1})$ can lie further than ϵ away from $L_{i,i+1}$. Due to the clearance assumption, this means that $L(n_i, n_{i+1})$ lies in the free configuration space, and, hence, n_i is graph-connected to n_{i+1}. By an analogous argument it follows that s is graph-connected to n_1 and g to n_k. See also Figure 7. □

Theorem 9 *Let $\mathcal{A}$ be a holonomic robot, L the general holonomic local planner (for $\mathcal{A}$), and $G = (V, E)$ a graph constructed by* $\text{PPP}_u(L)$*. Assume (s, g) is a problem of ϵ-complexity ζ. Then, the probability that s and g are not graph-connected in G is at most*

$$(\zeta - 1)\left(1 - \frac{\mathcal{V}_n^1}{\mathcal{V}(\mathcal{C}_f)}\epsilon^n\right)^{|V|}$$

Proof

Let $x_1, x_2, \ldots, x_{\zeta-1}$ be the internal vertices of P. Let us denote the set $\{B_\epsilon(x_1), \ldots, B_\epsilon(x_{\zeta-1})\}$ by $\mathcal{B}$. Lemma 5 states that $\Pr(\neg S(V, \mathcal{B}))$ is at most

$$(\zeta - 1)\left(1 - \frac{\mathcal{V}_n^1}{\mathcal{V}(\mathcal{C}_f)}\epsilon^n\right)^{|V|}$$

Lemma 8 concludes the proof. ◻

So we now have a linear dependence of the failure probability, and a logarithmic dependence of $|V|$, on the *complexity* ζ of the path P, that is, on the ϵ-complexity of the problem. We note that the existence of a path of a certain clearance $\epsilon > 0$ and implies the existence of a piecewise linear path of a similar clearance.

Extension of the result stated by Theorem 9 for nonholonomic robots is possible. However, a local planner is required that satisfies a property as described in Lemma 4. I.e., a local planner L is needed such that:

$$\forall \epsilon > 0 : \exists \delta > 0 : \forall (\tilde{a}, \tilde{b}) \in B_\delta(a) \times B_\delta(b) : L(\tilde{a}, \tilde{b}) \text{ lies within distance } \epsilon \text{ of } Q$$

The local planners for nonholonomic robots as presented in this paper however do not satisfy this property.

5 Conclusions

In this paper we reviewed and extended previous theoretical results regarding probabilistic completeness and expected running times for *PPP*, a general probabilistic scheme for robot path planning.

We have shown that probabilistic completeness, for holonomic as well as nonholonomic robots, can be obtained by the use of local planners that respect certain general topological properties. Furthermore, we have presented results (existing and new) that, under certain geometric assumptions on the free configuration space, link the expected running time and failure probability of the planner to the size of the roadmap and characteristics of paths solving the particular problem. For example, under one such assumption, we have shown that the expected size of a probabilistic roadmap required for solving a problem grows only logarithmically in the complexity of the problem.

A weak point of the theory presented in this paper is that the free configuration space characteristics that lead to the presented running time estimates can only be assumed, and, in general, not measured. Very useful future work would be the formulation of assumptions on the *workspace* (instead of the configuration space) that enable estimates of *PPP*'s running time, since these could most likely be measured at beforehand. However, at present, the formulation of such workspace characteristic is still an open problem.

References

[BKL+95] J. Barraquand, L. Kavraki, J.-C. Latombe, T.-Y. Li, R. Motwani, and P. Raghavan. A random sampling scheme for path planning. *Submitted to Intern. Journal of Rob. Research*, 1995.

[HA92] Y. Hwang and N. Ahuja. Gross motion planning—a survey. *ACM Comput. Surv.*, 24(3):219–291, 1992.

[HST94] Th. Horsch, F. Schwarz, and H. Tolle. Motion planning for many degrees of freedom - random reflections at C-space obstacles. In *Proc. IEEE Internat. Conf. on Robotics and Automation*, pages 3318–3323, San Diego, USA, 1994.

[JLT90] P. Jacobs, J.-P. Laumond, and M. Taïx. A complete iterative motion planner for a car-like robot. *Journees Geometrie Algorithmique*, 1990.

[Kav95] L. Kavraki. *Random networks in configuration space for fast path planning.* Ph.D. thesis, Department of Computer Science, Stanford University, Stanford, California, USA, January 1995.

[KKL96] L. Kavraki, M.N. Kolountzakis, and J.-C. Latombe. Analysis of probabilistic roadmaps for path planning. In *IEEE International Conference on Robotics and Automation*, pages 3020–3026, Minneapolis, MN, USA, 1996.

[KL94] L. Kavraki and J.-C. Latombe. Randomized preprocessing of configuration space for fast path planning. In *Proc. IEEE Internat. Conf. on Robotics and Automation*, pages 2138–2145, San Diego, USA, 1994.

[KLMR95] L. Kavraki, J.-C. Latombe, R. Motwani, and P. Raghavan. Randomized query processing in robot path planning. In *Proc. 27th Annual ACM Symp. on Theory of Computing (STOC)*, pages 353–362, Las Vegas, NV, USA, 1995.

[KŠLO95] L. Kavraki, P. Švestka, J.-C. Latombe, and M.H. Overmars. Probabilistic roadmaps for path planning in high dimensional configuration spaces. *To appear in IEEE Trans. Robot. Autom.*, 1995.

[Lat91] J.-C. Latombe. *Robot Motion Planning.* Kluwer Academic Publishers, Boston, USA, 1991.

[LTJ90] J.-P. Laumond, M. Taïx, and P. Jacobs. A motion planner for car-like robots based on a mixed global/local approach. In *IEEE IROS*, July 1990.

[OŠ94] M.H. Overmars and P. Švestka. A probabilistic learning approach to motion planning. In *Proc. The First Workshop on the Algorithmic Foundations of Robotics*, pages 19–37. A. K. Peters, Boston, MA, 1994.

[Ove92] M.H. Overmars. A random approach to motion planning. Technical Report RUU-CS-92-32, Dept. Comput. Sci., Utrecht Univ., Utrecht, the Netherlands, October 1992.

[SL96] S. Sekhavat and J.-P. Laumond. Topological property of trajectories computed from sinusoidal inputs for nonholonomic chained form systems. In *Proc. IEEE Internat. Conf. on Robotics and Automation*, April 1996.

[ŠO95a] P. Švestka and M.H. Overmars. Coordinated motion planning for multiple car-like robots using probabilistic roadmaps. In *Proc. IEEE Internat. Conf. on Robotics and Automation*, pages 1631–1636, Nagoya, Japan, 1995.

[ŠO95b] P. Švestka and M.H. Overmars. Motion planning for car-like robots using a probabilistic learning approach. *To appear in Intern. Journal of Rob. Research*, 1995.

[ŠO95c] P. Švestka and M.H. Overmars. Probabilistic path planning. Technical Report RUU-CS-1995-22, Dept. Comput. Sci., Utrecht Univ., Utrecht, the Netherlands, May 1995.

[ŠO96] P. Švestka and M.H. Overmars. Multi-robot path planning with super-graphs. In *Proc. CESA'96 IMACS Multiconference*, Lille, France, July 1996.

[SŠLO95] S. Sekhavat, P. Švestka, J.-P. Laumond, and M.H. Overmars. Probabilistic path planning for tractor-trailer robots. Technical Report 96007, LAAS-CNRS, Toulouse, France, 1995.

[SŠLO96] S. Sekhavat, P. Švestka, J.-P. Laumond, and M.H. Overmars. Multi-level path planning for nonholonomic robots using semi-holonomic subsystems. In *Proc. The Second Workshop on the Algorithmic Foundations of Robotics*, Toulouse, France, July 1996.

[Sus83] H.J. Sussmann. Lie brackets, real analyticity and geometric control. In R.W. Brockett, R.S. Millman, and H.J. Sussmann, editors, *Differential Geometric Control Theory*. Birkhauser, 1983.

[Sus87] H.J. Sussmann. A general theorem on local controllability. *SIAM Journal on Control and Optimization*, 25(1):158–194, January 1987.

[ŠV95] P. Švestka and J. Vleugels. Exact motion planning for tractor-trailer robots. In *Proc. IEEE Internat. Conf. on Robotics and Automation*, pages 2445–2450, Nagoya, Japan, 1995.

[Šve93] P. Švestka. A probabilistic approach to motion planning for car-like robots. Technical Report RUU-CS-93-18, Dept. Comput. Sci., Utrecht Univ., Utrecht, the Netherlands, April 1993.

[TMS93] D. Tilbury, R. Murray, and S. Sastry. Trajectory generation for the n-trailer problem using goursat normal form. Technical Report UCB/ERL M93/12, Univ. California, Berkeley, CA, USA, February 1993.

[Van94] A. Vandame. Planification de tâches de manipulations par méthodes de recherches aléatoires. master thesis, LAAS, Toulouse, France, June 1994.