

ISOBARIC-SPIN RELATIONSHIPS BETWEEN NUCLEAR SPECTRA

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Abstract: The simple fact that a one-body energy describes the interaction of a nucleon with a closed neutron subshell is used to establish sets of equations connecting the spectra of nuclei which are related by isobaric-spin when described by means of the nuclear shell model. Certain formal questions about isobaric spin are incidentally considered.

The interaction of a neutron or proton with a closed neutron subshell is representable by a one-body energy. Thus if the nucleons outside closed neutron and proton subshells (which we shall call the “active” nucleons) belong to a single configuration ††, and if a certain neutron subshell is empty we can add neutrons to fill it without changing the relative energy levels. Alternatively we can take away a filled neutron subshell without changing the level structure. However, it will often happen that, if we start with a physically realizable state of definite isobaric spin (we assume charge independence), the state which results from the addition or subtraction of the neutron subshell will not have a definite T . Expanding the resultant wave function in states of definite T we have then

$$\Psi_{JT} \rightarrow \phi_J = \sum_{T'} A_{T'} \phi_{JT'}, \quad (1)$$

where Ψ_{JT} is the initial state and the expansion coefficients $A_{T'}$ of the resultant state ϕ_J may also depend on T , etc. The initial and final *relative* energies are now given by

$$E_{JT}^{(i)} = \sum_{T'} (A_{T'})^2 E_{JT'}^{(f)}, \quad (2)$$

and similar relationships will hold for other quantities.

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†† Or more generally for mixed configurations of the active nucleons if a nucleon in the various contributing single-particle orbits has the same interaction energy with the neutron subshell. And of course throughout this paper the words “neutron” and “proton” can be interchanged.

Four cases of interest, involving inequivalent nucleons are illustrated † in figs. 1a . . . 1d. They refer to two subshells of spin j and j' , occupied by n and m nucleons, respectively. In fig. 1a and fig. 1b all the configurations are excited ones. Thus the cases $m = 1$, $n = (2j+1) - 1 \equiv N-1$, will be most important, the excited states being various kinds of single-particle or single-holes states; besides that these values lead to an eq. (1) in which the sum involves only two

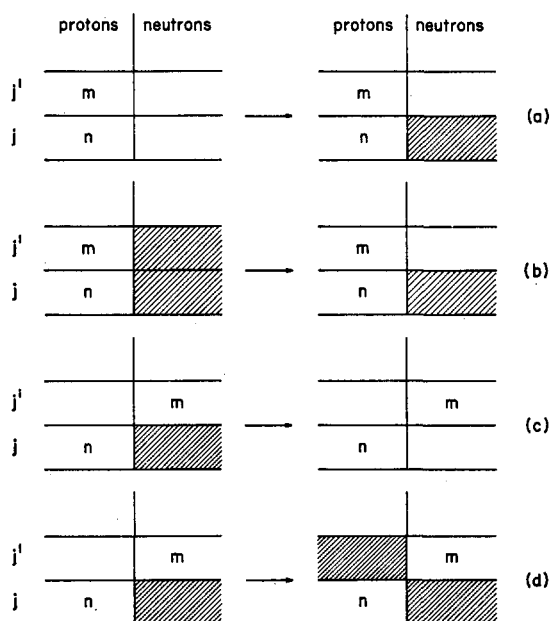


Fig. 1. Each diagram shows the arrangement of neutrons and protons in the two active orbits, all others being empty or completely filled. The cross-hatching denotes filled subshells. Shown on the left are the initial n-p configurations, all with definite $T = |T_z|$, and on the right the n-p configurations, not having definite T , which result when a closed subshell of identical particles is added or taken away.

† If, in the configuration diagrams of fig. 1 it is not possible to transfer a nucleon horizontally (fixed j) in the direction of increasing T_z it follows that T is automatically definite and $T = |T_z|$; for if $T \neq |T_z|$ the states have at least a part with $T > |T_z|$ and this can be produced by starting with a configuration of higher $|T_z|$. Thus T will be definite for an (n-p) configuration with a neutron excess if every orbit which contains protons is filled for neutrons and indeed these (n-p)_T configurations, as we shall call them, are the only ones with a definite T . (If the active neutrons and protons were equivalent, as with a proton and equivalent neutron, certain states would automatically have a definite T which would however vary from state to state.) Thus we see that all the initial configurations shown have $T = |T_z|$. Alternatively we see that the number of T values in an (n-p) configuration is equal to the number of different configurations which we can produce by making simultaneous $j \rightarrow j'$ (vertical) shifts for protons and $j' \rightarrow j$ shifts for neutrons. It is for example because such shifts are possible in the right side of 1a that the closed subshell is *not* inert and the T value not definite. Of course such counting can be replaced by the simple vector coupling of eq. (4).

different T values. In fig. 1c and fig. 1d the initial configuration is a ground-state one and the simplest and physically most interesting cases are $m = 1$ (fig. 1c) and $n = N-1$ (fig. 1d). Our problem now is to compute the expansion coefficients.

If we combine two separate nuclear systems, each with definite T, T_z values, our first reaction is that the resultant function has a T expansion

$$\chi_J = \sum_T C_{T_1 T_2}^{T_1 T_2 T} \chi_{JT}. \quad (3)$$

This is not generally correct, for the simple vector coupling will not produce wave-functions which are antisymmetrized [†].

We therefore must multiply through eq. (3) by the antisymmetrizer $A = \sum (-1)^{P_i} P_i$ and, rewriting it in terms of the normalized antisymmetrized functions, we have

$$\phi_J = \sum_T C_{T_1 T_2}^{T_1 T_2 T} \frac{d_{JT}}{d_J} \phi_{JT}, \quad (4)$$

where $d_J^2 = \langle \chi_J^* A^2 \chi_J \rangle$ and similarly for d_{JT} . Only in the special case when d_{JT} is independent of T (when obviously $d_{JT} = d_J$) will eq. (4) have the simple Clebsch-Gordan form. The one general circumstance in which this occurs is when the diagonal matrix elements of $A^2 = (n_1 + n_2)! A$ depend only on n_1 and n_2 , the number of particles in each group. This will be satisfied when $\langle \chi_{JT}^* P_i \chi_{JT} \rangle = 0$ for every χ_{JT} and for every P_i which exchanges particles between the groups, and this in turn requires that *every single-particle state for one group be orthogonal to every one for the other group*.

The resultant wave functions in all the cases considered above are of this type; we can take the j -nucleons to form one system with $T = -T_z = \frac{1}{2}n$ in fig. 1c and $T = +T_z = \frac{1}{2}(N-n)$ in the other cases ($t_z = \frac{1}{2}$ for a neutron). Then, *assuming for the moment that taking away or adding the neutron subshell does not disturb the manner in which the nucleons in the separate groups are coupled*, the expansion coefficients all have the form of the obvious Clebsch-Gordan coefficient $C_{T_1 T_2}^{T_1 T_2 T}$, where T_1, T_2 are the isobaric spins of the separate j, j' groups, and, except in fig. 1c where the signs are reversed, $T_{z_1} = +T_1$ and $T_{z_2} = -T_2$. We are interested in cases in which one of the groups is a single particle or hole and the configurations can be reached from a ground state by excitation of a single particle. These cases are: $m = 1$ or $n = N-1$ in figs. 1a, 1b; $m = 1$ in fig. 1c; and $n = N-1$ in fig. 1d. Then labelling the

[†] We prefer not to admit the concept of a definite T for a function which is not antisymmetrized. If we did allow such a concept eq. (3) would be correct but not in general pertinent. An illustration of the point involved is that in the coupling of an $s_{\frac{1}{2}}$ neutron to a $J = 0$ pair of $s_{\frac{1}{2}}$ protons the $T = \frac{3}{2}$ state does not exist. Also it is a little simpler to consider the comments which follow eq. (3) in terms of an isobaric-spin formalism and this we temporarily adopt.

isobaric spin of the other group as T_0 ($= T_1$ or T_2) we have

$$(A_{T_<})^2 = \frac{2T_0}{2T_0+1}, \quad (A_{T_>})^2 = \frac{1}{2T_0+1}. \quad (5)$$

With a new notation we now rewrite the energy equations for the cases of special interest. We use $(j^n|j'^m)$ to denote the configuration j^n for protons and j'^m for neutrons, every other subshell being either empty or fully closed (for neutrons *and* protons); $(j^n|j'^m)$ will be the same but with the j -neutron shell filled and similarly for $(j^n|j'^m)$, etc. In contrast to these (n—p) configurations, $(j^n j'^m)$ will denote n j -nucleons and m j' -nucleons *each separately coupled to maximum T* but not necessarily to maximum $|T_z|$. For example the configurations occurring on the left-hand side of figs. 1a, 1b and 1c are respectively $(j^n j'^m|)$, $(j^n j'^m|)$, $(j^n|j'^m)$. The configuration on the right-hand side of fig. 1a is $(j^n j'^m|)$; its states do not have a definite T but can be expanded in terms of the states of the configuration $(j^{N+n} j'^m)$. The symbols $(j^{-n}|)$ and $(j^{-n}|)$ will indicate n holes in the proton j -shell, the neutron j -shell being respectively empty and filled. The symbol (j^{-n}) will indicate n nucleon holes in the j shell (a very different thing). The equations are now †

$$E_J(j^{-r} j'|) = E_J(j^{-r} j'|) = \frac{r}{r+1} E_{JT_<}(j^{-r} j') + \frac{1}{r+1} E_{JT_>}(j^{-r} j'), \quad (6)$$

$$E_J(j^{-1} j'^r|) = E_J(j^{-1} j'^r|) = \frac{r}{r+1} E_{JT_<}(j^{-1} j'^r) + \frac{1}{r+1} E_{JT_>}(j^{-1} j'^r), \quad (7)$$

$$E_J(j^r | j') = \frac{r}{r+1} E_{JT_<}(j^r j') + \frac{1}{r+1} E_{JT_>}(j^r j'), \quad (8)$$

$$E_J(j^{-1} | j'^{-r}) = \frac{r}{r+1} E_{JT_<}(j^{-1} j'^{-r}) + \frac{1}{r+1} E_{JT_>}(j^{-1} j'^{-r}), \quad (9)$$

where $T_< = \frac{1}{2}(r-1)$ and $T_> = \frac{1}{2}(r+1)$. Eqs. (8) and (9) are related by $j \leftrightarrow j'$ together with a (hole \leftrightarrow particle) transformation but since we assume that j' is a higher orbit than j they are physically distinct unless $r = 1$. The right hand sides of eqs. (6) and (8) cannot be related by $r \rightarrow -r$ since $r \leq 2j+1$ and similarly for eqs. (7), (9).

† In these cases we have only two orbits which are neither completely empty nor fully closed (for both neutrons and protons). If we have also on the left-hand side one or more subshells which are filled for neutrons and empty for protons (or vice versa) the energy equations have a slightly different form but still follow immediately from eqs. (2), (5). The equations have an obvious structure in the isobaric-spin formalism. In the n-p formalism consider for example $n = m = 1$ in fig. 1c. The effect of the n-p interaction when the closed subshell is absent is to spread apart the frequencies of the two otherwise-degenerate oscillation modes ($\omega \rightarrow \omega \pm \delta$) but the average value is unchanged and has the value ω . On the other hand when the subshell is filled there is only one oscillation mode and this once again has frequency ω . Similarly for the more complicated cases, always provided that the isobaric spin admixture of the r -group can be ignored, as discussed below.

Eqs. (6)—(9), which are formally correct equations connecting certain shell-model diagonal matrix elements, will apply to relative energy levels of nuclei provided that certain conditions are satisfied. An obvious one is that the configuration admixtures in the states should not be too large. Besides this the r -group should be coupled to maximum isobaric spin (in the states on the right-hand side of the equations) and, in the various terms of an equation, should be coupled to the same angular momentum J and seniority or to the same linear combination of these.

For $r = 1$ the latter questions do not of course arise and there are several cases in which we would expect the equations to apply with some precision. In this case eqs. (6) and (7) are equivalent and by using the (hole \leftrightarrow particle) relationship we see that the terms appearing in eqs. (8) and (9) are equal. With $j \equiv d_{\frac{3}{2}}$ and $j' \equiv f_{\frac{7}{2}}$ the $r = 1$ equations then read

$$E_J(S^{36}) = \frac{1}{2}\{E_{J,T=0}(\text{Ca}^{40}) + E_{J,T=1}(\text{Ca}^{40})\} = E_J(\text{Ca}^{48}), \quad (10)$$

$$\begin{aligned} E_J(\text{Cl}^{38}) &= \frac{1}{2}\{E_{J,T=0}(\text{Cl}^{34}) + E_{J,T=1}(\text{Cl}^{34})\} \\ &= E_J(\text{K}^{46}) = \frac{1}{2}\{E_{J,T=0}(\text{Co}^{54}) + E_{J,T=1}(\text{Co}^{54})\}, \end{aligned} \quad (11)$$

where in all cases except Cl^{38} and K^{46} the states involved belong to excited configurations. The $T = 1$ levels of Ca^{40} could be located instead in K^{40} and similarly for the other cases.

For $r \neq 1$ the r -group isobaric-spin and angular-momentum admixtures must be considered. For the T -admixture the question which arises is as follows; if we have an equivalent group of particles coupled to maximum isobaric spin T_0 , is then T_0 preserved when we couple an inequivalent particle or hole to the group to form a resultant $T = T_0 - \frac{1}{2}$? The available information is insufficient to answer this question. If $r = 4n + 2$, states of different T_0 will in some cases be close together and then large admixtures might occur; this is not necessarily so because the interaction with the added nucleon could in some cases widely separate the states of different T_0 . If $r \neq 4n + 2$, states of different T_0 should be well separated but, on the other hand, a Majorana interaction in particular may give a large matrix element non-diagonal in T_0 . It seems unprofitable to speculate about this at present. The energy equations with $r = 2j$ in eqs. (6), (8) and $r = 2j'$ in the others could inform us about this question; for in these cases, if T_0 is good, there is no possibility of angular-momentum admixtures. An example involving $d_{\frac{3}{2}}$ $f_{\frac{7}{2}}$ interactions is

$$E_J(S^{34}) = \frac{3}{4}E_{J,T=1}(A^{38}) + \frac{1}{4}E_{J,T=2}(A^{38}). \quad (12)$$

Let us now consider cases in which, though $r \neq 1$, the T_0 admixtures are small enough to be ignored. When we have an excitable odd group of identical particles or holes (r odd $\neq 1$, $\neq 2j$ or $2j'$) the state of the r -group, in low-lying states of the system, should usually be the seniority one $J = j$ (or j') state

though, since the r -group energy spacings are small, there may be appreciable J -admixture[†]. If the admixtures were the same on both sides of an energy equation, the equation would still be valid but we cannot expect this to be so. However, if we construct the energy matrices in a representation which labels the excitation of the r -group we note that the energy equations (6)–(9) apply to the entire matrices (since eq. (2) applies equally to diagonal and off-diagonal elements). The energies are not linear functions of the matrix elements, and therefore we do indeed expect a departure from the energy equations; this will often be small since the $T_>$ matrix will be combined with the $T_<$ matrix with a relative coefficient which is never larger than $\frac{1}{2}$ and often much smaller; it often represents therefore a simple correction to the $T_<$ matrix. In these cases of course the E_J spectrum will be quite similar to the $E_{JT_<}$ spectrum. Moreover, since the interaction of a closed subshell with a group of particles may depend on their configuration but does not otherwise depend on their state, we see that the equations may be applied to the complete spectrum arising from a given $(n-p)$ configuration. For example with $(d_{\frac{5}{2}}^2|p_{\frac{3}{2}})$ we have states $(\frac{3}{2})$, $(\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2})$, $(\frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \frac{11}{2})$ arising respectively from the $J_0 = 0, 2, 4$ coupling of the $d_{\frac{5}{2}}$ particles. The equations apply separately for the states of fixed J_0 or for the entire spectrum.

Very few data are available to test the above equations. For $A = 14$ reasonable assignments¹⁾ for the $(p_{\frac{1}{2}} s_{\frac{1}{2}})$ levels are 4.91, 5.69, 8.70, 8.06 MeV for $JT = 00, 10, 01, 11$ respectively and these, by eq. (8) with $r = 1$, give 6.80, 6.87 MeV for the $J = 0, 1$ levels of $(\hat{p}_{\frac{1}{2}}|s_{\frac{1}{2}})$. The +70 keV splitting here may be compared with the experimental value +280 keV observed²⁾ in N^{16} . For the $(p_{\frac{1}{2}} d_{\frac{5}{2}})$ levels we similarly have 5.10, 5.83, 9.50, 8.90 MeV for $JT = 20, 30, 21, 31$ which give 7.30 and 7.36 MeV for the $J = 2, 3$ levels of $(\hat{p}_{\frac{1}{2}}|d_{\frac{5}{2}})$, the 60 keV splitting here to be compared with the experimental value 295 keV. In view of the crudity of the jj assumption in $A = 14$ these agreements seem satisfactory but they really tell us nothing new since the $A = 14, 16$ case is quite trivial and has already been considered by many authors in a different manner. We observe incidentally that the binding to the $p_{\frac{1}{2}}$ neutron subshell is about 600 keV larger for a $d_{\frac{5}{2}}$ neutron than for an $s_{\frac{1}{2}}$ neutron. For heavier nuclei there appears to be nowhere enough data for an application of the equations. The $(\hat{d}_{\frac{5}{2}}|f_{\frac{7}{2}})$ levels are known and a study of excited states in that region of the periodic table ($A \approx 40$) would seem worthwhile. Of course the possible complications which arise when we deal with an excitable r -group should be borne in mind and also the fact that "size" effects (and other such effects not easily taken into account in a model) may be significant when we consider together nuclei of quite different atomic number; conversely the energy equations might tell us something about these effects.

[†] No such admixtures if the admixing interaction is of odd rank. Note however that the $n-p$ interaction which determines the $d_{\frac{5}{2}} f_{\frac{7}{2}}$ spectrum of Cl^{36} is predominantly of quadrupole nature.

The energy equations can of course be derived in many other ways [†] and much more detailed ones can in some cases be given. There are also some cases of interest involving *equivalent* active nucleons outside closed subshells. It is easy to see for example that the excited states of Cl^{38} which arise from a single $d_{3/2} \rightarrow f_{7/2}$ excitation should include a $T = 2$ and a $T = 3$ spectrum, each identical with the even- J $(f_{7/2})^2$ spectrum of Ca^{42} , and a $T = 2$ odd- J spectrum identical with the odd- J $(f_{7/2})^2$ spectrum. However, if the mutual isobaric-spin polarization of an $(f_{7/2})^2$ pair and a closed $d_{3/2}$ neutron subshell were large the $T = 2$ states would be admixed with states in which the $(d_{3/2})^4$ core would involve both neutrons and protons and the $T = 2$ spectrum would correspondingly be changed. These effects could be easily calculated if the entire $(d_{3/2} f_{7/2})$ two-particle spectrum were known.

Finally if we consider the effects due to a smaller $J = 0$ group than a closed subshell, such as for example a $J = 0$ pair, other results can be derived which might be pertinent to a study of generalized symplectic symmetry in a heavier nucleus; these are particularly simple if the two-body interaction is of odd rank since such an interaction cannot excite a $J = 0$ pair (this is of course the physical reason for the connection between good seniority and odd-rank interactions); but these questions we do not consider here.

[†] One instructive way is to eliminate the explicit $\mathbf{t}_i \cdot \mathbf{t}_j$ dependence of the two-particle interaction (which may always be written as $A + B \mathbf{t}_i \cdot \mathbf{t}_j$) by observing that a configuration such as $(j^n | j^N)$, where $N = (2j+1)$ is the hole \leftrightarrow particle complement of the configuration (j^{N-n}) for identical particles only. In this way one learns that, even with multiply-excited configurations (the sum in eq. (1) having then more than two terms), energy equations involving only two T values are always possible, this coming about because only two tensorial forms are possible for the isobaric-spin dependence of the two body interaction.

References

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