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THE EXCHANGE MAGNETIC FORM FACTOR OF ^{3}He AND ^{3}H

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Abstract: The contribution to the exchange magnetic form factors of ^{3}He and ^{3}H from two of the important diagrams is calculated for realistic wave functions such as for the Reid potential. It is found that the exchange magnetic moment depends little on the type of wave function used. The q^2 dependence for small q^2, where q is the photon momentum, shows a spatial extension of the exchange magnetic moment density, due to these terms, that is considerably less than that of the nucleons. This will make the square radius of the total density decrease with a few percent.

1. Introduction

The magnetic moments of the three-nucleon states ^{3}He and ^{3}H, neglecting exchange effects, can be determined if the percentages of S, S' and D-states present in the ground state are known. Experimentally one has $P(\mathrm{S}') = (0.5\text{–}4)\,\%$ and $P(\mathrm{D}) = (4\text{–}10)\,\%$. As is well-known with the above range for $P(\mathrm{S}')$ and $P(\mathrm{D})$ there still remains a discrepancy of 0.3–0.4 nuclear magnetons between the experimental value of the magnetic moments and the theoretically predicted values.

The differences generally are ascribed to the so-called exchange effects. Already Villars [1]) estimated these effects but used in the calculations unrealistic parameters. Since then, several authors [2, 3]) dealt with the problem using different types of three-nucleon wave functions. In all cases, at least within the restriction of two-body forces, the most important contributions come from the effects already discussed by Villars. They are given by the diagrams where the photon interacts with a pion which is exchanged between two nucleons and the so-called contact interaction.

In view of the recent progress made in the solution of the three-body problem it is interesting to study the effect of the use of realistic wave functions on the contributions of these diagrams. In this paper we will be concerned with the calculation of these diagrams using three-nucleon wave functions which have been determined by solving the Faddeev equations for a given set of two-body local potentials. Since the iso-scalar contributions of these diagrams are small, we will restrict ourselves here to the study of the iso-vector part of the exchange current. The potentials which have been used are from ref. [4]) and the more realistic soft-core potential of Reid [5]). In sect. 2 an appropriate expression is derived for the magnetic moment form factor in terms of the three-nucleon wave function. This expression is computed in sect. 3 for the two relevant diagrams described above. Finally the results for the exchange magnetic moment are given in the last section, together with the q^2 dependence of the exchange magnetic form factor and the resulting magnetic radii.

2. The magnetic form factor

For a spin-$\frac{1}{2}$ object such as the three-nucleon ground state, the interaction with an electromagnetic field can in general be described by a charge- and magnetic form factor if we suppose that it satisfies the Dirac equation. The first problem to be solved is how to determine the contribution of a given Feynman diagram to the magnetic form factor. Such a Feynman diagram in general gives rise to an additional form factor in the matrix element of the electromagnetic current, since a single diagram does not have to be gauge invariant, i.e.

$$\tilde{J}_\mu(q) = ie\bar{u}^{(s')}(p')\left[\gamma_\mu F_1(q^2) - \frac{\sigma_{\mu\nu}q_\nu}{2m}F_2(q^2) + \frac{q_\mu}{2m}F_3(q^2)\right]u^{(s)}(p), \tag{1}$$

where $q = p'-p$ and m is the mass of the spin-$\frac{1}{2}$ object. The magnetic form factor we are interested in, is simply given by

$$G_{\text{mag}}(q^2) = F_1(q^2)+F_2(q^2). \tag{2}$$

Expressing the Dirac matrices in terms of the Pauli matrices and taking the non-relativistic limit, the space components of the current become

$$\tilde{\boldsymbol{J}}(q) = ie\chi_{s'}^+\left[-i(\boldsymbol{p}+\boldsymbol{p}')\frac{F_1(q^2)}{2m} - (\boldsymbol{q}\wedge\boldsymbol{\sigma})\frac{G_{\text{mag}}(q^2)}{2m} + \boldsymbol{q}\frac{F_3(q^2)}{2m}\right]\chi_s. \tag{3}$$

We now take the initial and final states with $s = s' = \frac{1}{2}$. Choosing the particular Lorentz frame with $\boldsymbol{p}+\boldsymbol{p}' = 0$ we obtain for the expression

$$-\frac{i}{2}[\nabla_q\wedge\tilde{\boldsymbol{J}}(q)]_z = \frac{e}{2m}[G_{\text{mag}}(q^2)+(q_x^2+q_y^2)G'_{\text{mag}}(q^2)]. \tag{4}$$

This procedure gives a direct way to find the contribution of a Feynman diagram to the magnetic form factor within the non-relativistic limit.

For the case of the ground state of three nucleons the matrix element of the electromagnetic current operator $\boldsymbol{j}_{\text{em}}$ is given by

$$\tilde{\boldsymbol{J}}(q) = \langle\psi_{\text{bound}}|\boldsymbol{j}_{\text{em}}(q)|\psi_{\text{bound}}\rangle, \tag{5}$$

where ψ_{bound} is the three-nucleon wave function. For simplicity we shall assume here that the bound state is only given by the totally symmetric S-state, i.e., in the c.m. system of the three particles we have in the momentum representation

$$\langle\boldsymbol{p}_1\boldsymbol{q}_1|\psi_{\text{bound}}\rangle = W^{\text{A}}\varphi_0(p_1, q_1), \tag{6}$$

where W^{A} is the totally anti-symmetric spin-isospin function and p_1, q_1 are the relative momenta related to the momenta k_i of the particles by

$$p_1 = \frac{1}{2\sqrt{M}}(k_2 - k_3),$$
$$q_1 = \frac{1}{2\sqrt{3M}}(k_2 + k_3 - 2k_1), \tag{7}$$

where M is the nucleon mass. Inserting eq. (6) into eq. (5) we obtain

$$\tilde{J}(q) = \int \mathrm{d}p_1\,\mathrm{d}q_1\,\mathrm{d}p_1'\,\mathrm{d}q_1'\,\varphi_0(p_1, q_1)\varphi_0(p_1', q_1')\langle k_1\,k_2\,k_3\,W^{\mathrm{A}}|j_{\mathrm{em}}(q)|k_1'\,k_2'\,k_3'\,W^{\mathrm{A}}\rangle, \tag{8}$$

with

$$k_1 = -2\sqrt{\tfrac{1}{3}M}q_1 + \tfrac{1}{3}P,$$
$$k_2 = \sqrt{M}p_1 + \sqrt{\tfrac{1}{3}M}q_1 + \tfrac{1}{3}P, \tag{9}$$
$$k_3 = -\sqrt{M}p_1 + \sqrt{\tfrac{1}{3}M}q_1 + \tfrac{1}{3}P,$$

and where P is the c.m. momentum of the three particles.

3. Contributions of the diagrams

We are now in a position to compute the contributions of the diagrams described in the introduction. They are shown in fig. 1.

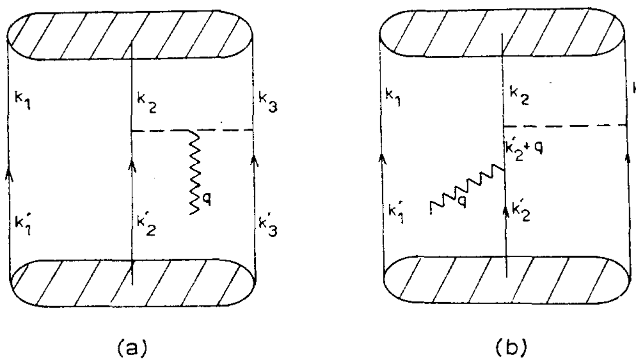

Fig. 1. The pionic-exchange current diagram (a) and the exchange current diagram (b) where the photon interacts with one of the nucleons. The blobs in both diagrams denote that the three nucleons form a bound state. k_i and k_i' are the nucleon momenta and q is the photon momentum.

Following Chemtob and Rho [3]) we consider a pseudo-scalar coupling theory of pions and nucleons. In contrast to the case of pseudo-vector coupling theory where the contact term occurs in the interaction Lagrangian, this term is found here by taking the negative frequency part of the nucleon propagator in fig. 1b.

After summing over all possible permutations, the contribution of the two types of diagrams may be written as

$$G_{\mathrm{mag}}^{\mathrm{exch}}(q^2) = \frac{g^2\sqrt{M}}{(2\pi)^3}[F_\pi(q^2)I(q^2) + F_{\mathrm{V}}(q^2)K(q^2)], \tag{10}$$

where g is the πN coupling constant, $F_\pi(q^2)$ is the pion charge form factor and $F_V(q^2)$ is the isovector charge form factor of the nucleon. The first term in eq. (10) is due to the fig. 1a. For $I(q^2)$ one readily finds

$$I(q^2) = \frac{1}{q^2}\int \mathrm{d}\boldsymbol{p}_1\,\mathrm{d}\boldsymbol{p}_1'\,\mathrm{d}\boldsymbol{q}_1\,\varphi_0(p_1, q_1)\varphi_0\left(p_1', \left|\boldsymbol{q}_1 - \frac{\boldsymbol{q}}{2\sqrt{3M}}\right|\right) \times \frac{(\boldsymbol{\Delta}\cdot\boldsymbol{q})^2 - \Delta^2 q^2}{[(\boldsymbol{\Delta}+\frac{1}{2}\boldsymbol{q})^2+\mu^2][(\boldsymbol{\Delta}-\frac{1}{2}\boldsymbol{q})^2+\mu^2]}, \tag{11}$$

with $\boldsymbol{\Delta} = (\boldsymbol{p}_1 - \boldsymbol{p}_1')\sqrt{M}$ and μ the pion mass. The second term in eq. (10) is due to the negative frequency part of fig. 1b, i.e., the nucleon propagator $[i\gamma(q+k_2)+M]^{-1}$ is replaced by

$$\frac{-1}{2\sqrt{|\boldsymbol{q}+\boldsymbol{k}_2|^2+M^2}}\,\frac{i\boldsymbol{\gamma}\cdot(\boldsymbol{q}+\boldsymbol{k}_2)+\gamma_4\sqrt{|\boldsymbol{q}+\boldsymbol{k}_2|^2+M^2}-M}{\sqrt{|q+k_2|^2+M^2}+q_0+k_{20}}.$$

As already noted by Chemtob and Rho the positive frequency part should be dropped, since it has already been taken into account in the one-particle matrix element of the magnetic form factor. In this way one gets

$$K(q^2) = \frac{1}{q^2}\int \mathrm{d}\boldsymbol{p}_1\,\mathrm{d}\boldsymbol{p}_1'\,\mathrm{d}\boldsymbol{q}_1\,\varphi_0(p_1, q_1)\varphi_0\left(p_1', \left|\boldsymbol{q}_1 - \frac{\boldsymbol{q}}{2\sqrt{3M}}\right|\right)\frac{q^2 - 2\boldsymbol{\Delta}\cdot\boldsymbol{q}}{(\boldsymbol{\Delta}-\frac{1}{2}\boldsymbol{q})^2+\mu^2}. \tag{12}$$

Taking $q^2 = 0$ the eqs. (10)–(12) reduce to the result of Chemtob *et al.* and of Padgett *et al.* and Villars for the magnetic moment due to these diagrams.

4. Results and concluding remarks

In the calculation of the exchange magnetic form factor we are dealing with the nine-dimensional integrals $I(q^2)$ and $K(q^2)$, which however can be simplified considerably. For the details we refer to the appendix. For the nucleon form factor $F_V(q^2)$ appearing in eq. (10) we have taken the analytic form given by Janssens *et al.* [6]) and the vector dominance model has been used for the pion form factor, i.e.

$$F_\pi(q^2) = \frac{m_\rho^2}{m_\rho^2+q^2}, \tag{13}$$

where m_ρ is the mass of the ρ-meson. For the various physical parameters we have used

$$M = 939\text{ MeV},\qquad \mu = 139.6\text{ MeV},\qquad m_\rho = 765\text{ MeV},\qquad g = 13.5.$$

The calculation was done for several wave functions. In the first place we have taken the solution of the Faddeev equations for the two-body soft-core potential of Reid. Also the central s-wave potentials of ref. [4]) have been considered. These consist of a singlet potential with core and a triplet one with and without a core respectively.

For details of how the wave functions are determined from the two-body interactions we refer the reader to refs. [4, 9]). For comparison we have also used the analytic wave functions of the Gauss and Irving type, which have been considered extensively by Schiff [7]). They are given respectively by

$$\varphi_{\text{Gauss}}(p_1, q_1) = N \exp\left[-\frac{M}{3\alpha^2}(p_1^2+q_1^2)\right], \tag{14}$$

$$\varphi_{\text{Irving}}(p_1, q_1) = N'\left[1+\frac{8M}{3\alpha^2}(p_1^2+q_1^2)\right]^{-\frac{7}{2}}, \tag{15}$$

where N and N' are normalization factors.

Writing the form factor for small q^2 as

$$G_{\text{mag}}^{\text{ex}}(q^2) = G_{\text{mag}}^{\text{ex}}(0)[1-\tfrac{1}{6}q^2\langle r^2\rangle_{\text{mag}}^{\text{ex}}], \tag{16}$$

the square radius of the magnetic moment distribution has also been computed. The results are summarized in table 1 and in fig. 2.

TABLE 1

The exchange magnetic moment in nuclear magnetons and magnetic square radii in fm^2 for various wave functions used

Wave function	$G_{\text{mag}}^{\text{ex}}(0)$ (n.m.)	$\langle r^2\rangle_{\text{mag}}^{\text{ex}}$ (fm^2)	$G^{(1)}(0)$ (n.m.)	$G^{(2)}(0)$ (n.m.)
Reid soft-core potential	0.15	1.16	−0.12	0.27
potential I-III from ref. [4])	0.16	0.99	−0.14	0.30
potential I-IV from ref. [4]) (soft core in singlet only)	0.14	0.84	−0.21	0.35
Gauss with $\alpha = 0.348$	0.16	0.79	−0.17	0.33
Irving with $\alpha = 1.27$	0.14	0.91	−0.18	0.32

Also the separate contributions $G^{(1)}$ and $G^{(2)}$ to the exchange magnetic moment from the figs. 1a and 1b respectively are given.

From table 1 we see that although the separate contributions of figs. 1a, 1b are very sensitive to the choice of the wave function, due to their opposite relative sign, the total contribution depends little on the type of wave function used. Also it is clear that these diagrams alone will not account for the total difference of 0.3–0.4 nuclear magnetons which one expects from exchange effects. Estimates have been made of contributions from heavy meson exchange and the presence of S′ and D-states in the three-nucleon wave functions. They turn out to be small. Other possible causes for the difference are three-body and normalization effects. Some recent attempts [2, 3]) have been made to study these effects, but since it is not clear how reliable they are we have not considered them here.

In conclusion, the value for $\langle r^2 \rangle_{\text{mag}}^{\text{exch}}$ is very small † compared to the square radius computed from the one-particle matrix element of the magnetic form factor [9]. In the latter case the square radius is about 4 fm^2. The exchange effect, calculated here,

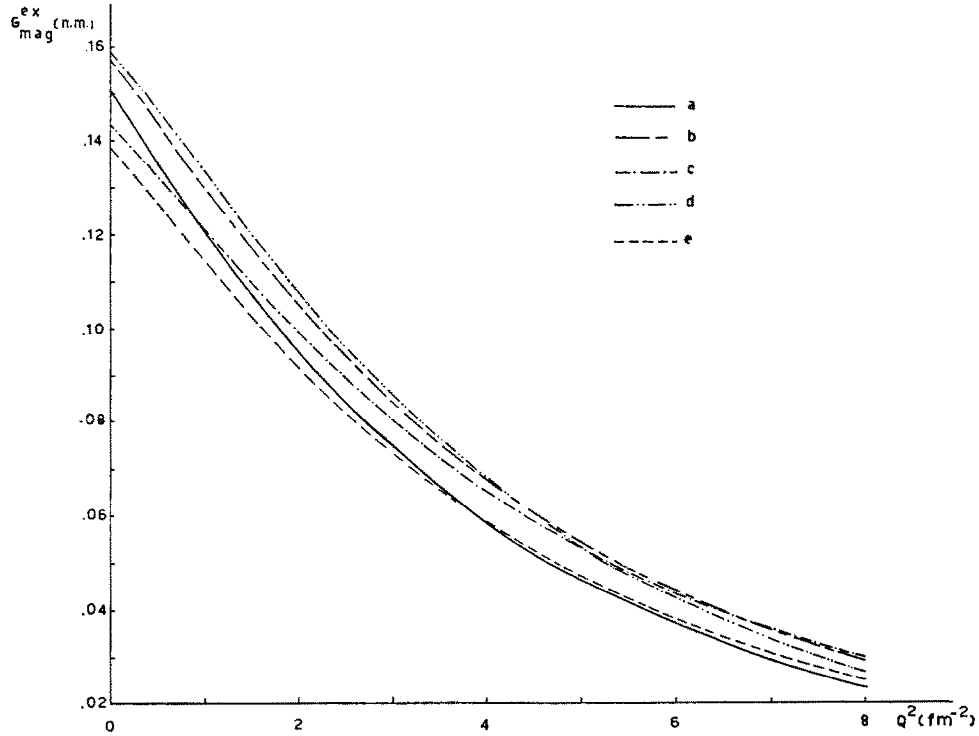

Fig. 2. The exchange magnetic form factor in nuclear magnetons as a function of momentum transfer in fm^{-2} for various wave functions. Curves (a), (b) and (c) are for the wave functions found from the Faddeev equations for the Reid potential, the potentials I-III and I-IV from ref. [4]). Curves (d) and (e) are for the Gauss and Irving wave function.

will decrease the values of $\langle r^2 \rangle$ given in ref. [9]) by only a few percent and hence will not affect the conclusions reached there about the magnetic radii of ^{3}He and ^{3}H.

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† A value of $\langle r^2 \rangle_{\text{mag}}^{\text{ex}} = 1.8$ fm^2 is mentioned in ref. [8]), which is larger than our result. This may be due to the wave function used.

Appendix

The functions $I(q^2)$ and $K(q^2)$ can be reduced to more manageable forms. Introducing the angle θ between the vectors $\boldsymbol{\Delta}$ and $\boldsymbol{q}$ we may write

$$I(q^2) = -\int \mathrm{d}\boldsymbol{p}_1\, \mathrm{d}\boldsymbol{p}_1'\, \mathrm{d}\boldsymbol{q}_1\, \varphi_0(p_1, q_1)\varphi_0\left(p_1', \left|\boldsymbol{q}_1 - \frac{\boldsymbol{q}}{2\sqrt{3M}}\right|\right) \times \frac{\Delta^2 \sin^2\theta}{(\Delta^2 + \frac{1}{4}q^2 + \mu^2)^2 - \Delta^2 q^2 \cos^2\theta}. \tag{A.1}$$

Defining the mixed Fourier transforms:

$$X(R, q_1) = \int \mathrm{d}\boldsymbol{p}_1 \exp(-i\boldsymbol{R}\cdot\boldsymbol{p}_1\sqrt{M})\varphi_0(p_1, q_1), \tag{A.2}$$

$$Y(\boldsymbol{R}, \boldsymbol{q}) = \int \mathrm{d}\boldsymbol{\Delta} \exp(i\boldsymbol{\Delta}\cdot\boldsymbol{R})\Delta^2 \sin^2\theta[(\Delta^2 + \tfrac{1}{4}q^2 + \mu^2)^2 - \Delta^2 q^2 \cos^2\theta]^{-1}, \tag{A.3}$$

the integral reduces to

$$I(q^2) = -\frac{1}{(2\pi)^3}\int \mathrm{d}\boldsymbol{q}_1 \int_0^\infty \mathrm{d}R\, R^2 X(R, q_1) X\left(R, \left|\boldsymbol{q}_1 - \frac{\boldsymbol{q}}{2\sqrt{3M}}\right|\right)\int \mathrm{d}\Omega_{\boldsymbol{R}}\, Y(\boldsymbol{R}, \boldsymbol{q}). \tag{A.4}$$

The integration over $\Omega_{\boldsymbol{R}}$ can be simplified to

$$\int \mathrm{d}\Omega_R\, Y(\boldsymbol{R}, \boldsymbol{q}) = 8\pi^2 \int_0^\infty \Delta^2 \mathrm{d}\Delta\, j_0(\Delta R)\left[\frac{2}{q^2} - \frac{(\Delta^2 + \frac{1}{4}q^2 + \mu^2)^2 - q^2\Delta^2}{q^3 \Delta(\Delta^2 + \frac{1}{4}q^2 + \mu^2)} \ln \frac{(\Delta + \frac{1}{2}q)^2 + \mu^2}{(\Delta - \frac{1}{2}q)^2 - \mu^2}\right] \equiv \frac{8\pi^2}{R} Z(R, q). \tag{A.5}$$

Finally $I(q^2)$ reduces to the form

$$I(q^2) = -2\int_0^\infty R\, \mathrm{d}R\, Z(R, q)\int_0^\infty q_1^2 \mathrm{d}q_1\, X(R, q_1)\int_{-1}^1 \mathrm{d}z \times X\left(R, \left(q_1^2 + \frac{q^2}{12M} - \frac{qq_1 z}{\sqrt{3M}}\right)^{\frac{1}{2}}\right). \tag{A.6}$$

Following the same procedure as for $I(q^2)$ one finds for the other function $K(q^2)$ in eq. (11):

$$K(q^2) = -\int \mathrm{d}\boldsymbol{q}_1 \int_0^\infty R^2 \mathrm{d}R\, X(R, q_1) X\left(R, \left|\boldsymbol{q}_1 - \frac{\boldsymbol{q}}{2\sqrt{3M}}\right|\right) W(R, q) \tag{A.7}$$

with

$$W(R, q) = \frac{\mathrm{e}^{-\mu R}}{R}\left(\mu + \frac{1}{R}\right)\frac{1}{q} j_1(\tfrac{1}{2}Rq), \tag{A.8}$$

where $j_1(x)$ is the spherical Bessel function. Hence $K(q^2)$ reduces to a three-dimensional integral.

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